Limit of the Lebesgue Stieltjes Integral and Change of Variable

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If a function is defined by a Lebegsue Stieltjes integral and turns out that it is of bounded variation, then we can define another Lebesgue Stieltjes integral with it as the integrator. More precisely, suppose $\phi: I \to \mathbb{R}$ is an absolutely continuous function on the closed and bounded interval $I = [a,b]$, where $a < b$. Suppose g is a Borel measurable function on I. Define $\Phi: I \to \mathbb{R}$, by $\Phi(x) = \int_a^x g\, d\lambda_\phi$ where λ_ϕ is the Lebesgue Stieljes measure associated with the function ϕ. Since ϕ is absolutely continuous on I, by Theorem 26 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*", $\Phi(x) = \int_a^x g\, d\lambda_\phi = \int_a^x g(y)\phi'(y)\,dy$. It follows that Φ is absolutely continuous on I and therefore of bounded variation. Suppose $f: I \to \mathbb{R}$ is a Borel measurable function. Then

$$\int_a^x f\, d\lambda_\Phi = \int_a^x f(y)\Phi'(y)\,dy = \int_a^x f(y)g(y)\phi'(y)\,dy = \int_a^x f(y)g(y)\,d\lambda_\phi .$$

Hence, we have proved:

Theorem 1. Suppose $\phi: I \to \mathbb{R}$ is an absolutely continuous function on the closed and bounded interval $I = [a,b]$, where $a < b$. Suppose g is a Borel measurable function on I. Define $\Phi: I \to \mathbb{R}$, by $\Phi(x) = \int_a^x g\, d\lambda_\phi$ where λ_ϕ is the Lebesgue Stieljes measure associated with the function ϕ. Suppose $f: I \to \mathbb{R}$ is a Borel measurable function. Then

$$\int_a^x f\, d\lambda_\Phi = \int_a^x f(y)g(y)\,d\lambda_\phi .$$

If a sequence of functions of bounded variation, whose total variation is uniformly bounded and converges to a function, is the limiting function also of bounded variation and if so, does the sequence of Lebesgue Stieltjes integrals defined using the given sequence of functions as integrators converges to the Lebesgue Stieltjes integral with the limiting function as integrator? The answer is "yes". We state this answer as Theorem 2 below.

Theorem 2. Suppose (g_n) is a sequence of functions defined on the closed and bounded interval $[a, b]$ whose total variations are uniformly bounded by a constant K, i.e., $V(g_n,[a,b]) \le K$, for all positive integer n and suppose the sequence (g_n) converges to a finite function g at every point of $[a,b]$. Let f be a continuous function on $[a,b]$. Then g is of finite variation and

$\lim_{n\to\infty}\int_a^b f d\lambda_{g_n} = \int_a^b f d\lambda_g$.

Proof.

We note that since the function f is continuous on [a, b], the Riemann Stieltjes integrals of the function f with integrators g and g_k exist and are equal to their respective Lebesgue Stieltjes integrals. Moreover, since f is continuous on [a, b], for any $a \le d < e < f \le b$, the Rieman Stieltjes integrals,

$$RS\int_d^e f d\lambda_g + RS\int_e^f f d\lambda_g = RS\int_d^f f d\lambda_g \text{ and } RS\int_d^e f d\lambda_{g_k} + RS\int_e^f f d\lambda_{g_k} = RS\int_d^f f d\lambda_{g_k} .$$

We shall show that the limiting function g is of finite variation.

Let $P: a = x_0 < x_1 < \cdots < x_n = b$ be a partition of the closed interval [a, b]. Given any $\varepsilon > 0$, since the sequence (g_m) converges to g pointwise, for each $0 \le k \le n$, there exists an integer $N_k > 0$ such that for $0 \le k \le n$,

$$m \ge N_k \Rightarrow |g_m(x_k) - g(x_k)| < \frac{\varepsilon}{2n} . \qquad \text{-------------------------- (1)}$$

Let $N = \max\{N_k : 0 \le k \le n\}$

Then it follows from (1) that for $0 \le k \le n$

$$m \ge N \Rightarrow |g_m(x_k) - g(x_k)| < \frac{\varepsilon}{2n} . \qquad \text{------------------ (2)}$$

Suppose $1 \le k \le n$. Then for $m \ge N$

$$\begin{aligned} |g(x_k) - g(x_{k-1})| &= |g(x_k) - g_m(x_k) + g_m(x_k) - g_m(x_{k-1}) + g_m(x_{k-1}) - g(x_{k-1})| \\ &\le |g(x_k) - g_m(x_k)| + |g_m(x_k) - g_m(x_{k-1})| + |g_m(x_{k-1}) - g(x_{k-1})| \\ &\le \frac{\varepsilon}{2n} + |g_m(x_k) - g_m(x_{k-1})| + \frac{\varepsilon}{2n} = \frac{\varepsilon}{n} + |g_m(x_k) - g_m(x_{k-1})| . \end{aligned}$$

Therefore,

$$\sum_{k=1}^{n} |g(x_k) - g(x_{k-1})| \le \sum_{k=1}^{n}\left(\frac{\varepsilon}{n} + |g_m(x_k) - g_m(x_{k-1})|\right) = \varepsilon + \sum_{k=1}^{n} |g_m(x_k) - g_m(x_{k-1})|$$

$$\le \varepsilon + V(g_m,[a,b]) \le \varepsilon + K .$$

Since $\varepsilon > 0$ is arbitrary, $V(g,[a,b]) \le K$. Hence, g is of bounded variation.

We shall now show that $\lim_{n\to\infty} \int_a^b f d\lambda_{g_n} = \int_a^b f d\lambda_g$. For the rest of the proof, all integrals are Riemann Stieltjes integrals.

We shall take an appropriate partition of the interval [a, b] and split the integrals into integrals on each sub intervals of the partition.

We note that since f is continuous on [a, b], f is uniformly continuous on [a, b]. Therefore, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3K} . \quad \text{--------------------- (3)}$$

Let $P: a = x_0 < x_1 < \cdots < x_n = b$ be a partition of [a, b] such that $\|P\| = \max\{|x_k - x_{k-1}| : 1 \le k \le n\} < \delta$.

Then

$$\int_a^b f\, d\lambda_g = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x) d\lambda_g = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(x) - f(x_k)) d\lambda_g + \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x_k) d\lambda_g . \quad \text{---------(4)}$$

And for each positive integer m,

$$\int_a^b f\, d\lambda_{g_m} = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x) d\lambda_{g_m} = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(x) - f(x_k)) d\lambda_{g_m} + \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x_k) d\lambda_{g_m} . \quad \text{--------- (5)}$$

(5) – (4) gives:

$$\begin{aligned}
\int_a^b f\, d\lambda_{g_m} - \int_a^b f\, d\lambda_g &= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(x) - f(x_k)) d\lambda_{g_m} - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(x) - f(x_k)) d\lambda_g \\
&\quad + \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x_k) d\lambda_{g_m} - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x_k) d\lambda_g \\
&= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(x) - f(x_k)) d\lambda_{g_m} - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(x) - f(x_k)) d\lambda_g \\
&\quad + \sum_{k=1}^n f(x_k)(g_m(x_k) - g_m(x_{k-1})) - \sum_{k=1}^n f(x_k)(g(x_k) - g(x_{k-1})) \\
&= \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(x) - f(x_k)) d\lambda_{g_m} - \sum_{k=1}^n \int_{x_{k-1}}^{x_k} (f(x) - f(x_k)) d\lambda_g \\
&\quad + \sum_{k=1}^n f(x_k)(g_m(x_k) - g(x_k)) - \sum_{k=1}^n f(x_k)(g_m(x_{k-1}) - g(x_{k-1})).
\end{aligned}$$

Then, $\left|\int_a^b f\,d\lambda_{g_m} - \int_a^b f\,d\lambda_g\right| \le \sum_{k=1}^{n}\left|\int_{x_{k-1}}^{x_k} \left(f(x)-f(x_k)\right)d\lambda_{g_m}\right| + \sum_{k=1}^{n}\left|\int_{x_{k-1}}^{x_k} \left(f(x)-f(x_k)\right)d\lambda_g\right|$

$$+\sum_{k=1}^{n}|f(x_k)||g_m(x_k)-g(x_k)| + \sum_{k=1}^{n}|f(x_k)||g_m(x_{k-1})-g(x_{k-1})|.$$

Let M be the maximum value of $|f(x)|$ on $[a, b]$. Then on account of (3) and Theorem 3 below,

$$\left|\int_a^b f\,d\lambda_{g_m} - \int_a^b f\,d\lambda_g\right| \le \frac{\varepsilon}{3K}\sum_{k=1}^{n} V_{g_m}[x_{k-1},x_k] + \frac{\varepsilon}{3K}\sum_{k=1}^{n} V_g[x_{k-1},x_k]$$

$$+M\sum_{k=1}^{n}|g_m(x_k)-g(x_k)| + M\sum_{k=1}^{n}|g_m(x_{k-1})-g(x_{k-1})|$$

$$\le \frac{\varepsilon}{3K}V_{g_m}[a,b] + \frac{\varepsilon}{3K}V_g[a,b]$$

$$+M\sum_{k=1}^{n}|g_m(x_k)-g(x_k)| + M\sum_{k=1}^{n}|g_m(x_{k-1})-g(x_{k-1})|$$

$$\le \frac{2\varepsilon}{3} + +M\sum_{k=1}^{n}|g_m(x_k)-g(x_k)| + M\sum_{k=1}^{n}|g_m(x_{k-1})-g(x_{k-1})|. \text{ ------ (6)}$$

For $0 \le k \le n$, $g_m(x_k) \to g(x_k)$. Therefore, for $0 \le k \le n$ there exists an integer $N_k > 0$ such that

$$m \ge N_k \Rightarrow |g_m(x_k)-g(x_k)| < \frac{\varepsilon}{6nM}. \text{ -------------------- (7)}$$

Let $N = \max\{N_k : 0 \le k \le n\}$. It follows from (6) and (7) that

$$m \ge N \Rightarrow \left|\int_a^b f\,d\lambda_{g_m} - \int_a^b f\,d\lambda_g\right| < \frac{2\varepsilon}{3} + M\sum_{k=1}^{n}\frac{\varepsilon}{6nM} + M\sum_{k=1}^{n}\frac{\varepsilon}{6nM} = \varepsilon.$$

It *follows* that $\lim_{n\to\infty}\int_a^b f d\lambda_{g_n} = \int_a^b f d\lambda_g$.

Remark. Theorem 2 is known as Helly's Second Theorem.

Theorem 3. Suppose $g:[a,b]\to\mathbb{R}$ is a function of bounded variation defined on the closed and bounded interval $I=[a,b]$. Suppose $f: I \to \mathbb{R}$ is a continuous function defined on I or f is Riemann Stieltjes integrable or Lebesgue Stieltjes integrable on [a, b]. Then $\left|\int_a^b f d\lambda_g\right| \le M(f)V(g,I)$, where $M(f)$ is the maximum value of $|f|$ on I and $V(g,I)$ is the total variation of g on I.

Proof. $\int_a^b f d\lambda_g = \int_a^b f d\mu_P - \int_a^b f d\mu_N$, where P and N are the positive and negative variation functions of g. Then,

$$\left|\int_a^b f d\lambda_g\right| \le \left|\int_a^b f d\mu_P\right| + \left|\int_a^b f d\mu_N\right| \le \int_a^b |f| d\mu_P + \int_a^b |f| d\mu_N$$

$$\le \int_a^b M(f) d\mu_P + \int_a^b M(f) d\mu_N = M(f)\big(P([a,b]) + N((la,b])\big) = M(f)V(g,I).$$

We present another proof of Theorem 2 using the Helly Selection Theorem.

Helly Theorem. A uniformly bounded sequence of increasing functions defined on a closed and bounded interval [a, b] contains a subsequence which converges at every point of [a, b] to an increasing function.

Another Proof of Theorem 2.

Suppose (g_n) is a sequence of functions defined on the closed and bounded interval [a, b] whose total variations are uniformly bounded by a constant K, i.e., $V(g_n,[a,b]) \le K$. Suppose the sequence (g_n) converges pointwise at every point of [a, b] to a function g on [a, b].

Let P_n and N_n be the positive and negative variation of functions g_n. Then $g_n(x) = g_n(a) + P_n(x) - N_n(x)$ and the total variation function of g_n is given by $V_g(x) = V_g[a,x] = P_n(x) + N_n(x)$. The function $g_n(a) + P_n(x)$ is an increasing function,

Note that $|g_n(a) + P_n(x)| \le |g_n(a)| + P_n(x) + N_n(x) \le |g_n(a)| + V(g_n[a,b]) \le |g_n(a)| + K$.
Since the sequence $(g_n(a))$ is convergent, it is bounded, that is there exists a constant $C > 0$ such that $|g_n(a)| \le C$ for all positive integer n. Hence, $|g_n(a) + P_n(x)| \le C + K$ for all positive integer n. Thus, the sequence $(g_n(a) + P_n(x))$ is uniformly bounded. Therefore, by the Helly selection Theorem it has a subsequence $\left(g_{n_k}(a) + P_{n_k}(x)\right)$ which converges pointwise to an increasing function $P^*(x)$. By replacing the sequence (g_n) with the subsequence $\left(g_{n_k}\right)$ we

may assume that $g_n(x)=g_n(a)+P_n(x)-N_n(x)$ and the sequence $\left(g_n(a)+P_n(x)\right)$ converges pointwise to $P^*(x)$. Similarly, since the sequence $\left(N_n(x)\right)$ is uniformly bounded by *K*, it has a convergent subsequence $\left(N_{n_k}(x)\right)$ converging pointwise to an increasing function $N^*(x)$ on [*a*, *b*]. By replacing the sequence $\left(g_n\right)$ with the subsequence $\left(g_{n_k}\right)$, we may assume that

$g_n(x)=g_n(a)+P_n(x)-N_n(x)$ converges pointwise to *g*, the sequence $\left(g_n(a)+P_n(x)\right)$ converges pointwise to an increasing function $P^*(x)$ and the sequence $\left(N_n(x)\right)$ converging pointwise to an increasing function $N^*(x)$ on [*a*, *b*]. It follows that the limiting function $g(x)=P^*(x)-N^*(x)$ is a function of bounded variation whose total variation is bounded by *C*+2*K*.

