Absolute Continuity, Lusin Condition and Banach's Conditions T1 and T2

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Let *I* be an interval and suppose $f : I \to \mathbb{R}$ is a function. We say *f* satisfies *Banach Condition T2*, if except for a set of measure zero in the image $f(I)$, every value *y* in $f(I)$ is assumed at most an enumerable number of times in *I*. We say *f* is a *Lusin function* or *f* satisfies *Lusin condition N* if it maps any set of measure zero to set of measure zero. We say *f* satisfies *Banach Condition T1* if, except for a set of measure zero in the image $f(I)$, every value *y* in $f(I)$ is assumed at most a finite number of times in *I*.

Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Banach Zarecki Theorem states that *f* is absolutely continuous if, and only if, *f* is of bounded variation and satisfies the Lusin condition *N*. A Banach's Theorem states that *f* is absolutely continuous if, and only if, *f* is a Lusin function and that f' is Lebesgue integrable on the set $P_+ = \{x : f \text{ is differentiable finitely at } x \text{ and } f'(x) \geq 0\}$. This result is proved in "*When is a continuous function on a closed and bounded interval be of bounded variation, absolutely continuous?" The answer and application to generalized change of variable for Lebesgue integral*" and involved the notion of Lusin condition *N* and Banach condition *T2*. We made use of *T2* and Banach's Theorem that if *f* is continuous and a Lusin function, then it satisfies Banach condition *T2*. Instead of checking that the function *f* satisfies *Lusin condition N* and that it is of bounded variation, we could check if *f* maps the set of points, *E*, where *f* is not differentiable finitely or infinitely to a set of measure zero, if also *f* maps the set E_∞ ,where the derivative is either $+\infty$ or $-\infty$ to a set of measure zero, and if the derivative f' is dominated by a Lebesgue integrable function on the set where the derivative is non-negative to conclude that *f* is absolutely continuous. (See Theorem 9 below.) This result is equivalent to Banach's theorem as *f* is a Lusin function if *f* maps $E \cup E_\infty$ to a set of measure zero. (This is because *f* is a Lusin function on $[a,b] - E \cup E_\infty$, see Theorem 12 of "*Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*" or *Denjoy Saks Young Theorem*".)

Following Banach, we shall prove Banach's Theorem concerning a continuous function on a closed interval satisfying Lusin condition and another concerning a continuous function of bounded variation satisfies Banach condition *T1*. This shall complement the paper "*When is a continuous function on a closed and bounded interval be of bounded variation, absolutely continuous?*" with a proof of the Banach Theorem. We explore these three notions of Lusin's condition, Banach's *T1* and *T2* in absolute continuity.

Theorem 1 (Banach). Suppose $I=[a,b]$ is a closed and bounded interval with $a<b$. Suppose $f:[a,b]\to\mathbb{R}$ is a continuous function satisfying Lusin's condition *N*, that is, *f* maps set of measure zero to set of measure zero. Then *f* satisfies Banach's condition *T2*, that is, except for a set of measure zero in the image $f(I)$, every value in $f(I)$ is assumed at most an enumerable number of times in *I*.

To prove Theorem 1, we shall use the following three results.

Theorem 2. Suppose $f:I\to\mathbb{R}$ is a continuous function, where $I=[a,b]$ is a closed and bounded interval. Then for any closed *E* in *I*, there exists a measurable subset *A* of *E* on which the function *f* assumes each value *y* in $f(E)$ exactly once.

Proof. Take a value *y* in $f(E)$. Since *f* is continuous, $f^{-1}(y)$ is a closed set in *I*.

Since *E* is a closed set, $f^{-1}(y)\cap E$ is also closed in *I*. Since *I* is bounded, the infimum of $f^{-1}(y)\cap E$ exists. Let $x_y=\inf f^{-1}(y)\cap E$. Since $f^{-1}(y)$ is closed, $x_y\in f^{-1}(y)$. Let $A=\{x_y : y\in f(E)\}$. Plainly, *f* assumes each value of *y* in $f(E)$, only once on *A*.

Next, we claim that *A* is measurable.

For each positive integer *n*, let

$E_n=\left\{x\in E:\text{ there exists } k\in E \text{ with } x-k\geq\frac{1}{n} \text{ and } f(k)=f(x)\right\}$. Note that $x\in E_n$ implies that *x* cannot be a lower bound of $f^{-1}(f(x))\cap E$. We next show that the set E_n is a closed set.

