Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals

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Introduction.

Following a series of articles on the de La Vallée Poussin Decomposition, we shall consider the Lebesgue Stieltjes measure generated by a function of bounded variation *g*. We have similar decomposition involving the measure of the image of the total variation function of *g*, the measure of the images under the positive and negative variation functions of *g* and of course the Lebesgue integral of the derived function of *g*. All Borel subsets are Lebesgue Stieltjes measurable. Not all continuous image of a measurable set is measurable. As the continuous image of a Borel set is measurable, we shall confine ourselves mostly on Borel sets. We have, in the previous articles, described the measure of a measurable set under the total variation function of the function of bounded variation. The present article will give a better picture of the measure of the image of a measurable set under a function of bounded variation or its total variation function or its positive variation function or its negative variation function, albeit with some constraint that the measurable set be Borel. Lebesgue Stieltjes integral is defined in the usual manner via Lebesgue Stieltjes measure. We present a generalized version of integration by parts (with correction term) for the Lebesgue Stieltjes integral for functions of bounded variation. We present versions of change of variable for the Lebesgue Stieltjes integral when the measure is generated by the composition of two increasing functions. Detail and complete proofs are presented.

Lebesgue Stieltjes Measure

We shall introduce the Lebesgue Stieltjes measure in stages. We begin by defining it for an increasing function and then proceed to define it for function of bounded variation.

Definition

Suppose *I* is an open interval and $g: I \to \mathbb{R}$ is an increasing function. Let Ω be the family of all intervals (*a*, *b*), with $a, b \in I$ and $a < b$. Define

$$\rho : \Omega \to [0, \infty)$$

by $\rho\big((a,b)\big)=g(b)-g(a)$ for $(a,b)\in\Omega$. When $a = b$, $(a, b) = \varnothing$, we define $\rho\big((a,b)\big)=\rho\big(\varnothing\big)=0=g(b)-g(a)$.

We define the *Lebesgue Stieltjes outer measure generated by g* on the collection $\wp(I)$ of all subsets of I, $\mu_g{*}:\wp(I)\to[0,\infty]$ by

$$\mu_g{*}(E)=\inf\left\{\sum_{n=1}^{\infty}\rho\big((a_n,b_n)\big):a_n,b_n\in I,a_n\le b_n,E\subseteq\bigcup_{n=1}^{\infty}(a_n,b_n)\right\}.$$

We define $\mu_g{*}\big(\varnothing\big)=0$.

We shall state some known results about the Lebesgue Stieltjes outer measure, $\mu_g{*}$.

Proposition 1. Suppose I is an open interval and $g:I\to\mathbb{R}$ is an increasing function. Then $\mu_g{*}$ is a *metric outer measure*. That is to say, if $E, F\subseteq I$ are *separated*, i.e., $d(E,F)=\inf\left\{|x-y|:x\in E,y\in F\right\}>0$, then

$$\mu_g{*}(E\cup F)=\mu_g{*}(E)+\mu_g{*}(F).$$

Proof. Obviously, $\mu_g{*}\ge 0$ and $\mu_g{*}\big(\varnothing\big)=0$. If $A\subseteq B$, then $\mu_g{*}\big(A\big)\le\mu_g{*}\big(B\big)$. This is obviously true if $A = \varnothing$ or $\mu_g{*}(B)=\infty$. Now we assume that $A\neq\varnothing$ and $\mu_g{*}(B)<\infty$. For any $\varepsilon > 0$, take a collection of $\{(a_n,b_n)\}$ such that $B\subseteq\bigcup_{n=1}^{\infty}(a_n,b_n)$ and $\sum_{n=1}^{\infty}\rho\big((a_n,b_n)\big)<\mu_g{*}(B)+\varepsilon$. It follows that $A\subseteq\bigcup_{n=1}^{\infty}(a_n,b_n)$ so that $\mu_g{*}(A)\le\sum_{n=1}^{\infty}\rho\big((a_n,b_n)\big)<\mu_g{*}(B)+\varepsilon$. Since ε is arbitrarily chosen, we conclude that $\mu_g{*}\big(A\big)\le\mu_g{*}\big(B\big)$.

We next show that $\mu_g{*}$ is σ sub-additive. Let $\{A_n\}$ be a countable collection of non-empty subsets of I. Let $A=\bigcup_{n=1}^{\infty}A_n$. If any one of $\mu_g{*}\big(A_n\big)$ is $+\infty$, then we have nothing to prove. We may thus assume that $\mu_g{*}\big(A_n\big)<+\infty$ for all n. Take coverings $\left\{(a_k^n,b_k^n)\right\}$ such that $A_n\subseteq\bigcup_{k=1}^{\infty}(a_k^n,b_k^n)$ and $\sum_{k=1}^{\infty}\rho\big((a_k^n,b_k^n)\big)<\mu_g{*}(A_n)+\varepsilon 2^{-n}$. Plainly, $A\subseteq\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}(a_k^n,b_k^n)$. It follows that

$$\mu_g{}^*(A) \le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\rho\left((a_k^n, b_k^n)\right) < \sum_{n=1}^{\infty}\left(\mu_g{}^*(A_n) + \varepsilon 2^{-n}\right) = \sum_{n=1}^{\infty}\mu_g{}^*(A_n) + \varepsilon .$$

Hence, we conclude that $\mu_g{}^*(A) \le \sum_{n=1}^{\infty}\mu_g{}^*(A_n)$. Thus, $\mu_g{}^*$ is an outer measure.

Next, we shall show that $\mu_g{}^*$ is a metric outer measure. Take subset *A*, *B* in *I* such that $d(A,B) = \inf\left\{|x-y| : x \in A, y \in B\right\} > 0$. Since *g* is increasing, for any open interval (a,b) with *a* < *b* and for any partition $a = a_0 < a_1 < \cdots < a_N = b$, we can write

$$\rho\left((a,b)\right) = g(b) - g(a) = \sum_{k=1}^{N} g(a_k) - g(a_{k-1}) = \sum_{k=1}^{N}\rho(a_{k-1}, a_k),$$

with $|a_k - a_{k-1}| < d(A,B)$, *k* = 1,2, …, *N*. This means we can write (a,b) as a union $\bigcup_{k=1}^{N}(a_{k-1}, a_k) \cup \bigcup_{k=1}^{N-1}\{a_k\}$. It follows that (a_{k-1}, a_k) can only meet one of *A* or *B* but not both and each a_k can only belong to one of *A* or *B* but not both.

For any given $\varepsilon > 0$, we can choose a covering $\{(a_k, b_k)\}$ such that $A \cup B \subseteq \bigcup_{k=1}^{\infty}(a_k, b_k)$ and $\sum_{k=1}^{\infty}\rho\left((a_k, b_k)\right) < \mu_g{}^*(A \cup B) + \varepsilon$. By the above deliberation the covering splits into two coverings one for *A* and one for *B*. It follows that $\mu_g{}^*(A) + \mu_g{}^*(B) \le \sum_{k=1}^{\infty}\rho\left((a_k, b_k)\right) < \mu_g{}^*(A \cup B) + \varepsilon$. As ε is arbitrarily chosen, we can conclude that $\mu_g{}^*(A) + \mu_g{}^*(B) \le \mu_g{}^*(A \cup B)$. It can be shown easily that $\mu_g{}^*(A \cup B) \le \mu_g{}^*(A) + \mu_g{}^*(B)$. Hence, $\mu_g{}^*(A \cup B) = \mu_g{}^*(A) + \mu_g{}^*(B)$. Therefore, $\mu_g{}^*$ is a metric outer measure.

Definition 2. Suppose *I* is an open interval and $g : I \to \mathbb{R}$ is an increasing function. We say a set $E \subseteq I$ is $\mu_g{}^*$ *measurable* if it is $\mu_g{}^*$ *measurable* in the sense of Caratheodory. That is to say *E* is $\mu_g{}^*$ *measurable* if

$$\mu_g{}^*(F) = \mu_g{}^*(F \cap E) + \mu_g{}^*(F - E),$$

for any set $F \subseteq I$.

Let $\mathcal{B}(I)$ be the Borel σ-algebra generated by the open sets of *I*. (See *Introduction to Measure Theory*). Members of $\mathcal{B}(I)$ are called Borel sets.

It turns out that the restriction of $\mu_g *$ to $\mathcal{B}(I)$ is a Radon measure, i.e. it is a positive measure, finite on compact set and is both *inner* and *outer regular*.

We recall the definition of a Radon measure.

Let X be a locally compact Hausdorff topological space. Suppose $\mathcal{M}$ is a σ-algebra of subsets of X, containing all the Borel sets of X and μ is a positive measure on $\mathcal{M}$.

μ is said to be *outer regular* if for all $E \in \mathcal{M}$,

$$\mu(E) = \inf\{\mu(V) : V \supseteq E \text{ and } V \text{ is open}\}.$$

μ is said to be *inner regular* if for all $E \in \mathcal{M}$, such that either E is open or $\mu(E) < \infty$, $\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\}$.

μ is said to be *regular* if it is both inner and outer regular.

Suppose $\mathcal{B}(X)$ is the σ- algebra generated by the Borel sets of X and $\mu : \mathcal{B}(X) \to [0, +\infty]$ is a positive measure. The positive measure μ is said to be a *Radon measure* if μ is regular and is finite on compact subsets of X.

Recall the following theorem from *Positive Borel Measure and Riesz Representation Theorem.*

Theorem 3. Let X be a locally compact Hausdorff topological space, in which every open subset is σ-compact. Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$ for any compact subset K in X. Then λ is regular. Hence, λ is a Radon measure on $\mathcal{B}(X)$.

(For a proof of this theorem, refer to Theorem 5 of *Positive Borel Measure and Riesz Representation Theorem*.)

Note that any open interval in $\mathbb{R}$ is a locally compact Hausdorff space, in which every open set is σ-compact.

Theorem 4. Suppose I is an open interval and $g : I \to \mathbb{R}$ is an increasing function. Every Borel set $E \in \mathcal{B}(I)$ is $\mu_g *$ *measurable*.

Theorem 4 is a consequence of the fact that $\mu_g *$ is a *metric outer measure*.
[See Theorem 11.5, page 283 of *Wheeden Zygmund*, *Measure and Integral*, *An Introduction to Real Analysis*.]

Theorem 5. Suppose I is an open interval and $g : I \to \mathbb{R}$ is an increasing function. The restriction of $\mu_g{}^*$ to $\mathcal{B}(I)$, $\mu_g : \mathcal{B}(I) \to [0, \infty]$ is a Radon measure. Every Borel set in I is both inner and outer regular.

Moreover, for any open interval $(a, b) \subseteq I$,

$$\mu_g\left((a,b)\right) \le \rho\left((a,b)\right) = g(b) - g(a).$$

$\mu_g\left([a,b]\right) = g_+(b) - g_-(a)$ for all $a, b \in I$ with $a \le b$, where $g_+(x)$ denotes the right limit of g at x and $g_-(x)$ denotes the left limit of g at x. We also have that $\mu_g\left((a,b]\right) = g_+(b) - g_+(a)$, $\mu_g\left((a,b)\right) = g_-(b) - g_+(a)$ and $\mu_g\left(\{a\}\right) = g_+(a) - g_-(a)$.

Furthermore, if g is bounded,

$$\mu_g(I) = \sup_{x \in I}(g(x)) - \inf_{x \in I}(g(x)) .$$

Thus, μ_g is finite if, and only if, g is bounded. μ_g is unique in the sense that if there is a Radon measure μ : $\mathcal{B}(I) \to [0, \infty]$ such that $\mu\left((a,b]\right) = g_+(b) - g_+(a)$, then $\mu = \mu_g$.

Proof.

Note that $\mu_g{}^*$ is a Borel outer measure by Theorem 4.

By definition, for any open interval (a, b) in I, $\mu_g\left((a,b)\right) \le \rho\left((a,b)\right) = g(b) - g(a) < \infty$. Suppose K is a compact subset of I. Then K is closed and bounded and there is a finite cover of K by open bounded intervals in I. It follows that $\mu_g{}^*(K)$ is finite. Hence, as I is locally compact, by Theorem 3, the restriction of $\mu_g{}^*$ to $\mathcal{B}(I)$ is a Radon measure. Therefore, every Borel set is inner and outer regular.

Suppose $[a, b] \subseteq I$. Since the interval I is open, there exists an integer M such that $[a,b] \subseteq \left(a - \frac{1}{M}, b + \frac{1}{M}\right)$. Therefore, for all integer $n \ge M$,

$$\mu_g([a,b]) \le \rho\left(\left(a - \frac{1}{n}, b + \frac{1}{n}\right)\right) = g\left(b + \frac{1}{n}\right) - g\left(a - \frac{1}{n}\right).$$

Letting n tends to infinity, we obtain $\mu_g([a,b]) \le g_+(b) - g_-(a)$.

Take any countable cover $\{(a_n, b_n)\}$ of [a, b] by open intervals in I. Then $[a,b] \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)$. Since $\bigcup_{n=1}^{\infty}(a_n, b_n)$ is open, [a, b] is contained in a union of finite number of these open intervals in the cover and so [a, b] is contained in a path connected open interval in the union of these finite number of open intervals. Hence, there exists an $\varepsilon > 0$ such that $[a-\varepsilon, b+\varepsilon]$ is contained in this open path component and so $[a-\varepsilon, b+\varepsilon] \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)$. We claim that the set $\bigcup_{n=1}^{\infty}[g(a_n), g(b_n)]$ covers $(g(a-\varepsilon), g(b+\varepsilon))$. Take $y \in (g(a-\varepsilon), g(b+\varepsilon))$. Then since g is increasing, there exists $x \in [a-\varepsilon, b+\varepsilon]$ such that $y \in [g_-(x), g_+(x)]$. As $[a-\varepsilon, b+\varepsilon] \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)$, $x \in (a_k, b_k)$ for some integer k. Therefore, we have

$$g(a_k) \le g_-(x) \le y \le g_+(x) \le g(b_k)$$

and so $y \in [g(a_k), g(b_k)]$ and $y \in \bigcup_{n=1}^{\infty}[g(a_n), g(b_n)]$. Therefore,

$$\sum_{n=1}^{\infty}\big(g(b_n) - g(a_n)\big) = \sum_{n=1}^{\infty} m\big([g(a_n), g(b_n)]\big) \ge m\left(\bigcup_{n=1}^{\infty}[g(a_n), g(b_n)]\right)$$

$$\ge m\big(g(a-\varepsilon), g(b+\varepsilon)\big) = g(b+\varepsilon) - g(a-\varepsilon) \ge g_+(b) - g_-(a).$$

Hence, by definition of $\mu_g([a,b])$, $\mu_g([a,b]) \ge g_+(b) - g_-(a)$. It follows that $\mu_g([a,b]) = g_+(b) - g_-(a)$.

If g is bounded, then using the fact that $\mu_g([a,b]) = g_+(b) - g_-(a)$ for [a, b] $\subseteq I$, by taking a sequence $\{a_n\}$ such that $a_n \searrow \inf I$ and a sequence $\{b_n\}$ such that $b_n \nearrow \sup I$, we conclude by the continuity from below property of outer measure, that

$$\mu_g(I) = \lim_{n\to\infty} \mu_g([a_n, b_n]) = \lim_{n\to\infty}\big(g_+(b_n) - g_-(a_n)\big) = \sup_{x\in I}(g(x)) - \inf_{x\in I}(g(x)).$$

Observe that if $a = b$, from $\mu_g([a,b]) = g_+(b) - g_-(a)$, we deduce that

$$\mu_g(\{a\}) = g_+(a) - g_-(a).$$

Since μ_g is a Borel measure, if a, $b \in I$ with $a < b$, then

$$\mu_g((a,b]) = \mu_g([a,b]) - \mu_g(\{a\}) = g_+(b) - g_-(a) - \big(g_+(a) - g_-(a)\big) = g_+(b) - g_+(a)$$

and

$$\mu_g((a,b)) = \mu_g((a,b]) - \mu_g(\{b\}) = g_+(b) - g_+(a) - \left(g_+(b) - g_-(b)\right) = g_-(b) - g_+(a).$$

Now the σ-algebra generated by the half open intervals $(a,b]$ in I is the Borel σ-algebra $\mathcal{B}(I)$. Note that μ and μ_g are both σ-finite. Therefore, since the collection of half open intervals $(a,b]$ in I is a π-system, by Corollary 4 of *Product Measure and Fubini's Theorem*, $\mu = \mu_g$.

Theorem 6. Suppose I is an open interval and $g : I \to \mathbb{R}$ is an increasing function. For any set $E \subseteq I$, $m^*(g(E)) \le \mu_g{}^*(E)$, where m^* denotes the Lebesgue outer measure on $\mathbb{R}$. Moreover, if $E \subseteq I$ is such that g is continuous at all points of E, then $m^*(g(E)) = \mu_g{}^*(E)$.

Proof.

Let $E \subseteq I$. If $\mu_g{}^*(E) = +\infty$, then plainly $m^*(g(E)) \le \mu_g{}^*(E)$ and we have nothing to prove.

Now, we assume that $\mu_g{}^*(E) < \infty$.

For a given arbitrary $\varepsilon > 0$, take a covering $\{(a_n, b_n)\}$ of E by open intervals in I such that $\sum_{n=1}^{\infty} \rho\left((a_n, b_n)\right) = \sum_{n=1}^{\infty} \left(g(b_n) - g(a_n)\right) \le \mu_g{}^*(E) + \varepsilon$. Since g is increasing, $\bigcup_{n=1}^{\infty} [g(a_n), g(b_n)] \supseteq g(E)$. Therefore,

$$m^*\left(g(E)\right) \le m^*\left(\bigcup_{n=1}^{\infty} [g(a_n), g(b_n)]\right)$$

$$\le \sum_{n=1}^{\infty} m\left([g(a_n), g(b_n)]\right) = \sum_{n=1}^{\infty} \left(g(b_n) - g(a_n)\right) \le \mu_g{}^*(E) + \varepsilon.$$

Since ε is arbitrarily chosen, we conclude that $m^*(g(E)) \le \mu_g{}^*(E)$.

Suppose now g is continuous at all points of E. We shall show that $\mu_g{}^*(E) \le m^*(g(E))$. If $m^*(g(E)) = +\infty$. Then we have nothing to show.

Now, assume that $m^*(g(E)) < +\infty$.

Take an open interval U in $\mathbb{R}$. Consider its preimage $F = \{x \in E : g(x) \in U\} = g^{-1}(U) \cap E$. We shall show that $\mu_g{}^*(F) \le m(U)$. If U is unbounded, then we have nothing to show. We now assume that U is bounded and $U = (c, d)$. Let $a = \inf F$ and $b = \sup F$ so that $F \subseteq [a, b]$. It is possible that

$a=-\infty$ and $b=+\infty$. For all *x* in *F*, $g(x)\in(c,d)$. Thus, $g_-(b)\le d$ and $g_+(a)\ge c$. As *g* is continuous at *x* in *F*, $\mu_g{}^*(\{x\})=0$. Therefore, if $a\in F$, $\mu_g{}^*(\{a\})=0$ and if $b\in F$, $\mu_g{}^*(\{b\})=0$. Hence,

$$\mu_g{}^*(F)\le\mu_g{}^*\big((a,b)\big)=g_-(b)-g_+(a)\le d-c=m(U)\ .\ \text{------------- (*)}$$

Since $m^*(g(E))<+\infty$, for any fixed $\varepsilon>0$, we can cover $g(E)$ by a countable collection of open intervals $\{U_j\}$ such that $g(E)\subseteq\bigcup_{j=1}^{\infty}U_j$ and

$\sum_{j=1}^{\infty}m(U_j)\le m^*(g(E))+\varepsilon$.

For each integer $j\ge 1$, let $F_j=\{x\in E:g(x)\in U_j\}=g^{-1}(U_j)\cap E$. Then by (*), we have, since $E\subseteq\bigcup_{j=1}^{\infty}F_j$,

$$\mu_g{}^*(E)\le\sum_{j=1}^{\infty}\mu_g{}^*(F_j)\le\sum_{j=1}^{\infty}m(U_j)\le m^*(g(E))+\varepsilon\ .$$

Thus, as ε is arbitrarily small, we conclude that $\mu_g{}^*(E)\le m^*(g(E))$. Therefore, if *g* is continuous at every point of *E*, then $\mu_g{}^*(E)=m^*(g(E))$.

Let *I* be an open interval. Denote $BV(I)$ to be the collection of all function on *I* which is of bounded variation on *I*.

Definition 7.

Suppose now $g:I\to\mathbb{R}$ is of bounded variation on *I*, i.e., $g\in BV(I)$, the collection of all functions of bounded variation on *I*. Then we know g is the difference of two increasing bounded functions (see Theorem 6 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*). Let v_g be the total variation function of *g* defined using an anchor point $a\in I$ as in *Functions of Bounded Variation and de La Vallée Poussin's Theorem*. Then

$$\varphi_1(x)=\frac{1}{2}\big(v_g(x)+g(x)\big)-\frac{g(a)}{2}\ \text{ and }\ \varphi_2(x)=\frac{1}{2}\big(v_g(x)-g(x)\big)+\frac{g(a)}{2}$$

are the positive and negative variation functions of *g*. More precisely,

$$v_g(x)=\varphi_1(x)+\varphi_2(x)=P(x)+N(x)-g(a)\,,$$

$$g(x)=P(x)-N(x)=g(a)+\varphi_1(x)-\varphi_2(x)\,,$$

where $P(x) = \varphi_1(x) + g(a)$ and $N(x) = \varphi_2(x)$. Note that φ_1 and φ_2 are as defined in *Functions of Bounded Variation and de La Vallée Poussin's Theorem*. Note that $P(x) = \frac{1}{2}\left(v_g(x) + g(x)\right) + \frac{1}{2}g(a)$ and $N(x) = \frac{1}{2}\left(v_g(x) - g(x)\right) + \frac{1}{2}g(a)$ are both increasing functions.

Thus, we can define the *signed Lebesgue Stieltjes measure,*

$$\lambda_g : \mathcal{B}(I) \to [-\infty, +\infty],$$

by $\lambda_g = \mu_P - \mu_N$.

Theorem 8. Suppose *I* is an open interval and $g : I \to \mathbb{R}$ is of bounded variation on *I*. Then λ_g is a unique finite signed Radon measure such that

$$\lambda_g\left((a,b]\right) = g_+(b) - g_+(a),$$

for all *a*, *b* $\in$ *I* with $a \le b$. Furthermore, $\left|\lambda_g\right| \le \mu_{v_g}$ and $\left|\lambda_g\right|(I) \le$ Total variation of *g* on *I*. If *g* is right continuous, then $\left|\lambda_g\right| = \mu_{v_g}$.

Proof. By Theorem 5, λ_g is a signed finite Radon measure and so it is a real Borel measure.

For any *a*, *b* in *I* with $a < b$ we have

$$\lambda_g\left((a,b]\right) = \mu_P\left((a,b]\right) - \mu_N\left((a,b]\right) = P_+(b) - P_+(a) - \left(N_+(b) - N_+(a)\right)$$

$$= P_+(b) - N_+(b) - \left(P_+(a) - N_+(a)\right) = g_+(b) - g_+(a).$$

Suppose λ: $\mathcal{B}(I) \to (-\infty, +\infty)$ is a signed finite Radon measure satisfying,

$\lambda\left((a,b]\right) = g_+(b) - g_+(a)$ for any *a*, *b* in *I* with $a < b$. Then take the Jordan decomposition of λ, $\lambda = \lambda^+ - \lambda^-$. Since λ is a finite measure, λ^+ and λ^- are positive finite measure. Therefore,

$$\lambda_g\left((a,b]\right) = \mu_P\left((a,b]\right) - \mu_N\left((a,b]\right) = \lambda^+\left((a,b]\right) - \lambda^-\left((a,b]\right) .$$

It follows that $\mu_P\left((a,b]\right) + \lambda^-\left((a,b]\right) = \lambda^+\left((a,b]\right) + \mu_N\left((a,b]\right)$ and so

$$\left(\mu_P + \lambda^-\right)\left((a,b]\right) = \left(\lambda^+ + \mu_N\right)\left((a,b]\right).$$

Since $\left(\mu_P + \lambda^-\right)$ and $\left(\lambda^+ + \mu_N\right)$ are positive finite measures which agree on any $(a,b]$ in *I*, we conclude that $\left(\mu_P + \lambda^-\right) = \left(\lambda^+ + \mu_N\right)$ and so $\lambda_g = \mu_P - \mu_N = \lambda^+ - \lambda^- = \lambda$. This proves the uniqueness of λ_g.

Now we claim that $\mu_P + \mu_N = \mu_{v_g}$. For any *a*, *b* in *I* with *a* < *b*.

$$\left(\mu_P + \mu_N\right)\left((a,b]\right) = \mu_P\left((a,b]\right) + \mu_N\left((a,b]\right) = P_+(b) - P_+(a) + N_+(b) - N_+(a)$$

$$= P_+(b) + N_+(b) - P_+(a) - N_+(a) = (P+N)_+(b) - (P+N)_+(a)$$

$$= v_{g+}(b) - v_{g+}(a) = \mu_{v_g}(a,b].$$

We deduce as before that $\mu_P + \mu_N = \mu_{v_g}$.

Therefore, for any Borel set *E* in *I*,

$$\left|\lambda_g(E)\right| = \left|\mu_P(E) - \mu_N(E)\right| \le \mu_P(E) + \mu_N(E) = \mu_{v_g}(E) \ .$$

It follows then, by the definition of the total variation measure $\left|\lambda_g\right|$, of λ_g, $\left|\lambda_g\right| \le \mu_{v_g}$. Hence, $\left|\lambda_g\right|(I) \le \mu_{v_g}(I) = \sup_{x\in I}(v_g(x)) - \inf_{x\in I}(v_g(x))$ = total variation of *g* on *I*.

Suppose now *g* is right continuous on *I*. Take *a*, *b* in *I* with *a* ≤ *b*.

Take any partition $a = x_0 < x_1 < \cdots < x_n = b$. Then

$$\sum_{i=1}^{n}\left|g(x_i) - g(x_{i-1})\right| = \sum_{i=1}^{n}\left|\lambda_g\left((x_{i-1}, x_i]\right)\right|, \text{ since } g \text{ is right continuous,}$$

$$\le \left|\lambda_g\right|\left((a,b]\right).$$

It follows that $v_g(b) - v_g(a) \le \left|\lambda_g\right|\left((a,b]\right)$. As *g* is right continuous, v_g is also right continuous. Hence, $\mu_{v_g}((a,b]) = v_g(b) - v_g(a) \le \left|\lambda_g\right|\left((a,b]\right)$. We have already shown that $\left|\lambda_g\right| \le \mu_{v_g}$ and so $\mu_{v_g}((a,b]) = \left|\lambda_g\right|\left((a,b]\right)$ and this implies that $\left|\lambda_g\right| = \mu_{v_g}$.

Theorem 9. Suppose *I* is an open interval and $g \in BV(I)$ is a right continuous function. Then $\lambda_g^+ = \mu_P$ and $\lambda_g^- = \mu_N$, where *P* and *N* are the positive and negative variation of *g*, $\lambda_g = \lambda_g^+ - \lambda_g^-$ is the Jordan decomposition of the measure λ_g. (See Theorem 13, *Complex Measure, Dual Space of L^p Space, Radon-Nikodym Theorem and Riesz Representation Theorems*.)

To prove this, we need the following result about associating a positive Radon measure to an increasing function.

Theorem 10. Suppose μ: $\mathcal{B}(I) \to [0, +\infty)$ is a finite Radon measure. Pick a point *e* in the open interval *I* and any point γ in $\mathbb{R}$. Define a function $g_\mu : I \to \mathbb{R}$ by

$$g_\mu(x) = \gamma + \begin{cases} \mu((e,x]), & \text{if } x \geq e \\ -\mu((x,e]), & \text{if } x < e \end{cases}.$$

Then $g_\mu : I \to \mathbb{R}$ is increasing and right continuous and

$$g_\mu(b) - g_\mu(a) = \mu((a,b])$$

for all a, $b \in I$, with $a \leq b$.