If $f:[a,b]\to\mathbb{R}$ is a continuous function, then as in the proof of Theorem 1 but not using Theorem 3, we can show that $\lim_{n\to\infty}\int_a^b fd\mu_{P_n+g(a)}=\int_a^b fd\mu_{P^*}$ and that $\lim_{n\to\infty}\int_a^b fd\mu_{N_n}=\int_a^b fd\mu_{N^*}$. Therefore,

$$\lim_{n\to\infty}\int_a^b fd\lambda_{g_n}=\lim_{n\to\infty}\int_a^b fd\lambda_{P_n+g_n(a)-N_n}=\lim_{n\to\infty}\left(\int_a^b fd\mu_{P_n+g_n(a)}-\int_a^b fd\mu_{N_n}\right)$$

$$=\lim_{n\to\infty}\int_a^b fd\mu_{P_n+g_n(a)}-\lim_{n\to\infty}\int_a^b fd\mu_{N_n}=\int_a^b fd\mu_{P^*}-\int_a^b fd\mu_{N^*}=\int_a^b fd\lambda_{P^*-N^*}=\int_a^b fd\lambda_g\ .$$

Now we shall investigate the relaxation of the condition of Theorem 1.

We shall do this in stages.

Theorem 4.

Suppose $\phi:I\to\mathbb{R}$ is an increasing function on the closed and bounded interval $I=[a,b]$, where *a* < *b*. Suppose ϕ is right continuous or left continuous. Suppose *g* is a Borel measurable non-negative function on *I*. Define $\Phi:I\to\mathbb{R}$, by $\Phi(x)=\int_a^x gd\mu_\phi$ where μ_ϕ is the Lebesgue Stieljes measure associated with the function ϕ. Suppose $f:[a,b]\to\mathbb{R}$ is a Borel measurable function. Then

$$\int_a^b fd\mu_\Phi=\int_a^b fgd\mu_\phi$$

Proof. Suppose ϕ is right continuous. Then by Theorem 45 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\Phi(x)=\int_a^x g d\mu_\phi=\int_{\varphi(a)}^{\phi(x)} g\circ v(y)dy,$$

where v is the generalised left continuous inverse of ϕ defined in Definition 38 of the above cited article. Note that v is an increasing left continuous function on *J*, where $J=[\phi(a),\phi(b)]$ is the smallest interval containing the image of ϕ.

Let $\Gamma: J\to\mathbb{R}$ be defined by $\Gamma(y)=\int_{\varphi(a)}^{y} g\circ v(t)dt$. Then Γ is absolutely continuous and increasing on *J*. Then $\Phi(x)=\Gamma\circ\phi(x)$.

$$\int_a^b f d\mu_\Phi=\int_a^b f d\mu_{\Gamma\circ\phi}.$$

By Theorem 58 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\int_a^b f d\mu_{\Gamma\circ\phi}=\int_{\varphi(a)}^{\phi(b)} f\circ v(y)d\mu_\Gamma,$$

where v is the generalised left continuous inverse of ϕ.

Since Γ is absolutely continuous,

$$\int_{\varphi(a)}^{\phi(b)} f\circ v(y)d\mu_\Gamma=\int_{\varphi(a)}^{\phi(b)} f\circ v(y)\cdot\Gamma'(y)dy=\int_{\varphi(a)}^{\phi(b)} f\circ v(y)\cdot g\circ v(y)dy=\int_a^b f(x)\cdot g(x)d\mu_\phi.$$

Suppose ϕ is left continuous. Then by Theorem 45 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\Phi(x)=\int_a^x g d\mu_\phi=\int_{\varphi(a)}^{\phi(x)} g\circ\eta(y)dy,$$

where η is the generalised right continuous inverse of ϕ defined in Definition 38 of the above cited article. Note that η is an increasing right continuous function on *J*, where $J=[\phi(a),\phi(b)]$ is the smallest interval containing the image of ϕ.

Let $\Gamma: J\to\mathbb{R}$ be defined by $\Gamma(y)=\int_{\varphi(a)}^{y} g\circ\eta(t)dt$. Then Γ is absolutely continuous and increasing on *J*. Then $\Phi(x)=\Gamma\circ\phi(x)$.

$$\int_a^b f d\mu_\Phi=\int_a^b f d\mu_{\Gamma\circ\phi}.$$

By Theorem 59 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\int_a^b f d\mu_{\Gamma\circ\phi} = \int_{\varphi(a)}^{\phi(b)} f \circ \eta(y) d\mu_{\Gamma} ,$$

where η is the generalised right continuous inverse of ϕ.

Since Γ is absolutely continuous,

$$\int_{\varphi(a)}^{\phi(b)} f \circ \eta(y) d\mu_{\Gamma} = \int_{\varphi(a)}^{\phi(b)} f \circ \eta(y) \cdot \Gamma'(y) dy = \int_{\varphi(a)}^{\phi(b)} f \circ \eta(y) \cdot g \circ \eta(y) dy = \int_a^b f(x) \cdot g(x) d\mu_{\phi} .$$

Remark. The requirement that the function *g* be non-negative can be lifted. This requirement implies that the function $\Gamma(y)$ is increasing and continuous so that we can apply Theorem 58 or Theorem 59 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*".

Theorem 5.

Suppose $\phi : I \to \mathbb{R}$ is an increasing function on the closed and bounded interval $I = [a,b]$, where $a < b$. Suppose ϕ is right continuous or left continuous. Suppose *g* is a Borel measurable function on *I*. Define $\Phi : I \to \mathbb{R}$, by $\Phi(x) = \int_a^x g d\mu_{\phi}$ where μ_{ϕ} is the Lebesgue Stieljes measure associated with the function ϕ. Suppose $f : [a,b] \to \mathbb{R}$ is a Borel measurable function. Then

$$\int_a^b f d\lambda_{\Phi} = \int_a^b fg d\mu_{\phi}$$

Proof.

Suppose ϕ is right continuous.

Then by Theorem 45 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\Phi(x) = \int_a^x g d\mu_{\phi} = \int_{\varphi(a)}^{\phi(x)} g \circ \nu(y) dy ,$$

where ν is the generalised left continuous inverse of ϕ defined in Definition 38 of the above cited article. Note that ν is an increasing left continuous function on *J*, where $J=[\phi(a),\phi(b)]$ is the smallest interval containing the image of ϕ.

Let $\Gamma: J \to \mathbb{R}$ be defined by $\Gamma(y)=\int_{\varphi(a)}^{y} g\circ\nu(t)dt$. Then Γ is absolutely continuous on *J* and so is a function of bounded variation. Then $\Phi(x)=\Gamma\circ\phi(x)$ is a function of bounded variation.

$$\int_a^b fd\lambda_\Phi=\int_a^b fd\lambda_{\Gamma\circ\phi}.$$

By Theorem 64 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*", as Γ is absolutely continuous,

$$\int_a^b fd\lambda_{\Gamma\circ\phi}=\int_{\varphi(a)}^{\phi(b)} f\circ\nu(y)\cdot\Gamma'(y)d\lambda_\Gamma,$$

where ν is the generalised left continuous inverse of ϕ.

Therefore, $\int_a^b fd\lambda_{\Gamma\circ\phi}=\int_{\varphi(a)}^{\phi(b)} f\circ\nu(y)\cdot g\circ\nu(y)dy=\int_a^b f(x)\cdot g(x)d\mu_\phi$.

Suppose ϕ is left continuous.

Then by Part (ii) of Theorem 45 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\Phi(x)=\int_a^x gd\mu_\phi=\int_{\varphi(a)}^{\phi(x)} g\circ\eta(y)dy,$$

where η is the generalised right continuous inverse of ϕ defined in Definition 38 of the above cited article. Note that η is an increasing right continuous function on *J*, where $J=[\phi(a),\phi(b)]$ is the smallest interval containing the image of ϕ.

Let $\Gamma: J \to \mathbb{R}$ be defined by $\Gamma(y)=\int_{\varphi(a)}^{y} g\circ\eta(t)dt$. Then Γ is absolutely continuous on *J* and so is a function of bounded variation. Then $\Phi(x)=\Gamma\circ\phi(x)$ is a function of bounded variation.

$$\int_a^b fd\lambda_\Phi=\int_a^b fd\lambda_{\Gamma\circ\phi}.$$

By Theorem 64 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*" and as Γ is absolutely continuous,

$$\int_a^b f d\lambda_{\Gamma\circ\phi} = \int_{\varphi(a)}^{\phi(b)} f\circ\eta(y)\cdot\Gamma'(y)d\lambda_\Gamma = \int_{\varphi(a)}^{\phi(b)} f\circ\eta(y)\cdot g\circ\eta(y)dy = \int_a^b f(x)\cdot g(x)d\mu_\phi\,,$$

We extend the result of Theorem 5, when the function $\phi: I \to \mathbb{R}$ is a function of bounded variation.

Corollary 6.

Suppose $\phi: I \to \mathbb{R}$ is a function of bounded variation on the closed and bounded interval $I = [a,b]$, where $a < b$. Suppose ϕ is right continuous or left continuous. Suppose g is a Borel measurable function on I. Define $\Phi: I \to \mathbb{R}$, by $\Phi(x) = \int_a^x g d\lambda_\phi$ where λ_ϕ is the Lebesgue Stieljes measure associated with the function ϕ. Suppose $f:[a,b]\to\mathbb{R}$ is a Borel measurable function. Then

$$\int_a^b f d\lambda_\Phi = \int_a^b fg d\lambda_\phi\,.$$

Proof. Let V_ϕ be the total variation of ϕ. Then V_ϕ and $V_\phi - \phi$ are both increasing functions.

Suppose ϕ is right continuous. It follows that V_ϕ is right continuous. Hence, $\phi_1 = V_\phi$ and $\phi_2 = V_\phi - \phi$ are both right continuous. Note that $\phi = \phi_1 - \phi_2$ Then $\Phi(x) = \int_a^x g d\lambda_\phi = \int_a^x g d\mu_{\phi_1} - \int_a^x g d\mu_{\phi_2}$ is a difference of two functions of bounded variation and so is of bounded variation. Let $\Phi_1(x) = \int_a^x g d\mu_{\phi_1}$ and $\Phi_2(x) = \int_a^x g d\mu_{\phi_2}$. Hence,

$$\int_a^b f d\lambda_\Phi = \int_a^b f d\lambda_{\Phi_1 - \Phi_2} = \int_a^b f d\lambda_{\Phi_1} - \int_a^b f d\lambda_{\Phi_2}\,. \text{ ------------------ } (1)$$

By Theorem 5, $\int_a^b f d\lambda_{\Phi_1} = \int_a^b fg d\mu_{\phi_1}$ and $\int_a^b f d\lambda_{\Phi_2} = \int_a^b fg d\mu_{\phi_2}$. It follows from (1) that

$$\int_a^b f d\lambda_\Phi = \int_a^b f d\lambda_{\Phi_1 - \Phi_2} = \int_a^b fg d\mu_{\phi_1} - \int_a^b fg d\mu_{\phi_2} = \int_a^b fg d\lambda_{\phi_1 - \phi_2} = \int_a^b fg d\lambda_\phi\,.$$

Suppose ϕ is left continuous. It follows $\phi_1 = V_\phi$ and $\phi_2 = V_\phi - \phi$ are both left continuous. It follows similarly as above that $\int_a^b f d\lambda_\Phi = \int_a^b fg d\lambda_\phi$.

More generally we have

Corollary 7.

Suppose $\phi: I \to \mathbb{R}$ is a function of bounded variation on the closed and bounded interval $I=[a,b]$, where $a < b$. Suppose ϕ is the difference or sum of two increasing functions ϕ_1 and ϕ_2. Suppose ϕ_1 and ϕ_2 are both right continuous or left continuous or ϕ_1 is right continuous and ϕ_2 is left continuous or ϕ_1 is left continuous and ϕ_2 is right continuous. Suppose *g* is a Borel measurable function on *I*. Define $\Phi: I \to \mathbb{R}$, by $\Phi(x)=\int_a^x g d\lambda_\phi$ where λ_ϕ is the Lebesgue Stieljes measure associated with the function ϕ. Suppose $f:[a,b] \to \mathbb{R}$ is a Borel measurable function. Then

$$\int_a^b f d\lambda_\Phi = \int_a^b fg d\lambda_\phi .$$

The proof of Corollary 7 is similar to that of Corollary 6 and is omitted.

We now only require the function $\phi: I \to \mathbb{R}$ to be of bounded variation.

Theorem 8. Suppose $\phi: I \to \mathbb{R}$ is a function of bounded variation on the closed and bounded interval $I=[a,b]$, where $a < b$. Suppose *g* is a Borel measurable function on *I*. Define $\Phi: I \to \mathbb{R}$, by $\Phi(x)=\int_a^x g d\lambda_\phi$ where λ_ϕ is the Lebesgue Stieltjes measure associated with the function ϕ. Suppose $f:[a,b] \to \mathbb{R}$ is a Borel measurable function. Then

$$\int_a^b f d\lambda_\Phi = \int_a^b fg d\lambda_\phi .$$

Proof. As detailed in the proof of Corollary 62 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*", an increasing function on the interval [*a*, *b*] can be decomposed as a sum of increasing continuous function, increasing right continuous function and an increasing left continuous function. More precisely, an increasing function ϕ on [*a*, *b*] can be written as

$$\phi = \Phi_{ac} + \Phi_c + \Phi_{ls} + \Phi_{rs} ,$$

where Φ_{ac} is an absolutely continuous increasing function with $\Phi_{ac}'(x) = \phi'(x)$ almost everywhere on [*a*, *b*], Φ_c is a continuous increasing singular function, i.e., $\Phi_c'(x) = 0$ almost everywhere, Φ_{ls} is a right continuous increasing saltus type function and Φ_{rs} is a left continuous increasing function. Let $\Phi_a = \Phi_{ac} + \Phi_c$. Then Φ_a is an increasing continuous function. Thus, $\phi = \Phi_a + \Phi_{ls} + \Phi_{rs}$.