Let *m* be a limit point of E_n. Then there exists a sequence of points (x_i) in E_n such that $x_i \to m$. For each x_i there exists a point $k_i \in E$ such that $x_i - k_i \geq \frac{1}{n}$ and $f(k_i) = f(x_i)$. Since the sequence (k_i) is bounded, by the Bolzano Weierstrass Theorem, it has a convergent subsequence (k_{n_i}) such that $k_{n_i} \to t$. Since *E* is closed, $t \in E$. We also have that $x_{n_i} \to m$. Then, $x_{n_i} - k_{n_i} \geq \frac{1}{n}$, $f(k_{n_i}) = f(x_{n_i})$. By the continuity of *f*, $f(t) = \lim_{i\to\infty} f(k_{n_i}) = \lim_{i\to\infty} f(x_{n_i}) = f(m)$ and $m - t = \lim_{i\to\infty}\left(x_{n_i} - k_{n_i}\right) \geq \frac{1}{n}$.
It follows that $m \in E_n$. Hence, E_n contains all its limit points and so is closed in *I*. By definition, $\bigcup_{n=1}^{\infty} E_n$ contains precisely all the non-lower bound of $f^{-1}(y)$ for each *y* in $f(E)$, $A = E - \bigcup_{n=1}^{\infty} E_n$. Since *E* is closed and each E_n is closed, *A* is measurable.

Lemma 3. Suppose $f : I \to \mathbb{R}$ is a continuous function, where $I = [a, b]$ is a closed and bounded interval. Suppose *f* satisfies Lusin's *Condition N*, that is, *f* maps set of measure zero to set of measure zero. Every measurable set *E* in *I* contains for each $\varepsilon > 0$, a measurable subset $Q \subseteq E$ such that $m\left(f(E) - f(Q)\right) < \varepsilon$, where *m* is the Lebesgue measure and the function *f* assumes each of its values at most once on *Q*.

Proof. Since $E \subseteq I$ is measurable, there exists a sequence of closed sets (F_n) in *E* such that $F_n \subseteq F_{n+1}$ for each positive integer *n* and $\bigcup_{n=1}^{\infty} F_n \subseteq E$ with $m\left(E - \bigcup_{n=1}^{\infty} F_n\right) = 0$. Then $E = F \cup N$, where $F = \bigcup_{n=1}^{\infty} F_n$ and $m(N) = m\left(E - \bigcup_{n=1}^{\infty} F_n\right) = 0$.

$f(E) = f(F \cup N) = f(F)$ since *f* is a Lusin function so that $f(N) = 0$. Then

$f\left(F\right) = f\left(\bigcup_{n=1}^{\infty} F_n\right) = \bigcup_{n=1}^{\infty} f\left(F_n\right)$ and $f\left(F_n\right) \subseteq f\left(F_{n+1}\right)$. Note that since *f* is a Lusin function, by Theorem 10 of "*Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*", each $f\left(F_n\right)$ is measurable. By the continuity from below property of Lebesgue measure, given $\varepsilon > 0$, there exists an integer n_0 such that for all integer $n \geq n_0$, $m(f(F)) - m(f(F_n)) < \varepsilon$. Hence,

$$m(f(F)-f(F_n))=m(f(F))-m(f(F_n))<\varepsilon .$$

Now,

$f(E)-f(F_n)=f(F\cup N)-f(F_n)\subseteq f(F)\cup f(N)-f(F_n)$

$$\subseteq \left(f(F)-f(F_n)\right)\cup\left(f(N)-f(F_n)\right).$$

Therefore, for $n\geq n_0$,

$m\left(f(E)-f(F_n)\right)\leq m\left(f(F)-f(F_n)\right)+m\left(f(N)-f(F_n)\right)=m\left(f(F)-f(F_n)\right)<\varepsilon .$

In particular, $m\left(f(E)-f(F_{n_0})\right)<\varepsilon$. Since F_{n_0} is closed in *E*, by Theorem 2, there exists a measurable subset *Q* of F_{n_0} on which the function *f* assumes each value *y* in $f(E)$ exactly once. Hence, $f(Q)=f(F_{n_0})$ and so $m\left(f(E)-f(Q)\right)<\varepsilon$ and *f* assumes each of its values at most once on *Q*.

Lemma 4. Suppose $f:I\rightarrow\mathbb{R}$ is a continuous function, where $I=[a,b]$ is a closed and bounded interval. Suppose *f* satisfies Lusin's *Condition N*, that is, *f* maps set of measure zero to set of measure zero. Every measurable set *E* in *I* contains a measurable subset $K\subseteq E$ such that $m\left(f(E)-f(K)\right)=0$, where *m* is the Lebesgue measure and the function *f* assumes each of its values in $f(K)$ at most an enumerable number of times in *K*. That is, for any *y* in *f*(*K*), the preimage $f^{-1}(y)$ is at most denumerable.

Proof.

By Lemma 3, for each positive integer *n*, there exists a measurable subset $Q_n\subseteq E$ such that $m\left(f(E)-f(Q_n)\right)<\dfrac{1}{n}$ and on which *f* assumes each of its values in $f(Q_n)$ exactly once in Q_n. Let $K=\bigcup_{n=1}^{\infty}Q_n$. Then for each positive integer n,

$$f(E)-f(K)=f(E)-f\left(\bigcup_{n=1}^{\infty}Q_n\right)\subseteq f(E)-f(Q_n) .$$

Thus, $m\left(f(E)-f(K)\right)\leq m\left(f(E)-f(Q_n)\right)<\dfrac{1}{n}$ for each positive integer *n*. Therefore, $m\left(f(E)-f(K)\right)=0$.

