National University of Singapore

Department of Mathematics

Semester 1 (2006/2007) **MA3110 Mathematical Analysis II** Tutorial 4

1. For each of the following statements, determine whether it is true or false and justify your answer.
   (a) If the function $f+g:\mathbf{R}\to\mathbf{R}$ is continuous, then the function $f:\mathbf{R}\to\mathbf{R}$ and $g:\mathbf{R}\to\mathbf{R}$ also are continuous.
   (b) If the function $f^{\,2}:\mathbf{R}\to\mathbf{R}$ is continuous, then so is the function $f:\mathbf{R}\to\mathbf{R}$.
   (c) If the function $f+g:\mathbf{R}\to\mathbf{R}$ and $g:\mathbf{R}\to\mathbf{R}$ are continuous, then so is the function $f:\mathbf{R}\to\mathbf{R}$.
   (d) Every function $f:\mathbf{N}\to\mathbf{R}$ is continuous **,** where **N** is the set of natural numbers.

2. Suppose that the function $f:[0, 1]\to\mathbf{R}$ is continuous and that $f(x)\geq 3$ for $0\leq x<1$.
   Show that $f(1)\geq 3$.

3. Suppose that the function $f:[0, 1]\to\mathbf{R}$ is continuous, $f(0)>0$, $f(1)=0$. Prove that there is a number x_0 in (0, 1] such that $f(x_0)=0$ and $f(x)>0$ for all $0\leq x<x_0$.

4. Prove that there is a solution of the equation
$$x^{179}+\frac{163}{1+x^2+\sin^2(x)}=119$$

5. Suppose $p:\mathbf{R}\to\mathbf{R}$ is a polynomial function of odd degree. Prove that there is a solution of the equation $p(x)=0$, x in **R**.

6. Suppose that the function $f:[a, b]\to\mathbf{R}$ is continuous. For a positive integer k, let $x_1, x_2, \ldots, x_k$ be points in [a, b]. Prove that there is a point z in $[a, b]$ such that
$$f(z)=\frac{f(x_1)+f(x_2)+\cdots+f(x_k)}{k}$$

7. Suppose that $f:[2,3]\to\mathbf{R}$ is continuous and that its image is a subset of the rational numbers. Prove that f is a constant function.

8. Show that there dose not exist a strictly increasing function $f;\ \mathbf{Q}\to\mathbf{R}$ such that $f(\mathbf{Q})=\mathbf{R}$. Here **Q** is the set of rational numbers.

9. Let $f:[0, 1]\to\mathbf{R}$ be defined by $f(x)=\sqrt{x}$.

   (a) Prove that f is continuous.

   (b) Show that f is uniformly continuous.

   (c) Show that f does not satisfy the Lipschitz condition.

   [f is said to satisfy a Lipschitz condition or is a Lipschitz function if there exists a constant C such that $|f(x)-f(y)|\leq C|x-y|$.]

10. Suppose that the function $f:(a, b)\to\mathbf{R}$ is uniformly continuous. Prove that then $f:(a, b)\to\mathbf{R}$ is bounded.

11. Prove that if $f:D\to\mathbf{R}$ and $g:D\to\mathbf{R}$ are uniformly continuous , then so is $f+g: D\to\mathbf{R}$ but it is not necessary that the product $fg: D\to\mathbf{R}$ be uniformly continuous.

12. Show that the function $f:\mathbf{R}\to\mathbf{R}$ defined by $f(x)=x^3$ is not uniformly continuous**.**