National University of Singapore

Department of Mathematics

Semester 1 (2006/2007) **MA3110 Mathematical Analysis II** Tutorial 10

1. Prove that for any two counting numbers n and m in $\boldsymbol{P}$,
$$\int_0^1 x^m(1-x)^n dx = \int_0^1 (1-x)^m x^n dx.$$
2. Suppose that the function $f:\mathbf{R}\to\mathbf{R}$ has a continuous second derivative. Prove that for any two numbers a and b,
$$\int_a^b x f''(x)dx = b f'(b) + f(a) - af'(a) - f(b).$$
3. Suppose that the function $f:\mathbf{R}\to\mathbf{R}$ has a continuous second derivative. Fix a number a. Prove that
$$\int_a^x f''(t)(x-t)dt = -(x-a)f'(a) + f(x) - f(a) \text{ for all } x.$$
4. Suppose that the function $f:\mathbf{[0,\infty)}\to\mathbf{R}$ is continuous and strictly increasing and that f is differentiable on $(0,\infty)$. Suppose $f(0)=0$. Consider the formula
$$\int_0^x f + \int_0^{f(x)} f^{-1} = x f(x) \quad \text{for all } x \ge 0.$$
Provide a geometric interpretation of this formula in terms of areas. Then prove this formula.
5. Suppose that the function $f:\mathbf{[0,\infty)}\to\mathbf{R}$ is continuous and strictly increasing with $f(0)=0$ and $f(\mathbf{[0,\infty)}) = \mathbf{[0,\infty)}$. Then define
$$F(x) = \int_0^x f \quad \text{and} \quad G(x) = \int_0^x f^{-1} \quad \text{for all } x \ge 0.$$
(i) Prove Young's Inequality:
$$ab \le F(a) + G(b) \quad \text{for all } a \ge 0 \text{ and } b \ge \text{o}.$$
(ii) Use Young's Inequality with $f(x) = x^{p-1}$ for all $x \ge 0$ and $p > 1$ fixed, to prove that if the number q is chosen to have the property $1/p + 1/q = 1$, then
$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for all } a \ge 0 \text{ and } b \ge \text{o}.$$

6. (a) Suppose that the function $f:\mathbf{R}\to\mathbf{R}$ is differentiable and $f' = cf$ for some constant c. Prove that there is a constant k such that $f(x) = k e^{cx}$ for all x in $\mathbf{R}$.
(b) Show that if $f(x) = \int_0^x f(t)dt$ for all real number x, then $f = 0$, the 0 constant function.

7. Suppose $f:[a,b]\to\mathbf{R}$ is continuously differentiable, i.e., f is differentiable and $f':[a,b]\to\mathbf{R}$ is continuous. Use integration by parts to prove that
$$\lim_{x\to\infty}\int_a^b f(t)\sin(xt)dt = 0.$$
8. Use the Second Mean Value Theorem for Integrals (Theorem 66) to prove that
$$\left|\int_1^{10}\frac{\sin(x)}{x}dx\right| < 2 \quad .$$

9. Suppose that the function $f:\mathbf{R}\to\mathbf{R}$ is continuous. Prove that
$$\int_0^x f(u)(x-u)du = \int_0^x\left(\int_0^u f(t)dt\right)du.$$
[Hint: Differentiate both sides.]

10. Evaluate (a) $\lim_{n\to\infty}\sum_{k=1}^n \frac{1}{2n}\sin(\frac{k\pi}{2n})$ (b) $\int_1^e (\ln(x))^2 dx$ (c) $\int_2^\pi x^2\cos(x)dx$.