Let $y\in f(K)$. Then $y\in f(Q_n)$ for some positive integer *n* and there exists precisely one point $x\in Q_n$ such that $y=f(x)$. Since *K* is a countable union of the

Q_n's, y can belong to at most an enumerable number of the Q_n's. Thus, f can assume the value y at most an enumerable number of times on K.

Proof of Theorem 1.

For each measurable set E in I, let

$$K_E = \left\{ \begin{array}{l} K \subseteq E : m\big(f(E) - f(K)\big) = 0 \text{ and} \\ \text{each } y \text{ in } f(E) \text{ is assumed at most enumerable of times in } K \end{array} \right\}.$$

By Lemma 4, $K_E \neq \varnothing$.

Then the set $\{m(K) : K \in K_E\}$ is bounded above by $m(I) = b - a$. Let $\mu_E = \sup\{m(K) : K \in K_E\}$.

Take $\mu_I = \sup\{m(K) : K \in K_I\}$. Then for any positive integer n, there exists a set $K_n \in K_I$ such that $\mu_I - \frac{1}{n} < m(K_n) \leq \mu_I$. It follows that $\lim_{n\to\infty} m(K_n) = \mu_I$. Let $H = \bigcup_{n=1}^{\infty} K_n \subseteq I$. Then $m(H) = m\left(\bigcup_{n=1}^{\infty} K_n\right) = \mu_I$. Note that $f(I) - f(H) = f(I) - f\left(\bigcup_{n=1}^{\infty} K_n\right) \subseteq f(I) - f(K_n)$ for any positive integer n. Thus,

$m\big(f(I) - f(H)\big) \leq m\big(f(I) - f(K_n)\big) = 0$ and so $m\big(f(I) - f(H)\big) = 0$. Take $y \in f(H)$. Then $y \in f(K_n)$ for some positive integer n and there exists at most denumerable number of points x in K_n such that $f(x) = y$. Since y can belong to at most denumerable number of the sets $f(K_n)$, there can be at most denumerable number of points x in H such that $f(x) = y$. It follows that $H \in K_I$. Suppose $H = I$. Then every value $y \in f(I) = f(H)$ is assumed at most denumerable number of times on I. This means f satisfies Banach Condition *T2*.

We assume now that $H \neq I$.

Consider the set K_{I-H} . By Lemma 4, K_{I-H} is not empty. Take a set $U \in K_{I-H}$.

Then $m(f(I-H) - f(U)) = 0$ and each value of y in $f(I-H)$ is assumed at most denumerable number of times in U. Take $y \in f(H \cup U)$. If $y \in f(U)$, then $y \in f(I-H)$ and y is assumed at most denumerable number of times on U. If y is also in $f(H)$ then y is assumed at most denumerable number of times on H and so it is assumed at most denumerable number of times on $H \cup U$. If $y \in f(U)$

and $y \notin f(H)$, then plainly *y* is assumed at most denumerable number of times on $H \cup U$. If $y \in f(H \cup U)$ and $y \notin f(U)$, then $y \in f(H)$ and so *y* is assumed at most denumerable number of times on *H* and hence on $H \cup U$. Now we have $m(f(I)-f(H))=0$ and so $m(f(I)-f(H \cup U))=0$. It follows that $H \cup U \in K_I$.

Then since *U* and *H* are disjoint, $m(H \cup U)=m(H)+m(U) \le \mu_I$. Since $m(H)=\mu_I$, $m(U)=0$.

Since $U \subseteq I-H$, $f(U) \subseteq f(I-H) \subseteq \left(f(I-H)-f(U)\right) \cup f(U)$. Therefore,

$m\left(f(U)\right) \le m\left(f(I-H)\right) \le m\left(f(I-H)-f(U)\right)+m\left(f(U)\right)=0+m\left(f(U)\right)=m\left(f(U)\right)$.
Thus, $m\left(f(I-H)\right)=m\left(f(U)\right)=0$ as $m(U)=0$ and *f* is a Lusin function. Since $f(I) \subseteq f(I-H) \cup f(H)$ and $m\left(f(I-H)\right)=0$, almost every *y* in $f(I)$, that is every *y* not in the image $f(I-H)$ belongs to $f(H)$ and so is assumed at most enumerable number of times only on *H* and so on *I*. Since $m\left(f(I-H)\right)=0$, this means *f* satisfies Banach Condition *T2*.

Note that Banach Condition *T1* implies Banach condition *T2*. The next theorem provides an easy way to determine Banach condition *T1* by examining the image of the set where the function is not differentiable finitely or not differentiable infinitely. Checking the cardinality of the preimage is somewhat arduous.

