National University of Singapore

Department of Mathematics

Semester 1 (2006/2007) **MA3110 Mathematical Analysis II** Tutorial 6

1. Suppose $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are functions such that $f(x) = x\, g(x)$ for all x in $\mathbf{R}$. Suppose g is continuous at 0. Prove that f is differentiable at 0 and find $f'(0)$ in terms of g..

2. Suppose $g : \mathbf{R} \to \mathbf{R}$ is a twice differentiable function with $g(0) = g'(0) = 0$ and $g''(0) = 31$. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \begin{cases} \frac{g(x)}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

   Prove that f is differentiable at $x = 0$ and find $f'(0)$. [Hint: Use L' Hôpital's Rule.]

3. Suppose $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Prove that if

   $m \leq f'(x) \leq M$ for all x in (a, b), then

   $$m(b - a) + f(a) \leq f(b) \leq f(a) + M(b - a).$$

4. (i) If $f(x) = x^3 + 1$, find $(f^{-1})'(y)$, $y \neq 1$.

   (ii) $f(x) = (x-1)^3$, find $(f^{-1})'(y)$, $y \neq 0$.

5. A point x_0 in D is said to be an isolated point of D provided that there is a $\delta > 0$ such that the only point of D in the interval $(x_0 - \delta, x_0 + \delta)$ is x_0 itself. Prove that a point x_0 is either an isolated point or a limit point of D.

6. Show that if $f : \mathbf{R} \to \mathbf{R}$ is differentiable at $x_0 = 1$

   (a) $\lim_{t \to 1} \frac{f(\sqrt{t}) - f(1)}{\sqrt{t} - 1} = f'(1)$ (b) $\lim_{x \to 1} \frac{f(x^2) - f(1)}{x^2 - 1} = f'(1)$

   (c) $\lim_{x \to 1} \frac{f(x^2) - f(1)}{x - 1} = 2f'(1)$ (d) $\lim_{x \to 1} \frac{f(x^3) - f(1)}{x - 1} = 3f'(1)$

7. Suppose that the function $f : \mathbf{R} \to \mathbf{R}$ is differentiable at a in $\mathbf{R}$. Prove that

   $$\lim_{x \to a} \frac{x f(a) - a f(x)}{x - a} = f(a) - a f'(a)$$

8. Suppose that the function $f : \mathbf{R} \to \mathbf{R}$ is differentiable at 0. Prove that

   $$\lim_{x \to 0} \frac{f(x^2) - f(0)}{x} = 0.$$

9. Suppose I is a neighbourhood of x_0. Suppose $f : I \to \mathbf{R}$ is a continuous, strictly monotone function differentiable at x_0. Assume that $f'(x_0) = 0$. Use the characteristic property of inverses,

   $$f^{-1}(f(x)) = x \quad \text{for } x \text{ in } I$$

   and the Chain Rule to prove that the inverse function $f^{-1} : f(I) \to \mathbf{R}$ is not differentiable at $f(x_0)$. Thus the assumption in Theorem 34 Chapter 4 is necessary.