Proof. Suppose E and F are Borel subsets in I and $E \subseteq F \subseteq I$. We know that $\mu(E) \leq \mu(F)$.

Thus, if $e \leq x \leq y$, $g_\mu(x) = \gamma + \mu((e,x]) \leq \gamma + \mu((e,y]) = g_\mu(x)$. If $x \leq y \leq e$, $g_\mu(x) = \gamma - \mu((x,e]) \leq \gamma - \mu((y,e]) = g_\mu(y)$. If $x \leq e \leq y$, then $g_\mu(x) \leq \gamma \leq g_\mu(y)$. We can now conclude that g_μ is increasing.

Now, we show that g_μ is right continuous. Take any x in I. Take a sequence (x_n) such that x_n tends to x on the right. Since g_μ is increasing we may assume that (x_n) is decreasing. Suppose $x \geq e$. Then $(e,x] = \bigcap_{n=1}^{\infty} (e, x_n]$. By the continuity from above property of measure, since $\mu((e,x_n]) < \infty$,

$$\lim_{n\to\infty} \mu((e,x_n]) = \mu\left(\bigcap_{n=1}^{\infty} (e,x_n]\right) = \mu((e,x]).$$

If $x < e$, then $(x,e] = \bigcup_{n=1}^{\infty} (x_n, e]$. Therefore, by the continuity from below property of measure,

$$\lim_{n\to\infty} \mu((x_n,e]) = \mu\left(\bigcup_{n=1}^{\infty} (x_n,e]\right) = \mu((x,e]).$$

This proves that g_μ is right continuous.

Suppose a, $b \in I$, with $a \leq b$. If $a = b$ then obviously $g_\mu(b) - g_\mu(a) = \mu((a,b])$. Now assume that $e \leq a < b$. Then $g_\mu(b) - g_\mu(a) = \mu((e,b]) - \mu((e,a]) = \mu((a,b])$. If $a < b \leq e$, then $g_\mu(b) - g_\mu(a) = -\mu((b,e]) - (-\mu((a,e]) = \mu((a,b])$. If $a < e < b$, then $g_\mu(b) - g_\mu(a) = \mu((e,b]) - (-\mu((a,e]) = \mu((a,b])$. Hence, we have for a, $b \in I$, with $a \leq b$, $g_\mu(b) - g_\mu(a) = \mu((a,b])$.

Proof of Theorem 9.

Suppose now $x_0 \in I$ is the anchor point used in the definition of the total variation function v_g of *g*. (For reference see under Total Variation in *Functions of Bounded Variation and de La Vallée Poussin's Theorem*.) Plainly the function *g* and $g - g(x_0)$ generate the same Lebesgue Stieltjes signed measure. Without loss of generality we may assume that $g(x_0) = 0$.

Let $\lambda_g = \lambda_g^+ - \lambda_g^-$ be the Jordan decomposition of the measure λ_g. Since *g* is of bounded variation, λ_g^+ and λ_g^- are positive finite Radon measures. Note that $\left|\lambda_g\right| = \lambda_g^+ + \lambda_g^-$. (See Theorem 13 of *Complex Measure, Dual Space of L^p Space, Radon-Nikodym, Theorem and Riesz Representation Theorem*.)

By Theorem 10, we may find two right continuous functions *u* and *w* defined on *I*, with $u(x_0) = w(x_0) = 0$ such that $\lambda_g^+ = \mu_u$ and $\lambda_g^- = \mu_w$ Therefore, by Theorem 8,

$$\mu_{v_g} = \left|\lambda_g\right| = \lambda_g{}^+ + \lambda_g{}^- = \mu_u + \mu_w$$

$$= \mu_{u+w},$$

by uniqueness of Lebesgue Stieltjes measure (Theorem 5).

Hence, for $x > x_0$,

$$v_g(x) - v_g(x_0) = \mu_{v_g}\left((x_0, x]\right) = \mu_{u+w}\left((x_0, x]\right) = \left(u+w\right)(x) - (u+w)(x_0).$$

As $v_g(x_0) = u(x_0) = w(x_0) = 0$, we get $v_g(x) = \left(u+w\right)(x)$ for all $x > x_0$.

Similarly, for $x < x_0$, from

$$v_g(x_0) - v_g(x) = \mu_{v_g}\left((x, x_0]\right) = \mu_{u+w}\left((x, x_0]\right) = \left(u+w\right)(x_0) - (u+w)(x),$$

we deduce that $v_g(x) = \left(u+w\right)(x)$. Hence $v_g = u + w$.

In a similar way using $\lambda_g = \lambda_g{}^+ - \lambda_g{}^- = \mu_u - \mu_w$, and that for any *a* < *b* in *I*, by Theorem 8,

$$\lambda_g\left((a,b]\right) = g_+(b) - g_+(a) = g(b) - g(a),$$

we can show that $g = u - w$.

It follows that $u = \frac{1}{2}\left(v_g + g\right) = P$ and $w = \frac{1}{2}\left(v_g - g\right) = N$, since $g(x_0) = 0$. Therefore, $\lambda_g^+ = \mu_P$ and $\lambda_g^- = \mu_N$.

Proposition 11. Suppose $g \in BV(I)$ and I is an open interval. For any set $E \subseteq I$, $m^*(g(E)) \le \mu_{v_g}{}^*(E)$.

Proof.

We show that for any set $E \subseteq I$, $m^*(g(E)) \le m^*\left(v_g(E)\right)$. Since g is a function of bounded variation, g is a bounded function and so v_g is bounded. Therefore, $m^*\left(v_g(E)\right) < \infty$. Thus, given any $\varepsilon > 0$, there exists an open set U such that $U \supseteq v_g(E)$ and $m^*(U) < m^*\left(v_g(E)\right) + \varepsilon$.

Since U is open, U is a disjoint union of at most countable number of open intervals, i.e., $U = \bigcup_n I_n$, I_n is an open interval and

$$m(U) = \sum_{i=1}^{\infty} m(I_i) < m^*\left(v_g(E)\right) + \varepsilon \ .$$

Moreover, $v_g{}^{-1}(U) \supseteq E$. Let $A_i = g\left(v_g{}^{-1}(I_i)\right)$. For any x, y in A_i, there exist $a, b \in v_g{}^{-1}(I_i)$ such that $x = g(a)$ and $y = g(b)$. Then

$$|x - y| = |g(a) - g(b)| \le |v_g(a) - v_g(b)| \le m(I_i) .$$

It follows that the diameter of A_i is less than or equal to $m(I_i)$. Hence, $m^*(A_i) \le m(I_i)$.

Now, $g(E) \subseteq g\left(v_g{}^{-1}(U)\right) = g\left(v_g{}^{-1}\left(\bigcup_i I_i\right)\right) = g\left(\bigcup_i v_g{}^{-1}(I_i)\right) = \bigcup_i g\left(v_g{}^{-1}(I_i)\right)$.

Therefore,

$$m^*\left(g(E)\right) \le \sum_i m^*\left(g\left(v_g{}^{-1}(I_i)\right)\right) = \sum_i m^*\left(A_i\right) \le \sum_i m(I_i) < m^*(v_g(E)) + \varepsilon \ .$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $m^*(g(E)) \le m^*\left(v_g(E)\right)$. Hence,

$m^*(g(E)) \le m^*\left(v_g(E)\right) \le \mu_{v_g}{}^*(E)$ by Theorem 6.

This completes the proof of our assertion.

Proposition 12. Suppose $g \in BV(I)$, I is an open interval and g is continuous. Then g is absolutely continuous if, and only if, $\lambda_g \ll m$, the Lebesgue measure, i.e., λ_g is *absolutely continuous with respect to the Lebesgue measure*, m. That is to say, $\lambda_g(E)=0$ for all Borel set $E \subseteq I$ with $m(E) = 0$. (We are being cautious here. Not all Lebesgue measurable set is λ_g measurable.)

Proof.

If g is absolutely continuous, then g is a Lusin function and so for any $E \subseteq I$ with $m(E) = 0$, $m(g(E))=0$ and so by Theorem 9 of *Function of Bounded Variation on Arbitrary Subset and Johnson's Indicatrix*, $m(v_g(E))=0$. It follows from Theorem 6 that $\mu_{v_g}{}^*(E)=m^*(v_g(E))=0$. If E is Borel, then $|\lambda_g|(E)=\mu_{v_g}(E)=\mu_{v_g}^*(E)=0$ and so $\lambda_g(E)=0$. Hence, $\lambda_g \ll m$.

Suppose λ_g is absolutely continuous with respect to the Lebesgue measure. Let E be a set of Lebesgue measure zero. Then there exists a G_δ set G such that $E \subseteq G$ and $m(G)=0$. Note that G is a Borel set. Therefore, $\lambda_g(G)=0$. This is true for all Borel subset of G. Thus, $|\lambda_g|(G)=\mu_{v_g}(G)=0$ and so $m(v_g(G))=\mu_{v_g}(G)=0$. It follows that $m(v_g(E))=0$. Hence, $m(g(E))=0$. Thus, g is a Lusin function. As g is a continuous function of bounded variation and the domain is an interval, by Theorem 15 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*, g is absolutely continuous. This completes the proof of Proposition 12.

Suppose I is an open interval. Denote the collection of all absolutely continuous functions on I by $AC(I)$.

Theorem 13. Suppose $g \in AC(I)$ and I is an open interval. Let $[a,b] \subseteq I$. Then for any Borel set $E \subseteq [a,b]$, $\lambda_g(E)=\int_E g'$, $\mu_{v_g}(E)=|\lambda_g|(E)=\int_E |g'|$ and $\mu_{v_g}(E)=m\left(v_g(E)\right)$.

Proof.

Since g is absolutely continuous, g is absolutely continuous on $[a, b]$. Therefore, by Lemma 2 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*, g is of bounded variation on $[a, b]$. Moreover, the total variation function of g, v_g, is also absolutely continuous on $[a, b]$. (See Theorem 15 below.) As $P(x)=\frac{1}{2}\left(v_g(x)+g(x)\right)+\frac{1}{2}g(x_0)$ and

$N(x)=\frac{1}{2}\left(v_g(x)-g(x)\right)+\frac{1}{2}g(x_0)$, it follows that the positive and negative variation functions, *P* and *N* of *g* are also absolutely continuous on [*a*, *b*]. Therefore,

$$\lambda_g(E)=\mu_P(E)-\mu_N(E)=m(P(E))-m(N(E)) \text{ , by Theorem 6,}$$

$$=\int_E g' \text{ ,}$$

by Theorem 2 of *A de La Vallée Poussin's Decomposition*, as *P* and *N* are Lusin functions by Lemma 3 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation* and are differentiable almost everywhere with respect to the Lebesgue measure.

By Theorem 8, since *g* is right continuous, $\left|\lambda_g\right|(E)=\mu_{v_g}(E)$ and so by Theorem 6,

$$\left|\lambda_g\right|(E)=\mu_{v_g}(E)=m(v_g(E))$$

$$=\int_E |g'| \text{ ,}$$

by Theorem 1 of *A de La Vallée Poussin's Decomposition*, since *g* is a Lusin functions and is differentiable almost everywhere with respect to the Lebesgue measure.

We state the properties of the nature of the sets of discontinuity, differentiability, non-differentiability and infinite differentiability. These are properties that help to understand the statements in the next few theorems. The results are scattered in the literature and a comprehensive account, where all the statements and their proofs are present, seems to be difficult to find or unavailable and some proofs are only found in Russian. The main secondary source is the article, *Derivatives*, by A. M. Bruckner and J. L. Leonard in the American Mathematical Monthly, Vol. 73, No. 4, Part 2: Papers in Analysis (Apr., 1966), pp. 24-56. Note that we shall use only the Borel property of the set involved and not necessary the Borel class it may or may not belong to.

Theorem 14.

(1) For a finite function, *f*, the set of points of discontinuity is a F_σ set.

(2) For a finite function *f*, the set of points, where *f* has infinite derivative $+\infty$ is $F_{\sigma\delta}$ and the set of points, where *f* has infinite derivative $-\infty$ is $F_{\sigma\delta}$ and so are Borel.

(3) The set of points where *f* has no finite derivative is of the form $G_\delta \cup G_{\delta\sigma}$, where $G_{\delta\sigma}$ is of measure zero and the set of points where *f* has no derivative finite or infinite is also of the form $G_\delta \cup G_{\delta\sigma}$, where $G_{\delta\sigma}$ is of measure zero. Hence, they are Borel.

(4) The set of points where a continuous *f* has finite differentiability is $F_{\sigma\delta}$ and hence Borel.

(5) The continuous image of a Borel set is Lebesgue measurable.

For the definitions of F_σ, G_δ, $G_{\delta\sigma}$ and $F_{\sigma\delta}$ sets, we refer to Chapter XII in *Set Theory*, by K Kuratowski and A Mostowski or Chapter Two Section 30 of Topology Volume 1 by K. Kuratowski.

Reference to (1) and (4) can be found in *Set Theory* by Felix Hausdorff.

A reference to (2) is V. M. Tsodyks, *On sets of points where the derivative is equal to* $+\infty$ *or* $-\infty$ *respectively*, (in Russian) Mat. Sb. (N.S.), 1957, Volume 43(85), Number 4, 429–450.

For (3), see Theorem 3.12 of the article by K M Garg, *A Unified Theory of Bilateral Derivates, Real Analysis Exchange, Volume 27, Number 1 (2001), 81-122*. This is attributed to Zahorski and Brudno.

For (5), we refer to Chapter 11, Theorem 11.18 of Thomas Jech's *Set Theory*.

It is also useful to note the following result concerning the absolute continuity of the total variation function of an absolutely continuous function, which is of bounded variation.

Theorem 15. Suppose *A* is a measurable closed and bounded subset of $\mathbb{R}$ or an interval and $f : A \to \mathbb{R}$ is a finite valued function of bounded variation on *A*. Then *f* is absolutely continuous, if and only if, $v_f : A \to \mathbb{R}$ is absolutely continuous on *A*.

Proof. By Theorem 13 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem, f* is continuous if, and only if v_f is continuous.
So, we assume that *f* is a continuous function of bounded variation.
Since $|f(y) - f(x)| \le |v_f(y) - v_f(x)|$ for any $x, y \in A$, it follows that if v_f is absolutely continuous, then *f* is absolutely continuous.

We note that ν_f is an increasing bounded function and so is of bounded variation.

Suppose now *f* is absolutely continuous. By Lemma 3 of *Absolutely Continuous Functions on Arbitrary Domain and Function of Bounded variation*, *f* is a Lusin function. Since *f* is a function of bounded variation, by Theorem 10 of *Function of Bounded Variation on Arbitrary Subset and Johnson's Indicatrix*, ν_f is also a Lusin Function.
If *A* is closed and bounded, by Theorem 4 of *Absolutely Continuous Functions on Arbitrary Domain and Function of Bounded variation*, ν_f is absolutely continuous since ν_f is a continuous function of bounded variation, which is also a Lusin function.
Suppose *A* is an interval. Since ν_f is a continuous bounded increasing Lusin function, by Theorem 15 of *Absolutely Continuous Functions on Arbitrary Domain and Function of Bounded variation*, ν_f is absolutely continuous.

Remark.

We have the following criterion:

If *I* is an interval, then a continuous function of bounded variation $g : I \to \mathbb{R}$ is absolutely continuous if, and only if, ν_g is absolutely continuous if, and only if, μ_{ν_g} is absolutely continuous with respect to the Lebesgue measure *m*, if and only if, λ_g is absolutely continuous with respect to the Lebesgue measure, *m*.

Theorem 16. Let *I* be an open interval and $g \in BV(I)$. Suppose *E* is a Borel set in *I* and *K* is any real number in $\mathbb{R}$ such that *g* is differentiable at every *x* in *E*, possibly infinitely and $g'(x) \geq K$ for all *x* in *E* (respectively, $g'(x) \leq K$). Then $\lambda_g(E) \geq Km(E)$(respectively, $\lambda_g(E) \leq Km(E)$). In particular, if there exists g' on a Borel set $E \subseteq I$ such that either $m(E) = 0$ or $g'(x) = 0$ for all *x* in *E*, then $\left|\lambda_g\right|(E) = 0$.

Proof.

If $g'(x) = 0$ for all *x* in *E*, then by Theorem 11 of *Arbitrary Functions, Limit Superior, Dini Derivative and Lebesgue Density Theorem*, $m(g(E)) = 0$. By

Theorem 10 of *Function of Bounded Variation on Arbitrary Subset and Johnson's Indicatrix*, $m(\nu_g(E)) = 0$. As $|\lambda_g|(E) \le \mu_{\nu_g}(E) = m^*(\nu_g(E)) = 0$, $|\lambda_g|(E) = 0$.

Suppose g' exists finitely on a Borel set *E* in *I*. Then by Theorem 12 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*, *g* is a Lusin Function on *E*. If *m*(*E*) = 0, then $m(g(E)) = 0$ and it follows that $m(\nu_g(E)) = 0$ and $|\lambda_g|(E) \le \mu_{\nu_g}(E) = m^*(\nu_g(E)) = 0$ so that $|\lambda_g|(E) = 0$.

Note that λ_g and μ_{ν_g} are finite Radon measures.

We begin by proving for the special case when *E* is contained in an open interval (*c*, *d*) in *I*.

Thus, $m(E) < \infty$.

Suppose *g* is differentiable at every *x* in *E*, possibly infinitely and $g'(x) \ge K$ for all *x* in *E*.

Let $x \in E$.

$$g'(x) = \liminf_{t \to x} \left\{ \frac{g(t) - g(x)}{t - x} : t \in (c, d) \right\} = \lim_{\delta \to 0^+} b_\delta ,$$

where $b_\delta = \inf \left\{ \dfrac{g(t) - g(x)}{t - x} : t \in (x - \delta, x + \delta) \cap (c, d) - \{x\} \right\}$.

Suppose $g'(x) = k$. Note that $b_\delta \nearrow g'(x)$ as $\delta \searrow 0^+$. Therefore, if $g'(x) = k < \infty$, given any $\varepsilon > 0$, there exists δ_ε such that for all $0 < \delta < \delta_\varepsilon$, $k \ge b_\delta > k - \varepsilon$. This means that for all $t \in (x - \delta, x + \delta) \cap (c, d) - \{x\}$ and $0 < \delta < \delta_\varepsilon$,

$$\frac{g(t) - g(x)}{t - x} \ge b_\delta > k - \varepsilon \ge K - \varepsilon .$$

Hence, for $t_1 \in (x - \delta, x + \delta) \cap (c, d)$ and $t_1 > x$, $\dfrac{g(t_1) - g(x)}{t_1 - x} \ge K - \varepsilon$ so that

$$g(t_1) - g(x) \ge (K - \varepsilon)(t_1 - x) \text{ ------------------- (1).}$$

Similarly, for $t_2 \in (x - \delta, x + \delta) \cap (c, d)$ and $t_2 < x$, $\dfrac{g(t_2) - g(x)}{t_2 - x} = \dfrac{g(x) - g(t_2)}{x - t_2} \ge K - \varepsilon$ so that

$$g(x) - g(t_2) \ge (K - \varepsilon)(x - t_2) \text{ ------------------- (2).}$$

Combining (1) and (2) we get for $t_1, t_2 \in (x-\delta, x+\delta) \cap (c,d)$ and $t_2 < x < t_1$, and $0 < \delta < \delta_\varepsilon = \delta_{\varepsilon,x}$

$$g(t_1) - g(t_2) \geq (K-\varepsilon)(t_1 - t_2) = (K-\varepsilon)|t_1 - t_2| \text{ --------------- (3)}$$

If $g'(x) = +\infty$, $b_\delta \nearrow +\infty$ as $\delta \searrow 0^+$. It follows that there exists $\delta_{\varepsilon,x}$ such that for all $0 < \delta < \delta_{\varepsilon,x}$, $b_\delta \geq K - \varepsilon$. We deduce similarly, that for $t_1, t_2 \in (x-\delta, x+\delta) \cap (c,d)$ with $t_2 < x < t_1$ and $0 < \delta < \delta_\varepsilon = \delta_{\varepsilon,x}$,

$$g(t_1) - g(t_2) \geq (K-\varepsilon)(t_1 - t_2) = (K-\varepsilon)|t_1 - t_2|.$$

For each integer $n \geq 1$, let

$$E_n = \left\{ x \in E : g(t_1) - g(t_2) \geq (K-\varepsilon)|t_1 - t_2|, t_1 > x > t_2, t_1, t_2 \in \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap (c,d) - \{x\} \right\}.$$

Plainly, $E_n \subseteq E_{n+1}$.

We claim that $E = \bigcup_{n=1}^{\infty} E_n$.

Take $x \in E$. Then there exists $\delta_{\varepsilon,x} > 0$ such that for $0 < \delta < \delta_\varepsilon = \delta_{\varepsilon,x}$, $t_1, t_2 \in (x-\delta, x+\delta) \cap (c,d)$ with $t_2 < x < t_1$, we have that

$$g(t_1) - g(t_2) \geq (K-\varepsilon)(t_1 - t_2) = (K-\varepsilon)|t_1 - t_2|.$$

Take any integer $n > 0$ such that $\frac{1}{n} < \delta_{\varepsilon,x}$, then we have for any $t_1, t_2 \in \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap (c,d) - \{x\}$ with $t_2 < x < t_1$,

$$g(t_1) - g(t_2) \geq (K-\varepsilon)(t_1 - t_2) = (K-\varepsilon)|t_1 - t_2|.$$

Therefore, $x \in E_n$. It follows that $E = \bigcup_{n=1}^{\infty} E_n$.

Now, as $m(E) < \infty$, by the continuity from below property of Lebesgue outer measure, $m(E) = m^*(E) = \lim_{n\to\infty} m^*(E_n) < \infty$.

Starting with E_1, since $m^*(E_1) < \infty$, we can find an open set U_1 containing E_1 such that $U_1 \subseteq (c,d)$ and

$$m(U_1) \leq m^*(E_1) + \varepsilon \frac{1}{2}.$$

By the definition of $\mu_{v_g}*(E_1)$ we may choose U_1 such that

$$\mu_{v_g}*(U_1) \le \mu_{v_g}*(E_1) + \varepsilon\frac{1}{2}.$$

Since $E_n \subseteq E_{n+1}$, we may assume that $U_n \subseteq U_{n+1}$.

Now, U_1 is a countable union of disjoint open intervals. To each of these open intervals, we can further partition it into at most countable number of non-overlapping intervals, each with length less than or equal to 1. Now, we collect all these intervals with non-empty intersection with E_1. These then form a countable covering of E_1. Let $\{I_k^1\}_{k=1}^{\infty}$ denote this countable covering. Note that each $I_k^1 \cap E_1 \neq \varnothing$. Thus, we have $\sum_k m(I_k^1) \le m(U_1) \le m*(E_1) + \varepsilon\frac{1}{2}$.

Suppose the end points of I_k^1 is a_k^1 and b_k^1 with $a_k^1 < b_k^1$. If a_k^1 or b_k^1 is equal to *x*, then $g(b_k^1) - g(a_k^1) \ge (K-\varepsilon)\left|b_k^1 - a_k^1\right|$. If *x* is in the interior of I_k^1, then $g(b_k^1) - g(a_k^1) \ge (K-\varepsilon)\left|b_k^1 - a_k^1\right|$. Since *g* is of bounded variation on *I*, the set of discontinuities of *g* is at most countable. We may thus assume that g is continuous at the end points of I_k^1. By Theorem 8,
$\lambda_g\left((a_k^1, b_k^1)\right) = \lambda_g\left((a_k^1, b_k^1]\right) = \lambda_g\left([a_k^1, b_k^1)\right) = \lambda_g\left([a_k^1, b_k^1]\right) = g\left(b_k^1\right) - g\left(a_k^1\right) \ge (K-\varepsilon)\left|b_k^1 - a_k^1\right|$.

This means, $\lambda_g\left(I_k^1\right) \ge (K-\varepsilon)m\left(I_k^1\right)$.

Let $V_1 = \bigcup_k I_k^1$. Then

$$\lambda_g(V_1) = \lambda_g\left(\bigcup_k I_k^1\right) = \sum_k \lambda_g\left(I_k^1\right)$$

$$\ge (K-\varepsilon)\sum_k m\left(I_k^1\right) = (K-\varepsilon)m\left(\bigcup_k I_k^1\right) = (K-\varepsilon)m\left(V_1\right). \text{ -------------------- (4)}$$

We have also that

$$m(V_1) \le m(U_1) \le m*(E_1) + \varepsilon\frac{1}{2}, \text{ --------------------------- (5)}$$

and

$$\mu_{v_g}(V_1) \le \mu_{v_g}*(E_1) + \varepsilon\frac{1}{2}. \quad \text{---------------------------------- (6)}$$

Now that $E_1 \subseteq E_2$. Consider $E_2 - V_1$. We can now take an open set U_2 such that $E_2 - V_1 \subseteq U_2$, $m(U_2) \le m^*(E_2 - V_1) + \varepsilon \frac{1}{2^2}$ and $\mu_{v_g}(U_2) \le \mu_{v_g}{}^*(E_2 - V_1) + \varepsilon \frac{1}{2^2}$.

As before we can write U_2 as a countable union of non-overlapping intervals $\bigcup_i J_i$ with length less than or equal to $\frac{1}{2}$ and discard those intervals with empty intersection with $E_2 - V_1$. We may assume that each of these intervals does not contain any of the intervals in the decomposition for V_1. Since any interval in V_1 has empty intersection with $E_2 - V_1$, if one of these intervals, say J_k, has a non-empty intersection with an interval I in V_1, then $J_k - I$ is at most a union of two disjoint intervals, at least one of which has non- empty intersection with $E_2 - V_1$. Discard the interval with empty intersection with $E_2 - V_1$ or proceed to select the other interval if it has nonempty intersection with $E_2 - V_1$. In this way we may assume that each interval J_i has non-empty intersection with $E_2 - V_1$ and does not contain any interval in the decomposition for V_1. Now let $U_2 = \bigcup_i J_i$. Then $\lambda_g(U_2) \ge (K - \varepsilon) m(U_2)$. Let $x \in U_2 - V_1$. Then $x \in J_k$ for some k and there exists $e_2 \in E_2 - V_1$ with $e_2 \in J_k$. If $x = e_2$, then x belongs to one of the non-overlapping intervals in $U_2 - V_1$. If $x \ne e_2$, then $[x, e_2]$ or $[e_2, x]$ has empty intersection with any one of the intervals of the decomposition for V_1 and so $[x, e_2]$ or $[e_2, x]$ is contained in one of the non-overlapping intervals in $U_2 - V_1$. It follows that the non-overlapping intervals in $U_2 - V_1$ has non-empty intersection with $E_2 - V_1$. Hence,

$$\lambda_g\left((U_2 - V_1)\right) \ge (K - \varepsilon) m\left(U_2 - V_1\right).$$

Let $V_2 = V_1 \cup U_2$. Then

$$\lambda_g(V_2) = \lambda_g(V_1 \cup (U_2 - V_1)) = \lambda_g(V_1) + \lambda_g((U_2 - V_1))$$

$$\ge (K - \varepsilon) m^*(V_1) + (K - \varepsilon) m(U_2 - V_1) = (K - \varepsilon) m(V_1 \cup U_2) = (K - \varepsilon) m(V_2) \text{ ---- } (7)$$

$$m(V_2) = m(V_1 \cup U_2) \le m(V_1) + m(U_2) \le m^*(E_1) + \varepsilon \frac{1}{2} + m^*(E_2 - V_1) + \varepsilon \frac{1}{2^2}$$

$$\le m^*(E_2 \cap V_1) + \varepsilon \frac{1}{2} + m^*(E_2 - V_1) + \varepsilon \frac{1}{2^2} = m^*(E_2) + \varepsilon \left(\frac{1}{2} + \frac{1}{2^2}\right). \text{ ---------(8)}$$

Similarly, we have,

$$\mu_{v_g}(V_2) = \mu_{v_g}(V_1 \cup U_2) \le \mu_{v_g}(V_1) + \mu_{v_g}(U_2) \le \mu_{v_g}*(E_1) + \frac{1}{2} + \mu_{v_g}*(E_2 - V_1) + \varepsilon\frac{1}{2^2}.$$

$$\le \mu_{v_g}*(E_2 \cap V_1) + \varepsilon\frac{1}{2} + \mu_{v_g}*(E_2 - V_1) + \varepsilon\frac{1}{2^2} = \mu_{v_g}*(E_2) + \varepsilon\left(\frac{1}{2} + \frac{1}{2^2}\right). \text{ ---- (9)}$$

Note that $V_1 \subseteq V_2$.