Theorem 5. Suppose $f: I \to \mathbb{R}$ is a continuous function, where $I=[a,b]$ is a closed and bounded interval. Let

$$E=\left\{x \in I : f \text{ has no derivative finitely or infinitely at } x\right\}.$$

Then *f* satisfies Banach Condition *T1* if, and only if, $m\left(f(E)\right)=0.$

Proof. Let $Z=\left\{y \in f(I) : f^{-1}(y) \text{ is infinite}\right\}$.

Suppose *f* satisfies Banach Condition *T1*. Then $m(Z)=0$.

Let $F=f^{-1}(Z)=\left\{x \in I : f(x) \in Z\right\}$. Then $m\left(f(F)\right)=m(Z)=0$.

Now consider $E-F$. Then $f\left(E-F\right) \cap Z=\varnothing$.

Take any point $x_0 \in E-F$. Then $f^{-1}(f(x_0))$ is finite and is a set of isolated points. Note that *f* is not differentiable finitely or infinitely on $E-F$. By Theorem 11 of "*When is a continuous function on a closed and bounded interval be of bounded variation, absolutely continuous? The answer and application to generalized change of variable for Lebesgue integral*", $m\big(f(E-F)\big)=0$. Since $f(E)\subseteq f(E-F)\cup f(F)$,

$$m\big(f(E)\big)\le m\big(f(E-F)\big)+m\big(f(F)\big)=0+0=0.$$

Hence, $m\big(f(E)\big)=0$.

Conversely, suppose $m\big(f(E)\big)=0$. Let

$$H=\{x\in I : f \text{ is differentiable at } x \text{ and } f'(x)=0\}.$$

Then by Theorem 3 of "*Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*", $m\big(f(H)\big)=0$.

We now examine the set $Z-f(E)$. Take any $y_0\in Z-f(E)$. Let $F_0=f^{-1}(y_0)$ is closed in *I* since *f* is continuous. Moreover, since $y_0\in Z$, F_0 is an infinite set. As $y_0\notin f(E)$, *f* is differentiable at each point of F_0. Let x_0 be an accumulation point of F_0. As F_0 is closed, $x_0\in F_0$ and so *f* is differentiable at x_0. Since x_0 is an accumulation of F_0, there exists a sequence of points (x_n) in F_0 such that $x_n\ne x_0$ $x_n\to x_0$ and as $f(x_n)=f(x_0)$, the difference quotient $\dfrac{f(x_n)-f(x_0)}{x_n-x_0}$ is always 0 and so it tends to 0 as $x_n\to x_0$. Thus, 0 is a derived number at x_0. Since *f* is differentiable at x_0, its derivative at x_0 must be 0. That is $f'(x_0)=0$. Therefore, $x_0\in H$. It follows that $y_0=f(x_0)\in f(H)$. Hence, $Z-f(E)\subseteq f(H)$. It follows that $m\big(Z-f(E)\big)=0$. Since $m\big(f(E)\big)=0$, $m\big(Z\big)=0$ and so *f* satisfies Banach Condition *T1*.

Theorem 6. Suppose $f: I\to\mathbb{R}$ is a continuous function of bounded variation, where $I=[a,b]$ is a closed and bounded interval. Then *f* satisfies Banach Condition *T1*.

Proof. By Theorem 14 (De La Vallée Poussin Theorem) of "*Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*", there exists a set *N* of measure zero in *I* such that $m(f(N)) = m(N) = m\left(v_f(N)\right) = 0$ and that for all *x* in $I - N$, *f* is differentiable finitely or infinitely at *x*. It follows by Theorem 5 that *f* satisfies Banach Condition *T1*.

Remark. It follows by Theorem 6, that if *f* is absolutely continuous, then *f* satisfies Banach condition *T1*.

Theorem 7. Suppose $f : I \to \mathbb{R}$ is a continuous function of bounded variation, where $I = [a,b]$ is a closed and bounded interval. Then *f* is absolutely continuous if, and only if, $m(f(E)) = 0$, where $E = \{x \in I : f'(x) = \infty \text{ or } f'(x) = -\infty\}$.

Proof. By Theorem 18 of my article "*Functions of Bounded Variation and de La Vallée Poussin's Theorem*", there exist a null set *N* in *I* such that $m(f(N)) = 0$ and for all *x* in *I*–*N*, *f* is differentiable finitely or infinitely at *x* and that *f* is a Lusin function on $I - N - E$, where $E = \{x \in I : f'(x) = \infty \text{ or } f'(x) = -\infty\}$. Since *f* is a continuous function of bounded variation, by Theorem 8 of "*Functions of Bounded Variation and de La Vallée Poussin's Theorem*" *f* is differentiable finitely almost everywhere on *I*. Thus, $m(E) = 0$.

Suppose *f* is absolutely continuous, then it is a Lusin function and so $m(f(E)) = 0$, since *E* is a null set. Conversely, if $m(f(E)) = 0$, then as *f* is a Lusin function on $I - N - E$, and $m\left(f(E \cup N)\right) = 0$, it follows that *f* is a Lusin function on *I*. Therefore, by the Banach Zarecki Theorem, *f* is absolutely continuous.

