National University of Singapore

Department of Mathematics

Semester 1 (2006/2007) **MA3110 Mathematical Analysis II** Tutorial 2

1. Let (a_n) be a real sequence. Suppose the subsequences (a_{2n}) and (a_{2n-1}) are convergent and converges to the same value a. Prove that $a_n \to a$.
2. Prove that a sequence cannot converge to two different limits.
3. Prove that if $a_n \to a$, then $|a_n| \to |a|$. If $(|a_n|)$ converges , show by a counter example that (a_n) need not converge.
4. (Existence of n-th root.). Suppose $a \geq 0$ and $n \in \mathbf{N}$, prove that there is a unique b in $\mathbf{R}$, $b \geq 0$ such that $b^n = a$. (Use the completeness property of $\mathbf{R}$.)

   Prove by induction or otherwise, that $h > 0 \Rightarrow (1 + h)^n \geq 1 + nh$ and deduce that

   $a > 1 \Rightarrow 1 < a^{1/n} \leq 1 + \dfrac{a-1}{n}$ and conclude that $a^{1/n} \to 1$.

   Show that $a > 1 \Rightarrow \lim\limits_{n\to\infty} a^n = +\infty$. I.e., for any $K > 0$, there exists an integer N such that $n \geq N \Rightarrow a_n > K$.

   Show that if $a_n \to +\infty$ and $a_n \neq 0$ for all n, then $1/a_n \to 0$.

   Using these results find $\lim\limits_{n\to\infty} a^n$ and $\lim\limits_{n\to\infty} a^{1/n}$ for $a = 1$, $0 < a < 1$ and $a = 0$.
5. Prove the following

   (i) $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$ (ii) $\lim\limits_{n\to\infty} \dfrac{n+1}{n^3+4} = 0$.
6. Use Squeeze Theorem or the Comparison test to prove

   (i) $\lim\limits_{n\to\infty} \dfrac{\sin(n)}{n} = 0$ (ii) $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$

   (iii) $\lim\limits_{n\to\infty} n^{1/n} = 1$ [Hint: write let $h_n = n^{1/n} - 1$ and show that $n = (1 + h_n)^n \geq 1 + \dfrac{n(n-1)}{2}h_n^2$]

   (iv) $\lim\limits_{n\to\infty} \dfrac{\alpha(n)}{n} = 0$, where $\alpha(n)$ = number of primes dividing n. [Hint: show $\alpha(n) \leq \sqrt{n}$.]
7. Show that if (a_n) converges to 0 and (b_n) is a bounded sequence, then $(a_n b_n)$ converges to 0. Hence, or otherwise, show that $\lim\limits_{n\to\infty} \left(\dfrac{2n-1}{3n+1}\right)^n = 0$.

Monotone Convergence Theorem

8. Show that $((1+1/n)^n)$ is an increasing sequence. Show that $(1+1/n)^n < 3$. Hence deduce that it is convergent.
9. (i) Suppose (a_n) is a decreasing (respectively increasing) sequence. Show that the sequence (U_n) , where $U_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ is also a decreasing (respectively increasing) sequence. Hence deduce that the sequence $(\frac{1}{n}(1 + \frac{1}{2} + \cdots + \frac{1}{n}))$ is convergent.

   (ii) Prove that $a_n \to a \Rightarrow U_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n} \to a$. Show that the converse is false.
10. Suppose (a_n) is a monotone sequence. Prove that (a_n) is convergent if and only if (a_n^2) is convergent.