Assuming that we have defined $V_n \supseteq E_n$ and a decomposition of V_n into non-overlapping intervals such that each intervals has non-empty intersection with E_n and satisfying

$$m(V_n) \le m*(E_n) + \varepsilon\sum_{k=1}^{n}\frac{1}{2^k}, \text{ --------------------------- (10)}$$

$$\mu_{v_g}(V_n) \le \mu_{v_g}*(E_n) + \varepsilon\sum_{k=1}^{n}\frac{1}{2^k} \text{ ------------------------- (11)}$$

and

$$\lambda_g(V_n) \ge (K-\varepsilon)m(V_n). \text{ ---------------------- (12)}$$

Then we can choose an open set U_{n+1}, such that $E_{n+1} - V_n \subseteq U_{n+1}$ and

$$m(U_{n+1}) \le m*(E_{n+1} - V_n) + \varepsilon\frac{1}{2^{n+1}}, \text{ ------------ (13)}$$

$$\mu_{v_g}(U_{n+1}) \le \mu_{v_g}*(E_{n+1} - V_n) + \varepsilon\frac{1}{2^{n+1}}. \text{ --------- (14)}$$

We note that U_{n+1} is a countable union of disjoint open intervals. To each of these open intervals, we can further partition it into at most countable number of non-overlapping intervals, each with length less than $\frac{1}{n+1}$. Now, we collect all these intervals with non-empty intersection with $E_{n+1} - V_n$ and discard those with empty intersection with $E_{n+1} - V_n$. Furthermore, as we deliberated above, we may assume that each of these intervals does not contain any intervals of the decomposition in V_n. Moreover,

$$\lambda_g(U_{n+1} - V_n) \ge (K-\varepsilon)m(U_{n+1} - V_n).$$

Now let $V_{n+1} = V_n \cup U_{n+1}$ so that $V_n \subseteq V_{n+1}$ and we can show similarly that

$m(V_{n+1}) \le m*(E_{n+1}) + \varepsilon\sum_{k=1}^{n+1}\frac{1}{2^k}$, $\mu_{v_g}(V_{n+1}) \le \mu_{v_g}*(E_{n+1}) + \varepsilon\sum_{k=1}^{n+1}\frac{1}{2^k}$ and

$\lambda_g(V_{n+1}) \geq (K-\varepsilon)m(V_{n+1})$.

We restate the argument below.

Let $\{I_k^{n+1}\}_{k=1}^{\infty}$ denote this special countable covering for $E_{n+1}-V_n$. We shall use the same symbol for the covering $U_{n+1}=\bigcup_k I_k^{n+1}$ Note that each $I_k^{n+1} \cap (E_{n+1}-V_n) \neq \varnothing$.

Thus, we have from (13), $m(U_{n+1})=\sum_k m(I_k^{n+1}) \leq m(E_{n+1}-V_n)+\varepsilon\frac{1}{2^{n+1}}$.

Suppose the end points of I_k^{n+1} is a_k^{n+1} and b_k^{n+1} with $a_k^{n+1}<b_k^{n+1}$. If a_k^{n+1} or b_k^{n+1} is equal to *x*, then $g(b_k^{n+1})-g(a_k^{n+1}) \geq (K-\varepsilon)\left|b_k^{n+1}-a_k^{n+1}\right|$. If *x* is in the interior of I_k^{n+1}, then $g(b_k^{n+1})-g(a_k^{n+1}) \geq (K-\varepsilon)\left|b_k^{n+1}-a_k^{n+1}\right|$. Since *g* is of bounded variation on *I*, the set of discontinuities of *g* is at most countable. We may thus assume that g is continuous at the end points of I_k^{n+1}. By Theorem 8,

$$\lambda_g\left((a_k^{n+1},b_k^{n+1})\right)=\lambda_g\left((a_k^{n+1},b_k^{n+1}]\right)=\lambda_g\left([a_k^{n+1},b_k^{n+1})\right)=\lambda_g\left([a_k^{n+1},b_k^{n+1}]\right)$$
$$=g\left(b_k^{n+1}\right)-g\left(a_k^{n+1}\right)\geq (K-\varepsilon)\left|b_k^{n+1}-a_k^{n+1}\right| \quad .$$

This means, $\lambda_g\left(I_k^{n+1}\right)\geq (K-\varepsilon)m\left(I_k^{n+1}\right)$. Thus $\lambda_g(U_{n+1})\geq (K-\varepsilon)m(U_{n+1})$. Similarly, we deduce that $\lambda_g(U_{n+1}-V_n)\geq (K-\varepsilon)m(U_{n+1}-V_n)$.

Thus, $\lambda_g(V_{n+1})=\lambda_g(V_n\cup(U_{n+1}-V_n))=\lambda_g(V_n)+\lambda_g((U_{n+1}-V_n))$

$\geq (K-\varepsilon)m(V_n)+(K-\varepsilon)m(U_{n+1}-V_n)=(K-\varepsilon)m(V_n\cup U_{n+1})=(K-\varepsilon)m(V_{n+1})$.

Let $V=\bigcup_{n=1}^{\infty}V_n$.

Then $\mu_{v_g}(V)=\lim_{n\to\infty}\mu_{v_g}(V_n)\leq \lim_{n\to\infty}\mu_{v_g}{}^*(E_n)+\lim_{n\to\infty}\varepsilon\sum_{k=1}^{n}\frac{1}{2^k}=\mu_{v_g}{}^*(E)+\varepsilon$ and

$m(V)=\lim_{n\to\infty}m(V_n)\leq \lim_{n\to\infty}m^*(E_n)+\lim_{n\to\infty}\varepsilon\sum_{k=1}^{n}\frac{1}{2^k}=m^*(E)+\varepsilon$.

Since $\lambda_g=\mu_P-\mu_N$, by the continuity from below property of positive measure,

$\lambda_g(V)=\lim_{n\to\infty}\lambda_g(V_n)\geq (K-\varepsilon)\lim_{n\to\infty}m(V_n)=(K-\varepsilon)m(V)$.

$$\mu_{v_g}(V-E)=\mu_{v_g}(V)-\mu_{v_g}(E)\leq\varepsilon \text{ ---------------------------- (15)}$$

Therefore, $|\lambda_g|(V-E)=\mu_{v_g}(V-E)\le\varepsilon$ and so $\lambda_g(V-E)\le\mu_{v_g}(V-E)\le\varepsilon$. It follows that

$$-\lambda_g(V-E)\ge-\varepsilon. \quad \text{-----------------------------} \ (16)$$

Similarly,

$$m(V-E)=m(V)-m(E)\le\varepsilon. \qquad \text{-----------------} \ (17)$$

Now, $\lambda_g(E)=\lambda_g(V)-\lambda_g(V-E)\ge(K-\varepsilon)m(V)-\varepsilon$.

If $(K-\varepsilon)\ge 0$, then $(K-\varepsilon)m(V)\ge(K-\varepsilon)m(E)\ge(K-\varepsilon)m(E)-|K-\varepsilon|\varepsilon$.

If $(K-\varepsilon)<0$, then

$(K-\varepsilon)m(V)\ge(K-\varepsilon)(m(E)+\varepsilon)=(K-\varepsilon)m(E)+(K-\varepsilon)\varepsilon=(K-\varepsilon)m(E)-|K-\varepsilon|\varepsilon$.

It follows that

$$\lambda_g(E)=\lambda_g(V)-\lambda_g(V-E)\ge(K-\varepsilon)m(E)-|K-\varepsilon|\varepsilon-\varepsilon.$$

Letting $\varepsilon\to 0^+$, we get

$$\lambda_g(E)\ge Km(E).$$

Suppose now E is any Borel subset of I and $g'(x)\ge K$ for all x in E. Let (c_n) be a sequence such that $c_n\searrow\inf I$ and (d_n) be a sequence such that $d_n\nearrow\sup I$. Then (c_n,d_n) is contained in the open interval I. Let $E_n=E\cap(c_n,d_n)$. Then $E_n\subset E_{n+1}$ and $E=\bigcup_{n=1}^{\infty}E_n$. Note that each E_n is Borel. By what we have just proved, $\lambda_g(E_n)\ge Km^*(E_n)$. Recall that $\lambda_g=\mu_P-\mu_N$. Then by the continuity from below property of the positive Radon measures, μ_P and μ_N, $\lim_{n\to\infty}(\mu_P(E_n)-\mu_N(E_n))=\lim_{n\to\infty}\mu_P(E_n)-\lim_{n\to\infty}\mu_N(E_n)=\mu_P(E)-\mu_N(E)=\lambda_g(E)$. Therefore,

$$\lambda_g(E)=\lim_{n\to\infty}\lambda_g(E_n)\ge K\lim_{n\to\infty}m^*(E_n)=Km(E).$$

This proves the first part of the Theorem.

For the case when E is a Borel subset of I such that $g'(x)\le K$ for all x in E, we start by considering the function $-g$. So, we have $(-g)'(x)\ge-K$ for all x in E. We may then conclude that $\lambda_{-g}(E)\ge-Km^*(E)$. But $\lambda_{-g}(E)=-\lambda_g(E)$ and so multiplying by -1 gives $\lambda_g(E)\le Km(E)$.

De La Vallée Poussin Decomposition of Lebesgue Stieltjes measure.

Now we state the de La Vallée Poussin theorem on the decomposition of the signed Lebesgue Stieltjes measure generated by a function of bounded variation on an open interval.

Theorem 17. Let *I* be an open interval and $g \in BV(I)$. Let

$I_{dis} = \{x \in I : g \text{ is discontinuous at } x\}$,

$I_{+\infty} = \{x \in I : g \text{ is continuous at } x \text{ and } g \text{ is differentiable at } x \text{ with } g'(x) = +\infty\}$ and

$I_{-\infty} = \{x \in I : g \text{ is continuous at } x \text{ and } g \text{ is differentiable at } x \text{ with } g'(x) = -\infty\}$.

Then for every Borel set $E \subseteq I - I_{dis}$,

$$\lambda_g(E) = \int_E g' + \lambda_g(E \cap I_{+\infty}) + \lambda_g(E \cap I_{-\infty}) \quad \text{and}$$

$$\mu_{v_g}(E) = |\lambda_g|(E) = \int_E |g'| + \lambda_g(E \cap I_{+\infty}) + |\lambda_g(E \cap I_{-\infty})|.$$

Moreover, there exists a Borel set $N \subseteq I - I_{dis}$ with $m(N) = \mu_{v_g}(N) = m\left(v_g(N)\right) = 0$ such that for all $x \in I - \left(I_{dis} \cup N\right)$, $v_g{}'(x) = |g'(x)|$. Note that $\lambda_g(E \cap I_{-\infty}) \leq 0$.

We shall need the following useful result on the additivity of total variation of two functions when one is absolutely continuous of bounded variation and the other a singular function of bounded variation.

Theorem 18. Suppose *I* is an open interval and $g : I \to \mathbb{R}$ is an absolutely continuous function of bounded variation and $h : I \to \mathbb{R}$ is a singular function of bounded variation on *I*. Then $v_{g+h} = v_g + v_h$.

Proof.

For $b \geq a$ with a, b in *I*, let $Var_g[a,b]$ be the total variation of g on the interval $[a,b]$. Then $v_{g+h}(x) = Var_{g+h}[x_0, x]$ for $x \geq x_0$ and $v_{g+h}(x) = -Var_{g+h}[x, x_0]$ if $x < x_0$. Therefore, for $x \geq x_0$,

$$v_{g+h}(x) = Var_{g+h}[x_0, x] \leq Var_g[x_0, x] + Var_g[x_0, x] = v_g(x) + v_h(x).$$

For $x < x_0$, $-v_{g+h}(x) = Var_{g+h}[x, x_0] \leq Var_g[x, x_0] + Var_h[x_0, x] = -v_g(x) - v_h(x)$ and so

$$\nu_{g+h}(x) \geq \nu_g(x) + \nu_h(x).$$

Similarly, for $x \geq x_0$,
$\nu_h(x) = Var_{(g+h)-g}[x_0, x] \leq Var_{g+h}[x_0, x] + Var_{-g}[x_0, x] = Var_{g+h}[x_0, x] + Var_g[x_0, x] = \nu_{g+h}(x) + \nu_g(x)$
and for $x < x_0$, $\nu_h(x) = \nu_{(g+h)-g}(x) \geq \nu_{g+h}(x) + \nu_g(x)$.

Thus, for $x \geq x_0$, $\nu_{g+h}(x) - \nu_h(x) \leq \nu_g(x)$ and for $x < x_0$, $\nu_h(x) - \nu_{g+h}(x) \leq -\nu_g(x)$ and that $x \geq x_0$, $\nu_h(x) - \nu_{g+h}(x) \leq \nu_g(x)$ and for $x < x_0$, $\nu_{g+h}(x) - \nu_h(x) \leq -\nu_g(x)$.

It follows that

$$\left|\nu_{g+h}(x) - \nu_h(x)\right| \leq \nu_g(x) \text{ for } x \geq x_0 \text{ and } \left|\nu_{g+h}(x) - \nu_h(x)\right| \leq -\nu_g(x) = \left|\nu_g(x)\right| \text{ for } x < x_0.$$

Let $f(x) = v_{g+h}(x) - v_h(x)$. Then for any *y* > *x*,

$$|f(y) - f(x)| = \left|v_{g+h}(y) - v_{g+h}(x) - \left(v_h(y) - v_h(x)\right)\right| = \left|Var_{g+h}[x, y] - Var_h[x, y]\right|.$$

Then taking $x_0 = x$, we see that

$$|f(y) - f(x)| = \left|Var_{g+h}[x, y] - Var_h[x, y]\right| \leq Var_g[x, y] = v_g(y) - v_g(x). \quad \text{---------- (1)}$$

Since *g* is absolutely continuous and of bounded variation, v_g is also absolutely continuous. It follows from the inequality (1) that *f* is absolutely continuous.

Now, $f'(x) = {v_{g+h}}'(x) - {v_h}'(x) = |(g+h)'(x)| - |h'(x)|$, almost everywhere on *I*,

$= |g'(x)|$, almost everywhere.

Since *f* and v_g are absolutely continuous, $f - v_g$ is absolutely continuous and

$$\left(f - v_g\right)'(x) = f'(x) - {v_g}'(x) = |g'(x)| - |g'(x)| = 0 \text{ almost everywhere on } I.$$

It then follows from Theorem 7 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*, that $f - v_g = 0$, since $f(x_0) = v_g(x_0) = 0$. Hence, $v_{g+h} = v_g + v_h$.

Before we embark on the proof of Theorem 17, we state and prove the next result, which facilitates the proof of Theorem 17.

Theorem 19. Suppose *I* is an open interval and $g: I \to \mathbb{R}$ is a function of bounded variation. Then *g* can be decomposed into a sum of three functions, $g = g_{ab} + g_c + g_s$, where g_{ab} is absolutely continuous , $g_{ab}'(x) = g'(x)$ almost everywhere on *I*, g_c is a continuous singular function, i.e., $g_c'(x) = 0$ almost everywhere and g_s is the difference of two Saltus functions. For any Borel set *E* in *I*,

(a) $$\lambda_{g_s}(E) = \sum_{y \in E} \left(g_+(y) - g_-(y) \right),$$

(b) $$\left| \lambda_{g_s} \right|(E) = \sum_{y \in E} \left| g_+(y) - g_-(y) \right|,$$

(c) $$\mu_{v_{g_s}}(E) = \sum_{y \in E} \left(\left| g_+(y) - g(y) \right| + \left| g(y) - g_-(y) \right| \right) \text{ and}$$

(d) $$\mu_{v_g}(E \cap I_{dis}) = \mu_{v_{g_s}}(E \cap I_{dis}) = \mu_{v_{g_s}}(E).$$

Furthermore,

$$\left| \lambda_{g_s} \right|(I - I_{dis}) = \mu_{v_{g_s}}(I - I_{dis}) = \left| \lambda_{g_{ab}} \right|(I_{dis}) = \left| \lambda_{g_c} \right|(I_{dis}) = \mu_{v_{g_{ab}}}(I_{dis}) = \mu_{v_{g_c}}(I_{dis}) = 0.$$

If *g* is continuous at every point of the Borel set *E*, $\left| \lambda_g \right|(E) = \mu_{v_g}(E)$ and

$$\mu_{v_{g_c + g_s}}(E) = \mu_{v_{g_c}}(E).$$

For any Borel set $E \subseteq I$, $\mu_{v_g}(E) = \mu_{v_{g_{ab}}}(E) + \mu_{v_{g_c}}(E) + \mu_{v_{g_s}}(E)$.

Proof.

We define the saltus function for unbounded increasing function $h: I \to \mathbb{R}$ in general as follows. Pick a reference point x_0. We assume that *h* is continuous at x_0. Then define

$$h_s(x) = \begin{cases} \sum\limits_{y \in I, x_0 \le y < x} \left(h_+(y) - h_-(y) \right) + h(x) - h_-(x), x > x_0 \\ -\sum\limits_{y \in I, x < y < x_0} \left(h_+(y) - h_-(y) \right) + h(x) - h_+(x), x < x_0 \\ 0, x = x_0 \end{cases}. \quad \text{----------- (1)}$$

Then $h_s'(x) = 0$ and $\left(h - h_s \right)'(x) = h'(x)$ almost everywhere on *I*.

It can be shown that $h - h_s$ is an increasing continuous function on *I*. A proof of this is given in Theorem 13 of *Arc Length, Functions of Bounded Variation and Total Variation*, when the domain is a closed and bounded interval. The proof there is also applicable when the domain is any interval. Now we assume that *h* is an increasing bounded function and so $\phi = h - h_s$ is also an increasing continuous bounded function. The function ϕ can be decomposed into a sum of an absolutely continuous function, h_{ab} and a continuous singular function, h_c such that $h_{ab}{}'(x) = h'(x)$ almost everywhere on *I*, $h_c{}'(x) = h_s{}'(x) = 0$ almost everywhere on *I*, that is, $h - h_s = \phi = h_{ab} + h_c$ and $h = h_{ab} + h_c + h_s$. Note that $\phi'(x) = h'(x)$ almost everywhere. The function h_{ab} may be defined by

$$h_{ab}(x) = \begin{cases} \int_{x_0}^{x} h'(t)dt, \; x \geq x_0, \\ -\int_{x}^{x_0} h'(t)dt, \; x < x_0 \end{cases}.$$

We note that since *h* is increasing and bounded, *h* is differentiable almost everywhere on *I* and h' is finite for almost all *x* in *I*. It follows then by Theorem 6 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation* that h' is Lebesgue integrable. It follows, as indicated in the proof of Theorem 8 in *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation* by applying Proposition 9 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*, that h_{ab} is absolutely continuous on *I*. Note that $h_c = (h - h_s) - h_{ab}$ and since $(h - h_s)$ and h_{ab} are continuous, h_c is continuous and $h_c{}'(x) = 0$ almost everywhere on *I*.

Now *g* is of bounded variation and so $g = P - N$ is a difference of two bounded increasing functions. So, taking the decomposition of *P* and *N* we get

$$g = P_{ab} + P_c + P_s - (N_{ab} + N_c + N_s) = (P_{ab} - N_{ab}) + (P_c - N_c) + (P_s - N_s).$$

Let $g_{ab} = P_{ab} - N_{ab}$, $g_c = P_c - N_c$ and $g_s = P_s - N_s$. Then g_{ab} is absolutely continuous as it is the sum of two absolutely continuous functions. g_c is the sum of two continuous singular functions and so it is a continuous singular function. The function g_s is the difference of two saltus type functions. We call it a *jump function.* Note that $g - g_s = g_{ab} + g_c$ is a continuous function.

By definition of the Saltus function and as *g* is continuous at *x* if, and only if, v_g is continuous, we have, using definition (1), that

$$g_s(x)=\begin{cases}\sum\limits_{y\in I, x_0\le y<x}\left(g_+(y)-g_-(y)\right)+g(x)-g_-(x), x>x_0\\ -\sum\limits_{y\in I, x<y<x_0}\left(g_+(y)-g_-(y)\right)+g(x)-g_+(x), x<x_0\\ 0, x=x_0\end{cases}. \text{ --------------- (2)}$$

Thus, using (2), we have, for $x_0<a<b$,

$$g_s(b)-g_s(a)=\sum_{x_0\le y<b}\left(g_+(y)-g_-(y)\right)+g(b)-g_-(b)-\left(\sum_{x_0\le y<a}\left(g_+(y)-g_-(y)\right)+g(a)-g_-(a)\right)$$

$$=\sum_{a\le y<b}\left(g_+(y)-g_-(y)\right)+g(b)-g_-(b)-(g(a)-g_-(a))$$

$$=\sum_{a<y<b}\left(g_+(y)-g_-(y)\right)+g(b)-g_-(b)-(g(a)-g_+(a))$$

For $a<b<x_0$,

$$g_s(b)-g_s(a)=-\sum_{y\in I, b<y<x_0}\left(g_+(y)-g_-(y)\right)+g(b)-g_+(b)+\left(\sum_{y\in I, a<y<x_0}\left(g_+(y)-g_-(y)\right)-\left(g(a)-g_+(a)\right)\right)$$

$$=\sum_{y\in I, a<y\le b}\left(g_+(y)-g_-(y)\right)+g(b)-g_+(b)-\left(g(a)-g_+(a)\right)$$

$$=\sum_{y\in I, a<y<b}\left(g_+(y)-g_-(y)\right)+g(b)-g_-(b)-\left(g(a)-g_+(a)\right).$$

For $a<x_0<b$,

$$g_s(b)-g_s(a)=\sum_{x_0\le y<b}\left(g_+(y)-g_-(y)\right)+g(b)-g_-(b)+\sum_{y\in I, a<y<x_0}\left(g_+(y)-g_-(y)\right)-\left(g(a)-g_+(a)\right)$$

$$=\sum_{a<y<b}\left(g_+(y)-g_-(y)\right)+g(b)-g_-(b)-\left(g(a)-g_+(a)\right).$$

Hence, for $a<b$ with $a,b\in I$, we have

$$g_s(b)-g_s(a)=\sum_{a<y<b}\left(g_+(y)-g_-(y)\right)+g(b)-g_-(b)-\left(g(a)-g_+(a)\right). \text{ ---------- (3)}$$

It follows that

$$\left|g_s(b)-g_s(a)\right|\le\sum_{a<y<b}\left(\left|g_+(y)-g(y)\right|+\left|g(y)-g_-(y)\right|\right)+\left|g(b)-g_-(b)\right|+\left|\left(g(a)-g_+(a)\right)\right|.$$

Thus, if we take any partition of $[a, b]$, $x_0=a<x_1<x_2<\cdots<x_n=b$, i =1, …, n–1, we see that

$$\sum_{1\le i\le n}\left|g_s(x_i)-g_s(x_{i-1})\right|\le\sum_{a<y<b}\left(\left|g_+(y)-g(y)\right|+\left|g(y)-g_-(y)\right|\right)+\left|g(b)-g_-(b)\right|+\left|\left(g(a)-g_+(a)\right)\right|.$$

Therefore,

$$Var_{g_s}[a,b] \le \sum_{a<y<b} \left(\left| g_+(y) - g(y) \right| + \left| g(y) - g_-(y) \right| \right) + \left| g(b) - g_-(b) \right| + \left| \left(g(a) - g_+(a) \right) \right| \ .$$

If there is only a finite number of discontinuities in [*a*, *b*], then

$Var_{g_s}[a,b] \ge \sum_{a<y<b} \left(\left| g_+(y) - g(y) \right| + \left| g(y) - g_-(y) \right| \right) + \left| g(b) - g_-(b) \right| + \left| \left(g(a) - g_+(a) \right) \right|$ and so

$$Var_{g_s}[a,b] = \sum_{a<y<b} \left(\left| g_+(y) - g(y) \right| + \left| g(y) - g_-(y) \right| \right) + \left| g(b) - g_-(b) \right| + \left| \left(g(a) - g_+(a) \right) \right| .$$

On the other hand, for any set *K* of finite number of discontinuities in [*a*, *b*],

$$Var_{g_s}[a,b] \ge \sum_{y \in K} \left(\left| g_+(y) - g(y) \right| + \left| g(y) - g_-(y) \right| \right) + \left| g(b) - g_-(b) \right| + \left| \left(g(a) - g_+(a) \right) \right| .$$

Therefore, since the number of discontinuities of *g* is at most countably infinite,

$$Var_{g_s}[a,b] \ge \sum_{a<y<b} \left(\left| g_+(y) - g(y) \right| + \left| g(y) - g_-(y) \right| \right) + \left| g(b) - g_-(b) \right| + \left| \left(g(a) - g_+(a) \right) \right| .$$

Thus,

$$Var_{g_s}[a,b] = \sum_{a<y<b} \left(\left| g_+(y) - g(y) \right| + \left| g(y) - g_-(y) \right| \right) + \left| g(b) - g_-(b) \right| + \left| \left(g(a) - g_+(a) \right) \right| \text{---- (4)}$$

Now take any $x \in I_{dis}$ in the interior of [*a*, *b*]. Then, by taking *b* such that g is continuous at *b* and $b \searrow x$, and *a* such that *g* is continuous at *a* with $a \nearrow x$, we get

$$\left(v_{g_s} \right)_+ (x) - \left(v_{g_s} \right)_- (x) = \left| g_+(x) - g(x) \right| + \left| g(x) - g_-(x) \right| \ . \text{------------------ (5)}$$

Let $G(x) = g(x) - g_s(x)$. Then *G*(*x*) is continuous on *I*. Therefore,

$$G_+(x) = g_+(x) - \left(g_s \right)_+ (x) = G(x) = g(x) - g_s(x) .$$

By identity (2), for $x > x_0$,

$$\left(g_s \right)_+ (x) = g_+(x) - g(x) + g_s(x) = \sum_{y \in I, x_0 \le y \le x} \left(g_+(y) - g_-(y) \right)$$

and for $x < x_0$,

$$\left(g_s \right)_+ (x) = g_+(x) - g(x) + g_s(x)$$
$$= g_+(x) - g(x) + g_s(x) - \sum_{y \in I, x < y < x_0} \left(g_+(y) - g_-(y) \right) + g(x) - g_+(x)$$

$$= -\sum_{x<y<x_0} \left(g_+(y) - g_-(y)\right).$$

Note that $\left(g_s\right)_+(x_0) = g_+(x_0) - g(x_0) + g_s(x_0) = 0$, since g is continuous at x_0.

Similarly, we have $G_-(x) = g_-(x) - \left(g_s\right)_-(x) = G(x) = g(x) - g_s(x)$ so that

$$\left(g_s\right)_-(x) = g_-(x) - g(x) + g_s(x)$$

For $x > x_0$, $\left(g_s\right)_-(x) = g_-(x) - g(x) + g_s(x)$

$$= g_-(x) - g(x) + \sum_{y\in I, x_0 \le y < x} \left(g_+(y) - g_-(y)\right) + g(x) - g_-(x)$$

$$= \sum_{y\in I, x_0 \le y < x} \left(g_+(y) - g_-(y)\right)$$

and for $x < x_0$, $\left(g_s\right)_-(x) = g_-(x) - g(x) + g_s(x)$

$$= g_-(x) - g(x) - \sum_{y\in I, x<y<x_0} \left(g_+(y) - g_-(y)\right) + g(x) - g_+(x)$$

$$= -\sum_{y\in I, x\le y<x_0} \left(g_+(y) - g_-(y)\right).$$

We have also that $\left(g_s\right)_-(x_0) = g_-(x_0) - g(x_0) + g_s(x_0) = 0$.