Suppose we know that a continuous function *f* maps its set of points, where *f* is not differentiable finitely or infinitely, to a set of measure zero. Then we only need to check if f' is Lebesgue integrable on the set where f' is finite and whether *f* maps the set where $|f'|$ is infinite to a set of measure zero.

Theorem 8. Suppose $I = [a,b]$ is a closed and bounded interval and $f : I \to \mathbb{R}$ is a continuous function. Let $E = \{x \in I : f \text{ is not differentiable finitely or infinitely at } x\}$. Suppose $m(f(E)) = 0$.

Then *f* is absolutely continuous if, and only if, f' is Lebesgue integrable on $I - E$ and $m(f(E_\infty)) = 0$, where $E_\infty = \{x \in I : f'(x) = \infty \text{ or } f'(x) = -\infty\}$.

Proof. $m(f(E)) = 0$ implies that *f* satisfies Banach condition *T1* and hence *T2*.

Suppose f' is Lebesgue integrable on $I-E$. Therefore, f' is Lebesgue integrable on the set of points, where *f* is differentiable finitely. Since *f* satisfies Banach condition *T2*, by Theorem 13 of the article "*When is a continuous function on a closed and bounded interval be of bounded variation, absolutely continuous? The answer and application to generalized change of variable for Lebesgue integral*.", *f* is of bounded variation. By the *de La Vallée Poussin's Theorem* (Theorem 15 of "*Functions of Bounded Variation and de La Vallée Poussin's Theorem*"), $m(f(E))=0$. Suppose $m(f(E_\infty))=0$. Since *f* is differentiable finitely on $I-E-E_\infty$, by Theorem 17 of "*Functions of Bounded Variation and de La Vallée Poussin's Theorem*", *f* is a Lusin function on $I-E-E_\infty$. As $m(f(E_\infty \cup E))=0$, *f* is a Lusin function on *I*. Therefore, by the Banach Zarecki Theorem, *f* is absolutely continuous.

Conversely, suppose *f* is absolutely continuous. Then *f* is of bounded variation and consequently f' is Lebesgue integrable on $I-E$. *f* is also a Lusin function and so $m(f(E_\infty))=0$ since $m(E_\infty)=0$.

Following Banach we also have the following:

Theorem 9. Suppose $I=[a,b]$ is a closed and bounded interval and $f:I\to\mathbb{R}$ is a continuous function. Let $E=\{x\in I: f \text{ is not differentiable finitely or infinitely at } x\}$. Suppose $m(f(E))=0$. Let $P_+=\{x\in I: f \text{ is differentiable finitely at } x \text{ and } f'(x)\geq 0\}$

Then *f* is absolutely continuous if, and only if, $m(f(E_\infty))=0$, where $E_\infty=\{x\in I: f'(x)=\infty \text{ or } f'(x)=-\infty\}$ and there exists a function $g:[a,b]\to\mathbb{R}$ such that $f'\leq g$ on P_+ and *g* is Lebesgue integrable on P_+.

Proof. If *f* is absolutely continuous, then *f* is of bounded variation and so f' is Lebesgue integrable and so it is Lebesgue integrable on P_+. Take *g* to be f' Moreover, *f* is a Lusin function, it follows that $m(f(E_\infty))=0$, since $m(E_\infty)=0$.

Conversely, suppose $m(f(E_\infty))=0$. As $m(f(E))=0$, $m(f(E_\infty \cup E))=0$ and *f* is differentiable finitely on $I-E-E_\infty$. Suppose there exists a function $g:[a,b]\to\mathbb{R}$ such that $f'\leq g$ on P_+ and *g* is Lebesgue integrable on P_+. Then by Theorem 17 of "*When is a continuous function on a closed and bounded interval be of bounded variation, absolutely continuous? The answer and application to generalized change of variable for Lebesgue integral*.", *f* is absolutely continuous.

Remark. Theorem 9 is a little easier to use for we only need to check two things: $m(f(E_\infty))=0$ and that f' be dominated above by a Lebesgue integrable function on P_+.

Given that $m(f(E))=0$, where $E=\{x\in I: f \text{ is not differentiable finitely or infinitely at } x\}$, by Theorem 6, *f* satisfies Banach condition *T1* and hence *T2*.

Suppose there exists a function $g:[a,b]\to\mathbb{R}$ such that $f'\le g$ on P_+ and *g* is Lebesgue integrable on P_+. Then by Theorem 13 of "*When is a continuous function on a closed and bounded interval be of bounded variation, absolutely continuous? The answer and application to generalized change of variable for Lebesgue integral*.", *f* is of bounded variation. Then by Theorem 15 of "*Functions of Bounded Variation and de La Vallée Poussin's Theorem*", there exists a subset $N\subseteq I$ such that *f* is differentiable finitely or infinitely on $I-N$, $m(N)=m(f(N))=0$. Since $E\subseteq N$, $m(f(E))=0$. Thus, *f* is differentiable finitely on the set $I-(E\cup E_\infty)$, where $E_\infty=\{x\in I: f'(x)=\infty \text{ or } f'(x)=-\infty\}$. By Theorem 12 of "*Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*", *f* is a Lusin function on $I-(E\cup E_\infty)$. If $m(f(E_\infty))=0$, then $m(f(E_\infty\cup E))=0$. It follows that *f* is a Lusin function on *I*. Therefore, by the Banach Zarecki Theorem, *f* is absolutely continuous.