Suppose $\phi : I \to \mathbb{R}$ is a function of bounded variation. Then $\phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are increasing functions. Then $\Phi(x) = \int_a^x g d\lambda_\phi = \int_a^x g d\mu_{\phi_1} - \int_a^x g d\mu_{\phi_2}$. Let $\Phi_1(x) = \int_a^x g d\mu_{\phi_1}$ and $\Phi_2(x) = \int_a^x g d\mu_{\phi_2}$. Hence,

$$\int_a^b f d\lambda_\Phi = \int_a^b f d\lambda_{\Phi_1 - \Phi_2} = \int_a^b f d\lambda_{\Phi_1} - \int_a^b f d\lambda_{\Phi_2}.$$

Now $\phi_1 = \Phi_{a,1} + \Phi_{ls,1} + \Phi_{rs,1}$ is a sum of continuous increasing function, left continuous increasing function and right continuous increasing function. Therefore, by Corollary 7, $\int_a^b f d\lambda_{\Phi_1} = \int_a^b fg d\mu_{\phi_1}$. Similarly, we deduce that $\int_a^b f d\lambda_{\Phi_2} = \int_a^b fg d\mu_{\phi_2}$. It follows that

$$\begin{aligned}\int_a^b f d\lambda_\Phi &= \int_a^b f d\lambda_{\Phi_1 - \Phi_2} = \int_a^b f d\lambda_{\Phi_1} - \int_a^b f d\lambda_{\Phi_2} \\ &= \int_a^b fg d\mu_{\phi_1} - \int_a^b fg d\mu_{\phi_2} = \int_a^b fg d\lambda_{\phi_1 - \phi_2} \\ &= \int_a^b fg d\lambda_\phi .\end{aligned}$$

Remark. In the proof of Theorem 8, we have that

$$\phi = \phi_1 - \phi_2,\ \phi_1 = \Phi_{a,1} + \Phi_{ls,1} + \Phi_{rs,1} \text{ and } \phi_2 = \Phi_{a,2} + \Phi_{ls,2} + \Phi_{rs,2}.$$

Thus,

$$\begin{aligned}\phi &= \Phi_{a,1} + \Phi_{ls,1} + \Phi_{rs,1} - \left(\Phi_{a,2} + \Phi_{ls,2} + \Phi_{rs,2}\right) \\ &= \left(\Phi_{a,1} - \Phi_{a,2} + \Phi_{ls,1} - \Phi_{ls,2}\right) + \left(\Phi_{rs,1} - \Phi_{rs,2}\right)\end{aligned}$$

is a sum of right continuous function of bounded variation and left continuous function of bounded variation. Let $\phi_3 = \Phi_{a,1} - \Phi_{a,2} + \Phi_{ls,1} - \Phi_{ls,2}$ and $\phi_4 = \Phi_{rs,1} - \Phi_{rs,2}$. Then $\phi = \phi_3 + \phi_4$. Let $\Phi_3(x) = \int_a^x g d\mu_{\phi_3}$ and $\Phi_4(x) = \int_a^x g d\mu_{\phi_4}$. Then $\Phi(x) = \int_a^x g d\lambda_\phi = \int_a^x g d\lambda_{\phi_3} + \int_a^x g d\lambda_{\phi_4} = \Phi_3(x) + \Phi_4(x)$. By Corollary 6,

$$\int_a^b f d\lambda_{\Phi_3} = \int_a^b fg d\lambda_{\phi_3} \text{ and } \int_a^b f d\lambda_{\Phi_4} = \int_a^b fg d\lambda_{\phi_4}.$$

Therefore,

$$\int_a^b f d\lambda_\Phi = \int_a^b f d\lambda_{\Phi_3+\Phi_4} = \int_a^b f d\lambda_{\Phi_3} + \int_a^b f d\lambda_{\Phi_4} = \int_a^b fg d\lambda_{\phi_3} + \int_a^b fg d\lambda_{\phi_4}$$

$$= \int_a^b fg d\lambda_{\phi_3+\phi_4} = \int_a^b fg d\lambda_\phi .$$

This gives another proof of Theorem 8.

We note that if a function *f* is Riemann Stieltjes integrable, then it is Lebesgue Stieltjes integrable and the integrals coincide. In what follows, we shall adopt the notation for Riemann Stieltjes integral in the same way as the Lebesgue Stieltjes integral. That is, if *g* the integrator is monotone, the Lebesgue Stieltjes integral of *f* is denoted by $\int_a^b f(x) d\mu_g$ and if *g* is of bounded variation, the Lebesgue Stieltjes integral of *f* is denoted by $\int_a^b f(x) d\lambda_g$ following the notation and convention in my article, "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*". Whenever the function is Riemann Stieltjes integrable, it will be stated so that the notation $\int_a^b f(x) d\mu_g$ or $\int_a^b f(x) d\lambda_g$ will mean the Riemann Stieltjes integrals as they are the same.

Next, we have a variation of Theorem 2 due to L. C. Young.

Theorem 9. Suppose $(g_n : [a,b] \to J)$ is a sequence of function converging uniformly to a continuous function $g : [a,b] \to J$, where *J* is a closed and bounded interval. Suppose $\phi : J \to \mathbb{R}$ is a function of bounded variation such that the total variation of the composition functions $\phi \circ g_n : [a,b] \to \mathbb{R}$, $V(\phi \circ g_n, [a,b]) < K$ for some $K > 0$ and for all positive integer *n*. Suppose either $f \circ g_n$ is Riemann Stieltjes integrable with respect to $\phi \circ g_n$ for all *n* or g_n is continuous for all *n*.

Suppose the function ϕ is continuous and $f : J \to \mathbb{R}$ is a continuous function. Then $\phi \circ g$ is of bounded variation and

$$\int_a^b f(g_n(x)) d\lambda_{\phi \circ g_n} \to \int_a^b f(g(x)) d\lambda_{\phi \circ g} .$$

Proof.

We note that since *f* is continuous on *J*, *f* is uniformly continuous on *J*. Therefore, given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all *x*, *y* in J,

$$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{K}\ . \quad \text{--------------------- (1)}$$

Since g_n converges to g uniformly, there exists an integer $N > 0$ such that

$$n > N \Rightarrow |g_n(x)-g(x)|<\delta \text{ for all } x \text{ in } [a, b].$$

It follows from (1) that

$$n > N \Rightarrow |g_n(x)-g(x)|<\delta \Rightarrow |f\circ g_n(x)-f\circ g(x)|<\frac{\varepsilon}{K} \text{ for all } x \text{ in } [a, b].$$

Therefore, for all $n > N$,

$$\left|\int_a^b \left(f(g_n(x))-f(g(x))\right)d\lambda_{\phi\circ g_n}\right|<\frac{\varepsilon}{K}V(\phi\circ g_n,[a,b]<\varepsilon\ .$$

It follows that $\lim_{n\to\infty}\left|\int_a^b \left(f(g_n(x))-f(g(x))\right)d\lambda_{\phi\circ g_n}\right|=0$ ------------------- (2)

Note that as ϕ is continuous, $\phi\circ g_n$ converges pointwise to $\phi\circ g$. Therefore, by Theorem 2, $\phi\circ g$ is of finite variation and $\lim_{n\to\infty}\int_a^b f(g(x))d\lambda_{\phi\circ g_n}=\int_a^b f(g(x))d\lambda_{\phi\circ g}$. In view of (2),

$$\lim_{n\to\infty}\int_a^b \left(f(g_n(x))\right)d\lambda_{\phi\circ g_n}=\int_a^b f(g(x))d\lambda_{\phi\circ g}\ .$$

We have next a change of variable theorem.

Theorem 10.

Suppose $g: I\to\mathbb{R}$ is a continuous function on the closed and bounded interval $I=[a,b]$, where $a < b$. Let the image of I under g be $J=[c, d]$. Let $\phi: J\to\mathbb{R}$ be a function of bounded variation. Let $f: J=[c,d]\to\mathbb{R}$ be a continuous function.

If (i) g is an increasing function, or a decreasing function, or

(ii) ϕ is continuous on J and $\phi\circ g$ is a function of bounded variation, then

$$\int_a^b f\circ g d\lambda_{\phi\circ g}=\int_{g(a)}^{g(b)} f d\lambda_\phi\ .$$

Proof.

(i) If $g: I\to\mathbb{R}$ is an increasing continuous function, Theorem 9 (i) follows from Corollary 62 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes*

Integrals". If $g: I \to \mathbb{R}$ is a decreasing continuous function, Theorem 9 (i) follows from an analogue result of Corollary 62 of the above cited article.

(ii) Suppose ϕ is continuous on J and $\phi \circ g$ is a function of bounded variation.

Take a sequence of piecewise linear continuous functions g_n such that $g_n(a) = g(a)$, $g_n(b) = g(b)$ and converging uniformly to g. Note that the total variation $V(\phi \circ g_n, [a,b]) \leq V(\phi \circ g, [a,b])$.

Corresponding to a g_n, we have a partition of $[a, b]$,

$P: a = x_0 < x_1 < \cdots < x_m = b$ such that g is linear on each subinterval $[x_{i-1}, x_i]$ and so is monotone on each subinterval $[x_{i-1}, x_i]$. Therefore, by part (i), for each integer i, $1 \leq i \leq m$,

$$\int_{x_{i-1}}^{x_i} f(g_n(x)) d\lambda_{\phi \circ g_n} = \int_{g_n(x_{i-1})}^{g_n(x_i)} f d\lambda_\phi = \int_{g_n(x_{i-1})}^{g_n(x_i)} f d\lambda_\phi \ .$$

It follows that

$$\int_a^b f(g_n(x)) d\lambda_{\phi \circ g_n} = \sum_{i=1}^m \int_{x_{i-1}}^{x_i} f(g_n(x)) d\lambda_{\phi \circ g_n}$$

$$= \sum_{i=1}^m \int_{g(x_{i-1})}^{g(x_i)} f d\lambda_\phi = \int_{g_n(a)}^{g_n(b)} f d\lambda_\phi = \int_{g(a)}^{g(b)} f d\lambda_\phi$$

Therefore, by Theorem 9, $\int_a^b f(g(x)) d\lambda_{\phi \circ g} = \int_{g(a)}^{g(b)} f d\lambda_\phi$.

If the function g is the uniform limit of continuous polygonal functions or piecewise linear functions with some condition on the total variation of the function involved, we have the next change of variable theorem.

Theorem 11. Suppose $(g_n : [a,b] \to J)$ is a sequence of continuous polygonal function converging uniformly to a continuous function $g: [a,b] \to J$, where J is a closed and bounded interval. Suppose $g_n(a) = g(a)$ and $g_n(b) = g(b)$. Suppose $\phi: J \to \mathbb{R}$ is a function of bounded variation such that the total variation of the composition functions $\phi \circ g_n : [a,b] \to \mathbb{R}$, $V(\phi \circ g_n, [a,b]) < K$ for some $K > 0$ and for all positive integer n. Suppose $f: J \to \mathbb{R}$ is a continuous function. Suppose $\phi \circ g_n$ converges to $\phi \circ g$ pointwise on $[a, b]$. Then $\phi \circ g$ is of bounded variation, $\int_a^b f(g_n(x)) d\lambda_{\phi \circ g_n} \to \int_a^b f(g(x)) d\lambda_{\phi \circ g}$ and $\int_a^b f(g(x)) d\lambda_{\phi \circ g} = \int_{g(a)}^{g(b)} f(y) d\lambda_\phi$.

Proof.

Since g_n is polygonal, there exists a partition $P: a = x_0 < x_1 < \cdots < x_m = b$ such that g is monotone on each subinterval $[x_{i-1}, x_i]$.

$$\int_a^b f(g_n(x))d\lambda_{\phi\circ g_n} = \sum_{i=1}^{m}\int_{x_{i-1}}^{x_i} f(g_n(x))d\lambda_{\phi\circ g_n} = \sum_{i=1}^{m}\int_{g_n(x_{i-1})}^{g_n(x_i)} f(y)d\lambda_\phi \text{ , by Theorem 10 (i),}$$

$$= \int_{g_n(a)}^{g_n(b)} fd\lambda_\phi = \int_{g(a)}^{g(b)} fd\lambda_\phi$$

We can show as in the proof of Theorem 9, using the fact that $\phi\circ g_n$ converges to $\phi\circ g$ that $\phi\circ g$ is of bounded variation.

We have shown in the proof of Theorem 9 that

$$\lim_{n\to\infty}\left|\int_a^b \big(f(g_n(x)) - f(g(x))\big)d\lambda_{\phi\circ g_n}\right| = 0 .$$

By Theorem 2, as $f\circ g$ is continuous and $V(\phi\circ g_n,[a,b]) < K$ for all n,

$$\int_a^b f(g(x))d\lambda_{\phi\circ g_n} \to \int_a^b f(g(x))d\lambda_{\phi\circ g} .$$

Hence, $\int_a^b f(g_n(x))d\lambda_{\phi\circ g_n} \to \int_a^b f(g(x))d\lambda_{\phi\circ g}$. Therefore,

$$\int_a^b f(g(x))d\lambda_{\phi\circ g} = \int_{g(a)}^{g(b)} f(y)d\lambda_\phi .$$

The next theorem is due to L.C. Young. It extends Theorem 2.

Theorem 12. Suppose $[a, b]$ is a closed and bounded interval with $a < b$. Suppose $(f_n : [a,b] \to \mathbb{R})$ is a sequence of functions converging uniformly to a continuous function $f : [a,b] \to \mathbb{R}$. Suppose $(g_n : [a,b] \to \mathbb{R})$ is a sequence of functions and g_n converges on a dense set E in $[a, b]$ with $\{a,b\} \subseteq E$ to a function g on E. That is to say, $g_n(x) \to g(x)$ for every x in E. Suppose the Riemann Stieltjes integral $\int_a^b f_n d\lambda_{g_n}$ exists for each positive integer n. Suppose each g_n is of bounded variation such that the total variation $V(g_n,[a,b]) < K$ for all positive integer n for some $K > 0$. Then there exists a subsequence $\left(g_{n_k} : [a,b] \to \mathbb{R}\right)$ such that $g_{n_k}(x)$ tends pointwise to a function h of bounded variation on $[a, b]$. Moreover, $h(x) = g(x)$ for every x in E. We extend g to all of $[a, b]$ by $g(x) = h(x)$ for $x \notin E$. Then

$$\int_a^b f_n(x)d\lambda_{g_n} \to \int_a^b f(x)d\lambda_g \ .$$

Proof. We show that we can find a subsequence $\left(g_{n_k}:[a,b]\to\mathbb{R}\right)$ which converges to an extension of g to $[a, b]$.