Thus, we have,

$$\left(g_s\right)_+(x) = \begin{cases} \sum_{y\in I, x_0\le y\le x} \left(g_+(y) - g_-(y)\right), \; x > x_0, \\ 0, \;\; x = x_0, \\ -\sum_{y\in I, x<y<x_0} \left(g_+(y) - g_-(y)\right), \; x < x_0 \end{cases}, \text{------------------------ (6)}$$

and

$$\left(g_s\right)_-(x) = \begin{cases} \sum_{y\in I, x_0\le y< x} \left(g_+(y) - g_-(y)\right), \; x > x_0, \\ 0, \;\; x = x_0, \\ -\sum_{y\in I, x\le y<x_0} \left(g_+(y) - g_-(y)\right), \; x < x_0 \end{cases}. \quad \text{-------------------- (7)}$$

It follows from (6) and (7) that for any x in I,

$$g_{s+}(x) - g_{s-}(x) = g_+(x) - g_-(x).$$

Therefore,

$$\lambda_{g_s}\left(\{x\}\right) = g_{s+}(x) - g_{s-}(x) = g_+(x) - g_-(x). \qquad \text{----------------- (8)}$$

From identity (5), we get for $x \in I_{dis}$,

$$\mu_{\nu_{g_s}}\left(\{x\}\right)=\left(\nu_{g_s}\right)_+(x)-\left(\nu_{g_s}\right)_-(x)=\left|g_+(x)-g(x)\right|+\left|g(x)-g_-(x)\right| \text{ ------- (9)}$$

Obviously, for $x \in I - I_{dis}$, $\mu_{\nu_{g_s}}\left(\{x\}\right)=\left(\nu_{g_s}\right)_+(x)-\left(\nu_{g_s}\right)_-(x)=0$ since ν_{g_s} is continuous at *x*.

Note that we use the same anchor point x_0 for the definition of total variation function as well as for the Saltus function. Recall that

$$\nu_g(x)=P(x)+N(x)-g(x_0) \text{ and } g(x)=P(x)-N(x)=g(x_0)+\varphi_1(x)-\varphi_2(x).$$

Now assume $x \neq x_0$. Let $y > x$ and $y \in I$.

Recall that x_0, a point of continuity of *g* is the anchor point for the total variation function of *g*.

If $[a,b]$ is an interval with end points in $I - I_{dis}$, then

$$\mu_{\nu_{g_s}}\left([a,b]\cap I_{dis}\right)=\sum_{x\in(a,b)\cap I_{dis}}\mu_{\nu_{g_s}}\left(\{x\}\right), \text{ since } \mu_{\nu_{g_s}} \text{ is finite and } I_{dis} \text{ is countable,}$$

$$=\sum_{x\in(a,b)\cap I_{dis}}\left|g_+(x)-g(x)\right|+\left|g(x)-g_-(x)\right| \text{ by identity (9).}$$

But $\mu_{\nu_{g_s}}\left([a,b]\right)=\left(\nu_{g_s}\right)_+(b)-\left(\nu_{g_s}\right)_-(a)=\nu_{g_s}(b)-\nu_{g_s}(a)$

$$=Var_{g_s}[a,b]$$

$$=\sum_{x\in(a,b)\cap I_{dis}}\left|g_+(x)-g(x)\right|+\left|g(x)-g_-(x)\right|, \text{ by identity (4).}$$

Hence, $\mu_{\nu_{g_s}}\left([a,b]-I_{dis}\right)=0$. It follows that $\mu_{\nu_{g_s}}\left(I-I_{dis}\right)=0$. Therefore, $\left|\lambda_{g_s}\right|\left(I-I_{dis}\right)=0$ and so $\lambda_{g_s}\left(I-I_{dis}\right)=0$.

Thus, for any Borel set *E* in *I*, we have

$$\lambda_{g_s}(E)=\lambda_{g_s}(E-I_{dis})+\lambda_{g_s}(E\cap I_{dis})=\lambda_{g_s}(E\cap I_{dis}), \text{ since } \lambda_{g_s}(E-I_{dis})=0,$$

$$=\sum_{x\in E\cap I_{dis}}\left(g_+(x)-g_-(x)\right)=\sum_{x\in E}\left(g_+(x)-g_-(x)\right), \text{ by identity (8).}$$

This proves part (a).

As I_{dis} is countable and $\left|\lambda_{g_s}\right|$ is finite, we have that

$$\left|\lambda_{g_s}\right|(E)=\left|\lambda_{g_s}\right|(E\cap I_{dis})+\left|\lambda_{g_s}\right|(E-I_{dis})=\left|\lambda_{g_s}\right|(E\cap I_{dis})$$

$$= \sum_{x\in E\cap I_{dis}} \left|g_+(x)-g_-(x)\right| = \sum_{x\in E}\left|g_+(x)-g_-(x)\right|. \quad \text{-------------------------} \ (*)$$

We may deduce this as follows:

Take any $x \in I_{dis}$. Pick any *a*, *b* in *I* such that g is continuous at *a* and *b* and $a < x < b$. Then

$$\left|\sum_{y\in(a,b)}\left(g_+(x)-g_-(x)\right)\right| = \left|\lambda_{g_s}([a,b])\right| \le \left|\lambda_{g_s}\right|([a,b]) \le \sum_{y\in(a,b)}\left|g_+(x)-g_-(x)\right|.$$

Letting $a \nearrow x$ and $b \searrow x$, we get

$$\left|g_+(x)-g_-(x)\right| = \left|\lambda_{g_s}(\{x\})\right| \le \left|\lambda_{g_s}\right|(\{x\}) \le \left|g_+(x)-g_-(x)\right|$$

and so $\left|\lambda_{g_s}\right|(\{x\}) = \left|g_+(x)-g_-(x)\right|$. Thus, (*) follows since I_{dis} is countable and *g* is of bounded variation so that $\sum_{y\in I_{dis}}\left|g_+(y)-g_-(y)\right| < \infty$.

This proves part (b).

Similarly, for any Borel set *E* in *I*,

$$\mu_{v_{g_s}}(E) = \mu_{v_{g_s}}(E - I_{dis}) + \mu_{v_{g_s}}(E\cap I_{dis}) = \mu_{v_{g_s}}(E\cap I_{dis}) = \sum_{x\in E\cap I_{dis}} \mu_{v_{g_s}}(\{x\})$$

$$= \sum_{x\in E\cap I_{dis}} \left|g_+(x)-g(x)\right| + \left|g(x)-g_-(x)\right|, \text{ by identity (9),}$$

$$= \sum_{x\in E}\left|g_+(x)-g(x)\right| + \left|g(x)-g_-(x)\right|$$

This proves part (c).

We shall show next that for $t \in I_{dis}$, $\mu_{v_{g_c+g_s}}\left(\{t\}\right) = \mu_{v_{g_s}}\left(\{t\}\right)$.

Suppose *x* is an isolated point of I_{dis}. Then, there exists *a*, *b* in $I - I_{dis}$ such that $a < b$ and $[a,b]\cap I_{dis} = \{x\}$. Let $\delta < \frac{1}{2}\min\{x-a, b-x\}$. Let $h = g_c + g_s$.

$$Var_h[a,b] = Var_h[a+\delta, x-\delta] + Var_h[x+\delta, b-\delta] + Var_h[x-\delta, x+\delta]$$

$$+ Var_h[a, a+\delta] + Var_h[b-\delta, b]$$

$$\ge Var_h[a+\delta, x-\delta] + Var_h[x+\delta, b-\delta] + \left(\left|h(x+\delta)-h(x)\right| + \left|h(x)-h(x-\delta)\right|\right)$$

$$+ \left|h(a+\delta)-h(a)\right| + \left|h(b)-h(b-\delta)\right|.$$

$$=Var_{g_c}[a+\delta,x-\delta]+Var_{g_c}[x+\delta,b-\delta]+\left(\left|h\left(x+\delta\right)-h(x)\right|+\left|h\left(x\right)-h(x-\delta)\right|\right)$$

$$+\left|g_c\left(a+\delta\right)-g_c(a)\right|+\left|g_c\left(b\right)-g_c(b-\delta)\right|.$$

Therefore, letting $\delta\to 0^+$, we get,

$$Var_h[a,b]\geq Var_{g_c}[a,x]+Var_{g_c}[x,b]+\left(\left|h_+\left(x\right)-h(x)\right|+\left|h\left(x\right)-h_-(x)\right|\right)+\left|\left(g_c\right)_+(a)-g_c(a)\right|+\left|g_c(b)-\left(g_c\right)_-(b)\right|$$

$$=Var_{g_c}[a,b]+\left(\left|\left(g_s\right)_+\left(x\right)-\left(g_s\right)(x)\right|+\left|\left(g_s\right)\left(x\right)-\left(g_s\right)_-(x)\right|\right)$$

Therefore, by the continuity from above property of measure,

$$\mu_{v_{g_c+g_s}}\left(\{x\}\right)\geq\mu_{v_{g_c}}\left(\{x\}\right)+\left(\left|\left(g_s\right)_+\left(x\right)-\left(g_s\right)(x)\right|+\left|\left(g_s\right)\left(x\right)-\left(g_s\right)_-(x)\right|\right)=\mu_{v_{g_s}}\left(\{x\}\right).$$

Hence, $\mu_{v_{g_c+g_s}}\left(\{x\}\right)=\mu_{v_{g_s}}\left(\{x\}\right)$ as $\mu_{v_{g_c+g_s}}\left(\{x\}\right)\leq\mu_{v_{g_c}}\left(\{x\}\right)+\mu_{v_{g_s}}\left(\{x\}\right)=\mu_{v_{g_s}}\left(\{x\}\right)$.

Let $x\in I_{dis}$. Suppose *x* is not an isolated point of I_{dis} but a limit point of I_{dis}.

Since v_{g_c} is continuous at *x*, given any $\varepsilon>0$, there exists a $\delta>0$ such that $|t-x|<\delta$ and $t\in I$ implies that $\left|v_{g_c}(t)-v_{g_c}(x)\right|<\varepsilon$. Thus, for all $0<x-t<\delta$ and $t\in I$, $Var_{g_c}[t,x]=v_{g_c}(x)-v_{g_c}(t)<\varepsilon$ and for all $0<t-x<\delta$, $Var_{g_c}[x,t]=v_{g_c}(t)-v_{g_c}(x)<\varepsilon$

Take $y\in I$ such that $y<x$ and $|y-x|<\delta$.

Take a partition of $[y,x]$ by non-overlapping intervals $\{I_i\}$ where $I_i=[a_i,b_i]$

Then, for each *i*,

$$\left|h(b_i)-h(a_i)\right|=\left|g_s(b_i)-g_s(a_i)+g_c(b_i)-g_c(a_i)\right|\geq\left|g_s(b_i)-g_s(a_i)\right|-\left|g_c(b_i)-g_c(a_i)\right|$$

Hence,

$$\sum_i\left|h(b_i)-h(a_i)\right|\geq\sum_i\left|g_s(b_i)-g_s(a_i)\right|-\sum_i\left|g_c(b_i)-g_c(a_i)\right|.$$

But $\sum_i\left|g_c(b_i)-g_c(a_i)\right|\leq Var_{g_c}[y,x]<\varepsilon$. It follows that

$$\sum_i\left|h(b_i)-h(a_i)\right|\geq\sum_i\left|g_s(b_i)-g_s(a_i)\right|-\varepsilon.$$

Therefore, $Var_h[y,x]\geq Var_{g_s}[y,x]-\varepsilon$ This means

$$v_h(x)-v_h(y)\geq v_{g_s}(x)-v_{g_s}(y)-\varepsilon$$

Letting $y\nearrow x$, we get

$$\nu_h(x)-(\nu_h)_-(x)\geq \nu_{g_s}(x)-\left(\nu_{g_s}\right)_-(x)-\varepsilon . \text{ ------------------ (10)}$$

Now we take $y\in I$ such that $y>x$ and $|y-x|<\delta$.

As before, take a partition of $[x,y]$ by non-overlapping intervals $\{I_i\}$ where $I_i=[a_i,b_i]$. We have as above that

$$\sum_i |h(b_i)-h(a_i)|\geq \sum_i |g_s(b_i)-g_s(a_i)|-\varepsilon .$$

It follows similarly that $\nu_h(y)-\nu_h(x)\geq \nu_{g_s}(y)-\nu_{g_s}(x)-\varepsilon$. Letting $y\searrow x$, we get

$$(\nu_h)_+(x)-\nu_h(x)\geq \left(\nu_{g_s}\right)_+(x)-\nu_{g_s}(x)-\varepsilon . \text{ ------------------ (11)}$$

Combining (10) and (11), we obtain

$$(\nu_h)_+(x)-(\nu_h)_-(x)\geq \left(\nu_{g_s}\right)_+(x)-\left(\nu_{g_s}\right)_-(x)-2\varepsilon .$$

That is to say, $\mu_{\nu_h}\left(\{x\}\right)=(\nu_h)_+(x)-(\nu_h)_-(x)\geq \left(\nu_{g_s}\right)_+(x)-\left(\nu_{g_s}\right)_-(x)-2\varepsilon=\mu_{\nu_{g_s}}\left(\{x\}\right)-2\varepsilon$.

Since ε is arbitrary, it follows that $\mu_{\nu_h}\left(\{x\}\right)\geq \mu_{\nu_{g_s}}\left(\{x\}\right)$ so that $\mu_{\nu_h}\left(\{x\}\right)=\mu_{\nu_{g_s}}\left(\{x\}\right)$.

This means, $\mu_{\nu_{g_c+g_s}}\left(\{t\}\right)=\mu_{\nu_h}\left(\{t\}\right)=\mu_{\nu_{g_s}}\left(\{t\}\right)$ for all $t\in I_{dis}$.

Hence,

$$\mu_{\nu_{g_c+g_s}}\left(E\cap I_{dis}\right)=\sum_{x\in E\cap I_{dis}}\mu_{\nu_{g_c+g_s}}\left(\{x\}\right)=\sum_{x\in E\cap I_{dis}}\mu_{\nu_{g_s}}\left(\{x\}\right)$$
$$=\sum_{x\in E\cap I_{dis}}\left(\left|g_+(x)-g(x)\right|+\left|g(x)-g_-(x)\right|\right).$$

Thus,

$$\mu_{\nu_g}\left(E\cap I_{dis}\right)=\mu_{\nu_{g_c+g_s}}\left(E\cap I_{dis}\right)=\mu_{\nu_{g_s}}\left(E\cap I_{dis}\right) \qquad \text{-------------------------------- (12)}$$
$$=\sum_{x\in E\cap I_{dis}}\left(\left|g_+(x)-g(x)\right|+\left|g(x)-g_-(x)\right|\right)=\sum_{x\in E}\left(\left|g_+(x)-g(x)\right|+\left|g(x)-g_-(x)\right|\right)$$
$$=\mu_{\nu_{g_s}}\left(E\right).$$

This proves (d).

Now, since I_{dis} is countable and g_{ab} is continuous on I_{dis}, by Theorem 6, $\mu_{\nu_{g_{ab}}}(I_{dis})=m^*\left(\nu_{g_{ab}}(I_{dis})\right)=0$. It follows by Theorem 8 that $\left|\lambda_{g_{ab}}\right|(I_{dis})=\lambda_{g_{ab}}\left(I_{dis}\right)=0$.

Similarly, as $\mu_{\nu_{g_c}}(I_{dis})=m^*\left(\nu_{g_c}(I_{dis})\right)=0$, $\left|\lambda_{g_c}\right|(I_{dis})=0$ and so $\lambda_{g_c}\left(I_{dis}\right)=0$.

For any Borel set E in $I-I_{dis}$, take any Borel subset $F\subseteq E$, since $\left|\lambda_{g_s}\right|(F)\le\left|\lambda_{g_s}\right|\left(I-I_{dis}\right)=0$, $\lambda_{g_s}(F)=0$. Similarly, $\mu_{v_{g_s}}(E)=0$.

Therefore, for $F\subseteq E$ and $E\subseteq I-I_{dis}$,

$$\lambda_g(F)=\lambda_{g_{ab}}(F)+\lambda_{g_c}(F)+\lambda_{g_s}(F)=\lambda_{g_{ab}}(F)+\lambda_{g_c}(F)$$

$$=\left(\lambda_{g_{ab}}+\lambda_{g_c}\right)(F)=\lambda_{g_{ab}+g_c}(F).$$

This implies that $\left|\lambda_g\right|(E)=\left|\lambda_{g_{ab}+g_c}\right|(E)$ for $E\subseteq I-I_{dis}$.

In particular, $\lambda_g(E)=\lambda_{g_{ab}+g_c}(E)$ for $E\subseteq I-I_{dis}$.

Thus, $\left|\lambda_g\right|(E)=\left|\lambda_{g_{ab}+g_c}\right|(E)=\mu_{v_{g_{ab}+g_c}}(E)=\mu_{v_{g_{ab}}+v_{g_c}}(E)$ by Theorem 8 and Theorem 18,

$$=\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c}}(E).$$

Now, $\mu_{v_g}(E)=\mu_{v_{g_{ab}+g_c+g_s}}(E)=\mu_{v_{g_{ab}+(g_c+g_s)}}(E)=\mu_{v_{g_{ab}}+v_{g_c+g_s}}(E)$, by Theorem 18,

$$=\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c+g_s}}(E)\le\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c}}(E)+\mu_{v_{g_s}}(E)=\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c}}(E),$$

since $E\subseteq I-I_{dis}$.

But, for $E\subseteq I-I_{dis}$,

$$\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c}}(E)=\left|\lambda_g\right|(E)\le\mu_{v_g}(E)\le\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c}}(E)$$

and so $\mu_{v_g}(E)=\left|\lambda_g\right|(E)=\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c}}(E)=\mu_{v_{g_{ab}+g_c}}(E)$.

But $\mu_{v_g}(E)=\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c+g_s}}(E)$ and so

$$\mu_{v_{g_c+g_s}}(E)=\mu_{v_{g_c}}(E),\text{ for } E\subseteq I-I_{dis}. \quad \text{--------------------} (13)$$

For $E\subseteq I$,

$\mu_{v_g}(E)=\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c+g_s}}(E)$, by Theorem 18,

$$=\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c+g_s}}(E-I_{dis})+\mu_{v_{g_c+g_s}}(E\cap I_{dis})$$

$=\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c}}(E-I_{dis})+\mu_{v_{g_c+g_s}}(E\cap I_{dis})$, by (13),

$=\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c}}(E)+\mu_{v_{g_c+g_s}}(E\cap I_{dis})$, since $\mu_{v_{g_c}}(E\cap I_{dis})=0$

$=\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c}}(E)+\mu_{v_{g_s}}(E)$, as

$$\mu_{\nu_{g_c+g_s}}(E \cap I_{dis}) = \mu_{\nu_g}(E \cap I_{dis}) = \mu_{\nu_{g_s}}(E) \text{ by (12).}$$

This concludes the proof of Theorem 19.

Proof of Theorem 17.

By Theorem 14, $I_{dis}, I_{+\infty}$ and $I_{-\infty}$ are Borel. Therefore, $I - I_{dis}$ is Borel and so for any Borel set $E \subseteq I - I_{dis}$, $E \cap I_{+\infty}$ and $E \cap I_{-\infty}$ are Borel.

$$\lambda_g(E) = \lambda_g(E \cap (I - (I_{+\infty} \cup I_{-\infty})) + \lambda_g(E \cap I_{+\infty}) + \lambda_g(E \cap I_{-\infty})$$

$$= \lambda_g(E \cap (I - (I_{dis} \cup I_{+\infty} \cup I_{-\infty})) + \lambda_g(E \cap I_{+\infty}) + \lambda_g(E \cap I_{-\infty}),$$

since E is contained in $I - I_{dis}$.

Now,

$$\lambda_g(E \cap (I - (I_{dis} \cup I_{+\infty} \cup I_{-\infty})) = \mu_P\left(E \cap (I - (I_{dis} \cup I_{+\infty} \cup I_{-\infty}))\right) - \mu_N\left(E \cap (I - (I_{dis} \cup I_{+\infty} \cup I_{-\infty}))\right)$$

$$= m\left(P\left(E \cap (I - (D_{disc} \cup I_{+\infty} \cup I_{-\infty}))\right)\right) - m\left(N\left(E \cap (I - (D_{disc} \cup I_{+\infty} \cup I_{-\infty}))\right)\right), \text{ ---- (1)}$$

by Theorem 6, since P and N are continuous on E.

Let

$E_{h,k}$ = {$x \in I$: there is a derived number of ν_g at x greater than k and a derived number of g at x, whose absolute value is less than h.} and

S = {$x \in I$: there is a positive derived number and a negative derived number of g at x.}.

Let $H = \bigcup \{E_{h,k} : 0 < h < k, h \text{ and } k \text{ are rational numbers.}\}$.

Then, H = {$x \in I$: there is a derived number of ν_f greater than the absolute value of a derived number of f at x.}.

Let $K = H \cup S$. We have already shown in the proof of Theorem 15 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, that

$$m(S) = m(g(S)) = m(\nu_g(S)) = 0,$$

$$m(H) = m(g(H)) = m(\nu_g(H)) = 0$$

and that for $x \in I - K$, $|g'(x)| = \nu_g{}'(x)$.

Let $\Delta^* = \{x \in I : g \text{ is differentiable at } x \text{ finitely or infinitely}\}$ and $\Delta = \{x \in I : g \text{ is differentiable finitely at } x\}$. Then Δ and Δ^* are Borel and

$$I - (I_{dis} \cup I_{+\infty} \cup I_{-\infty}) \subseteq (I - \Delta^*) \cup \Delta \subseteq (I - \Delta^*) \cup (\Delta - K) \cup K . \quad \text{---------- (2)}$$

By Theorem 19,

$$\begin{aligned} |\lambda_g|\big(E \cap (I - (I_{+\infty} \cup I_{-\infty}))\big) &= |\lambda_g|\big(E \cap (I - (I_{dis} \cup I_{+\infty} \cup I_{-\infty}))\big) = \mu_{v_g}\big(E \cap (I - (I_{dis} \cup I_{+\infty} \cup I_{-\infty}))\big) \\ &\le \mu_{v_g}\big(E \cap (I - \Delta^*)\big) + \mu_{v_g}(E \cap \Delta) \\ &= m^*\big(v_g(E \cap (I - \Delta^*))\big) + m^*\big(v_g(E \cap \Delta)\big). \end{aligned}$$

Since $I - K \subseteq \Delta^*$, $I - \Delta^* \subseteq K$ and so $m^*\big(v_g(E \cap (I - \Delta^*))\big) \le m^*\big(v_g(K)\big) = 0$. It follows that $\mu_{v_g}\big(E \cap (I - \Delta^*)\big) = m^*\big(v_g(E \cap (I - \Delta^*))\big) = 0$. Hence

$$|\lambda_g|\big(E \cap (I - (I_{+\infty} \cup I_{-\infty}))\big) \le \mu_{v_g}(E \cap \Delta) = |\lambda_g|(E \cap \Delta) .$$

But $E \cap \Delta \subseteq E \cap (I - (I_{+\infty} \cup I_{-\infty}))$ so that $|\lambda_g|(E \cap \Delta) \le |\lambda_g|\big(E \cap (I - (I_{+\infty} \cup I_{-\infty}))\big)$. Therefore,

$$|\lambda_g|\big(E \cap (I - (I_{+\infty} \cup I_{-\infty}))\big) = |\lambda_g|(E \cap \Delta).$$

As $m^*\big(v_g(K)\big) = 0$, $\mu_{v_g}{}^*(E \cap K) = m^*\big(v_g(E \cap K)\big) = 0$.

Therefore, $\mu_{v_g}(E \cap \Delta) = \mu_{v_g}{}^*\big(E \cap (\Delta - K)\big) = m^*\big(v_g(E \cap (\Delta - K))\big)$.

But by Theorem 6 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*,

$$\int_{E \cap \Delta} |g'| = m^*\big(v_g(E \cap \Delta)\big) = m^*\big(v_g(E \cap (\Delta - K))\big).$$

Hence, $|\lambda_g|\big(E \cap (I - (I_{+\infty} \cup I_{-\infty}))\big) = \mu_{v_g}\big(E \cap (I - (I_{+\infty} \cup I_{-\infty}))\big) = |\lambda_g|(E \cap \Delta) = \mu_{v_g}(E \cap \Delta)$

$$= \int_{E \cap \Delta} |g'| .$$

Since $m(I - \Delta) = 0$, $\int_E |g'| = \int_{E \cap \Delta} |g'|$. Thus,

$$|\lambda_g|\big(E \cap (I - (I_{+\infty} \cup I_{-\infty}))\big) = \mu_{v_g}\big(E \cap (I - (I_{+\infty} \cup I_{-\infty}))\big) = \int_E |g'| . \quad \text{--------------- (3)}$$

Now observe that $m^*\big(v_g((E \cap K))\big) = 0$ implies that

$$\mu_P*(E\cap K)+\mu_N*(E\cap K)=\mu_{v_g}*(E\cap K)=0$$

so that $\mu_P*(E\cap K)=\mu_N*(E\cap K)=0$. Since g is continuous on E, P and N are continuous on E. Therefore,

$$m*\big(P((E\cap K))\big)=\mu_P*(E\cap K)=0 \text{ and } m*\big(N((E\cap K))\big)=\mu_N*(E\cap K)=0.$$

Similarly, $m*\big(v_g(E\cap(I-\Delta*))\big)=0$ implies that

$$m*\big(P(E\cap(I-\Delta*))\big)=m*\big(N(E\cap(I-\Delta*))\big)=0.$$

It follows from (2) that $m\big(P\big(E\cap\big(I-(D_{disc}\cup I_{+\infty}\cup I_{-\infty})\big)\big)\big)\leq m\big(P(E\cap\Delta)\big)$. Since $E\cap\Delta\subseteq E\cap\big(I-(D_{disc}\cup I_{+\infty}\cup I_{-\infty})\big)$, $m\big(P(E\cap\Delta)\big)\leq m\big(P\big(E\cap\big(I-(D_{disc}\cup I_{+\infty}\cup I_{-\infty})\big)\big)\big)$. Therefore, $m\big(P\big(E\cap\big(I-(D_{disc}\cup I_{+\infty}\cup I_{-\infty})\big)\big)\big)=m\big(P(E\cap\Delta)\big)$. We deduce similarly that $m\big(N\big(E\cap\big(I-(D_{disc}\cup I_{+\infty}\cup I_{-\infty})\big)\big)\big)=m\big(N(E\cap\Delta)\big)$.

Hence,

$$m\big(P\big(E\cap\big(I-(D_{disc}\cup I_{+\infty}\cup I_{-\infty})\big)\big)\big)-m\big(N\big(E\cap\big(I-(D_{disc}\cup I_{+\infty}\cup I_{-\infty})\big)\big)\big)$$

$$= m*\big(P(E\cap\Delta)\big)-m*\big(N(E\cap\Delta)\big)$$

$$=m*\big(P(E\cap(\Delta-K))\big)-m*\big(N(E\cap(\Delta-K)\big)$$

$$=\int_{E\cap(\Delta-K)}P'-\int_{E\cap(\Delta-K)}N'$$

$$=\int_{E\cap\Delta}P'-\int_{E\cap\Delta}N',$$

by Theorem 6 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*,

$$=\int_{E\cap(\Delta_+)}P'-\int_{E\cap(\Delta_-)}N',$$

where $\Delta_+=\{x\in I: g \text{ is differentiable finitely at } x \text{ and } g'(x)\geq 0\}$ and $\Delta_-=\{x\in I: g \text{ is differentiable finitely at } x \text{ and } g'(x)<0\}$,

by the proof of Theorem 2 in *A de La Vallée Poussin's Decomposition*,

$$=\int_{E\cap(\Delta_+)}g'-\int_{E\cap(\Delta_-)}(-g')=\int_E g'.$$

Hence, $\lambda_g(E\cap\big(I-(D_{disc}\cup I_{+\infty}\cup I_{-\infty})\big)=\int_E g'$.