If *f* is absolutely continuous, then *f* is of bounded variation. Then by Theorem 15 of "*Functions of Bounded Variation and de La Vallée Poussin's Theorem*", there exists a subset $N\subseteq I$ such that *f* is differentiable finitely or infinitely on $I-N$, $m(N)=m(f(N))=0$. Since $E\subseteq N$, $m(f(E))=0$. Thus, the condition $m(f(E))=0$ is necessary for a function *f* to be absolutely continuous. If *f* is absolutely continuous, then it is a Lusin function. Since E_∞ is a null set, $m(f(E_\infty))=0$. Since *f* is also of bounded variation on *I*, by Theorem 6 of "*Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*", *f* is differentiable almost everywhere in *I* and f' is Lebesgue integrable. We can take *g* to be f' in Theorem 9. This gives another proof of Theorem 9.

Remark. We have shown that if the continuous function $f:I\to\mathbb{R}$ is absolutely continuous, then it must satisfy Lusin's Condition *N* and Banach's Condition *T1*.

Banach had shown that Lusin *Condition N* and Banach *Condition T1* is equivalent to Banach's condition *S*.

A function $f: I \to \mathbb{R}$, where *I* is the closed interval $[a,b]$ with $a < b$, is said to satisfy *Condition S*, if given any $\varepsilon > 0$, there exists $\delta > 0$ such that for any measurable set $E \subseteq I$, $m(E) < \delta \Rightarrow m(f(E)) < \varepsilon$.

Theorem 10. Suppose $I = [a,b]$ is a closed and bounded interval and $f: I \to \mathbb{R}$ is a continuous function. Then *f* satisfies *Condition S* if, and only if, *f* satisfies Lusin *Condition N* and Banach *Condition T1*.

Proof.

Suppose *f* satisfies *Condition S*.

We shall show that *f* satisfies Lusin *Condition N* and Banach *Condition T1*.

As *f* satisfies *Condition S*, for any positive integer *n* there exists $\delta > 0$ such that $m(E) < \delta \Rightarrow m(f(E)) < \frac{1}{n}$. Therefore, for any measurable set *E* with $m(E) = 0$, $m(f(E)) < \frac{1}{n}$ for any positive integer *n*. It follows that $m(f(E)) = 0$. Hence, *f* satisfies Lusin *Condition N*.

We shall show that *f* satisfies Banach *Condition T1*, by using a contradiction argument.

Suppose on the contrary, *f* does not satisfy Banach *Condition T1*. Then there exists a set of values, $H \subseteq f(I)$, where each $y \in H$ is assumed infinitely often on *I* and which has positive Lebesgue outer measure, i.e., $m^*(H) > 0$. Since *f* is continuous, the Banach indicatrix function N_f of *f* is measurable. (A theorem of Banach, see Theorem 3, Chapter VIII page 225 of Natanson, *Theory of functions of a real variable volume 1*.) It follows that *H* is measurable and $m(H) = m^*(H) > 0$. The Banach indicatrix function is defined on the range $f(I)$, which is a closed and bounded interval, by

$$N_f(y) = \begin{cases} \text{The number of points in } f^{-1}(y), \text{ if } f^{-1}(y) \text{ is finite,} \\ +\infty, \text{ if } f^{-1}(y) \text{ is infinite} \end{cases}.$$

Then, we claim that there exists a closed measurable set $Y \subseteq H$ such that $m(Y) > 0$. We shall define a sequence of measurable sets (X_i) such that

(i) $X_i \cap X_j = \varnothing$ for $i \neq j$,

(ii) $m(f(X_i)) \geq \dfrac{m(Y)}{2}$, for all *i* and

(iii) For each positive integer *i, f* assumes each of its values in $f(X_i)$ exactly once on X_i.

We shall define this sequence (X_i) inductively.

Since the set *H* is measurable, there exists a F_σ set, *F*, which is a countable union of closed sets (F_n), i.e., $F = \left(\bigcup_{n=1}^{\infty} F_n\right)$ and such that $m\left(H - \bigcup_{n=1}^{\infty} F_n\right) = 0$ and $H = \left(\bigcup_{n=1}^{\infty} F_n\right) \cup N$, where $m(N) = 0$.

