Department of Mathematics

Semester 1 (2006/2007) **MA3110 Mathematical Analysis II** Tutorial 5

1. Suppose that the function $f: \mathbf{R} \to \mathbf{R}$ satisfies $f(x+y) = f(x) + f(y)$ for all x and y in **R** and that f is continuous at some point a in **R**. Prove that.
   (a) f is continuous everywhere and
   (b) there is a constant C such that $f(x) = Cx$ for all x in **R**.

2. Prove that for any constant b, $x^3 - 3x + b = 0$ has at most one root in [−1. 1].

3. Suppose that the function $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ are continuous, $f^2 = g^2$ and that $f(x) \neq 0$ for all x in **R**. Prove that either $f(x) = g(x)$ for all x or else $f(x) = -g(x)$ for all x.

4. Let the function $f: [a, b] \to \mathbf{R}$ be continuous and injective and such that $f(a) < f(b)$. For any c in (a, b), prove that $f(a) < f(c) < f(b)$.

5. Assuming that temperature varies continuously along the equator of the earth, prove that there are at any time antipodal points on the equator with the same temperature. [Hint: Let f be a continuous function on $[0, 2\pi]$ such that $f(0) = f(2\pi)$. Define g on $[0, \pi]$ by $g(x) = f(x) - f(x+\pi)$]

Definition.

1. We write $\lim_{x\to+\infty} f(x) = L \in \mathbf{R}$ if, given $\varepsilon > 0$, there exists $K > 0$ such that $x > K \Rightarrow |f(x) - L| < \varepsilon$.
2. We write $\lim_{x\to-\infty} f(x) = L$ if, given $\varepsilon > 0$, there exists $K < 0$ such that $x < K \Rightarrow |f(x) - L| < \varepsilon$

6. Suppose $f: \mathbf{R} \to \mathbf{R}$ is a continuous function with $f(x) > 0$ for all x in **R**. Suppose $\lim_{x\to+\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$. Prove that there is a point c in **R** such that $f(c) \geq f(x)$ for all x in **R**.

7. Suppose $f: \mathbf{R} \to \mathbf{R}$ is a function such that $f'(a)$ exists. Determine which of the following statements are true. Justify your answer.
   (i) $f'(a) = \lim_{h\to a} \dfrac{f(h) - f(a)}{h - a}$; (ii) $f'(a) = \lim_{h\to 0} \dfrac{f(a+2h) - f(a)}{h}$
   (iii)) $f'(a) = \lim_{h\to 0} \dfrac{f(a) - f(a-h)}{h}$; (iv) $f'(a) = \lim_{h\to 0^+} \dfrac{f(a+h) - f(a-h)}{2h}$

8. Suppose that $f: \mathbf{R} \to \mathbf{R}$ is a function such that $f(x+y) = f(x) f(y)$ for all x and y in **R.** If $f(0) = 1$ and $f'(0)$ exists, prove that $f'(x) = f'(0) f(x)$ for all x in **R**.