Thus,

$$\lambda_g(E)=\int_E g'+\lambda_g(E\cap I_{+\infty})+\lambda_g(E\cap I_{-\infty})\,. \quad \text{---------------------} \ (4)$$

For any Borel set B in $E\cap I_{+\infty}$, by Theorem 16, $\lambda_g(B)\geq Lm(B)\geq 0$, by taking any positive $L > 0$. Therefore, $\left|\lambda_g\right|(E\cap I_{+\infty})=\lambda_g(E\cap I_{+\infty})$. Similarly, for any Borel set B in $E\cap I_{-\infty}$, by Theorem 16, $\lambda_g(B)\leq -Lm(B)\leq 0$ and so $\left|\lambda_g\right|(E\cap I_{-\infty})=-\lambda_g(E\cap I_{-\infty})$.

Since $\left|\lambda_g\right|$ is a positive Borel measure,

$$\begin{aligned}\left|\lambda_g\right|(E)&=\left|\lambda_g\right|(E\cap\left(I-\left(I_{+\infty}\cup I_{-\infty}\right)\right)+\left|\lambda_g\right|(E\cap I_{+\infty})+\left|\lambda_g\right|(E\cap I_{-\infty})\\ &=\left|\lambda_g\right|(E\cap\left(I-\left(I_{+\infty}\cup I_{-\infty}\right)\right)+\lambda_g(E\cap I_{+\infty})-\lambda_g(E\cap I_{-\infty})\\ &=\int_E\left|g'\right|+\lambda_g(E\cap I_{+\infty})-\lambda_g(E\cap I_{-\infty})\,, \text{------------} \ (5)\end{aligned}$$

by (3).

Since g is continuous on E, it follows from Theorem 19 that

$$\begin{aligned}\mu_{v_g}(E)=\left|\lambda_g\right|(E)&=\int_E\left|g'\right|+\lambda_g(E\cap I_{+\infty})-\lambda_g(E\cap I_{-\infty})\\ &=\int_E\left|g'\right|+\mu_{v_g}(E\cap I_{+\infty})+\mu_{v_g}(E\cap I_{-\infty})\,. \quad \text{--------------------------} \ (6)\end{aligned}$$

Note that

$$\lambda_g(E\cap I_{-\infty})=\mu_P(E\cap I_{-\infty})-\mu_N(E\cap I_{-\infty})\leq 0 \quad \text{-------------------------} \ (7)$$

and $\quad \lambda_g(E\cap I_{+\infty})=\mu_P(E\cap I_{+\infty})-\mu_N(E\cap I_{+\infty})\geq 0$. -------------------------- (8)

Hence, $\mu_P(E\cap I_{-\infty})\leq\mu_N(E\cap I_{-\infty})$ and $\mu_P(E\cap I_{+\infty})\geq\mu_N(E\cap I_{+\infty})$.

$$\begin{aligned}\lambda_g(E\cap I_{+\infty})=\left|\lambda_g\right|(E\cap I_{+\infty})=\mu_{v_g}(E\cap I_{+\infty})&=\mu_{v_g+g(x_0)}(E\cap I_{+\infty})=\mu_{P+N}(E\cap I_{+\infty})\\ &=\mu_P(E\cap I_{+\infty})+\mu_N(E\cap I_{+\infty})\,. \qquad \text{-----------------------------} \ (9)\end{aligned}$$

From (8) and (9),

$$\mu_N(E\cap I_{+\infty})=0\,. \ \text{----------------------------------} \ (10)$$

Similarly, since $\left|\lambda_g\right|(E\cap I_{-\infty})=-\lambda_g(E\cap I_{-\infty})=\mu_N(E\cap I_{-\infty})-\mu_P(E\cap I_{-\infty})$ and

$\mu_{v_g}(E\cap I_{-\infty})=\mu_P(E\cap I_{-\infty})+\mu_N(E\cap I_{-\infty})$, we get

$$\mu_P(E\cap I_{-\infty})=0\,. \ \text{--------------------------} \ (11)$$

Thus,

$$\lambda_g(E\cap I_{+\infty})=\mu_P(E\cap I_{+\infty}) \text{ and } \lambda_g(E\cap I_{-\infty})=-\mu_N(E\cap I_{-\infty}). \text{ ---- (12)}$$

It follows from (4) and (12) that

$$\begin{aligned}\lambda_g(E)&=\int_E g'+\mu_P(E\cap I_{+\infty})-\mu_N(E\cap I_{-\infty})\\&=\int_E g'+m\big(P(E\cap I_{+\infty})\big)-m\big(N(E\cap I_{-\infty})\big)\end{aligned}$$

and from (5) and (12) that,

$$\begin{aligned}\left|\lambda_g\right|(E)&=\int_E|g'|+\mu_P(E\cap I_{+\infty})+\mu_N(E\cap I_{-\infty})\\&=\int_E|g'|+m\big(P(E\cap I_{+\infty})\big)+m\big(N(E\cap I_{-\infty})\big).\end{aligned}$$

Let $K=H\cup S$. Then $m(K)=0$. Let V be a Borel set such that $K\subseteq V$ and $m(V)=m(K)=0$. Now let $B=V-\left(I_{+\infty}\cup I_{-\infty}\right)$. Note that if $g'(x)=+\infty$ or $g'(x)=-\infty$, then $\nu_g{}'(x)=|g'(x)|=\infty$ and so $I_{+\infty}\cup I_{-\infty}\subseteq I-K$. Let

$B=K\cup\big(V\cap\big(I-K\cup I_{+\infty}\cup I_{-\infty}\big)\big)$. As $m(K)=0$ and $m\big(V\cap\big(I-K\cup I_{+\infty}\cup I_{-\infty}\big)\big)=0$,

$m(B)=m\Big(K\cup\big(V\cap\big(I-K\cup I_{+\infty}\cup I_{-\infty}\big)\big)\Big)=m(K)+m\big(V\cap\big(I-K\cup I_{+\infty}\cup I_{-\infty}\big)\big)=0$.

Note that $m\big(g(B)\big)\le m\big(g(K)\big)+m\Big(g\big(V\cap\big(I-K\cup I_{+\infty}\cup I_{-\infty}\big)\big)\Big)$. Now, $m\big(g(K)\big)=0$ and g is a Lusin function on $\left(I-K\cup I_{+\infty}\cup I_{-\infty}\right)$ so that $m\Big(g\big(V\cap\big(I-K\cup I_{+\infty}\cup I_{-\infty}\big)\big)\Big)=0$ because $m\big(V\cap\big(I-K\cup I_{+\infty}\cup I_{-\infty}\big)\big)=0$. It follows that $m\big(g(B)\big)=0$. Similarly, as $m\big(\nu_g(K)\big)=0$ and by Theorem 10 of *Function of Bounded Variation on Arbitrary Subset and Johnson's Indicatrix*, $m\Big(\nu_g\big(V\cap\big(I-K\cup I_{+\infty}\cup I_{-\infty}\big)\big)\Big)=0$. Therefore, $m\big(\nu_g(B)\big)=0$. Now, B is Borel and $m(B)=m(g(B))=m(\nu_g(B))=0$ and as $K\subseteq B$, $I-B\subseteq I-K$, and so by Theorem 18 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, $|g'(x)|=\left(\nu_g\right)'(x)$ for all x in $I-B$. Now let $N=B-I_{dis}$. Then N is Borel, g is continuous on N and so $\left|\lambda_g\right|(N)=\mu_{\nu_g}(N)=m\big(\nu_g(N)\big)=0$. Therefore, $\lambda_g(N)=0$. Obviously, $m(N)=m\big(g(N)\big)=0$. Thus, for all $x\notin N\cup I_{dis}$, $\nu_g{}'(x)=|g'(x)|$, finitely or infinitely.

This completes the proof of Theorem 17.

As a consequence of Theorem 17 we have:

Proposition 20.

The continuous image of a Borel set is Lebesgue measurable. If *E* is a Borel set in the open interval *I* and $g: I \to \mathbb{R}$ is a function of bounded variation on *I*, then

$$\lambda_g(E \cap I_{+\infty}) = \mu_P(E \cap I_{+\infty}) = m\left(P(E \cap I_{+\infty})\right),$$
$$\lambda_g(E \cap I_{-\infty}) = -\mu_N(E \cap I_{-\infty}) = -m\left(N(E \cap I_{-\infty})\right),$$
$$\mu_P(E \cap I_{-\infty}) = m\left(P(E \cap I_{-\infty})\right) = 0 \text{ and}$$
$$\mu_N(E \cap I_{+\infty}) = m\left(N(E \cap I_{+\infty})\right) = 0,$$

where *P* and *N* are respectively the positive and negative variation functions of *g*, which are increasing on *I*.

Therefore, if $E \subseteq I - I_{dis}$,

$$\lambda_g(E) = \int_E g'(x)dx + m\left(P(E \cap I_{+\infty})\right) - m\left(N(E \cap I_{-\infty})\right)$$

and $\left|\lambda_g\right|(E) = \mu_{v_g}(E) = m\left(v_g(E)\right) = \int_E \left|g'(x)\right| dx + m\left(P(E \cap I_{+\infty})\right) + m\left(N(E \cap I_{-\infty})\right)$.

There exists a Borel set $N \subseteq I - I_{dis}$ with $\lambda_g(N) = \mu_{v_g}(N) = 0$ and $m(N) = \mu_{v_g}(N) = m\left(v_g(N)\right) = 0$ such that for all $x \in I - N \cup I_{dis}$, $v_g{}'(x) = \left|g'(x)\right|$.

Proof.

$\lambda_g(E \cap I_{+\infty}) = \mu_P(E \cap I_{+\infty}) = m\left(P(E \cap I_{+\infty})\right)$ follows from (12) in the proof of Theorem 17 and the fact that *P* is continuous on $I_{+\infty}$. We deduce in like manners that $\lambda_g(E \cap I_{-\infty}) = -\mu_N(E \cap I_{-\infty}) = -m\left(N(E \cap I_{-\infty})\right)$. Since *P* and *N* are continuous on $I_{+\infty}$ and $I_{-\infty}$ respectively , by (10) and (11) in the proof of Theorem 17, $m\left(P(E \cap I_{-\infty})\right) = \mu_P(E \cap I_{-\infty}) = 0$ and $m\left(N(E \cap I_{+\infty})\right) = \mu_N(E \cap I_{+\infty}) = 0$. The remaining assertions are given by Theorem 17.

The following is an application of Theorem 17 to absolutely continuous function of bounded variation.

Proposition 21.

Let *I* be an open interval, $g \in BV(I)$ and suppose *g* is absolutely continuous. Then for any Borel set *E* in *I*,

$$\lambda_g(E) = m(P(E)) - m(N(E)) = \int_E g',$$

and $$m(v_g(E)) = \mu_{v_g}(E) = |\lambda_g|(E) = m(P(E)) + m(N(E)) = \int_E |g'|.$$

There exists a Borel set $N \subseteq I$ with $\lambda_g(N) = 0$ and

$m(N) = \mu_{v_g}(N) = m(g(N)) = m(v_g(N)) = 0$ such that for all $x \in I - N$, $v_g{}'(x) = |g'(x)|$.

Proof,

For any Borel set *E* in *I*, $\lambda_g(E) = \mu_P(E) - \mu_N(E)$, where *P* and *N* are the positive and negative variation functions of *g* as defined in Definition 7**.** Since *g* is absolutely continuous, by Theorem 15, the total variation function of *g*, v_g , is also absolutely continuous on *I*. It follows that *P* and *N* are absolutely continuous on *I*. Therefore, by Theorem 6,

$$\lambda_g(E) = \mu_P(E) - \mu_N(E) = m(P(E)) - m(N(E)).$$

By Theorem 17, $\lambda_g(E) = \int_E g' + \lambda_g(E \cap I_{+\infty}) + \lambda_g(E \cap I_{-\infty})$.

By Theorem 18 of Functions of Bounded Variation and de La Vallée Poussin's Theorem, $m(E \cap I_{-\infty}) = m(E \cap I_{+\infty}) = 0$. Since *P* and *N* are Lusin functions, it follows that $m(P(E \cap I_{+\infty})) = m(N(E \cap I_{+\infty})) = m(P(E \cap I_{-\infty})) = m(N(E \cap I_{-\infty})) = 0$.

Hence, $\lambda_g(E \cap I_{+\infty}) = m(P(E \cap I_{+\infty})) - m(N(E \cap I_{+\infty})) = 0$ and

$$\lambda_g(E \cap I_{-\infty}) = m(P(E \cap I_{-\infty})) - m(N(E \cap I_{-\infty})) = 0.$$

.

Therefore, $\lambda_g(E) = \mu_P(E) - \mu_N(E) = m(P(E)) - m(N(E)) = \int_E g'$.

By Theorem 17,

$$\mu_{v_g}(E) = |\lambda_g|(E) = \int_E |g'| + \lambda_g(E \cap I_{+\infty}) + |\lambda_g(E \cap I_{-\infty})|$$

$$= \int_E |g'|,$$

as we have just shown that $\lambda_g(E \cap I_{+\infty}) = \lambda_g(E \cap I_{-\infty}) = 0$.

Now, $v_g(x) = P(x) + N(x) - g(a)$, where $a \in I$ is the anchor point used in the definition of the total variation function of *g*. Note that $P(x) - g(a)$ is absolutely continuous on *I* and hence is a Lusin function on *I*. Therefore,

$\mu_{v_g}(E) = \mu_{P-g(a)+N}(E) = \mu_{P-g(a)}(E) + \mu_N(E)$

$= m(P^*(E)) + m(N(E))$, where $P^*(x) = P(x) - g(a)$,

$= m(P(E)) + m(N(E))$.

Therefore, as v_g is continuous on *I*, by Theorem 6,

$$m\left(v_g(E)\right) = \mu_{v_g}(E) = \left|\lambda_g\right|(E) = m\left(P\left(E\right)\right) + m\left(N\left(E\right)\right) = \int_E |g'|.$$

The last assertion is from Theorem 17.

This completes the proof of Proposition 21.

Finally,

Theorem 22. Let *I* be an open interval and $g \in BV(I)$. Let

$I_{dis} = \{x \in I : g \text{ is discontinuous at } x\}$,

$I_{+\infty} = \{x \in I : g \text{ is continuous at } x \text{ and } g \text{ is differentiable at } x \text{ with } g'(x) = +\infty\}$ and

$I_{-\infty} = \{x \in I : g \text{ is continuous at } x \text{ and } g \text{ is differentiable at } x \text{ with } g'(x) = -\infty\}$.

Then for every Borel set $E \subseteq I$,

$$\lambda_g(E) = \int_E g' + \lambda_g\left(E \cap I_{+\infty}\right) + \lambda_g\left(E \cap I_{-\infty}\right) + \sum_{y \in E \cap I_{dis}} \left(g_+(y) - g_-(y)\right),$$

$$\left|\lambda_g\right|(E) = \int_E |g'| + \lambda_g\left(E \cap I_{+\infty}\right) + \left|\lambda_g\left(E \cap I_{-\infty}\right)\right| + \sum_{y \in E \cap I_{dis}} \left|g_+(y) - g_-(y)\right| \text{ and}$$

$$\mu_{v_g}(E) = \int_E |g'| + \lambda_g\left(E \cap I_{+\infty}\right) + \left|\lambda_g\left(E \cap I_{-\infty}\right)\right| + \sum_{x \in E \cap I_{dis}} \left(\left|g_+(x) - g(x)\right| + \left|g(x) - g_-(x)\right|\right).$$

Moreover, there exists a Borel set $N \subseteq I - I_{dis}$ with $m(N) = \mu_{v_g}(N) = m\left(v_g(N)\right) = 0$ such that for all $x \in I - \left(I_{dis} \cup N\right)$ $v_g'(x) = |g'(x)|$. Note that $\lambda_g(E \cap I_{-\infty}) \leq 0$.

Proof.

$$\lambda_g(E) = \lambda_g\left(\left(E - I_{dis}\right)\right) + \lambda_g\left(E \cap I_{dis}\right)$$

$$= \int_{E - I_{dis}} g' + \lambda_g\left(\left(E - I_{dis}\right) \cap I_{+\infty}\right) + \lambda_g\left(\left(E - I_{dis}\right) \cap I_{-\infty}\right)$$

$+\lambda_{g_{ab}}(E \cap I_{dis}) + \lambda_{g_c}(E \cap I_{dis}) + \lambda_{g_s}(E \cap I_{dis})$, by Theorem 17,

$= \int_E g' + \lambda_g\left(E \cap I_{+\infty}\right) + \lambda_g\left(E \cap I_{-\infty}\right) + 0 + 0 + \sum_{y \in E \cap I_{diis}} \left(g_+(y) - g_-(y)\right)$, by Theorem 19,

$$= \int_E g' + \lambda_g\left(E \cap I_{+\infty}\right) + \lambda_g\left(E \cap I_{-\infty}\right) + \sum_{y \in E}\left(g_+(y) - g_-(y)\right).$$

$$\left|\lambda_g\right|(E) = \left|\lambda_g\right|\left(\left(E - I_{dis}\right)\right) + \left|\lambda_g\right|\left(E \cap I_{dis}\right)$$

$= \int_{E - I_{dis}} |g'| + \lambda_g\left(\left(E - I_{dis}\right) \cap I_{+\infty}\right) + \left|\lambda_g\left(\left(E - I_{dis}\right) \cap I_{-\infty}\right)\right| + \left|\lambda_g\right|\left(E \cap I_{dis}\right)$, by Theorem 17,

$= \int_E |g'| + \lambda_g\left(E \cap I_{+\infty}\right) - \lambda_g\left(E \cap I_{-\infty}\right) + \left|\lambda_g\right|\left(E \cap I_{dis}\right)$. ---------------- (1)

Now for any $F \subseteq E \cap I_{dis}$,

$\lambda_g(F \cap I_{dis}) = \lambda_{g_{ab}}(F \cap I_{dis}) + \lambda_{g_c}(F \cap I_{dis}) + \lambda_{g_s}(F \cap I_{dis}) = \lambda_{g_s}(F \cap I_{dis})$, by Theorem 19.

Therefore,

$\left|\lambda_g\right|\left(E \cap I_{dis}\right) = \left|\lambda_{g_s}\right|\left(E \cap I_{dis}\right) = \sum_{y \in E \cap I_{dis}} \left|g_+(y) - g_-(y)\right|$. -------------------- (2)

It follows that

$\left|\lambda_g\right|(E) = \int_E |g'| + \lambda_g\left(E \cap I_{+\infty}\right) - \lambda_g\left(E \cap I_{-\infty}\right) + \sum_{y \in E \cap I_{dis}} \left|g_+(y) - g_-(y)\right|$. -------------- (3)

$\mu_{v_g}(E) = \mu_{v_g}(E - I_{dis}) + \mu_{v_g}(E \cap I_{dis}) = \left|\lambda_g\right|(E - I_{dis}) + \mu_{v_g}(E \cap I_{dis})$, by Theorem 19,

$= \int_{E - I_{dis}} |g'| + \lambda_g\left(E \cap I_{+\infty}\right) + \left|\lambda_g\left(E \cap I_{-\infty}\right)\right| + \mu_{v_g}(E \cap I_{dis})$, by Theorem 17,

$= \int_E |g'| + \lambda_g\left(E \cap I_{+\infty}\right) + \left|\lambda_g\left(E \cap I_{-\infty}\right)\right| + \mu_{v_{g_s}}(E \cap I_{dis})$ by Theorem 19 (d)

$$= \int_E |g'| + \lambda_g\left(E \cap I_{+\infty}\right) + \left|\lambda_g\left(E \cap I_{-\infty}\right)\right| + \sum_{y \in E}\left(\left|g_+(y) - g(y)\right| + \left|g(y) - g_-(y)\right|\right).$$

The last assertion came from Theorem 17.

Remark 23.

1. Note that if $g: I \to \mathbb{R}$ is absolutely continuous and *I* is a bounded interval, then by Lemma 2 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*, *g* is of bounded variation. Therefore, Proposition 21 applies when $g: I \to \mathbb{R}$ is absolutely continuous and *I* is a bounded interval.

2. We have, in the proof of Theorem 17, deduced that for any Borel set *E* in *I*,

$$\lambda_g\left(E \cap I_{+\infty}\right) = \left|\lambda_g\right|\left(E \cap I_{+\infty}\right)$$

$= \mu_{v_g}\left(E \cap I_{+\infty}\right) = m\left(v_g\left(E \cap I_{+\infty}\right)\right)$, by Theorem 19 and Theorem 6.

By Proposition 20, $\lambda_g(E\cap I_{+\infty})=\mu_P(E\cap I_{+\infty})=m\big(P(E\cap I_{+\infty})\big)$.

Therefore, $m\big(\nu_g(E\cap I_{+\infty})\big)=m\big(P(E\cap I_{+\infty})\big)$.

Similarly,

$$\left|\lambda_g(E\cap I_{-\infty})\right|=\left|\lambda_g\right|(E\cap I_{-\infty})=\mu_{\nu_g}(E\cap I_{-\infty})=m\big(\nu_g(E\cap I_{-\infty})\big),$$ by Theorem 6.

We deduce similarly as above that

$$\lambda_g(E\cap I_{-\infty})=-\mu_N(E\cap I_{-\infty})=-m\big(N(E\cap I_{-\infty})\big),$$ where N is the negative variation of g, and so we have $m\big(\nu_g(E\cap I_{-\infty})\big)=m\big(N(E\cap I_{-\infty})\big)$.

Hence,

$$\begin{aligned}\mu_{\nu_g}(E)&=\int_E|g'|+m\big(\nu_g(E\cap I_{+\infty})\big)+m\big(\nu_g(E\cap I_{-\infty})\big)+\sum_{x\in E\cap I_{dis}}\big(|g_+(x)-g(x)|+|g(x)-g_-(x)|\big)\\&=\int_E|g'|+m\big(P(E\cap I_{+\infty})\big)+m\big(N(E\cap I_{-\infty})\big)+\sum_{x\in E\cap I_{dis}}\big(|g_+(x)-g(x)|+|g(x)-g_-(x)|\big)\\&=\int_E|g'|+m\big(P(E\cap I_{+\infty})\big)+m\big(N(E\cap I_{-\infty})\big)+\sum_{x\in E}\big(|g_+(x)-g(x)|+|g(x)-g_-(x)|\big).\end{aligned}$$

$$\begin{aligned}\lambda_g(E)&=\int_E g'+m\big(\nu_g(E\cap I_{+\infty})\big)-m\big(\nu_g(E\cap I_{-\infty})\big)+\sum_{y\in E\cap I_{dis}}(g_+(y)-g_-(y))\\&=\int_E g'+m\big(P(E\cap I_{+\infty})\big)-m\big(N(E\cap I_{-\infty})\big)+\sum_{y\in E}(g_+(y)-g_-(y))\end{aligned}$$

By Theorem 22 and Remark 23 (2), In terms of measure of the images we have:

Corollary 24. Let I be an open interval and $g\in BV(I)$. Let

$I_{dis}=\{x\in I: g \text{ is discontinuous at } x\}$,

$I_{+\infty}=\{x\in I: g \text{ is continuous at } x \text{ and } g \text{ is differentiable at } x \text{ with } g'(x)=+\infty\}$ and

$I_{-\infty}=\{x\in I: g \text{ is continuous at } x \text{ and } g \text{ is differentiable at } x \text{ with } g'(x)=-\infty\}$.

Then for every Borel set $E\subseteq I$,

$$\begin{aligned}\lambda_g(E)&=\int_E g'+m\big(\nu_g(E\cap I_{+\infty})\big)-m\big(\nu_g(E\cap I_{-\infty})\big)+\sum_{y\in E\cap I_{dis}}(g_+(y)-g_-(y))\\&=\int_E g'+m\big(P(E\cap I_{+\infty})\big)-m\big(N(E\cap I_{-\infty})\big)+\sum_{y\in E}(g_+(y)-g_-(y)),\end{aligned}$$

$$\begin{aligned}\left|\lambda_g\right|(E)&=\int_E|g'|+m\big(\nu_g(E\cap I_{+\infty})\big)+m\big(\nu_g(E\cap I_{-\infty})\big)+\sum_{y\in E\cap I_{dis}}|g_+(y)-g_-(y)|\\&=\int_E|g'|+m\big(P(E\cap I_{+\infty})\big)+m\big(N(E\cap I_{-\infty})\big)+\sum_{y\in E}|g_+(y)-g_-(y)|\end{aligned}$$

and

$$\mu_{v_g}(E)=\int_E|g'|+m\left(v_g\left(E\cap I_{+\infty}\right)\right)+m\left(v_g\left(E\cap I_{-\infty}\right)\right)+\sum_{x\in E\cap I_{dis}}\left(\left|g_+(x)-g(x)\right|+\left|g(x)-g_-(x)\right|\right)$$
$$=\int_E|g'|+m\left(P\left(E\cap I_{+\infty}\right)\right)+m\left(N\left(E\cap I_{-\infty}\right)\right)+\sum_{x\in E}\left(\left|g_+(x)-g(x)\right|+\left|g(x)-g_-(x)\right|\right).$$

Moreover, there exists a Borel set $N\subseteq I-I_{dis}$ with $m(N)=\mu_{v_g}(N)=m\left(v_g(N)\right)=0$ such that for all $x\in I-\left(I_{dis}\cup N\right)$, $v_g{}'(x)=|g'(x)|$. Note that $\lambda_g(E\cap I_{-\infty})\le 0$.

If g is continuous at every point of E, then

$$m\left(v_g(E)\right)=\mu_{v_g}(E)=\left|\lambda_g\right|(E)=\int_E|g'|+m\left(v_g\left(E\cap I_{+\infty}\right)\right)+m\left(v_g\left(E\cap I_{-\infty}\right)\right).$$

Remark 25.

1. Suppose I is an open interval and g is a function of bounded variation on I. Note that $\mu_{v_g}=\mu_P+\mu_N$, where P and N are respectively the positive and negative variations of g. Therefore, if g is continuous on E,

$$m\left(v_g(E)\right)=\mu_{v_g}(E)=m\left(P(E)\right)+m\left(N(E)\right)=\int_E|g'|+m\left(v_g\left(E\cap I_{+\infty}\right)\right)+m\left(v_g\left(E\cap I_{-\infty}\right)\right)$$
$$=\int_E|g'|+m\left(P\left(E\cap I_{+\infty}\right)\right)+m\left(N\left(E\cap I_{-\infty}\right)\right).$$

We also have that

$$\lambda_g(E)=\mu_P(E)-\mu_N(E)=m\left(P(E)\right)-m\left(N(E)\right)=\int_E g'+m\left(v_g\left(E\cap I_{+\infty}\right)\right)-m\left(v_g\left(E\cap I_{-\infty}\right)\right)$$
$$=\int_E g'+m\left(P\left(E\cap I_{+\infty}\right)\right)-m\left(N\left(E\cap I_{-\infty}\right)\right).$$

Thus, if g is continuous on I and if $\int_I|g'|=m\left(v_g(I)\right)=$total variation of g on I, then $m\left(v_g\left(I_{+\infty}\right)\right)=m\left(v_g\left(I_{-\infty}\right)\right)=0$. It follows that v_g is a Lusin Function. Therefore, by Theorem 15 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*, v_g is absolutely continuous and so g is absolutely continuous.