Then $m(F) = m(H) > 0$. By the continuity from below property of the Lebesgue measure, $m(F) = \lim_{k\to\infty} m\left(\bigcup_{i=1}^{k} F_i\right)$ and so there exists a positive integer n_0 such that for all $n \geq n_0$, $m\left(\bigcup_{i=1}^{n} F_i\right) \geq \frac{m(F)}{2} = \frac{m(H)}{2}$. Let $Y = \left(\bigcup_{i=1}^{n_0} F_i\right)$. Then Y is closed in $f(I)$. As *f* is continuous, $X = f^{-1}(Y)$ is closed in *I*. Then by Theorem 2, there exists a measurable set X_1 in $f^{-1}(Y)$ such that *f* assumes each value of Y exactly once.

Moreover, $m(f(X_1)) = m(Y) \geq \frac{m(Y)}{2}$.

Suppose now that the first *k* sets of the (X_i) is defined with the properties,

(i) $X_i \cap X_j = \varnothing$ for $i \neq j$,

(ii) $m(f(X_i)) \geq \frac{m(Y)}{2}$, for all *i* and

(iii) For each positive integer *i, f* assumes each of its values in $f(X_i)$ exactly once on X_i.

Note that $X_1 \subseteq X$ and $m(f(X_1)) \geq \frac{m(Y)}{2}$.

Let $E_k = I - \bigcup_{i=1}^{k} X_i$. Now *f* can only assume each of its values on $\bigcup_{i=1}^{k} X_i$ (at most k times) a finite number of times. It follows that each value $y \in Y = \left(\bigcup_{i=1}^{n_0} F_i\right)$ is in the image of $E_k = I - \bigcup_{i=1}^{k} X_i$, since $f^{-1}(y)$ is infinite, that is, $Y \subseteq f(E_k)$. Therefore, $m\big(f(E_k)\big) \geq m(Y) > 0$. Since *f* satisfies Lusin's *Condition N*, by Lemma 3, there exists a measurable set $X_{k+1} \subseteq E_k = I - \bigcup_{i=1}^{k} X_i$ such that *f* assumes each of its values on X_{k+1} exactly once and $m(f(E_k) - f(X_{k+1})) \le \frac{m\left(f(E_k)\right)}{2}$. Note that $X_{k+1} \cap X_j = \varnothing$ for $1 \le j \le k$. Since $f(E_k) = \left(f\left(E_k\right) - f\left(X_{k+1}\right)\right) \cup f\left(X_{k+1}\right)$, $m\left(f(E_k)\right) \le m(f(E_k) - f(X_{k+1})) + m\left(f\left(X_{k+1}\right)\right)$ so that $m\left(f(X_{k+1})\right) \ge m\left(f(E_k)\right) - m(f(E_k) - f(X_{k+1})) \ge \frac{m\left(f(E_k)\right)}{2} \ge \frac{m(Y)}{2}$. In this way, we define the sequence (X_i) inductively. Plainly, this sequence is a sequence of disjoint measurable subsets in *I* satisfying properties (i), (ii) and (iii).

We shall now derive a contradiction.

By the Bolzano Weierstrass Theorem, as the sequence $(m(X_n))$ is bounded it has a convergence subsequence $\left(m(X_{n_i})\right)$. We claim that $\lim_{i\to\infty} m(X_{n_i}) = 0$. Suppose on the contrary, $\lim_{i\to\infty} m(X_{n_i}) > 0$. Then there exists an integer *N*, such that for all $i \ge N$, $m(X_{n_i}) > \frac{K}{2} > 0$, where $K = \lim_{i\to\infty} m(X_{n_i})$. Therefore, as the sequence (X_i) consists of disjoint sets, $\sum_{i=N}^{\infty} m(X_{n_i}) = \infty$. But as the sets $\{X_i\}$ are mutually disjoint subsets of *I*, $\sum_{i=N}^{\infty} m(X_{n_i}) \le m(I) < \infty$ and so $\sum_{i=N}^{\infty} m(X_{n_i}) = \infty$ is not valid. This contradiction shows that $\lim_{i\to\infty} m(X_{n_i}) = 0$. Since *f* satisfies *Condition S*, this implies that for any positive integer *n*, there exists $\delta_n > 0$ such that, for any measurable set *B* in *I*, $m(B) < \delta_n \Rightarrow m(f(B)) < \frac{1}{n}$. As $\lim_{i\to\infty} m(X_{n_i}) = 0$, there exists an integer N_n such that $m(X_{n_i}) < \delta_n$ for all $i > N_n$. It follows that $m(f(X_{n_i})) < \frac{1}{n}$ for all $i > N_n$. Therefore, $\lim_{i\to\infty} m(f(X_{n_i})) = 0$. But $m(f(X_{n_i})) \ge \frac{m(Y)}{2} > 0$ and so the sequence $\left(m(f(X_{n_i}))\right)$ cannot converge to zero as it is bounded below by $\frac{m(Y)}{2} > 0$. This contradiction shows that $m(H) = 0$. Therefore, *f* must satisfy Banach Condition *T1*.