Let P_n and N_n be the positive and negative variation of g_n with $P_n(a)=N_n(a)=0$. Then $g_n(x)=g_n(a)+P_n(x)-N_n(x)$ for x in $[a, b]$. Therefore,

$$|g_n(x)|\le|g_n(a)|+P_n(x)+N_n(x)=|g_n(a)|+V(g_n,[a,x])\le|g_n(a)|+V(g_n,[a,b]) \ .$$

Since $g_n(a)$ is convergent, $\{|g_n(a)|\}$ is bounded, say by B. Therefore,

$|g_n(x)|\le B+K=C$ for all x in $[a, b]$ and for all positive integer n.

Thus, $|g_n(x)|\le C$ for all x in $[a, b]$ and $V(g_n,[a,b])<C$ for all positive integer n.

Therefore, by Helly's First Theorem, there exists a subsequence $\left(g_{n_k}:[a,b]\to\mathbb{R}\right)$ such that g_{n_k} converges at every point of $[a, b]$ to a function h of bounded variation on $[a, b]$. Plainly, $h(x)=g(x)$ for x in E. Thus, h is an extension of g to a function of bounded variation on $[a, b]$. Note that the total variation of h is also less than K. (See the proof of Theorem 2.)

We now rename h as g.

Since f is continuous on $[a, b]$ and $[a, b]$ is compact, f is uniformly continuous. Therefore, given $\varepsilon>0$, there exists $\delta>0$ such that for all $x,y\in[a,b]$,

$$|x-y|<\delta\Rightarrow|f(x)-f(y)|<\frac{\varepsilon}{6K} \ . \quad \text{--------------------------------} \quad (1).$$

We now consider subdivide the interval $[a, b]$ into subintervals so that the oscillation of f on each subinterval is less than ε. We further want the partition points of the subinterval to be points in the dense subset E. Because E is dense in $[a, b]$, we can choose the partition $P: a=x_0<x_1<\cdots<x_N=b$ such that $x_i\in E$ for $0\le i\le N$, $|x_i-x_{i-1}|<\delta$ for $1\le i\le N$ so that

$$|f(x_i)-f(x_{i-1})|<\frac{\varepsilon}{6K} \text{ for } 1\le i\le N. \quad \text{---------------------} \ (2)$$

In view of (1) for each integer i with $1\le i\le N$, we select a point $a_i\in f([x_{i-1},x_i])$ such that

$$|f(x)-a_i|<\frac{\varepsilon}{6K} \text{ for all } x\in(x_{i-1},x_i) \ . \quad \text{--------------------} \ (3)$$

Since f is continuous, we shall assume all Stieltjes integrals are Riemann Stieltjes integrals.

$$\int_a^b f(x)d\lambda_{g_n-g} = \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} f(x)d\lambda_{g_n-g}$$

$$= \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}\left(f(x)-a_i\right)d\lambda_{g_n-g} + \sum_{i=1}^{N} a_i\int_{x_{i-1}}^{x_i} d\lambda_{g_n-g} \;. \text{ -------------------- (4)}$$

$$\sum_{i=1}^{N} a_i\int_{x_{i-1}}^{x_i} d\lambda_{g_n-g} = \sum_{i=1}^{N} a_i\left((g_n-g)(x_i)-(g_n-g)(x_{i-1})\right)$$

$$= \sum_{i=1}^{N} a_i\left((g_n-g)(x_i)\right) - \sum_{i=1}^{N} a_i\left((g_n-g)(x_{i-1})\right)$$

$$= \sum_{i=1}^{N-1} a_i\left((g_n-g)(x_i)\right) + a_N\left((g_n-g)(x_N)\right) - \sum_{i=0}^{N-1} a_{i+1}\left((g_n-g)(x_i)\right)$$

$$= \sum_{i=1}^{N-1} a_i\left((g_n-g)(x_i)\right) + a_N\left((g_n-g)(x_N)\right) - \sum_{i=1}^{N-1} a_{i+1}\left((g_n-g)(x_i)\right) - a_1\left((g_n-g)(x_0)\right)$$

$$= \sum_{i=1}^{N-1} (a_i-a_{i+1})\left((g_n-g)(x_i)\right) + a_N\left((g_n-g)(x_N)\right) - a_1\left((g_n-g)(x_0)\right)$$

$$= \sum_{i=0}^{N} b_i\left((g_n-g)(x_i)\right), \qquad \text{------------------------------------- (5)}$$

where $b_i = (a_i - a_{i+1})$, $1 \le i \le N-1$, $b_0 = -a_1$ and $b_N = a_N$.

Let $B = \sum_{i=0}^{N} |b_i|$.

For each integer i, $0 \le i \le N$, since $g_n(x_i)$ converges to $g(x_i)$, there exists positive integer M_i such that

$$n > M_i \Rightarrow |(g_n-g)(x_i)| < \frac{\varepsilon}{3B} \;. \text{ --------------------- (6)}$$

Let $M = \max\{M_0, M_1, \cdots, M_N\}$. Then

$$n > M \Rightarrow \left|\sum_{i=1}^{N} a_i\int_{x_{i-1}}^{x_i} d\lambda_{g_n-g}\right| = \left|\sum_{i=0}^{N} b_i\left((g_n-g)(x_i)\right)\right|$$

$$\le \sum_{i=0}^{N} |b_i|\left|\left((g_n-g)(x_i)\right)\right| < \sum_{i=0}^{N} |b_i|\frac{\varepsilon}{3B} = \frac{\varepsilon}{3} \;. \text{ ------- (7)}$$

For each i, with $0 \le i \le N$,

$$\left|\int_{x_{i-1}}^{x_i}\left(f(x)-a_i\right)d\lambda_{g_n-g}\right| \le \frac{\varepsilon}{6K}Var(g_n-g,[x_{i-1},x_i]) \text{ by Theorem 3,}$$

$$\le \frac{\varepsilon}{6K}\left(Var(g_n,[x_{i-1},x_i])+Var(g,[x_{i-1},x_i])\right).$$

It follows that

$$\left|\int_a^b\left(f(x)-a_i\right)d\lambda_{g_n-g}\right| = \left|\sum_{i=1}^N\int_{x_{i-1}}^{x_i}\left(f(x)-a_i\right)d\lambda_{g_n-g}\right| \le \sum_{i=1}^N\left|\int_{x_{i-1}}^{x_i}\left(f(x)-a_i\right)d\lambda_{g_n-g}\right|$$

$$\le \frac{\varepsilon}{6K}\sum_{i=1}^N\left(Var(g_n,[x_{i-1},x_i])+Var(g,[x_{i-1},x_i])\right) = \frac{\varepsilon}{6K}\left(Var(g_n,[a,b])+Var(g,[a,b])\right)$$

$$< \frac{\varepsilon}{3}. \text{ ------------------------- (8)}$$

Therefore, it follows from (7) and (8) that for $n > M$,

$$\left|\int_a^b f(x)d\lambda_{g_n-g}\right| \le \left|\sum_{i=1}^N\int_{x_{i-1}}^{x_i}\left(f(x)-a_i\right)d\lambda_{g_n-g}\right| + \left|\sum_{i=1}^N a_i\int_{x_{i-1}}^{x_i}d\lambda_{g_n-g}\right| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\frac{2\varepsilon}{3}. \text{ ------ (9)}$$

Since f_n converges to a continuous f uniformly on $[a, b]$, there exists a positive integer N_0 such that

$$n > N_0 \Rightarrow |f_n(x)-f(x)| < \frac{\varepsilon}{3K} \text{ for all } x \text{ in } [a, b]. \text{ ------------------- (10).}$$

Now we show that the original sequence $\left(\int_a^b f_n(x)d\lambda_{g_n}\right)$ is a Cauchy sequence.

$$\left|\int_a^b f_n(x)d\lambda_{g_n} - \int_a^b f_m(x)d\lambda_{g_m}\right|$$

$$= \left|\int_a^b f_n(x)d\lambda_{g_n} - \int_a^b f(x)d\lambda_{g_n} - \int_a^b f_m(x)d\lambda_{g_m} + \int_a^b f(x)d\lambda_{g_m} + \int_a^b f(x)d\lambda_{g_n} - \int_a^b f(x)d\lambda_{g_m}\right|$$

$$\le \left|\int_a^b f_n(x)d\lambda_{g_n} - \int_a^b f(x)d\lambda_{g_n}\right| + \left|\int_a^b f(x)d\lambda_{g_m} - \int_a^b f_m(x)d\lambda_{g_m}\right| + \left|\int_a^b f(x)d\lambda_{g_n} - \int_a^b f(x)d\lambda_{g_m}\right|$$

$$\le \left|\int_a^b\left(f_n(x)-f(x)\right)d\lambda_{g_n}\right| + \left|\int_a^b\left(f_m(x)-f(x)\right)d\lambda_{g_m}\right| + \left|\int_a^b f(x)d\lambda_{g_n-g} - \int_a^b f(x)d\lambda_{g_m-g}\right|$$

$$\le \left|\int_a^b\left(f_n(x)-f(x)\right)d\lambda_{g_n}\right| + \left|\int_a^b\left(f_m(x)-f(x)\right)d\lambda_{g_m}\right| + \left|\int_a^b f(x)d\lambda_{g_n-g}\right| + \left|\int_a^b f(x)d\lambda_{g_m-g}\right|$$

$$\le \frac{\varepsilon}{3K}V(g_n,[a,b]) + \frac{\varepsilon}{3K}V(g_m,[a,b]) + \frac{2\varepsilon}{3} + \frac{2\varepsilon}{3} \quad \text{if } n,m > \max\{M, N_0\},$$

by (10) and Theorem 3 and (9)

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{4\varepsilon}{3} = 2\varepsilon$ if, $n, m > \max\{M, N_0\}$.

Since ε is arbitrarily chosen, this shows that $\left(\int_a^b f_n(x) d\lambda_{g_n}\right)$ is a Cauchy sequence. It follows that it is convergent.

Next, we show that $\int_a^b f_{n_k}(x) d\lambda_{g_{n_k}} \to \int_a^b f(x) d\lambda_g$

Now there exists a positive integer L such that $k > L \Rightarrow n_k > \max\{M, N_0\}$.

Thus, for $k > L$

$$\left|\int_a^b f_{n_k}(x) d\lambda_{g_{n_k}} - \int_a^b f(x) d\lambda_g\right| = \left|\int_a^b \left(f_{n_k}(x) - f(x)\right) d\lambda_{g_{n_k}} + \int_a^b f(x) d\lambda_{g_{n_k}} - \int_a^b f(x) d\lambda_g\right|$$

$$= \left|\int_a^b \left(f_{n_k}(x) - f(x)\right) d\lambda_{g_{n_k}} + \int_a^b f(x) d\lambda_{g_{n_k} - g}\right|$$

$$\leq \left|\int_a^b \left(f_{n_k}(x) - f(x)\right) d\lambda_{g_{n_k}}\right| + \left|\int_a^b f(x) d\lambda_{g_{n_k} - g}\right|$$

$$\leq \frac{\varepsilon}{3K} V(g_{n_k}, [a,b]) + \left|\int_a^b f(x) d\lambda_{g_{n_k} - g}\right| \text{ by Theorem 3}$$

$$< \frac{\varepsilon}{3K} K + \frac{2\varepsilon}{3} = \varepsilon \text{ by (9).}$$

Hence, $\int_a^b f_{n_k}(x) d\lambda_{g_{n_k}} \to \int_a^b f(x) d\lambda_g$. Thus, $\left(\int_a^b f_n(x) d\lambda_{g_n}\right)$ is a Cauchy sequence that has a convergent subsequence that converges to $\int_a^b f(x) d\lambda_g$ and so $\int_a^b f_n(x) d\lambda_{g_n} \to \int_a^b f(x) d\lambda_g$.

Alternatively, we may use Theorem 2 to show that $\int_a^b f_{n_k}(x) d\lambda_{g_{n_k}} \to \int_a^b f(x) d\lambda_g$

$$\left|\int_a^b f_{n_k}(x) d\lambda_{g_{n_k}} - \int_a^b f(x) d\lambda_g\right| = \left|\int_a^b f_{n_k}(x) d\lambda_{g_{n_k}} - \int_a^b f(x) d\lambda_{g_{n_k}} + \int_a^b f(x) d\lambda_{g_{n_k}} - \int_a^b f(x) d\lambda_g\right|$$

$$\leq \left|\int_a^b f_{n_k}(x) d\lambda_{g_{n_k}} - \int_a^b f(x) d\lambda_{g_{n_k}}\right| + \left|\int_a^b f(x) d\lambda_{g_{n_k}} - \int_a^b f(x) d\lambda_g\right|$$

$$\leq \left|\int_a^b \left(f_{n_k}(x) - f(x)\right) d\lambda_{g_{n_k}}\right| + \left|\int_a^b f(x) d\lambda_{g_{n_k}} - \int_a^b f(x) d\lambda_g\right|.$$

Since f_{n_k} converges uniformly to f, there exists integer $L1$ such that

$$k > L1 \Rightarrow \left| f_{n_k}(x) - f(x) \right| < \frac{\varepsilon}{2K} \text{ for all } x \text{ in } [a, b]$$

Therefore, by Theorem 3, $\left| \int_a^b \left(f_{n_k}(x) - f(x) \right) d\lambda_{g_{n_k}} \right| \le \frac{\varepsilon}{2K} V(g_{n_k}, [a,b]) < \frac{\varepsilon}{2}$.

By Theorem 2, $\int_a^b f(x) d\lambda_{g_{n_k}} \to \int_a^b f(x) d\lambda_g$. Therefore, there exists integer *L2* such that $k > L1 \Rightarrow \left| \int_a^b f(x) d\lambda_{g_{n_k}} - \int_a^b f(x) d\lambda_g \right| < \frac{\varepsilon}{2}$.