2. Suppose I is an open interval and g is a function of bounded variation on I. By Theorem 19, for any Borel set $E\subseteq I$, $\mu_{v_g}(E)=\mu_{v_{g_{ab}}}(E)+\mu_{v_{g_c}}(E)+\mu_{v_{g_s}}(E)$.

Since $v_{g_{ab}}$ is absolutely continuous, by Proposition 21, we have

$$m\left(v_{g_{ab}}(E)\right)=\mu_{v_{g_{ab}}}(E)=\int_E\left(v_{g_{ab}}\right)'=\int_E\left|\left(g_{ab}\right)'\right|=\int_E|g'|.$$

By Theorem 19 (c), $\mu_{v_{g_s}}(E)=\sum_{x\in E}\left(\left|g_+(x)-g(x)\right|+\left|g(x)-g_-(x)\right|\right)$.

Now, $\mu_{v_{g_c}}(E)=\mu_{v_{g_c}}(E-I_{dis})+\mu_{v_{g_c}}(E\cap I_{dis})=\mu_{v_{g_c}}(E-I_{dis})$

$$=\mu_{v_{g_c}}(E-I_{dis}-I_{+\infty}-I_{-\infty})+\mu_{v_{g_c}}(E\cap I_{+\infty})+\mu_{v_{g_c}}(E\cap I_{-\infty}).$$

Note that $\mu_{v_{g_c}}(E\cap I_{+\infty})=\mu_{v_{g_c+g_s}}(E\cap I_{+\infty})=\mu_{v_{g_c+g_s}}(E\cap I_{+\infty})+\mu_{v_{g_{ab}}}(E\cap I_{+\infty})=\mu_{v_g}(E\cap I_{+\infty})$,

since $\mu_{v_{g_{ab}}}(E\cap I_{+\infty})=m\left(v_{g_{ab}}(E\cap I_{+\infty})\right)=0$ as $v_{g_{ab}}$ is absolutely continuous and $m(E\cap I_{+\infty})=0$. We deduce in the same manner that $\mu_{v_{g_c}}(E\cap I_{-\infty})=\mu_{v_g}(E\cap I_{-\infty})$. We show below that $\mu_{v_{g_c}}\left(E-\left(I_{dis}\cup I_{+\infty}\cup I_{-\infty}\right)\right)=0$. Let $N\subseteq I-I_{dis}$ be the subset as given in the proof of Theorem 17 such that $m(N)=m\left(v_g(N)\right)=\mu_{v_g}(N)=0$. Then it follows that $\mu_{v_{g_{ab}}}(N)=\mu_{v_{g_c}}(N)=\mu_{v_{g_c+g_s}}(N)=\mu_{v_{g_s}}(N)=0$ as $\mu_{v_g}(N)=\mu_{v_{g_{ab}}}(N)+\mu_{v_{g_c}}(N)+\mu_{v_{g_s}}(N)$. Thus,

$$\mu_{v_{g_c}}\left(E-\left(I_{dis}\cup I_{+\infty}\cup I_{-\infty}\right)\right)=\mu_{v_{g_c}}\left(E-\left(I_{dis}\cup I_{+\infty}\cup I_{-\infty}\cup N\right)\right).$$

Note that $m\left(I_{dis}\cup I_{+\infty}\cup I_{-\infty}\cup N\right)=0$. Let $B=E-\left(I_{dis}\cup I_{+\infty}\cup I_{-\infty}\cup N\right)$.

Let $H=\left\{x\in B:\left(g_c\right)'(x)=0\right\}$. Then by Theorem 16, $\left|\lambda_{g_c}\right|(H)=\mu_{v_{g_c}}(H)=0$. Since $\left(g_c\right)'(x)=0$ on H. Since $\left(g_c\right)'(x)=0$ almost everywhere, $m(B-H)=0$. Note that g is differentiable finitely on B and so on B–H. By Theorem 16, since $m(B-H)=0$, $\left|\lambda_g\right|(B-H)=0$. Hence, $\mu_{v_g}(B-H)=\left|\lambda_g\right|(B-H)=0$. As $\mu_{v_{g_{ab}}}(B-H)=0$, $\mu_{v_{g_c+g_s}}(B-H)=0$. It follows that $\mu_{v_{g_c}}(B-H)=\mu_{v_{g_c+g_s}}(B-H)=0$. Therefore, as $\mu_{v_{g_c}}(N)=0$, $\mu_{v_{g_c}}\left(E-\left(I_{dis}\cup I_{+\infty}\cup I_{-\infty}\right)\right)=0$ and $\mu_{v_{g_c}}(E)=\mu_{v_g}(E\cap I_{+\infty})+\mu_{v_g}(E\cap I_{-\infty})$. Thus, $\mu_{v_{g_c}}(E)=m\left(P\left(E\cap I_{+\infty}\right)\right)+m\left(N\left(E\cap I_{-\infty}\right)\right)$. This gives another proof of Theorem 17.

3. Suppose I is an open interval and g is a function of bounded variation on I.

$$\lambda_g(E)=\lambda_g(E\cap(I-I_{dis}))+\lambda_g(E\cap I_{dis})$$
$$=\mu_P(E\cap(I-I_{dis}))-\mu_N(E\cap(I-I_{dis}))+\lambda_g(E\cap I_{dis})$$
$$=m\left(P(E\cap(I-I_{dis}))\right)-m\left(N(E\cap(I-I_{dis}))\right)+\lambda_g(E\cap I_{dis}),$$

by Theorem 6, since P and N are continuous on $I-I_{dis}$,

$$=\int_{E\cap(I-I_{dis})}g'+m\left(P\left(E\cap(I_{+\infty}\cup I_{-\infty}\right)\right)-m\left(N\left(E\cap(I_{+\infty}\cup I_{-\infty}\right)\right)+\lambda_g(E\cap I_{dis}),$$

by Theorem 2 of *A de La Vallée Poussin's Decomposition*,

$$= \int_E g' + m\big(P\left(E \cap I_{+\infty}\right)\big) - m\big(N\left(E \cap I_{-\infty}\right)\big) + \lambda_g(E \cap I_{dis}),$$

since $m\big(P\left(E \cap I_{-\infty}\right)\big) = 0$ and $m\big(N\left(E \cap I_{+\infty}\right)\big) = 0$,

$$= \int_E g' + m\big(P\left(E \cap I_{+\infty}\right)\big) - m\big(N\left(E \cap I_{-\infty}\right)\big) + \sum_{y \in E}\left(g_+(y) - g_-(y)\right)$$, by Theorem 19.

Similarly,

$$\mu_{v_g}(E) = \mu_{v_g}(E \cap (I - I_{dis})) + \mu_{v_g}(E \cap I_{dis})$$
$$= \mu_P(E \cap (I - I_{dis})) + \mu_N(E \cap (I - I_{dis})) + \mu_{v_g}(E \cap I_{dis})$$
$$= m\big(P(E \cap (I - I_{dis}))\big) + m\big(N(E \cap (I - I_{dis}))\big) + \mu_{v_g}(E \cap I_{dis})$$
$$= \int_{E \cap (I - I_{dis})} |g'| + m\big(P\left(E \cap (I_{+\infty} \cup I_{-\infty}\right)\big) + m\big(N\left(E \cap (I_{+\infty} \cup I_{-\infty}\right)\big) + \mu_{v_g}(E \cap I_{dis}),$$

by Corollary 3 of *A de La Vallée Poussin's Decomposition*,

$$= \int_E |g'| + m\big(P\left(E \cap I_{+\infty}\right)\big) + m\big(N\left(E \cap I_{-\infty}\right)\big) + \mu_{v_g}(E \cap I_{dis}),$$

since $m\big(P\left(E \cap I_{-\infty}\right)\big) = 0$ and $m\big(N\left(E \cap I_{+\infty}\right)\big) = 0$,

$$= \int_E |g'| + m\big(P\left(E \cap I_{+\infty}\right)\big) + m\big(N\left(E \cap I_{-\infty}\right)\big) + \sum_{x \in E}\big(\left|g_+(x) - g(x)\right| + \left|g(x) - g_-(x)\right|\big),$$

by Theorem 19.

$$\left|\lambda_g\right|(E) = \left|\lambda_g\right|(E \cap (I - I_{dis})) + \left|\lambda_g\right|(E \cap I_{dis})$$
$$= \mu_{v_g}(E \cap (I - I_{dis})) + \left|\lambda_g\right|(E \cap I_{dis})$$
$$= \int_E |g'| + m\big(P\left(E \cap I_{+\infty}\right)\big) + m\big(N\left(E \cap I_{-\infty}\right)\big) + \left|\lambda_g\right|(E \cap I_{dis})$$

by using Corollary 3 of *A de La Vallée Poussin's Decomposition* as above,

$$= \int_E |g'| + m\big(P\left(E \cap I_{+\infty}\right)\big) + m\big(N\left(E \cap I_{-\infty}\right)\big) + \sum_{y \in E}\left|g_+(y) - g_-(y)\right|$$, by Theorem 19.

This gives another proof of Corollary 24.

Lebesgue Stieltjes Integral

Suppose $f : I \to \mathbb{R}$ is a Borel function or more precisely Borel measurable function. Suppose $g : I \to \mathbb{R}$ is an increasing function. Then we have the Lebesgue Stieltjes measure, μ_g, which is a positive Radon measure. In the standard way we can define the Lebesgue integral of a non-negative function *f* with respect to the Lebesgue Stieltjes measure μ_g, $\int_I f d\mu_g$. This is called the Lebesgue Stieltjes integral. For the characteristic function of a Borel set in *I*, $\int_I \chi_B d\mu_g = \mu_g(B)$. Thus, we can define the Lebesgue Stieltjes integral for a simple Borel function. For a non-negative Borel function, *f*, the Lebesgue

Stieltjes integral, $\int_I fd\mu_g$, is just the Lebesgue integral of *f* with respect to the Lebesgue Stieltjes measure μ_g . (See Definition 19 in *Introduction To Measure Theory*.) Note that for a non-negative Borel function, *f*, there exists a monotone increasing sequence of simple Borel functions converging pointwise to *f* and by the Lebesgue Monotone Convergence Theorem, the Lebesgue Stieltjes integral of *f* is the limit of the sequence of the Lebesgue Stieltjes integral of the Borel simple functions of the sequence. This may be infinite. In general, for a Borel measurable function *f*, we can define $\int_B fd\mu_g$ as $\int_B f^+d\mu_g - \int_B f^-d\mu_g$, whenever it is not of the form $(+\infty)-(+\infty)$.

Suppose *g* is a function of bounded variation on *I*. Note that λ_g is a finite signed Radon measure and $|\lambda_g|$ and μ_{v_g} are finite Radon measures. (See Theorem 8 and its proof.) Then, following *Introduction To Measure Theory*, we can define in the usual manner, the following Lebesgue Stieltjes integrals for any Borel set *E* in *I*.

$$\int_E fd\lambda_g \ , \ \int_E fd|\lambda_g| \text{ and } \int_E fd\mu_{v_g} .$$

Suppose *g* is a right continuous function of bounded variation. By Theorem 9, in the Jordan decomposition of the finite real Borel measure, $\lambda_g = \lambda_g^+ - \lambda_g^-$, $\lambda_g^+ = \mu_P$ and $\lambda_g^- = \mu_N$. Therefore, for any Borel measurable function, $f: I \to \mathbb{R}$, and any Borel set *E* in *I*,

$$\int_E fd\lambda_g = \int_E fd\lambda_g^+ - \int_E fd\lambda_g^- = \int_E fd\mu_P - \int_E fd\mu_N .$$

We assume that $\int_E fd\mu_P - \int_E fd\mu_N$ is not of the form $+\infty-(+\infty)$ or $-\infty-(-\infty)$.

We now assume *g* is absolutely continuous. It follows that *P* and *N* are also absolutely continuous. Therefore, by Theorem 6, for any Borel set *E* in *I*,

$$\begin{aligned}\mu_P(E) &= m(P(E)) \\ &= \int_E P'dm ,\end{aligned}$$

by Theorem 13 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*.

Now, $\int_E fd\mu_P = \int_E f^+d\mu_P - \int_E f^-d\mu_P$. We assume that $\int_E f^+d\mu_P - \int_E f^-d\mu_P$ is not of the form $+\infty-(+\infty)$.

Therefore, by Proposition 28 of *Introduction to Measure Theory*,
$\int_E f^+ d\mu_P - \int_E f^- d\mu_P = \int_E f^+ P' dm - \int_E f^- P' dm = \int_E f\, P' dm$. Thus $\int_E f d\mu_P = \int_E f\, P' dm$.
Similarly, assuming that $\int_E f^+ d\mu_N - \int_E f^- d\mu_N$ is not of the form $+\infty - (+\infty)$, we deduce that $\int_E f d\mu_N = \int_E f\, N' dm$. It follows that
$\int_E f d\lambda_g = \int_E f\, P' dm - \int_E f\, N' dm = \int_E f\,(P' - N')\, dm = \int_E f\,(P-N)'\, dm = \int_E f g' dm$. Hence,

$$\int_E f d\lambda_g = \int_E f g' dm. \qquad \text{------------------------ (1)}$$

Now for any Borel set *E* in *I*,

$|\lambda_g|(E) = \mu_{v_g}(E) = m(v_g(E))$, since *g* is continuous on *I*,

$= \int_E v_g' dm$,

by Theorem 13 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*. It follows, by Proposition 28 of *Introduction to Measure Theory*, that

$$\int_E f d|\lambda_g| = \int_E f d\mu_{v_g} = \int_E f\; v_g' dm = \int_E f\, |g'|\, dm. \quad \text{---------------------- (2)}$$

We note that we may define $g'(x) = 0$ when *g* is not differentiable finitely since the set $\{x \in I : g \text{ is not differentiable finitely at } x\}$ is of measure zero and that both *g* and v_g are Lusin functions so that the definition will not affect the identities (1) and (2).

Thus, we have

Theorem 26. Suppose $f : I \to \mathbb{R}$ is a Borel function and $g : I \to \mathbb{R}$ is a right continuous function of bounded variation. We assume that $\int_E f d\mu_P - \int_E f d\mu_N$ is not of the form $+\infty - (+\infty)$ or $-\infty - (-\infty)$, where *P* and *N* are the positive and negative variation functions of g. Then we can define

$$\int_E f d\lambda_g = \int_E f d\lambda_g^+ - \int_E f d\lambda_g^- = \int_E f d\mu_P - \int_E f d\mu_N .$$

If $\int_E f d\mu_P + \int_E f d\mu_N$ is not of the form $+\infty + (-\infty)$ or $-\infty + (+\infty)$, we can define

$$\int_E f d|\lambda_g| = \int_E f d\lambda_g^+ + \int_E f d\lambda_g^- = \int_E f d\mu_P + \int_E f d\mu_N .$$

Suppose *g* is absolutely continuous. Then assuming that $\int_E f^+ d\mu_P - \int_E f^- d\mu_P$ and $\int_E f^+ d\mu_N - \int_E f^- d\mu_N$ are not of the form $+\infty - (+\infty)$, we have that

$$\int_E fd\lambda_g = \int_E fg'dm \text{ and } \int_E fd\left|\lambda_g\right| = \int_E fd\mu_{v_g} = \int_E f\, v_g{}'dm = \int_E f\,|g'|\,dm.$$

Theorem 27. Suppose $g: I \to \mathbb{R}$ is an increasing bounded function. Then the Lebesgue Stieltjes measure μ_g is a finite positive Borel measure. Suppose $f: I \to \mathbb{R}$ is a continuous function on the closed and bounded interval [*a*, *b*], where *a* < *b* and $a, b \in I$. Then *f* is Lebesgue-Stieltjes integrable on [*a*, *b*] and there exists $c \in [a,b]$ such that

$$\int_a^b fd\mu_g = f(c)(g_+(b) - g_-(a)).$$

Proof. If g is a constant function, or $g_+(b) = g_-(a)$ then we have nothing to prove, for we can just take any *c* in [*a*, *b*].

Now we assume that $g_+(b) > g_-(a)$. Since *f* is continuous on [*a*, *b*], *f* is Borel on [*a*, *b*], it follows by a standard argument that *f* is Lebesgue-Stieltjes integrable on [*a*, *b*]. Since *f* is continuous on [*a*, *b*], there exists *M* and *m* such that $m \le f(x) \le M$ for all *x* in [*a*, *b*]. Therefore, $\int_a^b md\mu_g \le \int_a^b fd\mu_g \le \int_a^b Md\mu_g$. Now, $\mu_g([a,b]) = g_+(b) - g_-(a)$ and so $m \le \dfrac{\int_a^b fd\mu_g}{g_+(b) - g_-(a)} \le M$. Therefore, by the Intermediate Value Theorem, there exists *c* in [*a*, *b*] such that $\dfrac{\int_a^b fd\mu_g}{g_+(b) - g_-(a)} = f(c)$. This completes the proof.

Theorem 28. Suppose $g: I \to \mathbb{R}$ is an increasing bounded function and *I* is an open interval. Suppose *a* < *b* and $a, b \in I$. Suppose $f: [a,b] \to \mathbb{R}$ is a Lebesgue Stieltjes integrable function with respect to the Lebesgue Stieltjes measure μ_g. Suppose *f* is bounded on [*a*, *b*]. Define for *x* in $[a,b]$, $F(x) = \int_a^x f\, d\mu_g$.

(i) If *g* is continuous at *c* in [*a*, *b*], *F* is continuous at *c*.

(ii) If *g* is continuous in [*a*, *b*] or in a neighbourhood of *c* in [*a*, *b*] and is differentiable at *c* and *f* is continuous at *c*, then *F* is differentiable at *c* and $F'(c) = f(c)g'(c)$.

Proof.

(i) Suppose $x, c \in [a,b]$, $c < b$ and $x > c$. $F(x) - F(c) = \int_c^x f\, d\mu_g$. Therefore,

$$\left|F(x)-F(c)\right| = \left|\int_c^x f\,d\mu_g\right| \le \int_c^x |f|\,d\mu_g .$$

Since f is bounded, there exists $M > 0$ such that $|f(x)| \le M$ for all x in $[a, b]$. It follows that

$$\left|F(x)-F(c)\right| \le \int_c^x M d\mu_g = M\mu_g\left([c,x]\right) = M(g_+(x)-g_-(c)) \text{ -------------- (1)}$$

Therefore,

$$\lim_{x\searrow c}\left|F(x)-F(c)\right| \le M\left(g_+(c)-g_-(c)\right).$$

It follows that if g is continuous at c, then $\lim_{x\searrow c}\left|F(x)-F(c)\right| = 0$.

Hence, if $c = a$ and g is continuous at c, then F is continuous at c.

Similarly, we can show that if $x, c \in [a,b]$, $c > a$ and $x < c$,

$$\left|F(x)-F(c)\right| \le \int_x^c M d\mu_g = M\mu_g\left([x,c]\right) = M(g_+(c)-g_-(x)) .$$

It follows that $\lim_{x\nearrow c}\left|F(x)-F(c)\right| \le M\left(g_+(c)-g_-(c)\right)$. Therefore, if g is continuous at c, then $\lim_{x\nearrow c}\left|F(x)-F(c)\right| = 0$. It follows that if $c = b$ and g is continuous at c, then F is continuous at c. Moreover if $a < c < b$ and g is continuous at c, then $\lim_{x\nearrow c}\left|F(x)-F(c)\right| = \lim_{x\searrow c}\left|F(x)-F(c)\right| = 0$. Therefore, F is continuous at c. This proves part (i).

(ii)

Suppose $a < c < b$. Suppose $x > c$ and $x \in [a,b]$. Let $m_x = \inf\left\{f(x) : x \in [c,x]\right\}$ and $M_x = \inf\left\{f(x) : x \in [c,x]\right\}$. Then we have

$$\int_c^x m_x d\mu_g \le \int_c^x f d\mu_g \le \int_c^x M_x d\mu_g .$$

Therefore, there exists $m_x \le L_x \le M_x$ such that

$$\int_c^x f d\mu_g = L_x \mu_g([c,x]) = L_x\left(g_+(x)-g_-(c)\right).$$

Therefore,

$$\lim_{x\searrow c}\frac{F(x)-F(c)}{x-c} = \lim_{x\searrow c}\frac{\int_c^x f d\mu_g}{x-c} = \lim_{x\searrow c} L_x \lim_{x\searrow c}\frac{g_+(x)-g_-(c)}{x-c} . \text{ --------------- (2)}$$

Since f is continuous at c, $\lim_{x \searrow c} L_x = f(c)$. As g is differentiable at c and continuous in a neighbourhood of c,

$$\lim_{x \searrow c} \frac{g_+(x) - g_-(c)}{x - c} = \lim_{x \searrow c} \frac{g_+(x) - g(c)}{x - c} = \lim_{x \searrow c} \frac{g(x) - g(c)}{x - c} = g'(c).$$

Hence, $\lim_{x \searrow c} \frac{F(x) - F(c)}{x - c} = f(c)g'(c)$. Similarly, we can show that $\lim_{x \nearrow c} \frac{F(x) - F(c)}{x - c} = f(c)g'(c)$. It follows that $F'(c) = f(c)g'(c)$.

Integration by Parts

The next result is a technical result that we shall use to prove a version of the integration by parts theorem.

Theorem 29. Suppose $\mu : \mathcal{B}((a, b)) \to [0, \infty)$ and $\nu : \mathcal{B}((a, b)) \to [0, \infty)$ are two finite Borel measures define on the Borel σ-algebra generated by the open sets of the open interval (a, b). Define for $x \in (a, b)$,

$$f(x) = \frac{1}{2}\big(\mu((a, x]) + \mu((a, x))\big) \text{ and}$$

$$g(x) = \frac{1}{2}\big(\nu((a, x]) + \nu((a, x))\big).$$

Then $\int_{(a,b)} f \, d\nu + \int_{(a,b)} g \, d\mu = \mu((a,b))\nu((a,b))$.

Proof. Note that both $\mu((a,x])$ and $\mu((a,x))$ are increasing and non-negative. Therefore, they are Borel measurable and bounded since μ is finite. Similarly, $\nu((a,x])$ and $\nu((a,x))$ are Borel measurable bounded increasing functions.

Let $E = \{(x, y) \in (a,b) \times (a,b) : x \ge y\}$. For each $x \in (a,b)$, let $E_x = \{y \in (a,b) : (x, y) \in E\}$ and for each $y \in (a,b)$, let $E^y = \{x \in (a,b) : (x, y) \in E\}$. By Theorem 8 of *Product Measure and Fubini's Theorem*,

$$\int_{(a,b)} \nu(E_x) d\mu(x) = \int_{(a,b)} \mu(E^y) d\nu(y) \big(= \mu \times \nu(E)\big). \text{ ---------------------- } \mathbf{(1)}$$

Observe that $E^y = \{x \in (a,b) : (x, y) \in E\} = \{x \in (a,b) : x \ge y\} = [y, b)$ and $E_x = \{y \in (a,b) : (x, y) \in E\} = \{y \in (a,b) : x \ge y\} = (a, x]$.

It follows from (1) that

$$\int_{(a,b)} \nu\big((a,x]\big) d\mu(x) = \int_{(a,b)} \mu([y,b)) d\nu(y) \text{. -------------------- (2)}$$

For $a < y < b$, $(a,b) = (a,y) \cup [y,b)$ and so

$$\mu(E^y) = \mu\big([y,b)\big) = \mu\big((a,b)\big) - \mu\big((a,y)\big) \text{ .}$$

Thus, $\int_{(a,b)} \mu([y,b)) d\nu(y) = \int_{(a,b)} \mu((a,b)) d\nu(y) - \int_{(a,b)} \mu((a,y)) d\nu(y)$

$$= \mu((a,b))\nu\big((a,b)\big) - \int_{(a,b)} \mu((a,y)) d\nu(y) \text{ . ------------- (3)}$$

Therefore,

$$\int_{(a,b)} \nu\big((a,x]\big) d\mu(x) = \int_{(a,b)} \mu([y,b)) d\nu(y) = \mu((a,b))\nu\big((a,b)\big) - \int_{(a,b)} \mu((a,y)) d\nu(y)$$

and so

$$\int_{(a,b)} \nu\big((a,x]\big) d\mu(x) + \int_{(a,b)} \mu((a,y)) d\nu(y) = \mu((a,b))\nu\big((a,b)\big) \text{. ------------- (4)}$$

Interchanging the role of μ and ν, we get from (4),

$$\int_{(a,b)} \mu\big((a,y]\big) d\nu(y) + \int_{(a,b)} \nu((a,x)) d\mu(x) = \mu((a,b))\nu\big((a,b)\big) \text{. -------------- (5)}$$

Thus, from (4) and (5), we obtain,

$$\frac{1}{2}\int_{(a,b)} \Big(\mu\big((a,y]\big) + \mu\big((a,y)\big)\Big) d\nu(y) + \frac{1}{2}\int_{(a,b)} \big(\nu((a,x]) + \nu((a,x))\big) d\mu(x) = \mu((a,b))\nu\big((a,b)\big) \text{.}$$

We note that (a, b) may be unbounded, i.e., (a, b) may be $(-\infty, b)$ or $(a, +\infty)$ or $(-\infty, +\infty)$.

This completes the proof of Theorem 29.

We have a similar result when the domain is a closed and bounded interval.

Theorem 30. Suppose $\mu : \mathcal{B}([a, b]) \to [0, \infty)$ and $\nu : \mathcal{B}([a, b]) \to [0, \infty)$ are two finite Borel measures define on the Borel sets of $[a, b]$. Define for $x \in [a, b]$,

$$f(x) = \frac{1}{2}\Big(\mu\big([a,x]\big) + \mu\big([a,x)\big)\Big) \text{ and}$$

$$g(x) = \frac{1}{2}\Big(\nu\big([a,x]\big) + \nu\big([a,x)\big)\Big).$$

Then $\int_{[a,b]} f \, d\nu + \int_{[a,b]} g \, d\mu = \mu\big([a,b]\big)\nu\big([a,b]\big)$.

Proof. The proof is similar to that for Theorem 29.

Let $E=\{(x,y)\in[a,b]\times[a,b]: x\geq y\}$. For each $x\in[a,b]$, let

$$E_x=\{y\in[a,b]:(x,y)\in E\}=[a,x]$$

and for each $y\in[a,b]$,

let $E^y=\{x\in[a,b]:(x,y)\in E\}=[y,b]$. By Theorem 8 of *Product Measure and Fubini's Theorem*,

$$\int_{[a,b]}\nu(E_x)d\mu(x)=\int_{[a,b]}\mu(E^y)d\nu(y)\left(=\mu\times\nu(E)\right). \text{ ---------------------- } \mathbf{(1)}$$

It follows from (1) that

$$\int_{[a,b]}\nu([a,x])d\mu(x)=\int_{[a,b]}\mu([y,b])d\nu(y). \text{ -------------------- } (2)$$

For $a\leq y\leq b$, $[a,b]=[a,y)\cup[y,b]$ and so

$$=\mu([y,b])=\mu([a,b])-\mu([a,y)) \ .$$

Thus, $\int_{[a,b]}\mu([y,b])d\nu(y)=\int_{[a,b]}\mu([a,b])d\nu(y)-\int_{[a,b]}\mu([a,y))d\nu(y)$

$$=\mu([a,b])\nu([a,b])-\int_{[a,b]}\mu([a,y))d\nu(y) \text{ . ------------- } (3)$$

Therefore,

$$\int_{[a,b]}\nu([a,x])d\mu(x)=\int_{[a,b]}\mu([y,b])d\nu(y)=\mu([a,b])\nu([a,b])-\int_{[a,b]}\mu([a,y))d\nu(y)$$

and so

$$\int_{[a,b]}\nu([a,x])d\mu(x)+\int_{[a,b]}\mu([a,y))d\nu(y)=\mu([a,b])\nu([a,b]). \text{ ------------- } (4)$$

Interchanging the role of μ and ν, we get from (4),

$$\int_{[a,b]}\mu([a,y])d\nu(y)+\int_{[a,b]}\nu([a,x))d\mu(x)=\mu([a,b])\nu([a,b]). \text{ -------------- } (5)$$

Thus, from (4) and (5), we obtain,

$$\frac{1}{2}\int_{[a,b]}\left(\mu([a,y])+\mu([a,y))\right)d\nu(y)+\frac{1}{2}\int_{[a,b]}\left(\nu([a,x])+\nu([a,x))\right)d\mu(x)=\mu([a,b])\nu([a,b]).$$

This completes the proof of Theorem 30.