Now we prove the converse by contrapositive argument. Suppose *f* satisfies Lusin *Condition N* and Banach C*ondition T1* but not Banach *Condition S*.

Therefore, there exists $\sigma > 0$ such that for any positive integer *n*, there exists a measurable set $E_n \subseteq I$ such that $m(E_n) < \dfrac{1}{2^n}$ but $m\left(f(E_n)\right) > \sigma$.

Let $E = \limsup_n E_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)$ and $A = \limsup_n \left(E_n\right) = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} f\left(E_k\right)\right)$.

If $y \in A$, then $y \in \bigcup_{k=n}^{\infty} f\left(E_k\right) = f\left(\bigcup_{k=n}^{\infty} E_k\right)$ for all $n \geq 1$. Then *y* must belong to infinite numbers of the $f\left(E_k\right)' s$, for if *y* belong to only finite number of the $f\left(E_k\right)' s$, then there exists an integer n_0 such that $y \notin f(E_k)$ for all $k \geq n_0$ and so $y \notin \bigcup_{k=n_0}^{\infty} f\left(E_k\right)$ and so $y \notin \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} f\left(E_k\right)\right) = A$, giving a contradiction.

Suppose $x \in \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right) = E$. Then *x* belongs to infinite number of the $E_k' s$. Therefore, $y = f(x)$ belongs to infinite numbers of the $f\left(E_k\right)' s$ and so $y \in \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} f\left(E_k\right)\right) = A$. Hence, $f(E) \subseteq A$.

Take $y \in \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} f\left(E_k\right)\right) = A$ and $y \notin f(E)$.

Then there exists a sequence of distinct integers (n_k) such that $y \in f\left(E_{n_k}\right)$. Therefore, there exists a sequence of points in *I*, (x_k) such that $x_k \in E_{n_k}$ and $f\left(x_k\right) = y$. The sequence (x_k) is a bounded sequence. Therefore, it has a monotone convergent subsequence $\left(x_{j_k}\right)$. Suppose $\left(x_{j_k}\right)$ is increasing. If the values the sequence takes is finite, then it has a constant subsequence whose limit is *x*, then $x \in \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right) = E$ and so $y \in f(E)$. Since $y \notin f(E)$, the values the sequence $\left(x_{j_k}\right)$ takes must be infinite and so $f^{-1}(y)$ is infinite. Suppose $\left(x_{j_k}\right)$ is decreasing. If the values the sequence $\left(x_{j_k}\right)$ takes is finite, then it has a constant subsequence whose limit is *x*, then $x \in \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right) = E$ and so $y \in f(E)$. As $y \notin f(E)$ the values the sequence takes must be infinite, and so $f^{-1}(y)$ is infinite. Thus, every $y \in A - f(E)$ is assumed infinite number of times by *f* in *I*.

Now $m(E) \le m\left(\bigcup_{k=n}^{\infty} E_k\right) \le \sum_{k=n}^{\infty} \frac{1}{2^k}$ for all positive integer *n*. Since $\sum_{k=n}^{\infty} \frac{1}{2^k} \to 0$ as $n \to \infty$, we conclude that $m(E) = 0$. Since *f* satisfies *Condition N*, $m\left(f(E)\right) = 0$. Now, note that $\bigcup_{k=n}^{\infty} f\left(E_k\right) \supseteq \bigcup_{k=n+1}^{\infty} f\left(E_k\right)$ for any positive integer *n*. By the continuity from above property of Lebesgue measure, $m(A) = m\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} f\left(E_k\right)\right)\right) = \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} f\left(E_k\right)\right)$.

As each $m\left(\bigcup_{k=n}^{\infty} f\left(E_k\right)\right) > \sigma$, $m(A) \ge \sigma$. Note that $f(E) \subseteq A$ and so

$m(A) = m(A - f(E)) + m\left(f(E)\right) = m(A - f(E))$. Hence, $m(A - f(E)) \ge \sigma$. As every value in $A - f(E)$ is assumed infinite number of times in *I* and $m(A - f(E)) \ge \sigma > 0$, this contradicts that *f* satisfies *Condition T1*.

Hence, if *f* satisfies Lusin *Condition N* and Banach C*ondition T1*, then *f* satisfies Banach *Condition S*.

This completes the proof of Theorem 10.

Corollary 11. Suppose $I = [a,b]$ is a closed and bounded interval and $f : I \to \mathbb{R}$ is a continuous function. If *f* is absolutely continuous on *I*, then *f* must satisfy Banach *Condition S*.

Proof. Suppose *f* is absolutely continuous on *I*. Then by the Banach Zarecki Theorem, *f* satisfies Lusin's *Condition N* and is of bounded variation. Since *f* is of bounded variation, by Theorem 6, *f* satisfies Banach *Condition T1* and so by Theorem 10, *f* satisfies Banach *Condition S*.

Corollary 11 says that Banach *Condition S* is a necessary condition for a continuous function on a closed and bounded interval to be absolutely continuous.