Let *L* be an integer such that $L > \max\{L1, L2\}$. Then

$$k > L \Rightarrow \left| \int_a^b f_{n_k}(x) d\lambda_{g_{n_k}} - \int_a^b f(x) d\lambda_g \right|$$

$$\le \left| \int_a^b \left(f_{n_k}(x) - f(x) \right) d\lambda_{g_{n_k}} \right| + \left| \int_a^b f(x) d\lambda_{g_{n_k}} - \int_a^b f(x) d\lambda_g \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, $\int_a^b f_{n_k}(x) d\lambda_{g_{n_k}} \to \int_a^b f(x) d\lambda_g$.

Theorem 13. Suppose $[a, b]$ is a closed and bounded interval with $a < b$. Suppose $(f_n : [a,b] \to \mathbb{R})$ is a sequence of functions converging uniformly to a bounded function $f : [a,b] \to \mathbb{R}$. Suppose $(g_n : [a,b] \to \mathbb{R})$ is a sequence of increasing functions and g_n converges on a dense set *E* in $[a, b]$ with $\{a,b\} \subseteq E$ to a bounded function *g* on *E*. That is to say, $g_n(x) \to g(x)$ for every *x* in *E*. Suppose the Riemann Stieltjes integral $\int_a^b f_n d\mu_{g_n}$ exists for each positive integer *n*. Then there exists a subsequence $(g_{n_k} : [a,b] \to \mathbb{R})$ such that $g_{n_k}(x)$ tends pointwise to an increasing function *h* on $[a, b]$. Moreover, $h(x) = g(x)$ for every *x* in *E*. We extend *g* to all of $[a, b]$ by $g(x) = h(x)$ for $x \notin E$. Suppose *f* is Riemann Stieltjes integrable with respect to *g*. Then

$$\int_a^b f_n(x) d\mu_{g_n} \to \int_a^b f(x) d\mu_g .$$

To prove Theorem 13, we shall need the following technical lemmas.

Suppose $P : a = x_0 < x_1 < \cdots < x_N = b$ is a partition. Suppose *g* is an increasing function on $[a, b]$. A function *H* defined on $[a, b]$ by

$$H(x) = c_i \text{ if } x \in (x_{i-1}, x_i)$$

is called a *step function.*

Suppose g is an increasing function defined on [a, b].

We say H is an *admissible step function* with respect to the increasing function g if g is continuous at x_i for $1 \le i \le N-1$ and H is continuous at a (i.e., $H(a) = c_1$) if g is discontinuous at a, and H is continuous at b ($H(b) = c_N$) if g is discontinuous at b. Note that H is Riemann Stieltjes integrable with respect to g on [a, b].

Lemma 14. Suppose $H : [a,b] \to \mathbb{R}$ is an admissible step function with respect to an increasing function g. Then given any $\varepsilon > 0$, there exists a continuous function $\alpha : [a,b] \to \mathbb{R}$ and a continuous function $\beta : [a,b] \to \mathbb{R}$ such that $H - \alpha \ge 0$, $H - \beta \le 0$, that is, $\alpha \le H \le \beta$,and such that

$$\int_a^b \left(H(x) - \alpha(x)\right) d\mu_g < \varepsilon \text{ and } \int_a^b \left(\beta(x) - H(x)\right) d\mu_g < \varepsilon ,$$

i.e., $$\int_a^b \beta(x) d\mu_g - \varepsilon < \int_a^b H(x) d\mu_g < \int_a^b \alpha(x) d\mu_g + \varepsilon ,$$

where the integrals are Riemann Stieltjes integrals with integrator g.

Proof. Let $P : a = x_0 < x_1 < \cdots < x_N = b$ be the admissible partition associated with the admissible step function H. We assume that H is continuous at a and at b. We define the function α on $[x_0, x_2]$ and extend it to [a, b]. Without loss of generality, we may assume that H is right continuous at x_i for $1 \le i \le N-1$. If $c_2 > c_1$, let $\Delta x_1 = p_1 > 0$ and $0 < p_1 < x_2 - x_1$, join (x_1, c_1) to $(x_1 + \Delta x_1, c_2)$ by a line and thus defined a continuous map on $[x_0, x_2)$ joining (x_0, c_1) to (x_1, c_1) by a line segment and (x_1, c_1) to $(x_1 + \Delta x_1, c_2)$ and with values c_2 on the interval $(x_1 + \Delta x_1, x_2]$. Note that the line segment is below the graph of H and so H is greater or equal to this continuous segment on $[x_0, x_2]$.

If $c_2 < c_1$, let $\Delta x_1 = -p_1 > 0$ and $0 < p_1 < x_1$, join $(x_1 + \Delta x_1, c_1)$ to (x_1, c_2) by a line and thus defined a continuous map on the interval $[x_0, x_2)$ joining (x_0, c_1) to $(x_1 + \Delta x_1, c_1)$ by a line segment and $(x_1 + \Delta x_1, c_1)$ to (x_1, c_2) by a line segment. Note that the line segment is below the graph of H and so H is greater or equal to this continuous segment on $[x_0, x_2]$.

We proceed in this manner to extend the definition of the continuous function to $[x_0, x_3]$ and progressively to [a, b]. Note that

$$\int_a^b \left(H(x)-\alpha(x)\right)d\mu_g \le \sum_{i=1}^{N-1}|c_{i+1}-c_i||g(x_i+\Delta x_i)-g(x_i)|.$$

Let $C=\sum_{i=1}^{N-1}|c_{i+1}-c_i|$. Since g is continuous at x_i, $1\le i\le N-1$ choose Δx_i so that $|g(x_i+\Delta x_i)-g(x_i)|<\dfrac{\varepsilon}{C+1}$ and so

$$\int_a^b \left(H(x)-\alpha(x)\right)d\mu_g < \varepsilon .$$

We can define the continuous function $\beta:[a,b]\to\mathbb{R}$ similarly to obtain the desired property $\int_a^b \left(\beta(x)-H(x)\right)d\mu_g<\varepsilon$. We note that in the definition of β, the line segment in the construction is always above the graph of *H*.

Suppose the function $f:[a,b]\to\mathbb{R}$ is Riemann Stieltjes integrable with respect to an increasing function $g:[a,b]\to\mathbb{R}$. We shall show that *f* can be approximated from above and below by continuous major function and minor function.

Lemma 15. Suppose [*a*, *b*] is a closed and bounded interval with *a* < *b*. Let $g:[a,b]\to\mathbb{R}$ be an increasing function and $f:[a,b]\to\mathbb{R}$, a bounded function, which is a Riemann Stieltjes integrable function with respect to *g* as the integrator. Then given $\varepsilon>0$, there exists continuous functions *L* and *M* such that for all *x* in [*a*, *b*], $M(x)\le f(x)\le L(x)$, $\int_a^b f(x)d\mu_g<\int_a^b M(x)d\mu_g+\dfrac{\varepsilon}{2}$, $\int_a^b f(x)d\mu_g>\int_a^b L(x)d\mu_g-\dfrac{\varepsilon}{2}$ and $\int_a^b \left(L(x)-M(x)\right)d\mu_g\le\varepsilon$.

Proof. Since the function *f* is bounded and Riemann Stieltjes integrable, it is Darboux Stieltjes integrable. That is to say, the upper and lower Darboux Stieltjes integals are the same. Thus, given $\varepsilon>0$, there exists a partition $P: a=x_0<x_1<\cdots<x_N=b$ of [*a*, *b*] such that the lower Darboux Stieltjes sum, $L(f,P)=\sum_{i=1}^{N} m_i(f,P)\left(g(x_i)-g(x_{i-1})\right)$ and the upper Darboux Stieltjes sum $U(f,P)=\sum_{i=1}^{N} M_i(f,P)\left(g(x_i)-g(x_{i-1})\right)$, where $m_i(f,P)=\inf\{f(x): x\in[x_{i-1},x_i]\}$ and $M_i(f,P)=\sup\{f(x): x\in[x_{i-1},x_i]\}$, satisfy

$$0 \le \int_a^b f(x)dg - L(f,P) < \frac{\varepsilon}{8} \text{ and } 0 \le U(f,P) - \int_a^b f(x)dg < \frac{\varepsilon}{8}.$$

That is,

$$\int_a^b f(x)dg - \frac{\varepsilon}{8} < L(f,P) \le \int_a^b f(x)dg \text{ , -------------- (1)}$$

$$\int_a^b f(x)dg \le U(f,P) < \int_a^b f(x)dg + \frac{\varepsilon}{8} \text{ --------------- (2)}$$

and $$0 \le U(f,P) - L(f,P) < \frac{\varepsilon}{4}, \text{ ------------------(3)}$$

Now, since *f* is Riemann Stieltjes integrable with respect to *g*, *f* and *g* cannot have the same point of discontinuity. Suppose $\|P\| < K$ for some $K > 0$. Therefore, by Lemma 16 below, there exists *admissible partition* $P1: a = y_0 < y_1 < \cdots < y_{N1} = b$ with respect to the function *g* such that $\|P1\| < K$ and

$$L(f,P) \le L(f,P1) + \frac{\varepsilon}{8} \text{ ------------------ (4)}$$

and $$U(f,P) \ge U(f,P1) - \frac{\varepsilon}{8}. \text{ ----------------- (5)}$$

We shall define an admissible step function *H* corresponding to the Lower Darboux sum $L(f,P1) = \sum_{i=1}^{N1} m_i(f,P1)(g(y_i) - g(y_{i-1}))$, where $m_i(f,P1) = \inf\{f(x) : x \in [y_{i-1}, y_i]\}$ for $1 \le i \le N1$.

Define $H(x) = m_i(f,P)$ for $x \in [y_{i-1}, y_i)$ for $1 \le i \le N1$ and $H(b) = m_{N1}(f,P)$. Note that *H* is right continuous at y_i $0 \le i \le N1-1$ and continuous at y_{N1}. Then $\int_a^b H(x)d\mu_g = L(f,P1)$.

By Lemma 14, there exists a continuous function $\alpha : [a,b] \to \mathbb{R}$ and continuous function $\beta : [a,b] \to \mathbb{R}$ such that $\alpha \le H \le \beta$ and

$$\int_a^b \beta(x)d\mu_g - \frac{\varepsilon}{4} < \int_a^b H(x)d\mu_g < \int_a^b \alpha(x)d\mu_g + \frac{\varepsilon}{4}$$

$$0 \le \int_a^b H(x)d\mu_g - \int_a^b \alpha(x)d\mu_g < \frac{\varepsilon}{4} \text{ . ----------------- (6)}$$

$$0 \le \int_a^b \beta(x)d\mu_g - \int_a^b H(x)d\mu_g < \frac{\varepsilon}{4}$$

Thus, $f - \alpha \ge H - \alpha \ge 0$ and so $f \ge \alpha$.

Define $H1(x) = M_i(f,P)$ for $x \in [y_{i-1}, y_i)$ for $1 \le i \le N1$ and $H1(b) = M_{N1}(f,P)$.
$H(b) = m_{N1}(f,P)$. Note that $H1$ is right continuous at y_i $0 \le i \le N1-1$ and continuous at y_{N1}. Then $\int_a^b H1(x)d\mu_g = U(f,P1)$. *H1* is an admissible step function with respect to *g*.

By Lemma 14, there exists a continuous function $\alpha 1:[a,b] \to \mathbb{R}$ and continuous function $\beta 1:[a,b] \to \mathbb{R}$ such that $\alpha 1 \le H1 \le \beta 1$ and
$\int_a^b \beta 1(x)d\mu_g - \frac{\varepsilon}{4} < \int_a^b H1(x)d\mu_g < \int_a^b \alpha 1(x)d\mu_g + \frac{\varepsilon}{4}$

By Lemma 14, there exists a continuous function $\beta 1:[a,b] \to \mathbb{R}$ such that $H1 \le \beta 1$ and

$$0 \le \int_a^b \beta 1(x)d\mu_g - \int_a^b H1(x)d\mu_g < \frac{\varepsilon}{4} \text{ . -------------------- (7)}$$

$$\int_a^b \beta 1(x)d\mu_g < \int_a^b H1(x)d\mu_g + \frac{\varepsilon}{4} = U(f,P1) + \frac{\varepsilon}{4}$$

$$< U(f,P) + \frac{\varepsilon}{8} + \frac{\varepsilon}{4} < \int_a^b f(x)d\mu_g + \frac{\varepsilon}{8} + \frac{3\varepsilon}{8} = \int_a^b f(x)d\mu_g + \frac{\varepsilon}{2} \text{, ---------- (8)}$$

by (5) and (2).

$$\int_a^b \alpha(x)d\mu_g > \int_a^b H(x)d\mu_g - \frac{\varepsilon}{4} = L(f,P1) - \frac{\varepsilon}{4}$$

$$> L(f,P) - \frac{\varepsilon}{8} - \frac{\varepsilon}{4} > \int_a^b f(x)d\mu_g - \frac{\varepsilon}{8} - \frac{3\varepsilon}{8} = \int_a^b f(x)d\mu_g - \frac{\varepsilon}{2} \text{, -----------(9)}$$

by (4) and (1).

Thus, $f - \beta 1 \le H1 - \beta 1 \le 0$ and so $f \le \beta 1$. It follows that $\alpha \le f \le \beta 1$.

$\int_a^b f(x)d\mu_g < \int_a^b \alpha(x)d\mu_g + \frac{\varepsilon}{2}$ and $\int_a^b f(x)d\mu_g > \int_a^b \beta 1(x)d\mu_g - \frac{\varepsilon}{2}$.

Therefore,

$$0 \le \int_a^b \beta 1(x)d\mu_g - \int_a^b \alpha(x)d\mu_g < \int_a^b f(x)d\mu_g + \frac{\varepsilon}{2} - \left(\int_a^b f(x)d\mu_g - \frac{\varepsilon}{2} \right) = \varepsilon,$$

by (8) and (9).