Remark. Theorem 30 holds if $[a,b]=(-\infty,b]$ or $[a,+\infty)$ or $(-\infty,+\infty)$.

The next theorem is a key result towards formulating an integration by parts formula.

Theorem 31. Let I be an open interval. Let $u,v \in BV(I)$. Then for any interval $(a,b) \subseteq I$,

$$\int_{(a,b)} \frac{u_+ + u_-}{2} d\lambda_v + \int_{(a,b)} \frac{v_+ + v_-}{2} d\lambda_u = u_-(b)v_-(b) - u_+(a)v_+(a).$$

In particular, if there are no points in (a, b) at which both u and v are discontinuous, then

$$\int_{(a,b)} u d\lambda_v + \int_{(a,b)} v d\lambda_u = u_-(b)v_-(b) - u_+(a)v_+(a).$$

Proof.

We shall prove the first part of the theorem when $u,v \in BV(I)$ are increasing functions.

Suppose u and $\boldsymbol{v}$ are increasing bounded functions on I. Then we have the associated Lebesgue Stieltjes measures, $\mu_u : \mathcal{B}(I) \to [0, \infty)$ and $\mu_v : \mathcal{B}(I) \to [0, \infty)$ are two finite Radon measures. Suppose $a, b \in I$ and $a < b$. For $a < x < b$, by Theorem 5, $\mu_u\left((a,x]\right) = u_+(x) - u_+(a)$, $\mu_u\left((a,x)\right) = u_-(x) - u_+(a)$, $\mu_v\left((a,x]\right) = v_+(x) - v_+(a)$, and $\mu_v\left((a,x)\right) = v_-(x) - v_+(a)$.

By Theorem 29,

$$\frac{1}{2}\int_{(a,b)} \left(u_+(x) - u_+(a) + u_-(x) - u_+(a)\right) d\mu_v(x) + \frac{1}{2}\int_{(a,b)} \left(v_+(x) - v_+(a) + v_-(x) - v_+(a)\right) d\mu_u(x)$$

$$= \mu_u((a,b))\mu_v\left((a,b)\right) = \left(u_-(b) - u_+(a)\right)\left(v_-(b) - v_+(a)\right).$$

Hence,

$$\frac{1}{2}\int_{(a,b)} \left(u_+(x) + u_-(x)\right) d\mu_v(x) + \frac{1}{2}\int_{(a,b)} \left(v_+(x) + v_-(x)\right) d\mu_u(x)$$

$$= \left(u_-(b) - u_+(a)\right)\left(v_-(b) - v_+(a)\right) + u_+(a)\mu_v(a,b) + v_+(a)\mu_u(a,b)$$

$$= \left(v_-(b) - v_+(a)\right)u_-(b) + v_+(a)\mu_u(a,b) = \left(v_-(b) - v_+(a)\right)u_-(b) + \left(u_-(b) - u_+(a)\right)v_+(a)$$

$$= u_-(b)v_-(b) - u_+(a)v_+(a).$$

This proves the first part of the theorem when u and v are increasing.

In general, we write $u = P_u - N_u$ and $v = P_v - N_v$, where P_u, P_v and N_u, N_v are the positive and negative variation functions of *u* and *v* respectively.

By the case for increasing functions, as positive and negative variation functions of a function of bounded variation are increasing, we have:

$$\int_{(a,b)} \frac{P_{u+} + P_{u-}}{2} d\mu_{P_v} + \int_{(a,b)} \frac{P_{v+} + P_{v-}}{2} d\mu_{P_u} = P_{u-}(b)P_{v-}(b) - P_{u+}(a)P_{v+}(a), \text{ --------------- } (1)$$

$$\int_{(a,b)} \frac{P_{u+} + P_{u-}}{2} d\mu_{N_v} + \int_{(a,b)} \frac{N_{v+} + N_{v-}}{2} d\mu_{P_u} = P_{u-}(b)N_{v-}(b) - P_{u+}(a)N_{v+}(a), \text{ ------------- } (2)$$

$$\int_{(a,b)} \frac{N_{u+} + N_{u-}}{2} d\mu_{P_v} + \int_{(a,b)} \frac{P_{v+} + P_{v-}}{2} d\mu_{N_u} = P_{v-}(b)N_{u-}(b) - P_{v+}(a)N_{u+}(a), \text{ -------------- } (3)$$

And

$$\int_{(a,b)} \frac{N_{u+} + N_{u-}}{2} d\mu_{N_v} + \int_{(a,b)} \frac{N_{v+} + N_{v-}}{2} d\mu_{N_u} = N_{u-}(b)N_{v-}(b) - N_{u+}(a)N_{v+}(a). \text{ ---------- } (4)$$

Subtracting (2) from (1), we obtain,

$$\int_{(a,b)} \frac{P_{u+} + P_{u-}}{2} \left(d\mu_{P_v} - d\mu_{N_v} \right) + \int_{(a,b)} \frac{P_{v+} - N_{v+} + P_{v-} - N_{v-}}{2} d\mu_{P_u}$$

$$= P_{u-}(b)\left(P_{v-}(b) - N_{v-}(b)\right) - P_{u+}(a)\left(P_{v+}(a) - N_{v+}(a)\right) = P_{u-}(b)v_-(b) - P_{u+}(a)v_+(a).$$

Thus, $\displaystyle\int_{(a,b)} \frac{P_{u+} + P_{u-}}{2} d\lambda_v + \int_{(a,b)} \frac{v_+ + v_-}{2} d\mu_{P_u} = P_{u-}(b)v_-(b) - P_{u+}(a)v_+(a).$ -------------- (5)

Subtracting (4) from (3) we get:

$$\int_{(a,b)} \frac{N_{u+} + N_{u-}}{2} \left(d\mu_{P_v} - d\mu_{N_v} \right) + \int_{(a,b)} \frac{P_{v+} - N_{v+} + P_{v-} - N_{v-}}{2} d\mu_{N_u}$$

$$= N_{u-}(b)\left(P_{v-}(b) - N_{v-}(b)\right) - N_{u+}(a)\left(P_{v+}(a) - N_{v+}(a)\right) = N_{u-}(b)v_-(b) - N_{u+}(a)v_+(a).$$

Hence, $\displaystyle\int_{(a,b)} \frac{N_{u+} + N_{u-}}{2} d\lambda_v + \int_{(a,b)} \frac{v_+ + v_-}{2} d\mu_{N_u} = N_{u-}(b)v_-(b) - N_{u+}(a)v_+(a).$ ----------- (6)

(5) – (6) gives:

$$\int_{(a,b)} \frac{u_+ + u_-}{2} d\lambda_v + \int_{(a,b)} \frac{v_+ + v_-}{2} \left(d\mu_{P_u} - d\mu_{N_u} \right)$$

$$= \left(P_{u-}(b) - N_{u-}(b)\right)v_-(b) - v_+(a)\left(P_{u+}(a) - N_{u+}(a)\right).$$

Therefore, $\displaystyle\int_{(a,b)} \frac{u_+ + u_-}{2} d\lambda_v + \int_{(a,b)} \frac{v_+ + v_-}{2} d\lambda_u = u_-(b)v_-(b) - u_+(a)v_+(a).$

This proves the first part of the theorem.

Suppose there are no points in (a, b) where u and v are both discontinuous. Note that since u and v are of bounded variation on (a, b), points of discontinuities of u and v are at most countable. Let S be the points of discontinuities of u in (a, b). Then v is continuous at all points in S. Therefore, $\lambda_v(S)=0$ and

$$\int_{(a,b)} \frac{u_+ + u_-}{2}\, d\lambda_v = \int_{(a,b)-S} \frac{u_+ + u_-}{2}\, d\lambda_v = \int_{(a,b)-S} u\, d\lambda_v = \int_{(a,b)} u\, d\lambda_v \,.$$

Similarly, we can deduce that

$$\int_{(a,b)} \frac{v_+ + v_-}{2}\, d\lambda_u = \int_{(a,b)} v\, d\lambda_u \,.$$

Hence, $\int_{(a,b)} u\, d\lambda_v + \int_{(a,b)} v\, d\lambda_u = u_-(b)v_-(b) - u_+(a)v_+(a)$.

This completes the proof of Theorem 31.

In general, u and v may have common points of discontinuities, then we shall have a correction term for the integration by parts formula.

Since the points of discontinuities of u and v are at most countable, the formula for singleton sets will be useful, for instance, at the end points of a Borel set, where the integral is to be taken.

For any $x \in I$, $\int_{\{x\}} u\, d\lambda_v = u(x)\lambda_v(\{x\}) = u(x)(v_+(x) - v_-(x))$,
$\int_{\{x\}} v\, d\lambda_u = v(x)\lambda_u(\{x\}) = v(x)(u_+(x) - u_-(x))$ and $\lambda_{uv}(\{x\}) = u_+(x)v_+(x) - u_-(x)v_-(x)$.

Therefore,

$$\int_{\{x\}} u\, d\lambda_v + \int_{\{x\}} v\, d\lambda_u - \lambda_{uv}(\{x\}) = u(x)\lambda_v(\{x\}) + v(x)\lambda_u(\{x\}) - (u_+(x)v_+(x) - u_-(x)v_-(x))$$

$$= u(x)\lambda_v(\{x\}) + v(x)\lambda_u(\{x\}) - \left(u_+(x)\lambda_v(\{x\}) + u_+(x)v_-(x) + v_-(x)\lambda_u(\{x\}) - u_+(x)v_-(x)\right)$$

$$= (u(x) - u_+(x))\lambda_v(\{x\}) + (v(x) - v_-(x))\lambda_u(\{x\}).$$

Also,

$$\int_{\{x\}} u\, d\lambda_v + \int_{\{x\}} v\, d\lambda_u - \lambda_{uv}(\{x\}) = (u(x) - u_-(x) + u_-(x) - u_+(x))\lambda_v(\{x\}) + (v(x) + v_+(x) - v_+(x) - v_-(x))\lambda_u(\{x\})$$

$$= (u(x) - u_-(x))\lambda_v(\{x\}) - \lambda_u(\{x\})\lambda_v(\{x\}) + (v(x) - v_+(x))\lambda_u(\{x\}) + \lambda_u(\{x\})\lambda_v(\{x\})$$

$$= (u(x) - u_-(x))\lambda_v(\{x\}) + (v(x) - v_+(x))\lambda_u(\{x\}).$$

It follows that

$$2\left(\int_{\{x\}} ud\lambda_v + \int_{\{x\}} vd\lambda_u - \lambda_{uv}\left(\{x\}\right)\right)$$
$$= \left(2u(x) - u_+(x) - u_-(x)\right)\lambda_v\left(\{x\}\right) + \left(2v(x) - v_+(x) - v_-(x)\right)\lambda_u\left(\{x\}\right).$$

Therefore,

$$\int_{\{x\}} ud\lambda_v + \int_{\{x\}} vd\lambda_u =$$
$$= \left(u(x) - \frac{1}{2}\left(u_+(x) + u_-(x)\right)\right)\lambda_v\left(\{x\}\right) + \left(v(x) - \frac{1}{2}\left(v_+(x) + v_-(x)\right)\right)\lambda_u\left(\{x\}\right) + \lambda_{uv}\left(\{x\}\right).$$

Thus, we have proved the following:

Theorem 32. Let I be an open interval. Let $u, v \in BV(I)$. Then for any $x \in I$,

$$\int_{\{x\}} ud\lambda_v + \int_{\{x\}} vd\lambda_u =$$
$$= \left(u(x) - \frac{1}{2}\left(u_+(x) + u_-(x)\right)\right)\lambda_v\left(\{x\}\right) + \left(v(x) - \frac{1}{2}\left(v_+(x) + v_-(x)\right)\right)\lambda_u\left(\{x\}\right) + \lambda_{uv}\left(\{x\}\right).$$

Theorem 33. Let I be an open interval. Let $u, v \in BV(I)$. Then for any interval $(a,b) \subseteq I$,

$$\int_{(a,b)} ud\lambda_v + \int_{(a,b)} vd\lambda_u$$
$$= \lambda_{uv}\left((a,b)\right) + \sum_{x\in D}\left(u(x) - \frac{1}{2}\left(u_+(x) + u_-(x)\right)\right)\lambda_v\left(\{x\}\right) + \sum_{x\in D}\left(v(x) - \frac{1}{2}\left(v_+(x) + v_-(x)\right)\right)\lambda_u\left(\{x\}\right),$$

where $D = \{x \in (a,b) : u \text{ and } v \text{ are discontinuous at } x.\}$.

Proof. Let S_u , S_v be the sets of discontinuities in (a, b) of u and v respectively. Let $D = S_u \cap S_v$ and $T = S_u - D$. Then T is at most countable and v is continuous at all points in T.

$$\int_{(a,b)-D} \frac{u_+ + u_-}{2} d\lambda_v = \int_{(a,b)-D-T} \frac{u_+ + u_-}{2} d\lambda_v = \int_{(a,b)-D-T} ud\lambda_v = \int_{(a,b)-D} ud\lambda_v \ ,$$

since $\lambda_v(T) = 0$.

Similarly, we can show that $\int_{(a,b)-D} \frac{v_+ + v_-}{2} d\lambda_u = \int_{(a,b)-D} vd\lambda_u$.

Therefore,

$$\int_{(a,b)-D} v d\lambda_u + \int_D \frac{v_+ + v_-}{2} d\lambda_u + \int_{(a,b)-D} u d\lambda_v + \int_D \frac{u_+ + u_-}{2} d\lambda_v$$

$$= \int_{(a,b)} \frac{v_+ + v_-}{2} d\lambda_u + \int_{(a,b)} \frac{u_+ + u_-}{2} d\lambda_v$$

$$= u_-(b)v_-(b) - u_+(a)v_+(a),$$

by Theorem 31.

Thus,

$$\int_{(a,b)-D} v d\lambda_u + \int_{(a,b)-D} u d\lambda_v = u_-(b)v_-(b) - u_+(a)v_+(a) - \int_D \frac{v_+ + v_-}{2} d\lambda_u - \int_D \frac{u_+ + u_-}{2} d\lambda_v .$$

Therefore,

$$\int_{(a,b)} v d\lambda_u + \int_{(a,b)} u d\lambda_v = \int_{(a,b)-D} v d\lambda_u + \int_{(a,b)-D} u d\lambda_v + \int_D v d\lambda_u + \int_D u d\lambda_v$$

$$= \lambda_{uv}\left((a,b)\right) - \int_D \frac{v_+ + v_-}{2} d\lambda_u - \int_D \frac{u_+ + u_-}{2} d\lambda_v + \int_D v d\lambda_u + \int_D u d\lambda_v$$

$$= \lambda_{uv}\left((a,b)\right) + \sum_{x\in D}\left(u(x) - \frac{1}{2}\left(u_+(x) + u_-(x)\right)\right)\lambda_v\left(\{x\}\right) + \sum_{x\in D}\left(v(x) - \frac{1}{2}\left(v_+(x) + v_-(x)\right)\right)\lambda_u\left(\{x\}\right)$$

This completes the proof of Theorem 33.

Remark 34.

With notation as in Theorem 33, by Theorem 32,

$$\sum_{x\in D}\left(u(x) - \frac{1}{2}\left(u_+(x) + u_-(x)\right)\right)\lambda_v\left(\{x\}\right) + \sum_{x\in D}\left(v(x) - \frac{1}{2}\left(v_+(x) + v_-(x)\right)\right)\lambda_u\left(\{x\}\right)$$

$$= \sum_{x\in D}\left(\int_{\{x\}} u d\lambda_v + \int_{\{x\}} v d\lambda_u - \lambda_{uv}\left(\{x\}\right)\right) = \int_D u d\lambda_v + \int_D v d\lambda_u - \lambda_{uv}(D).$$

It follows that

$$\int_{(a,b)-D} v d\lambda_u + \int_{(a,b)-D} u d\lambda_v = \lambda_{uv}\left((a,b)\right) - \lambda_{uv}(D) = \lambda_{uv}\left((a,b) - D\right).$$

From Theorem 32 and Theorem 33, we have the following variation of Theorem 33.

Theorem 35. Let *I* be an open interval. Let $u, v \in BV(I)$. Then for any $a < b$ in *I*,

$$\int_{[a,b)} u\,d\lambda_v + \int_{[a,b)} v\,d\lambda_u$$

$$= \lambda_{uv}([a,b)) + \sum_{x\in D\cup\{a\}} \left(u(x) - \frac{1}{2}(u_+(x)+u_-(x)) \right) \lambda_v(\{x\})$$

$$+ \sum_{x\in D\cup\{a\}} \left(v(x) - \frac{1}{2}(v_+(x)+v_-(x)) \right) \lambda_u(\{x\}),$$

$$\int_{(a,b]} u\,d\lambda_v + \int_{(a,b]} v\,d\lambda_u$$

$$= \lambda_{uv}((a,b]) + \sum_{x\in D\cup\{b\}} \left(u(x) - \frac{1}{2}(u_+(x)+u_-(x)) \right) \lambda_v(\{x\})$$

$$+ \sum_{x\in D\cup\{b\}} \left(v(x) - \frac{1}{2}(v_+(x)+v_-(x)) \right) \lambda_u(\{x\}) \text{ and}$$

$$\int_{[a,b]} u\,d\lambda_v + \int_{[a,b]} v\,d\lambda_u$$

$$= \lambda_{uv}([a,b]) + \sum_{x\in D\cup\{a,b\}} \left(u(x) - \frac{1}{2}(u_+(x)+u_-(x)) \right) \lambda_v(\{x\})$$

$$+ \sum_{x\in D\cup\{a,b\}} \left(v(x) - \frac{1}{2}(v_+(x)+v_-(x)) \right) \lambda_u(\{x\}),$$

where $D = \{x \in (a,b) : u \text{ and } v \text{ are discontinuous at } x.\}$.

Remark. If I is a closed and bounded interval, $I = [a,b]$ and $u, v \in BV([a,b])$, then we can use Theorem 30 to deduce the following,

$$\int_{[a,b]} \frac{u_+ + u_-}{2} d\lambda_v + \int_{[a,b]} \frac{v_+ + v_-}{2} d\lambda_u = u_+(b)v_+(b) - u_-(a)v_-(a),$$

and that $\int_{[a,b]} u\,d\lambda_v + \int_{[a,b]} v\,d\lambda_u$

$$= \lambda_{uv}([a,b]) + \sum_{x\in D} \left(u(x) - \frac{1}{2}(u_+(x)+u_-(x)) \right) \lambda_v(\{x\})$$

$$+ \sum_{x\in D} \left(v(x) - \frac{1}{2}(v_+(x)+v_-(x)) \right) \lambda_u(\{x\}),$$

where $D = \{x \in [a,b] : u \text{ and } v \text{ are discontinuous at } x.\}$.

Corollary 36. Let I be an open interval. Let $u, v \in BV(I)$ and $a < b$ be in I. Let $D = \{x \in (a,b) : u \text{ and } v \text{ are discontinuous at } x.\}$

(a) If D is empty or if $u(x)=\frac{1}{2}\left(u_+(x)+u_-(x)\right)$ and $v(x)=\frac{1}{2}\left(v_+(x)+v_-(x)\right)$ for all x in D, then $\int_{(a,b)} ud\lambda_v+\int_{(a,b)} vd\lambda_u=\lambda_{uv}\left((a,b)\right)$.

(b) If u and v are continuous on the right at all points of D, then

$$\int_{(a,b)} ud\lambda_v+\int_{(a,b)} vd\lambda_u=\lambda_{uv}\left((a,b)\right)+\sum_{x\in D}\lambda_u\left(\{x\}\right)\lambda_v\left(\{x\}\right).$$

(c) If u and v are continuous on the left at all points of D, then

$$\int_{(a,b)} ud\lambda_v+\int_{(a,b)} vd\lambda_u=\lambda_{uv}\left((a,b)\right)-\sum_{x\in D}\lambda_u\left(\{x\}\right)\lambda_v\left(\{x\}\right).$$

(d) If the function $w:I\to\mathbb{R}$ is continuously differentiable in I and with compact support in I and $v\in BV(I)$, then $\int_I v\cdot w'dy+\int_I wd\lambda_v=0$.

Proof.

Parts (a), (b) and (c) follow from Theorem 33.

Suppose w is continuously differentiable with compact support. Then the support of w is a closed set in I and I is open and so it is a closed set contained in an open interval, say, (a, b), in I. Then by continuity of w' and the compactness of the support of w, $|w'|\le K$, for some $K > 0$. It follows by the Mean Value Theorem, that w is Lipschitz with constant K. Hence, w is absolutely continuous. By Theorem 33, $\int_{(a,b)} vd\lambda_w+\int_{(a,b)} wd\lambda_v=0$ since $w_+(a)=w_-(b)=0$. Since w is absolutely continuous, $\int_{(a,b)} vd\lambda_w=\int_{(a,b)} vw'dx$. Now, $\int_{I-(a,b)} vd\lambda_w=0$ since $\lambda_w(I-(a,b))=0$ and $w'(x)=0$ for $x\in I-(a,b)$, we have that

$$\int_I vw'dx+\int_I wd\lambda_v=0.$$

Remark 37.

Suppose I is an open interval. Theorem 31 and Theorem 33 hold when the domain of the integral is taken to be the whole of I. If $I=(c,d)$, then we can take a sequence of nested intervals, $\{(a_n,b_n)\}$, with $a_n,b_n\in(c,d), a_n<b_n$ such that $a_n\searrow c$ and $b_n\nearrow d$. Apply the theorems with $(a,b)=(a_n,b_n)$ and take limits.

The same argument applies when $I=(-\infty,d)$ or $(c,+\infty)$ or $(-\infty,+\infty)$.

Thus, we have that for an open interval I and $u,v\in BV(I)$,

$$\int_I ud\lambda_v+\int_I vd\lambda_u$$

$$= \lambda_{uv}(I) + \sum_{x \in D}\left(u(x) - \frac{1}{2}(u_+(x) + u_-(x)) \right)\lambda_v(\{x\}) + \sum_{x \in D}\left(v(x) - \frac{1}{2}(v_+(x) + v_-(x)) \right)\lambda_u(\{x\}),$$

where $D = \{x \in I : u \text{ and } v \text{ are discontinuous at } x.\}$.

Change of Variable

We shall next present several versions of a change of variable theorem for Lebesgue Stieltjes integral.

Firstly, we introduce the notion of a generalized inverse of an increasing function. We introduce the two most common types of inverse. We shall use both of them, but give more elaboration to the left continuous inverse.

Definition 38.

Suppose *I* is an interval and $g : I \to \mathbb{R}$ is an increasing function. Let *J* be the smallest interval containing the image of *g*, *g*(*I*).

If the interval *I* is bounded from below, then we define the generalized inverse $v : J \to \mathbb{R}$ by $v(y) = \inf\{x \in I : g(x) \geq y\}$ for *y* in *J*. Note that since the interval *I* is bounded from below, v is well defined and does not take the value $-\infty$.

If the interval *I* is bounded from above, then we define the generalized inverse $\eta : J \to \mathbb{R}$ by $\eta(y) = \sup\{x \in I : g(x) \leq y\}$ for *y* in *J*. Likewise, since the interval *I* is bounded from above, η is well defined and does not take the value ∞.

Properties of the inverses v, η.

Proposition 39. Suppose *I* is an interval bounded from below and $g : I \to \mathbb{R}$ is an increasing function. Let *J* be the smallest interval containing the image of *g*, *g*(*I*). Let $v : J \to \mathbb{R}$ be defined by $v(y) = \inf\{x \in I : g(x) \geq y\}$ for *y* in *J*.

(i) v is an increasing left continuous function on *J*,

(ii) *v* has a discontinuity jump at some point $y_0 \in J - \sup g(I)$ if, and only if, $g(x) = y_0$ for all *x* in some interval $(x_1, x_2) \subseteq I$ with $x_1 < x_2$,

(iii) $v(g(x)) \leq x$ for every *x* in *I*. $v(g(x)) < x$ if, and only if, *g* is constant on some closed interval, $[z, x] \subseteq I$ with $z < x$,

(iv) $v(y)=x_0$ for all y in some open interval $(y_1,y_2)\subseteq J$ with $y_1<y_2$ and for some x_0 in the interior of *I* if, and only if, *g* has a discontinuity jump at x_0 and $(y_1,y_2)\subseteq(g_-(x_0),g_+(x_0))$. In particular, if *g* is strictly increasing, then v is a left inverse of *g* and is continuous.

(v) For any $y\in J$, $g_-(v(y))\le y\le g_+(v(y))$.

Proof.

(i) Take $y_1,y_2\in J$ with $y_1\le y_2$. Then $\{x\in I:g(x)\ge y_2\}\subseteq\{x\in I:g(x)\ge y_1\}$ and so $\inf\{x\in I:g(x)\ge y_1\}\le\inf\{x\in I:g(x)\ge y_2\}$. It follows that $v(y_1)=\inf\{x\in I:g(x)\ge y_1\}\le\inf\{x\in I:g(x)\ge y_2\}=v(y_2)$. Hence, v is an increasing function on *J*.

Now we shall show that v is left continuous. Take $y_0\in J$. If $y_0=\inf g(I)=\inf J$, then we have nothing to prove. Assume now $y_0>\inf g(I)=\inf J$. $v(y_0)=\inf\{x\in I:g(x)\ge y_0\}$. Note that if $z\in\{x\in I:g(x)\ge y_0\}$, then $z\ge\inf\{x\in I:g(x)\ge y_0\}=v(y_0)$. Thus,

if $z\in I$ and $z<v(y_0)$, then $g(z)<y_0$. -------------------------- (1)

We note that

for any $z\in I$, $v(g(z))\le z$, ----------------------------------- (2)

since $v(g(z))=\inf\{x\in I:g(x)\ge g(z)\}$ and $z\in\{x\in I:g(x)\ge g(z)\}$.

If $z<v(y_0)$ and $g(z)\ge y_0$, then $z\ge v(g(z))\ge v(y_0)$ contradicting $z<v(y_0)$. This proves assertion (1) above.

Take any $\varepsilon>0$. Since $y_0>\inf g(I)=\inf J$, there exists z$\in$ *I* such that

$g(z)<y_0$. If $v(g(z))=v(y_0)$, then *v* is constant on $[g(z),y_0]$ and so v is left continuous at y_0. We now assume that *v* is not left constant at some left neighbourhood of y_0. Therefore, we may assume that $v(g(z))<v(y_0)$ for all $g(z)<y_0$. That is to say, $v(y_0)>\inf I$. Then $I\cap[v(y_0)-\varepsilon,v(y_0))\ne\varnothing$.

Take a point $z_0\in I\cap[v(y_0)-\varepsilon,v(y_0))$. Thus, since $v(y_0)-\varepsilon\le z_0<v(y_0)$, by (1), $g(z_0)<y_0$. For any $y\in(g(z_0),y_0)$, $v(y)=\inf\{x\in I:g(x)\ge y\}\ge z_0$. This is because if $v(y)<z_0$, then there exists $x_0<z_0$ such that $g(x_0)\ge y$ and so $y\le g(x_0)\le g(z_0)$,

contradicting $y > g(z_0)$. It follows that $v(y_0) \geq v(y) \geq z_0 \geq v(y_0) - \varepsilon$. This means that v is left continuous at y_0.

