National University of Singapore

Department of Mathematics

Semester 1 (2006/2007) **MA3110 Mathematical Analysis II** Tutorial 3

1. For each of the following statements, determine whether it is true or false and justify your answer.
   (a) Every bounded sequence converges.
   (b) A convergent positive sequence of positive numbers has a positive limit.
   (c) A convergent sequence of rational numbers has a rational limit.
   (d) The limit of a convergent sequence in the interval (a, b) also belongs to (a, b)
   (e) The set of irrational numbers is a closed subset of **R**.
   (f) The set of rational numbers in the interval [0, 1] is (countably) compact.
   (g) A subset of a (countably) compact set is also (countably) compact.
   (h) Every closed set is compact.
   (i) Every bounded set in **R** is a closed subset of **R**.
   (j) Every sequence of rational numbers has a convergent subsequence.
   (k) Every sequence in (0, 1) has a convergent subsequence.

2. Let S be the interval [1, 5).
   (a) Using the definition of sequential compactness, show that S is not sequentially compact.
   (b) Using the definition of countably compactness, show that S is not countably compact.
   (c) Using the definition of closedness, show that S is not closed.

3. Let S be the set of rational numbers in [0,1].
   (a) Using the definition of sequential compactness, show that S is not sequentially compact.
   (b) Using the definition of countably compactness, show that S is not countably compact.
   (c) Using the definition of closedness, show that S is not closed.

4. For $c > 0$, consider the quadratic equation
   $$x^2 - x - c = 0, \quad x > 0.$$
   Define the sequence (x_n) recursively by fixing $x_1 > 0$ and then, if n is an index for which x_n is defined, defining
   $$x_{n+1} = \sqrt{c + x_n}\,.$$
   Prove that the sequence (x_n) converges monotonically to the solution of the above equation.

5. Suppose (b_n) is a bounded sequence of nonnegative numbers and r is a number such that $0 \le r < 1$. Define $s_n = b_1 r + b_2 r^2 + \ldots + b_n r^n$ for each n in **P**. Prove that (s_n) is convergent.

6. Show that $[2, 3] \cup [4,5]$ is (countably) compact.

7. Let A and B be compact subsets of **R**. Show that $A \cup B$ and $A \cap B$ are (countably) compact.

8. If $A \cup B$ is (countably) compact, does it follow that both A and B are (countably) compact?

9. Suppose $f : [0,1] \to \mathbf{R}$ is defined by $f(x) = \begin{cases} x, & x \text{ is rational} \\ 1 - x, & x \text{ is irrational} \end{cases}$. Determine the points of continuity of f.

10. If $f : D \to \mathbf{R}$ is continuous, prove that $|f| : D \to \mathbf{R}$ is also continuous.

11. Suppose $g : \mathbf{R} \to \mathbf{R}$ is continuous and that $g(x) = x^2$ for all rational x. Prove that $g(x) = x^2$ for all x in **R**.