Let $M(x)$ be the continuous function α and $L(x)$ be the continuous function $\beta 1$. Then we have $M(x) \le f(x) \le L(x)$ and

$$\int_a^b \left(L(x)-M(x)\right)d\mu_g \le \varepsilon .$$

$\int_a^b f(x)d\mu_g < \int_a^b M(x)d\mu_g + \frac{\varepsilon}{2}$ and $\int_a^b f(x)d\mu_g > \int_a^b L(x)d\mu_g - \frac{\varepsilon}{2}$.

Note that we start with a partition P such that $0 \le \int_a^b f(x)d\mu_g - L(f,P) < \frac{\varepsilon}{8}$ and $0 \le U(f,P) - \int_a^b f(x)d\mu_g < \frac{\varepsilon}{8}$. Then we refine P to $P1$ such that $L(f,P) \le L(f,P1) + \frac{\varepsilon}{8}$ and $U(f,P) \ge U(f,P1) - \frac{\varepsilon}{8}$. Therefore, $U(f,P1) \le \int_a^b f(x)d\mu_g + \frac{\varepsilon}{4}$ and $L(f,P1) \ge L(f,P) - \frac{\varepsilon}{8} \ge \int_a^b f(x)d\mu_g - \frac{\varepsilon}{4}$.

Lemma 16. Suppose $[a, b]$ is a closed and bounded interval with $a < b$. Let $g:[a,b] \to \mathbb{R}$ be an increasing function and $f:[a,b] \to \mathbb{R}$ a bounded function, which is a Riemann Stieltjes integrable function with respect to g as the integrator. Suppose $P: a = x_0 < x_1 < \cdots < x_N = b$ is a partition of the interval $[a, b]$, with $\|P\| < \delta$ for some $\delta > 0$. Then given any $\varepsilon > 0$, there exist an admissible partition Q of $[a, b]$, which is a refinement of P such that

$$L(f,g,P) \le L(f,g,Q) + \varepsilon(g(b) - g(a)) \text{ and}$$

$$U(f,g,P) \ge U(f,g,Q) - \varepsilon(g(b) - g(a)),$$

where $L(f,g,P)$ and $L(f,g,Q)$ are the lower Darboux Stieltjes sums of f with respect to the partitions P and Q and integrator g and $U(f,g,P)$ and $U(f,g,Q)$ are the upper Darboux Stieltjes sums of f with respect to the partitions P and Q and integrator g.

Proof.

Take the partition $P: a = x_0 < x_1 < \cdots < x_N = b$.

Since f is bounded and Riemann Stieltjes integrable with respect to g, if g is right discontinuous at x_i or left discontinuous at x_i for $i \ne 0, N$, then f must be continuous at x_i. This is because if g is right discontinuous at x_i and f is right discontinuous at x_i *or* g is left discontinuous at x_i and f is left discontinuous at x_i, then f cannot be Riemann Stieltjes integrable on $[a, b]$. Moreover, if g is right discontinuous at x_i and f is right continuous at x_i and left discontinuous at x_i, then f cannot be Riemann Stieltjes integrable on $[a, b]$ and that if g is left discontinuous at x_i and f is left continuous at x_i and right discontinuous at x_i, then f cannot be Riemann Stieltjes integrable on [a, b].

Therefore, when g is discontinuous at x_i, there exists $\delta_i > 0$ such that

$$|f(x) - f(x_i)| < \frac{\varepsilon}{3} \quad \text{for } |x - x_i| < \delta_i \text{ .} \quad \text{--------------------------} \quad (1).$$

Let $S = \{\delta_i : g \text{ is discontinuous at } x_i, 0 < i < N\}$ and $M_P = \min\{x_i - x_{i-1} : 1 \le i \le N\}$. Since g is an increasing function on [a, b], the set of points of discontinuity of g is at most countable. We can add points near points of discontinuity of g in the partition P so that g is continuous at these points and so that the norm of the resulting partition is less than δ_i for each point of discontinuity of g in the partition P and also less than $\min\{x_i - x_{i-1} : 1 \le i \le N\}$ of P. Let $P2 : a = y_0 < y_1 < \cdots < y_s = b$ be this refinement. Then $\|P2\| = \delta < \min\{S \cup M_p\}$

Note that $P \subseteq P2$ and $\|P2\| = \delta < \delta_i$ for each $\delta_i \in S$.

The partition point y_j is either a point of continuity of g when it does belong to P or a point of discontinuity of g in P.

Suppose g is discontinuous at y_j . Note that $j \ne 0, s$. Consider the intervals $(y_{j-1}, y_{j-1} + \delta)$ and (y_j, y_{j+1}). Plainly, $(y_{j-1}, y_{j-1} + \delta) \cap (y_j, y_{j+1}) \ne \varnothing$. Choose a point $z_j \in (y_{j-1}, y_{j-1} + \delta) \cap (y_j, y_{j+1}) \ne \varnothing$ so that $z_j \in (y_j, y_{j+1})$ and is near to y_j and g is continuous at z_j. That is, we shall replace the point y_j by z_j If g is continuous at y_j, let $z_j = y_j$. Since f is Riemann Stieltjes integrable on [a, b], if g is not continuous at a, then f is continuous at a and if g is not continuous at b, then f is continuous at b. Let $z_0 = y_0 = a$ and $z_s = y_s = b$.

In this way, we define a partition $Q : a = z_0 < z_1 < \cdots < z_s = b$ and Q is admissible with respect to g. That is, g is continuous at z_i for $1 \le i \le s - 1$.

Then we have $[z_0, z_1] = [y_0, y_1]$ and $[z_{s-1}, z_s] = [y_{s-1}, y_s]$.

Let $m_{j,2} = \inf\{f(x) : x \in [y_{j-1}, y_j]\}$ and $m_j = \inf\{f(x) : x \in [z_{j-1}, z_j]\}$ for $1 \le j \le s$.

Then $m_{1,2} = m_1$ and $m_{s,2} = m_s$

Let $g_j = g(z_j), g_{j,2} = g(y_j)$ and $\Delta g_j = g_j - g_{j,2} = g(z_j) - g(y_j)$.

Since $z_j \ge y_j$ and g is increasing, $\Delta g_j \ge 0$ for $2 \le j \le s - 1$, $\Delta g_1 = \Delta g_s = 0$.

Suppose now $2 \le j \le s-2$.

Note that by construction, *g* must be continuous at y_1 and at y_{s-1} and so $\Delta g_1 = \Delta g_{s-1} = 0$.

Suppose $\Delta g_j > 0$, then $z_j > y_j$.

As $z_j > y_j$, *g* is discontinuous at y_j and so *f* is continuous at y_j. This means

$$x \in [y_{j-1}, y_{j+1}] \Rightarrow \left| f(x) - f(y_j) \right| < \frac{\varepsilon}{3} \text{ or}$$

$$f(y_j) - \frac{\varepsilon}{3} < f(x) < f(y_j) + \frac{\varepsilon}{3} . \qquad \text{--------------------------} \quad (2)$$

Note that if $\Delta g_j > 0$ and $1 \le j \le s-1$, then *g* is discontinuous at y_j for $1 \le j \le s-1$ and *g* is continuous at y_{j-1} and y_{j+1}. It follows that

$$\left[z_{j-1}, z_j \right] = \left[y_{j-1}, z_j \right] \subseteq \left[y_{j-1}, y_{j+1} \right] \text{ and } \left[z_{j-1}, z_j \right] = \left[y_{j-1}, z_j \right] \supseteq \left[y_{j-1}, y_j \right].$$

Since $\left[z_{j-1}, z_j \right] \subseteq \left[y_{j-1}, y_{j+1} \right]$, it follows from (2) that for all $x \in [z_{j-1}, z_j]$,

$$f(y_j) - \frac{\varepsilon}{3} \le m_j = \inf\left\{ f(x) : x \in [z_{j-1}, z_j] \right\} \le f(y_j) < f(y_j) + \frac{\varepsilon}{3}, \text{ ------------ } (3)$$

and for all $x \in [y_{j-1}, y_j]$,

$$f(y_j) - \frac{\varepsilon}{3} \le m_{j+1,2} = \inf\left\{ f(x) : x \in [y_j, y_{j+1}] \right\} \le f(y_j) < f(y_j) + \frac{\varepsilon}{3}. \text{ ------- } (4)$$

$$m_{j,2} - m_{j+1,2} = \inf\left\{ f(x) : x \in [y_{j-1}, y_j] \right\} - \inf\left\{ f(x) : x \in [y_j, y_{j+1}] \right\}$$

$$= \inf\left\{ f(x) : x \in [y_{j-1}, y_j] \right\} + \sup\left\{ -f(x) : x \in [y_j, y_{j+1}] \right\}$$

$$\le f(y_j) + \frac{\varepsilon}{3} - f(y_j) + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} .$$

$$m_j - m_{j+1,2} = \inf\left\{ f(x) : x \in [z_{j-1}, z_j] \right\} - \inf\left\{ f(x) : x \in [y_j, y_{j+1}] \right\}$$

$$= \inf\left\{ f(x) : x \in [z_{j-1}, z_j] \right\} + \sup\left\{ -f(x) : x \in [y_j, y_{j+1}] \right\}$$

$$\le f(y_j) - f(y_j) + \frac{\varepsilon}{3} = \frac{\varepsilon}{3} .$$

Similarly,

$$m_{j+1,2} - m_{j,2} = \inf\left\{ f(x) : x \in [y_j, y_{j+1}] \right\} - \inf\left\{ f(x) : x \in [y_{j-1}, y_j] \right\}$$

$$= \inf\left\{f(x): x\in[y_j, y_{j+1}]\right\} + \inf\left\{-f(x): x\in[y_{j-1}, y_j]\right\}$$

$$\leq f(y_j) + \frac{\varepsilon}{3} - f(y_j) + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

$$m_{j+1,2} - m_j = \inf\left\{f(x): x\in[y_j, y_{j+1}]\right\} - \inf\left\{f(x): x\in[z_{j-1}, z_j]\right\}$$

$$= \inf\left\{f(x): x\in[y_j, y_{j+1}]\right\} + \inf\left\{-f(x): x\in[z_{j-1}, z_j]\right\}$$

$$\leq f(y_j) - f(y_j) + \frac{\varepsilon}{3} = \frac{\varepsilon}{3}.$$

Hence, if $\Delta g_j > 0$ and $2 \leq j \leq s-2$, $\left|m_{j+1,2} - m_{j,2}\right| \leq \frac{2\varepsilon}{3}$. ------------------------ (5)

If $\Delta g_j > 0$ and $2 \leq j \leq s-2$, then $\left[z_{j-1}, z_j\right] = \left[y_{j-1}, z_j\right] \supseteq \left[y_{j-1}, y_j\right]$ and so $m_j \leq m_{j,2}$.

If $\Delta g_j > 0$ and $2 \leq j \leq s-2$ we have that $\left[z_{j-1}, z_j\right] \subseteq \left[y_{j-1}, y_{j+1}\right]$,

$$m_{j,2} \leq f(y_j) < f(x) + \frac{\varepsilon}{3} \text{ for all } x\in[z_{j-1}, z_j] \text{ by (2).}$$

It follows that $m_{j,2} \leq m_j + \frac{\varepsilon}{3}$ and $m_j \leq m_{j,2} \leq m_j + \frac{\varepsilon}{3}$.

$$m_{j,2} \leq m_j + \frac{\varepsilon}{3}. \quad \text{--------------------- (6).}$$

Hence, if $\Delta g_j > 0$ and $2 \leq j \leq s-2$, $m_{j,2} \leq m_j + \frac{\varepsilon}{3}$.

If $\Delta g_j = 0$ and $2 \leq j \leq s-1$, then $z_j = y_j$, $\left[z_{j-1}, z_j\right] = \left[z_{j-1}, y_j\right] \subseteq [y_{j-1}, y_j]$ and so $m_{j,2} \leq m_j \leq m_j + \frac{\varepsilon}{3}$.

Note that $\Delta g_{s-1} = \Delta g_1 = 0$. Since $m_{1,2} = m_1$ and $m_{s,2} = m_s$, $m_{1,2} \leq m_1 + \frac{\varepsilon}{3}$ $m_{s,2} \leq m_s + \frac{\varepsilon}{3}$

Hence, $m_{j,2} \leq m_j + \frac{\varepsilon}{3}$ for $1 \leq j \leq s$.

Now,

$$g_{j,2} - g_{j-1,2} = g(y_j) - g(y_{j-1}) = g_j - g_{j-1} + (g_{j-1} - g_{j-1,2}) - (g_j - g_{j,2})$$

$$= g_j - g_{j-1} + \Delta(g_{j-1}) - \Delta(g_j). \quad \text{----------------------------- (7)}$$

$$L(f,g,P) \leq L(f,g,P1) = \sum_{j=1}^{s} m_{j,2}(g_{j,2} - g_{j-1,2})$$

$$= \sum_{j=1}^{s} m_{j,2}(g_j - g_{j-1}) + \sum_{j=1}^{s} m_{j,2}(\Delta(g_{j-1}) - \Delta(g_j))$$

$$\le \sum_{j=1}^{s} \left(m_j + \frac{\varepsilon}{3} \right)(g_j - g_{j-1}) + \sum_{j=1}^{s} m_{j,2}(\Delta(g_{j-1}) - \Delta(g_j))$$

$$\le L(f,g,Q) + \frac{\varepsilon}{3}(g(b) - g(a)) + \sum_{j=1}^{s} m_{j,2}(\Delta(g_{j-1}) - \Delta(g_j))$$

$$\le L(f,g,Q) + \frac{\varepsilon}{3}(g(b) - g(a)) + \sum_{j=1}^{s-1} \left(m_{j+1,2} - m_{j,2} \right) \Delta(g_j)$$

$$\le L(f,g,Q) + \frac{\varepsilon}{3}(g(b) - g(a)) + \sum_{j=1}^{s-1} \left| m_{j+1,2} - m_{j,2} \right| \Delta(g_j) .$$

Now if $\Delta(g_j) \ne 0$, then by (5) $\left| m_{j+1,2} - m_{j,2} \right| \le \frac{2\varepsilon}{3}$. Therefore,

$$\sum_{j=1}^{s-1} \left| m_{j+1,2} - m_{j,2} \right| \Delta(g_j) \le \frac{2\varepsilon}{3} \sum_{j=1}^{s-1} \Delta(g_j) \le \frac{2\varepsilon}{3}(g(b) - g(a)) .$$

Hence, $L(f,g,P) \le L(f,g,Q) + \varepsilon(g(b) - g(a))$.

Now, we consider the upper Darboux sum.

Note that $\Delta g_0 = \Delta g_1 = \Delta g_{s-1} = \Delta g_s = 0$

Let $M_{j,2} = \sup\{ f(x) : x \in [y_{j-1}, y_j] \}$ and $M_j = \sup\{ f(x) : x \in [z_{j-1}, z_j] \}$.

Since $[z_0, z_1] = [y_0, y_1]$ and $[z_{s-1}, z_s] = [y_{s-1}, y_s]$, $M_{1,2} = M_1$ and $M_{s,2} = M_s$.

If $\Delta g_j > 0$ and $2 \le j \le s-2$, then $z_j > y_j$, and *g* is discontinuous at y_j and so *f* is continuous at y_j. Recall (2) says $x \in [y_{j-1}, y_{j+1}] \Rightarrow \left| f(x) - f(y_j) \right| < \frac{\varepsilon}{3}$ or

$$f(y_j) - \frac{\varepsilon}{3} < f(x) < f(y_j) + \frac{\varepsilon}{3} .$$

If $\Delta g_j > 0$ and $2 \le j \le s-2$, then *g* is continuous at y_{j-1} so that $\Delta g_{j-1} = 0,\ z_{j-1} = y_{j-1}$, $\left[z_{j-1}, z_j \right] = \left[y_{j-1}, z_j \right] \subseteq \left[y_{j-1}, y_{j+1} \right]$,

for all $x \in [z_{j-1}, z_j]$,

$$f(y_j) + \frac{\varepsilon}{3} \ge M_j = \sup\{ f(x) : x \in [z_{j-1}, z_j] \} \ge f(x) > f(y_j) - \frac{\varepsilon}{3} , \text{------------ (8)}$$

and for all $x \in [y_{j-1}, y_j]$,

$$f(y_j)+\frac{\varepsilon}{3}\geq M_{j+1,2}=\sup\left\{f(x):x\in[y_j,y_{j+1}]\right\}\geq f(x)>f(y_j)-\frac{\varepsilon}{3}\text{ . ------- (9)}$$

Thus,

$$M_{j,2}-M_{j+1,2}=\sup\left\{f(x):x\in[y_{j-1},y_j]\right\}-\sup\left\{f(x):x\in[y_j,y_{j+1}]\right\}$$

$$=\sup\left\{f(x):x\in[y_{j-1},y_j]\right\}+\inf\left\{-f(x):x\in[y_j,y_{j+1}]\right\}$$

$$\geq f(y_j)-\frac{\varepsilon}{3}-f(y_j)-\frac{\varepsilon}{3}=-\frac{2\varepsilon}{3}\ .$$

Similarly,

$$M_{j+1,2}-M_{j,2}=\sup\left\{f(x):x\in[y_j,y_{j+1}]\right\}-\sup\left\{f(x):x\in[y_{j-1},y_j]\right\}$$

$$=\inf\left\{f(x):x\in[y_j,y_{j+1}]\right\}+\inf\left\{-f(x):x\in[y_{j-1},y_j]\right\}$$

$$\geq f(y_j)-\frac{\varepsilon}{3}-f(y_j)-\frac{\varepsilon}{3}=-\frac{2\varepsilon}{3}.$$

Hence, for $\Delta g_j>0$ and $2\leq j\leq s-2$,

$$\left|M_{j+1,2}-M_{j,2}\right|\leq\frac{2\varepsilon}{3}.\text{ -- (10)}$$

If $\Delta g_j>0$ and $2\leq j\leq s-2$, then $\left[z_{j-1},z_j\right]=\left[y_{j-1},z_j\right]\subseteq\left[y_{j-1},y_{j+1}\right]$,
$M_{j,2}\geq f(y_j)>f(x)-\frac{\varepsilon}{3}$ for all $x\in[z_{j-1},z_j]$ by (3) and so $M_{j,2}\geq M_j-\frac{\varepsilon}{3}$

If $\Delta g_j>0$ and $2\leq j\leq s-2$, then $\left[z_{j-1},z_j\right]=\left[y_{j-1},z_j\right]\supseteq\left[y_{j-1},y_j\right]$ and so $M_j\geq M_{j,2}$.

Hence, $M_j\geq M_{j,2}\geq M_j-\frac{\varepsilon}{3}$, if $\Delta g_j>0$ and $2\leq j\leq s-2$.

If $\Delta g_j=0$ and $1\leq j\leq s-1$ so that $z_j=y_j$, then $\left[z_{j-1},z_j\right]=\left[z_{j-1},y_j\right]\subseteq\left[y_{j-1},y_j\right]$ and so $M_{j,2}\geq M_j\geq M_j-\frac{\varepsilon}{3}$.

Since $M_{s,2}=M_s$, we conclude that if $\Delta g_j=0$, then $M_{j,2}\geq M_j\geq M_j-\frac{\varepsilon}{3}$.

It follows that for $1\leq j\leq s$

$$M_{j,2}\geq M_j-\frac{\varepsilon}{3}.\quad\text{-------------------- (11)}$$

Recall from (7), for $1\leq j\leq s$, $g_{j,2}-g_{j-1,2}==g_j-g_{j-1}+\Delta(g_{j-1})-\Delta(g_j)$.

$$U(f,g,P) \geq U(f,g,P1) = \sum_{j=1}^{s} M_{j,2}(g_{j,2} - g_{j-1,2})$$

$$= \sum_{j=1}^{s} M_{j,2}(g_j - g_{j-1}) + \sum_{j=1}^{s} M_{j,2}(\Delta(g_{j-1}) - \Delta(g_j))$$

$$\geq \sum_{j=1}^{s} \left(M_j - \frac{\varepsilon}{3} \right)(g_j - g_{j-1}) + \sum_{j=1}^{s} M_{j,2}(\Delta(g_{j-1}) - \Delta(g_j))$$

$$\geq U(f,g,Q) - \frac{\varepsilon}{3}(g(b) - g(a)) + \sum_{j=1}^{s} M_{j,2}(\Delta(g_{j-1}) - \Delta(g_j))$$

$$\geq U(f,g,Q) - \frac{\varepsilon}{3}(g(b) - g(a)) + \sum_{j=1}^{s-1} \left(M_{j+1,2} - M_{j,2} \right) \Delta(g_j)$$

$$\geq U(f,g,Q) - \frac{\varepsilon}{3}(g(b) - g(a)) - \sum_{j=1}^{s-1} \left| M_{j+1,2} - M_{j,2} \right| \Delta(g_j) .$$

Mow if $\Delta(g_j) \neq 0$, then by (5), we have $\left| M_{j+1,2} - M_{j,2} \right| \leq \frac{2\varepsilon}{3}$.

Therefore, $\sum_{j=1}^{s-1} \left| M_{j+1,2} - M_{j,2} \right| \Delta(g_j) \leq \frac{2\varepsilon}{3} \sum_{j=1}^{s-1} \Delta(g_j) \leq \frac{2\varepsilon}{3}(g(b) - g(a))$.

Hence, $U(f,g,P) \geq U(f,g,Q) - \varepsilon(g(b) - g(a))$.

Remark. Replacing ε by $\dfrac{\varepsilon}{g(b) - g(a)}$, if g is an increasing function and for a partition *P*, we can find a refinement to an admissible partition Q such that $L(f,g,P) \leq L(f,g,Q) + \varepsilon$ and $U(f,g,P) \geq U(f,g,Q) - \varepsilon$,

Proof of Theorem 13.

Note that each g_n is an increasing function. As $b \in E$ and the sequence $\left(g_n(b)\right)$ is convergent and so the sequence is bounded above by *K* for some *K* > 0. It follows that for all $x \in [a,b]$ and for all positive integer *n*, $g_n(x) < K$. As the sequence $\left(g_n(a)\right)$ is convergent it is bounded below by a constant *L*. It follows that for all positive integer *n*, $g_n(x) > L$. Therefore, $|g_n(x)| < \max\{|L|, K\}$. Hence, by Helly Selection Theorem (see page 221 of Natanson, Theory of function of a real variable, Vol 1, Lemma 2), there exists a subsequence $\left(g_{n_k} : [a,b] \to \mathbb{R}\right)$

such that $g_{n_k}(x)$ tends pointwise to an increasing function h on $[a, b]$. Plainly $h(x) = g(x)$ for all x in E. We extend g to all of $[a, b]$ by $g(x) = h(x)$ for $x \notin E$.

Note that $g_n(b) - g_n(a) \le K + |L|$ and so the total variation of g_n is uniformly bounded.

By Lemma 15, since f is Riemann Stieltjes integrable with respect to g, given $\varepsilon > 0$, there exists continuous functions L and M such that for all x in $[a, b]$, $M(x) \le f(x) \le L(x)$ and

$$\int_a^b \left(L(x) - M(x)\right) d\mu_g \le \varepsilon .$$

$\int_a^b f(x) d\mu_g < \int_a^b M(x) d\mu_g + \frac{\varepsilon}{2}$ and $\int_a^b f(x) d\mu_g > \int_a^b L(x) d\mu_g - \frac{\varepsilon}{2}$.

Let $G1(x) = L(x) + \frac{3\varepsilon}{4(1 + g(b) - g(a))}$ and $G2(x) = M(x) - \frac{3\varepsilon}{4(1 + g(b) - g(a))}$ and

$$G2(x) + \frac{3\varepsilon}{4(1 + g(b) - g(a))} \le f(x) \le G1(x) - \frac{3\varepsilon}{4(1 + g(b) - g(a))} .$$

Since f_n converges uniformly to a bounded function f, there exists an integer $N2 > 0$ such that $|f_n(x) - f(x)| < \frac{3\varepsilon}{4(1 + g(b) - g(a))}$ for $n > N2$ and for all $x \in [a,b]$, i.e.,

$$f(x) - \frac{3\varepsilon}{4(1 + g(b) - g(a))} < f_n(x) < f(x) + \frac{3\varepsilon}{4(1 + g(b) - g(a))} .$$

That is, $G2(x) < f_n(x) < G1(x)$.

Hence,

$$\int_a^b G2(x) d\mu_{g_n} \le \int_a^b f_n(x) d\mu_{g_n} \le \int_a^b G1(x) d\mu_{g_n} \quad \text{----------------------} \quad (1)$$

Since $G1$ and $G2$ are continuous, by Theorem 12 (with (f_n) a constant sequence),

$$\lim_{n\to\infty} \int_a^b G1(x) d\mu_{g_n} = \int_a^b G1(x) d\mu_g \text{ and } \lim_{n\to\infty} \int_a^b G2(x) d\mu_{g_n} = \int_a^b G2(x) d\mu_g .$$

Therefore, given $\varepsilon > 0$, there exists $N1 > 0$ such that

$$n > N1 \Rightarrow \int_a^b G1 d\mu_g - \varepsilon < \int_a^b G1 d\mu_{g_n} < \int_a^b G1 d\mu_g + \varepsilon$$

and there exists $N2 > 0$ such that

$$n > N2 \Rightarrow \int_a^b G2d\mu_g - \varepsilon < \int_a^b G2d\mu_{g_n} < \int_a^b G2d\mu_g + \varepsilon \ .$$

Thus, it follows from (1) that for $n > \max\{N1, N2\}$,

$$\int_a^b G2(x)d\mu_g - \varepsilon \le \int_a^b f_n(x)d\mu_{g_n} \le \int_a^b G1(x)d\mu_g + \varepsilon \text{ ----------- (2)}$$

Now, $\int_a^b G1(x)d\mu_g = \int_a^b L(x)d\mu_g + \int_a^b \frac{3\varepsilon}{4(1+g(b)-g(a))} d\mu_g \le \int_a^b L(x)d\mu_g + \frac{3\varepsilon}{4}$

$$\le \int_a^b f(x)d\mu_g + \frac{\varepsilon}{2} + \frac{3\varepsilon}{4} = \int_a^b f(x)d\mu_g + \frac{5\varepsilon}{4}, \text{ ----------------- (3)}$$

and $\int_a^b G2(x)d\mu_g = \int_a^b M(x)d\mu_g - \int_a^b \frac{3\varepsilon}{4(1+g(b)-g(a))} d\mu_g \ge \int_a^b M(x)d\mu_g - \frac{3\varepsilon}{4}$

$$\ge \int_a^b f(x)d\mu_g - \frac{\varepsilon}{2} - \frac{3\varepsilon}{4} = \int_a^b f(x)d\mu_g - \frac{5\varepsilon}{4}. \text{ -------------------- (4)}$$

Therefore, for $n > \max\{N1, N2\}$, it follows from (2), (3) and (4) that

$$\int_a^b f_n(x)d\mu_{g_n} \le \int_a^b f(x)d\mu_g + \frac{5\varepsilon}{4} + \varepsilon = \int_a^b f(x)d\mu_g + \frac{9\varepsilon}{4} \text{ and}$$

$$\int_a^b f_n(x)d\mu_{g_n} \ge \int_a^b G2(x)d\mu_g - \varepsilon \ge \int_a^b f(x)d\mu_g - \frac{5\varepsilon}{4} - \varepsilon = \int_a^b f(x)d\mu_g - \frac{9\varepsilon}{4}.$$

Thus, for $n > \max\{N1, N2\}$,

$$\int_a^b f(x)d\mu_g - \frac{9\varepsilon}{4} \le \int_a^b f_n(x)d\mu_{g_n} \le \int_a^b f(x)d\mu_g + \frac{9\varepsilon}{4}.$$

Hence, $\lim_{n\to\infty} \int_a^b f_n(x)d\mu_{g_n} = \int_a^b f(x)d\mu_g$.

The next Theorem is a variation of Theorem 9.

Theorem 17. Suppose $(g_n : [a,b] \to J)$ is a sequence of bounded function converging uniformly to a continuous function $g : [a,b] \to J$, where *J* is a closed and bounded interval. Suppose $\phi : J \to \mathbb{R}$ is a function such that the total variation of the composition functions $\phi \circ g_n : [a,b] \to \mathbb{R}$, $V(\phi \circ g_n, [a,b]) < K$ for some $K > 0$ and for all positive integer *n*. Suppose $g_n(x) = g(x)$ for $x \in E_n$ $E_n \subseteq E_{n+1}$ and $E = \bigcup_{n=1}^{\infty} E_n$ is an everywhere dense set in [*a*, *b*] with $\{a,b\} \subseteq E$.
Suppose either g_n is continuous for all *n* or $f \circ g_n$ is Riemann Stieltjes

integrable with respect to $\phi \circ g_n$, that is, $\int_a^b f(g_n(x))d\lambda_{\phi\circ g_n}$ exists for all n.
Suppose $f : J \to \mathbb{R}$ is a continuous function and $\phi \circ g$ is of bounded variation. Then

$$\int_a^b f(g_n(x))d\lambda_{\phi\circ g_n} \to \int_a^b f(g(x))d\lambda_{\phi\circ g} .$$

Proof. Since *f* is continuous and g_n converges uniformly to *g*, $f \circ g_n$ converges uniformly to $f \circ g$.

We note that since *f* is continuous on *J*, *f* is uniformly continuous on *J*. Therefore, given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all *x*, *y* in J,

$$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{K} \quad . \quad \text{--------------------- (3)}$$

Since g_n converges to *g* uniformly, there exists an integer $N > 0$ such that

$$n > N \Rightarrow |g_n(x)-g(x)| < \delta \text{ for all } x \text{ in } [a, b].$$

It follows from (3) that

$$n > N \Rightarrow |g_n(x)-g(x)| < \delta \Rightarrow |f\circ g_n(x)-f\circ g(x)| < \frac{\varepsilon}{K} \text{ for all } x \text{ in } [a, b].$$

Therefore, for all $n > N$,

$$\left|\int_a^b \left(f(g_n(x))-f(g(x))\right)d\lambda_{\phi\circ g_n}\right| < \frac{\varepsilon}{K}V(\phi\circ g_n,[a,b] < \varepsilon .$$

It follows that $\lim_{n\to\infty}\left|\int_a^b \left(f(g_n(x))-f(g(x))\right)d\lambda_{\phi\circ g_n}\right| = 0$

Therefore, $\lim_{n\to\infty}\int_a^b f(g_n(x))d\lambda_{\phi\circ g_n} = \lim_{n\to\infty}\int_a^b f(g(x))d\lambda_{\phi\circ g_n}$

Now, $\phi\circ g_n : [a,b] \to \mathbb{R}$ is a sequence of function of bounded variation such that $V(\phi\circ g_n,[a,b]) < K$ for all integer $n > 1$. Since $f\circ g$ is continuous, by Theorem 12 , there exists a subsequence $\phi\circ g_{n_k} : [a,b] \to \mathbb{R}$ such that $\phi\circ g_{n_k}$ converges to a function Φ of bounded variation such that $\Phi(x) = \phi(g(x))$ for every *x* in *E* and

$$\lim_{n\to\infty}\int_a^b f(g(x))d\lambda_{\phi\circ g_n} = \int_a^b f(g(x))d\lambda_{\Phi} .$$

Since $\int_a^b f(g(x))d\lambda_\Phi$ exists, given $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that for any partition $P: a = x_0 < x_1 < \cdots < x_N = b$ with $\|P\| < \delta_1$, any Riemann Stieltjes sum $\sum_{i=1}^{N} f(g(u_i))\left(\Phi(x_i) - \Phi(x_{i-1})\right)$ satisfies

$$\left| \int_a^b f(g(x))d\lambda_\Phi - \sum_{i=1}^{N} f(g(u_i))\left(\Phi(x_i) - \Phi(x_{i-1})\right) \right| < \varepsilon .$$

Similarly, there exists a $\delta_2 > 0$ such that for any partition $P: a = x_0 < x_1 < \cdots < x_N = b$ with $\|P\| < \delta_2$, any Riemann Stieltjes sum $\sum_{i=1}^{N} f(g(u_i))\left(\phi \circ g(x_i) - \phi \circ g(x_{i-1})\right)$ satisfies

$$\left| \int_a^b f(g(x))d\lambda_{\phi\circ g} - \sum_{i=1}^{N} f(g(u_i))\left(\phi \circ g(x_i) - \phi \circ g(x_{i-1})\right) \right| < \varepsilon .$$

Take any subdivision $P: a = x_0 < x_1 < \cdots < x_N = b$ of $[a, b]$ with norm sufficiently small so that $\|P\| < \min(\delta_1, \delta_2)$.

Since E is dense in $[a, b]$, we can choose a subdivision $P: a = x_0 < x_1 < \cdots < x_N = b$ such that $\|P\| < \min(\delta_1, \delta_2)$ and $x_i \in E$ for $0 \le i \le N$. Then we have

$$\left| \int_a^b f(g(x))d\lambda_\Phi - \sum_{i=1}^{N} f(g(u_i))\left(\Phi(x_i) - \Phi(x_{i-1})\right) \right|$$

$$= \left| \int_a^b f(g(x))d\lambda_\Phi - \sum_{i=1}^{N} f(g(u_i))\left(f(g(x_i)) - f(g(x_{i-1}))\right) \right| < \varepsilon$$

and $\left| \int_a^b f(g(x))d\lambda_{\phi\circ g} - \sum_{i=1}^{N} f(g(u_i))\left(\phi \circ g(x_i) - \phi \circ g(x_{i-1})\right) \right| < \varepsilon$. Therefore,

$$\left| \int_a^b f(g(x))d\lambda_\Phi - \int_a^b f(g(x))d\lambda_{\phi\circ g} \right| < 2\varepsilon .$$

As ε is arbitrarily small, we conclude that $\int_a^b f(g(x))d\lambda_\Phi = \int_a^b f(g(x))d\lambda_{\phi\circ g}$.

Thus, $\lim_{n\to\infty} \int_a^b f(g_n(x))d\lambda_{\phi\circ g_n} = \lim_{n\to\infty} \int_a^b f(g(x))d\lambda_{\phi\circ g_n} = \int_a^b f(g(x))d\lambda_\Phi = \int_a^b f(g(x))d\lambda_{\phi\circ g}$.

We can relax the condition of convergence of the sequence of function $\phi \circ g_n : [a,b] \to \mathbb{R}$ in Theorem 11. We require only this sequence be convergent on a dense subset of $[a, b]$ in the next theorem.

Theorem 18. Suppose $(g_n:[a,b]\to J)$ is a sequence of continuous polygonal function converging uniformly to a continuous function $g:[a,b]\to J$, where J is a closed and bounded interval. Suppose $g_n(a)=g(a)$ and $g_n(b)=g(b)$. Suppose $\phi:J\to\mathbb{R}$ is a function of bounded variation such that the total variation of the composition functions $\phi\circ g_n:[a,b]\to\mathbb{R}$, $V(\phi\circ g_n,[a,b])<K$ for some $K>0$ and for all positive integer n. Suppose $f:J\to\mathbb{R}$ is a continuous function and $\phi\circ g$ is of bounded variation. Suppose $\phi\circ g_n$ converges to $\phi\circ g$ at every point in a dense subset E in $[a, b]$. Then $\int_a^b f(g(x))d\lambda_{\phi\circ g}=\int_{g(a)}^{g(b)} f(y)d\lambda_\phi$

Proof.

Note that $\{a,b\}\subseteq E$.

As in the proof of Theorem 17, we have $\lim_{n\to\infty}\int_a^b f(g_n(x))d\lambda_{\phi\circ g_n}=\lim_{n\to\infty}\int_a^b f(g(x))d\lambda_{\phi\circ g_n}$.

Since g_n is polygonal, we have shown in the proof of Theorem11, that $\int_a^b f(g_n(x))d\lambda_{\phi\circ g_n}=\int_{g(a)}^{g(b)} f(x)d\lambda_\phi$. We have shown in the proof of Theorem 17 that $\lim_{n\to\infty}\int_a^b f(g(x))d\lambda_{\phi\circ g_n}=\int_a^b f(g(x))d\lambda_{\phi\circ g}$. Therefore,

$$\int_a^b f(g(x))d\lambda_{\phi\circ g}=\int_{g(a)}^{g(b)} f(x)d\lambda_\phi .$$

When the function $\phi:J\to\mathbb{R}$ is monotone, we have the following change of variable theorem.

Theorem 19. Suppose $g:[a,b]\to J$ and $f:J\to\mathbb{R}$ are continuous functions. Suppose $\phi:J\to\mathbb{R}$ is monotone and $\phi\circ g:[a,b]\to\mathbb{R}$ is a function of bounded variation. Then we have $\int_a^b f(g(x))d\lambda_{\phi\circ g}=\int_{g(a)}^{g(b)} f(y)d\mu_\phi$.

Proof.

Let $P: a=x_0<x_1<\cdots<x_N=b$ be a sub division of [a, b].

$$\int_{g(a)}^{g(b)} f(y)d\lambda_\phi-\sum_{i=1}^{N} f\circ g(u_i)(\phi(g)(x_i)-\phi(g)(x_{i-1}))\ ,\ u_i\in[x_{i-1},x_i]$$

$$=\sum_{i=1}^{N}\int_{g(x_{i-1})}^{g(x_i)} f(y)d\mu_\phi-\sum_{i=1}^{N}\int_{g(x_{i-1})}^{g(x_i)} f(g(u_i)d\mu_\phi$$

$$= \sum_{i=1}^{N} \int_{g(x_{i-1})}^{g(x_i)} \left(f(y) - f(g(u_i)) \right) d\mu_\phi$$

Since $f \circ g$ is continuous on [a, b] and so given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f \circ g(x) - f \circ g(y)| < \varepsilon$ whenever $|x - y| < \delta$. Let $\|P\| < \delta$.

Note that $u_i \in [x_{i-1}, x_i]$ and for $y \in g([x_{i-1}, x_i])$, $|f(y) - f(g(u_i)| < \varepsilon$.

$$\left| \sum_{i=1}^{N} \int_{g(x_{i-1})}^{g(x_i)} \left(f(y) - f(g(u_i)) \right) d\mu_\phi \right| \leq \sum_{i=1}^{N} \left| \int_{g(x_{i-1})}^{g(x_i)} \left(f(y) - f(g(u_i)) \right) d\mu_\phi \right|$$

$$\leq \varepsilon \sum_{i=1}^{N} \left| \int_{g(x_{i-1})}^{g(x_i)} d\mu_\phi \right| \leq \varepsilon \sum_{i=1}^{N} V(\phi, [g(x_{i-1}), g(x_i)]),$$

where $[g(x_{i-1}), g(x_i)]$ denotes the closed interval determined

by the end points $\{g(x_{i-1}), g(x_i)\}$,

$$\leq \varepsilon \sum_{i=1}^{N} |\phi(g(x_i)) - \phi(g(x_{i-1}))| \leq \varepsilon V(\phi \circ g, [a, b]),$$

since ϕ is monotone.

Therefore, $\left| \int_{g(a)}^{g(b)} f(y) d\lambda_\phi - \sum_{i=1}^{N} f \circ g(u_i)(\phi(g)(x_i) - \phi(g)(x_{i-1}) \right| \leq \varepsilon V(\phi \circ g, [a, b])$.

Since ε is arbitrarily small, it follows that $\int_a^b f(g(x)) d\lambda_{\phi \circ g} = \int_{g(a)}^{g(b)} f(y) d\lambda_\phi$.

This concludes the proof of Theorem 19.

Remark.

The function $\phi : J \to \mathbb{R}$ in Theorem 19 need not be monotone. Using Theorem 18, we can extend Theorem 19 to the case when ϕ is of bounded variation.

Suppose $g : [a, b] \to J$ is a continuous function. Then g is uniformly continuous and we can define a sequence of piecewise linear functions g_n by joining points on the graph of g by lines. By uniform continuity of g, we can construct the sequence of piecewise linear functions g_n such that it converges uniformly to g. Let E_n be the set of points in [a, b] that defines the piecewise linear functions g_n. That is, $E_n = \{x_i : x_0 = a, x_{s_n} = b, x_i < x_{i+1} \text{ for } 1 \leq i \leq s_n - 1\}$, $g_n(x_i) = g(x_i)$ for $0 \leq i \leq s_n$

, g_n is linear on $[x_{i-1}, x_i]$, joining the points $(x_{i-1}, g(x_{i-1}))$ to $(x_i, g(x_i))$ by a straight line for $1 \le i \le s_n$. Then $g_n(x) = g(x)$ for all x in E_n. Moreover, we may assume that $E_n \subseteq E_{n+1}$ and $E = \bigcup_{n=1}^{\infty} E_n$ is everywhere dense in [a, b]. Therefore, $\phi \circ g_n$ converges pointwise to $\phi \circ g$ in E.

Suppose $\phi \circ g$ is of bounded variation.

If g_n is linear on the interval $[x_{i-1}, x_i]$, where $g_n(x_i) = g(x_i)$ and $g_n(x_{i-1}) = g(x_{i-1})$, then the variation $V(\phi \circ g_n, [x_{i-1}, x_i]) = \begin{cases} V(\phi, [g_n(x_{i-1}), g_n(x_i)]), \text{ if } g_n(x_{i-1}) \le g_n(x_i) \\ V(\phi, [g_n(x_i), g_n(x_{i-1})]), \text{ if } g_n(x_i) \le g_n(x_{i-1}) \end{cases}$

$\le V(\phi \circ g, [x_{i-1}, x_i])$.

Therefore, $V(\phi \circ g_n, [a, b_i]) \le V(\phi \circ g, [a, b_i])$.

Therefore, by Theorem 18, $\int_a^b f(g(x)) d\lambda_{\phi \circ g} = \int_{g(a)}^{g(b)} f(y) d\lambda_\phi$.

Thus, we have proved the following theorem.

Theorem 20. Suppose $g : [a,b] \to J$ and $f : J \to \mathbb{R}$ are continuous functions. Suppose $\phi : J \to \mathbb{R}$ is a function of bounded variation and $\phi \circ g : [a,b] \to \mathbb{R}$ is also a function of bounded variation. Then we have $\int_a^b f(g(x)) d\lambda_{\phi \circ g} = \int_{g(a)}^{g(b)} f(y) d\lambda_\phi$.

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