(ii) Suppose v has a discontinuity jump at $y_0 \in J - \sup g(I) = J - \sup J$. By part (i) the discontinuity jump at y_0 must be a right jump, i.e., $v(y_0) < v_+(y_0)$. Now recall that $v(y_0) = \inf\{x \in I : g(x) \geq y_0\}$. If $z \in I$ and $z > v(y_0)$, then $g(z) \geq y_0$. We deduce this as follows. $z > v(y_0)$ implies that there exists $x_0 \in \{x \in I : g(x) \geq y_0\}$ such that $x_0 < z$ and $g(x_0) \geq y_0$. Thus, $g(z) \geq g(x_0) \geq y_0$. For any x in I such that $v(y_0) < x < v_+(y_0)$, for all $y > y_0$, $v(y) \geq v_+(y_0) > x$ since v is increasing. Therefore, by (1), $g(x) < y$. It follows that $g(x) \leq y_0$. Since $x > v(y_0)$, $g(x) \geq y_0$. It follows that $g(x) = y_0$ for all $x \in (v(y_0), v_+(y_0))$. Take $x_1 = v(y_0)$ and $x_2 = v_+(y_0)$.

Conversely, suppose $g(x) = y_0$ in some interval $(x_1, x_2) \subseteq I$ with $x_1 < x_2$. Then $v(y_0) = \inf\{x \in I : g(x) \geq y_0\} \leq x_1$ as $(x_1, x_2) \subseteq \{x \in I : g(x) \geq y_0\}$. If $y \in (y_0, \sup J)$, then for every $x \in (x_1, x_2)$, we have that $g(x) = y_0 < y$ and so

$$v(y) = \inf\{x \in I : g(x) \geq y\} \geq x.$$

We deduce this as follows. If $v(y) < x$ for $x \in (x_1, x_2)$, then there exists $x_0 \in \{x \in I : g(x) \geq y\}$ such that $x_0 < x$ and $g(x_0) \geq y$. Thus, $v(y) \leq x_0 < x$ and $y \leq g(x_0) \leq g(x) = y_0$, contradicting $y > y_0$. It follows that $v(y) \geq x$ for all $x \in (x_1, x_2)$ and so $v(y) \geq x_2$. This means $v(y) \geq x_2$ for all $y \in (y_0, \sup J)$. Taking the limit as $y \searrow y_0$, we have that $v_+(y_0) \geq x_2$. Hence, $v(y_0) \leq x_1 < x_2 \leq v_+(y_0)$. It follows that v has a jump discontinuity at y_0.

(iii) For every $x \in I$, $v(g(x)) = \inf\{z \in I : g(z) \geq g(x)\} \leq x$, as $x \in \{z \in I : g(z) \geq g(x)\}$.

Suppose $v(g(x)) < x$ for some $x \in I$. Then there exists $x_0 \in \{z \in I : g(z) \geq g(x)\}$ such that $x_0 < x$ and $g(x_0) \geq g(x)$. Since g is increasing, $g(x_0) = g(x)$ and g is constant on $[x_0, x] \subseteq I$. Conversely, suppose g is constant on $[z, x] \subseteq I$, with $z < x$. Then $v(g(x)) \leq z$ as $[z, x] \subseteq \{y \in I : g(y) \geq g(x)\}$. It follows that $v(g(x)) < x$.

(iv) Suppose v is constant in some open interval $(y_1, y_2) \subseteq J$ with $v(y) = x_0$ for all y in (y_1, y_2) and for some x_0 in the interior of I.

Note that if $z > x_0 = v(y)$, then $g(z) \geq y$. ------------------- (3)

This is because there exists $z_0 \in \{x \in I : g(x) \geq y\}$ such that $z_0 < z$ so that $g(z) \geq g(z_0) \geq y$. This means that for any $z > x_0 = v(y)$ and for all $y \in (y_1, y_2)$, $g(z) \geq y$. Therefore, for any $z > x_0$, $g(z) \geq y_2$. Thus, taking limit as $z \searrow x_0$, we obtain $g_+(x_0) \geq y_2$. On the other hand, if $z < x_0 = v(y)$, then by (1), $g(z) < y$. Thus, for any $z < x_0$ and for any $y \in (y_1, y_2)$, $g(z) < y$. It follows that $g(z) \leq y_1$. Now, letting $z \nearrow x_0$, we deduce that $g_-(x_0) \leq y_1$. Hence $g_-(x_0) \leq y_1 < y_2 \leq g_+(x_0)$. This means $(y_1, y_2) \subseteq (g_-(x_0), g_+(x_0))$ and g has a jump discontinuity at x_0.

Conversely, suppose *g* has a jump discontinuity at x_0 for some x_0 in the interior of *I*, that is to say, $g_-(x_0) < g_+(x_0)$. Take any $y \in (g_-(x_0), g_+(x_0))$. Then $v(y) \geq x_0$. This is because if $v(y) < x_0$, then there exists $z_0 \in \{x \in I : g(x) \geq y\}$ with $z_0 < x_0$ so that $y \leq g(z_0) \leq g_-(x_0)$, contradicting that $y > g_-(x_0)$. Thus, for all $x > x_0$, since *v* is increasing, together with part (iii) we get,

$$v(y) \leq v\big(g_+(x_0)\big) \leq v\big(g(x)\big) \leq x.$$

Hence, $v(y) \leq x_0$. Therefore, $v(y) = x_0$ for all $y \in (g_-(x_0), g_+(x_0))$.

Suppose now *g* is strictly increasing. By part (i), ν is increasing and left continuous. By part (ii), *v* cannot have a right jump in $J - \sup J$. If $\sup J \in J$, then plainly *v* is continuous at $\sup J$. Hence v is continuous in *J*. By part (iii), $v(g(x)) = x$ for all *x* in *I*, since *g* is non-constant in any subinterval of *I*. Thus, *v* is a left inverse of *g*.

(v) By definition, $\nu(y) = \inf\{x \in I : g(x) \geq y\}$, so for any $x > \nu(y)$, there exists $z \in I$ such that $z < x$ and $g(z) \geq y$ so that $g(x) \geq y$. Therefore, $g_+(\nu(y)) \geq y$. We have shown that if $z < \nu(y)$, then $g(z) < y$. (See (1).) Hence, $g_-(\nu(y)) \leq y$. It follows that $g_-(\nu(y)) \leq y \leq g_+(\nu(y))$.

This completes the proof of Proposition 39.

We now state the corresponding result for the other generalized inverse η.

Proposition 40. Suppose *I* is an interval bounded from above and $g : I \to \mathbb{R}$ is an increasing function. Let *J* be the smallest interval containing the image of *g*, *g*(*I*). Let $\eta : J \to \mathbb{R}$ be defined by $\eta(y) = \sup\{x \in I : g(x) \leq y\}$ for *y* in *J*.

(i) η is an increasing right continuous function on *J*,

(ii) η has a discontinuity jump at some point $y_0 \in J - \inf g(I)$ if, and only if, $g(x) = y_0$ for all x in some interval $(x_1, x_2) \subseteq I$ with $x_1 < x_2$,

(iii) $\eta(g(x)) \geq x$ for every x in I. $\eta(g(x)) > x$ if, and only if, g is constant on some closed interval, $[x,z] \subseteq I$ with $x < z$,

(iv) $\eta(y) = x_0$ for all y in some open interval $(y_1, y_2) \subseteq J$ with $y_1 < y_2$ and for some x_0 in the interior of I if, and only if, g has a discontinuity jump at x_0 and $(y_1, y_2) \subseteq (g_-(x_0), g_+(x_0))$. In particular, if g is strictly increasing, then η is a left inverse of g and is continuous.

(v) For any $y \in J$, $g_-(\eta(y) \leq y \leq g_+(\eta(y))$.

Proof.

The proof is similar to that of Proposition 39. Nevertheless, we shall present the proof.

(i) Take $y_1, y_2 \in J$ with $y_1 \leq y_2$. Then $\{x \in I : g(x) \leq y_1\} \subseteq \{x \in I : g(x) \leq y_2\}$ and so $\sup\{x \in I : g(x) \leq y_1\} \leq \sup\{x \in I : g(x) \leq y_2\}$. It follows that $\eta(y_1) = \sup\{x \in I : g(x) \leq y_1\} \leq \sup\{x \in I : g(x) \leq y_2\} = \eta(y_2)$. Hence, η is an increasing function on J.

Now we shall show that η is right continuous. Take $y_0 \in J$. If $y_0 = \sup g(I) = \sup J$, then we have nothing to prove. Assume now $y_0 < \sup g(I) = \sup J$. $\eta(y_0) = \sup\{x \in I : g(x) \leq y_0\}$. Note that if $z \in \{x \in I : g(x) \leq y_0\}$, then $z \leq \sup\{x \in I : g(x) \leq y_0\} = \eta(y_0)$. Thus,

if $z \in I$ and $z > \eta(y_0)$, then $g(z) > y_0$. -------------------------- (1)

We note that

for any $z \in I$, $\eta(g(z)) \geq z$, ---------------------------------- (2)

since $\eta(g(z)) = \sup\{x \in I : g(x) \leq g(z)\}$ and $z \in \{x \in I : g(x) \leq g(z)\}$.

If $z > \eta(y_0)$ and $g(z) \leq y_0$, then $z \leq \eta(g(z)) \leq \eta(y_0)$ contradicting $z > \eta(y_0)$. This proves assertion (1) above.

Take any $\varepsilon > 0$. Since $y_0 < \sup g(I) = \sup J$, there exists z$\in$ I such that

$g(z) > y_0$. If $\eta(g(z)) = \eta(y_0)$, then η is constant on $[y_0, g(z)]$ and so ν is right continuous at y_0. We now assume that η is not right constant at some right neighbourhood of y_0 . Therefore, we may assume that $\eta(g(z)) > \eta(y_0)$ for all $g(z) > y_0$. That is to say, $\eta(y_0) < \sup I$. Then $I \cap (\eta(y_0), \eta(y_0)+\varepsilon] \neq \varnothing$.

Take a point $z_0 \in I \cap (\eta(y_0), \eta(y_0)+\varepsilon]$. Since $\eta(y_0) < z_0 \le \eta(y_0)+\varepsilon$, by (1), $g(z_0) > y_0$. For any $y \in (y_0, g(z_0))$, $\eta(y) = \sup\{x \in I : g(x) \le y\} \le z_0$. This is because if $\eta(y) > z_0$, then there exists $x_0 > z_0$ such that $g(x_0) \le y$ and so $y \ge g(x_0) \ge g(z_0)$, contradicting $y < g(z_0)$. It follows that $\eta(y_0) \le \eta(y) \le z_0 \le \eta(y_0)+\varepsilon$. This means that η is right continuous at y_0 .

(ii) Note that if $\inf g(I) = \inf J \in J$, η is continuous at $\inf J$ by part (i).

Suppose η has a discontinuity jump at $y_0 \in J - \inf g(I) = J - \inf J$. By part (i), the discontinuity jump at y_0 must be a left jump, i.e., $\eta_-(y_0) < \eta(y_0)$. Now recall that $\eta(y_0) = \sup\{x \in I : g(x) \le y_0\}$. If $z \in I$ and $z < \eta(y_0)$, then $g(z) \le y_0$. We deduce this as follows. $z < \eta(y_0)$ implies that there exists $x_0 \in \{x \in I : g(x) \le y_0\}$ such that $x_0 > z$ and $g(x_0) \le y_0$. Thus, $g(z) \le g(x_0) \le y_0$. For any *x* in *I* such that $\eta_-(y_0) < x < \eta(y_0)$, for all $y < y_0$, $\eta(y) \le \eta_-(y_0) < x$, since η is increasing. Therefore, by (1), $g(x) > y$, for all $y < y_0$. It follows that $g(x) \ge y_0$. Since $x < \eta(y_0)$, $g(x) \le y_0$. Hence, $g(x) = y_0$ for all $x \in (\eta_-(y_0), \eta(y_0))$. Take $x_1 = \eta_-(y_0)$ and $x_2 = \eta(y_0)$.

Conversely, suppose $g(x) = y_0$ in some interval $(x_1, x_2) \subseteq I$ with $x_1 < x_2$. Then

$\eta(y_0) = \sup\{x \in I : g(x) \le y_0\} \ge x_2$ as $(x_1, x_2) \subseteq \{x \in I : g(x) \le y_0\}$. If $y \in (\inf J, y_0)$, then for every $x \in (x_1, x_2)$, we have that $g(x) = y_0 > y$ and so

$$\eta(y) = \sup\{z \in I : g(z) \le y\} \le x .$$

We deduce this as follows. If $\eta(y) > x$ for $x \in (x_1, x_2)$, then there exists $x_0 \in \{z \in I : g(z) \le y\}$ such that $x_0 > x$ and $g(x_0) \le y$. Thus, $\eta(y) \ge x_0 > x$ and $y \ge g(x_0) \ge g(x) = y_0$, contradicting $y < y_0$. It follows that $\eta(y) \le x$ for all $x \in (x_1, x_2)$ and so $\eta(y) \le x_1$. This means $\eta(y) \le x_1$ for all $y \in (\inf J, y_0)$. Taking the limit as $y \nearrow y_0$, we have that $\eta_-(y_0) \le x_1$. Hence, $\eta_-(y_0) \le x_1 < x_2 \le \eta(y_0)$. It follows that η has a jump discontinuity at y_0 .

(iii) For every $x \in I$, $\eta(g(x)) = \sup\{z \in I : g(z) \le g(x)\} \ge x$, as $x \in \{z \in I : g(z) \le g(x)\}$.

Suppose $\eta(g(x)) > x$ for some $x \in I$. Then there exists $x_0 \in \{z \in I : g(z) \le g(x)\}$ such that $x_0 > x$ and $g(x_0) \le g(x)$. Since *g* is increasing, $g(x_0) = g(x)$ and *g* is constant on $[x, x_0] \subseteq I$. Conversely, suppose g is constant on $[x, z] \subseteq I$, with $z > x$. Then $\eta(g(x)) \geq z$ as $[x, z] \subseteq \{y \in I : g(y) \leq g(x)\}$. It follows that $\eta(g(x)) > x$.

(iv) Suppose η is constant in some open interval $(y_1, y_2) \subseteq J$ with $\eta(y) = x_0$ for all y in (y_1, y_2) and for some x_0 in the interior of *I*.

Note that if $z < x_0 = \eta(y)$, then $g(z) \leq y$. ------------------- (3)

This is because there exists $z_0 \in \{x \in I : g(x) \leq y\}$ such that $z_0 > z$ so that $g(z) \leq g(z_0) \leq y$. This means that for any $z < x_0 = \eta(y)$ and for all $y \in (y_1, y_2)$, $g(z) \leq y$. Therefore, for any $z < x_0$, $g(z) \leq y_1$. Thus, taking limit as $z \nearrow x_0$, we obtain $g_-(x_0) \leq y_1$. On the other hand, if $z > x_0 = \eta(y)$, then by (1), $g(z) > y$. Thus, for any $z > x_0$ and for any $y \in (y_1, y_2)$, $g(z) > y$. It follows that $g(z) \geq y_2$. Now, letting $z \searrow x_0$, we deduce that $g_+(x_0) \geq y_2$. Hence $g_-(x_0) \leq y_1 < y_2 \leq g_+(x_0)$. This means $(y_1, y_2) \subseteq (g_-(x_0), g_+(x_0))$ and g has a jump discontinuity at x_0.

Conversely, suppose g has a jump discontinuity at x_0 for some x_0 in the interior of *I*, that is to say, $g_-(x_0) < g_+(x_0)$. Take any $y \in (g_-(x_0), g_+(x_0))$. Then $\eta(y) \leq x_0$. This is because if $\eta(y) > x_0$, then there exists $z_0 \in \{x \in I : g(x) \leq y\}$ with $z_0 > x_0$ so that $y \geq g(z_0) \geq g_+(x_0)$, contradicting that $y < g_+(x_0)$. Thus, for all $x < x_0$, since η is increasing, together with part (iii) we get,

$$\eta(y) \geq \eta\left(g_-(x_0)\right) \geq \eta\left(g(x)\right) \geq x.$$

Hence, $\eta(y) \geq x_0$. Therefore, $\eta(y) = x_0$ for all $y \in (g_-(x_0), g_+(x_0))$.

Suppose now g is strictly increasing. By part (i), η is increasing and right continuous. By part (ii), η cannot have a left jump in $J - \inf J$. Thus, η is continuous in $J - \inf J$. If $\inf J \in J$, then plainly, η is continuous at $\inf J$. It follows that η is continuous in *J*. By part (iii) $\eta(g(x)) = x$ for all x in *I* since g is non-constant in any subinterval of *I*. Thus, η is a left inverse of g.

(v) By definition, $\eta(y) = \sup\{x \in I : g(x) \leq y\}$, so for any $x < \eta(y)$, there exists $z \in I$ such that $z > x$ and $g(z) \leq y$ so that $g(x) \leq y$. Therefore, $g_-(\eta(y)) \leq y$. We have shown that if $z > \eta(y)$, then $g(z) > y$. Hence, $g_+(\eta(y)) \geq y$. It follows that $g_-(\eta(y)) \leq y \leq g_+(\eta(y))$.

Remark 41.

In Proposition 39 (iv) and Proposition 40 part (iv), we have required that the generalised inverses, ν and η, take a value in the interior of the domain, *I*. This is not a necessary requirement.

For the various possible situation when η may take the value of the infimum or the supremum of *I*, part (iv) actually holds.

For the definition of η, we require that *I* be bounded above. If $g: I \to \mathbb{R}$ is an increasing function, then *J* is the smallest interval containing *g*(*I*). Note that *J* need not be bounded. Suppose $\inf(g(I)) = \inf(J) \notin J$. Then either $\inf(J) = -\infty$ or $\inf(J) > -\infty$. If $\inf(J) = -\infty$, then $\inf I \notin I$. If $\inf(J) > -\infty$, then either $\inf(J) \in J$ or $\inf(J) \notin J$.

We now assume that the interval *I* is bounded above.

Supremum of *J*.

(1) Suppose $\sup I \notin I$.

If $\sup(g(I)) = \sup(J) \notin J$, then there does not exists $y_0 \in J$ such that $\eta(y_0) = \sup I$. We deduce this as follows. Suppose

$$\eta(y_0) = \sup\{x \in I : g(x) \le y_0\} = \sup I .$$

Then, for all *x* in *I*, $x < \eta(y_0)$, $g(x) \le y_0$. Hence, $\sup g(I) = \sup J \le y_0$ contradicting $\sup J > y_0$.

If $\sup(g(I)) \in g(I)$, then there exists $x_0 \in I$ such that $\sup(g(I)) = g(x_0) = y_0$. Note that $\eta(y_0) = \sup\{x \in I : g(x) \le y_0\} = \sup I > x_0$. For all $z \in I$ such that $x_0 < z < \sup I$, as *g* is increasing, $g(x_0) = g(z) = y_0$ for $x_0 < z < \sup I$. That is to say, *g* is constant on $[x_0, \sup I)$.

(2) Suppose $\sup I \in I$.

Then $\sup(g(I)) = \sup(J) = g(\sup I) \in J$. Let $x_0 = \sup I$. Then $y_0 = \sup g(I) = \sup J = g(x_0)$ since *g* is increasing.

Moreover $\eta(y_0) = \sup\{x \in I : g(x) \le y_0\} = x_0$.

Suppose $\eta(y) = x_0$ for all *y* in $(y_1, y_0]$ for some $y_1 \in J$ with $y_1 < y$. For any $z \in I$, $z < x_0 = \eta(y)$ for any *y* in $(y_1, y_0]$. Then $g(z) \le y$. Therefore, $g(z) \le y_1$. It follows that $g_-(x_0) \le y_1 < y_0 = g(x_0)$. This shows that g is discontinuous at x_0.

Conversely, suppose g has a jump at x_0. That is $g_-(x_0) < y_0 = g(x_0)$. Take $y \in J$ such that $g_-(x_0) < y < y_0 = g(x_0)$. For any $z \in I$, $z < x_0$ implies that $x_0 \geq \eta(y) \geq \eta\left(g_-(x_0)\right) \geq \eta\left(g(z)\right) \geq z$. Letting $z \nearrow x_0$, we conclude that $\eta(y) = x_0$ for all $y \in (g_-(x_0), g(x_0)] = (g_-(x_0), y_0]$.

Infimum of *J*.

(3) Suppose $\inf I = -\infty$. Then as $\eta(y) > -\infty$, parts (1) to (2) cover the situation when $\eta(y) = \sup I$.

(4) Suppose $\inf I > -\infty$.

If $\inf g(I) = \inf J \notin J$, then $\inf(I) \notin I$. For any $y \in J$, $y > \inf J$ so that there exist $x_0 \in I$ such that $y > g(x_0)$ and so $\eta(y) \geq x_0 > \inf I$. Parts (1) to (2) cover the situation when $\eta(y) = \sup I$.

If $\inf g(I) = \inf J \in J$, then $\inf g(I) = g(\inf I) = \inf J$. Let $x_0 = \inf I$ and $y_0 = g(x_0)$. Now, $\eta(y_0) = \eta(g(x_0) \geq x_0$. If $\eta(y_0) = \eta(g(x_0) > x_0$, then for $x \in I$ with $x_0 < x < \eta(y_0)$ $g(x) \leq y_0$ and so $g(x) = y_0$ as $g(x) \geq y_0$. That means g is constant on $[x_0, \eta(y_0))$ taking the value y_0. Let $\eta(y_0) = z_0$. Then $z_0 \geq x_0$. Moreover, $g_-(z_0) = y_0$, if $z_0 > x_0$. Suppose now η is constant on $[y_0, y_1)$ with $y_0 < y_1$ for some $y_1 \in J$ and $\eta(y) = z_0$ for all $y \in [y_0, y_1)$. For all $z > z_0 = \eta(y)$ and any $y \in [y_0, y_1)$, $g(z) > y$. Therefore, $g(z) \geq y_1$. It follows that $g_+(z_0) \geq y_1 > y_0 = g_-(z_0)$. Hence, g is discontinuous at z_0. Conversely, suppose g is discontinuous at z_0. I.e., $g_+(z_0) > g_-(z_0) = y_0 = \inf J$. Then for all $y \in (g_-(z_0), g_+(z_0))$, $\eta(y) \leq z_0$. This is because if $\eta(y) > z_0$, then there exists $x_1 \in \{x \in I : g(x) \leq y\}$ with $x_1 > z_0$ so that $y \geq g(x_1) \geq g_+(z_0)$, contradicting that $y < g_+(z_0)$. Thus, for all $x < z_0$, since η is increasing, we get, $\eta(y) \geq \eta\left(g_-(z_0)\right) \geq \eta\left(g(x)\right) \geq x$. Hence, $\eta(y) \geq z_0$. Therefore, $\eta(y) = z_0$ for all $y \in (g_-(z_0), g_+(z_0))$.

Remark 42.

The situation with the extreme values of ν in part (iv) of Proposition 39 is elaborated as follows.

For the definition of ν, we require that I be bounded below. If $g : I \to \mathbb{R}$ is an increasing function, then J is the smallest interval containing $g(I)$. Note that J need not be bounded. Suppose $\sup(g(I)) = \sup(J) \notin J$. Then either $\sup(J) = \infty$ or $\sup(J) < \infty$. If $\sup(J) = \infty$, then $\sup I \notin I$. If $\sup(J) < \infty$, then either $\sup(J) \in J$ or $\sup(J) \notin J$.

We now assume that the interval I is bounded below.

Infimum of J

(1) Suppose $\inf I \notin I$.

If $\inf(g(I)) = \inf(J) \notin g(I)$, then $\inf(g(I)) \notin J$. There does not exist $y_0 \in J$ such that $\nu(y_0) = \inf I$. We deduce this as follows. Suppose

$$\nu(y_0) = \inf\{x \in I : g(x) \geq y_0\} = \inf I.$$

Then, for all x in I, $x > \nu(y_0)$, $g(x) \geq y_0$. Hence, $\inf g(I) = \inf J \geq y_0$ contradicting $\inf J < y_0$.

If $\inf(g(I)) \in g(I)$, then there exists $x_0 \in I$ such that $\inf(g(I)) = g(x_0) = y_0$. Note that $\inf g(I) = \inf J$. Now $\nu(y_0) = \inf\{x \in I : g(x) \geq y_0\} = \inf I < x_0$. For all $z \in I$ such that $x_0 > z > \inf I$, as g is increasing, $g(x_0) = g(z) = y_0$. That is to say, g is constant on $(\inf I, x_0]$. Note that $\nu(y_0) \notin I$.

(2) Suppose $\inf I \in I$.

Then $\inf(g(I)) = \inf(J) = g(\inf I) \in J$. Let $x_0 = \inf I$ and $y_0 = \inf g(I) = \inf J = g(x_0)$.

Moreover $\nu(y_0) = \inf\{x \in I : g(x) \geq y_0\} = x_0$.

Suppose $\nu(y) = x_0$ for all y in $[y_0, y_1)$ for some $y_1 \in J$ and $y_1 > y_0$. For any $z \in I$, $z > x_0 = \nu(y)$ for any y in $[y_0, y_1)$. Then $g(z) \geq y$. Therefore, $g(z) \geq y_1$. It follows that $g_+(x_0) \geq y_1 > y_0 = g(x_0)$. This shows that g is discontinuous at x_0.

Conversely, suppose g has a jump at x_0. That is, $g_+(x_0) > y_0 = g(x_0)$. Take $y \in J$ such that $y_0 = g(x_0) < y < g_+(x_0)$. For any $z \in I$, $z > x_0$ implies that $x_0 \leq \nu(y) \leq \nu(g_+(x_0)) \leq \nu(g(z)) \leq z$. Letting $z \searrow x_0$, we conclude that $\nu(y) = x_0$ for all $y \in [g(x_0), g_+(x_0) = [y_0, g_+(x_0))$.

Supremum of *J*.

(3) Suppose $\sup I = \infty$.

As $\nu(y) < \infty$, parts (1) to (2) cover the situation when $\nu(y) = \inf I$.

(4) Suppose now $\sup(I) < +\infty$.

If $\sup g(I) \notin g(I)$, then it follows that $\sup g(I) \notin J$ and that $\sup I \notin I$. As remark above, for all $y \in J$, there exist $x_0 \in I$ such that $y < g(x_0)$ and so $\nu(y) \leq x_0 < \sup I$.

If $\sup g(I) \in g(I)$, then there exists $x_0 \in I$ such that $g(x_0) = \sup g(I) = \sup J = y_0$ and so $v(y_0) \le v(g(x_0)) \le x_0$. Thus, for all $y \in J$, $v(y) \le v(y_0) \le x_0$. If $x_0 = \sup I \in I$, and if there exists $y_1 \in I$ with $y_1 < y_0$ such that for all $y \in (y_1, y_0]$, $v(y) = x_0$. Then for all $z \in I$ with $z < x_0 = v(y)$, $g(z) \le y$ and so $g(z) \le y_1$. It follows that $g_-(x_0) \le y_1 < y_0 = g(x_0)$ and so g is discontinuous at x_0. Conversely, suppose g is discontinuous at $x_0 = \sup I$. That means $g_-(x_0) < y_0 = g(x_0)$. Now, for $y \in (g_-(x_0), g(x_0)]$, $y \le g(x_0)$ and so $v(y) \le x_0$. For $z < x_0$, $g(z) \le g(x_0) < y$ for all $y \in (g_-(x_0), g(x_0)]$. Therefore, $z \le v(y)$ and so $v(y) \ge x_0$. Hence, $v(y) = x_0$ for $y \in (g_-(x_0), g(x_0)]$.

Proposition 43. Suppose I is an open interval and $g : I \to \mathbb{R}$ is an increasing function. Let $\alpha, \beta \in I$ with $\alpha < \beta$.

(i) Suppose I is bounded below.

Then $\alpha < v(y) \le \beta$ if, and only if, $g_+(\alpha) < y \le g_+(\beta)$. In particular, if g is also right continuous, we have $g(\alpha) < y \le g(\beta) \Leftrightarrow \alpha < v(y) \le \beta$.

(ii) Suppose I is bounded above.

Then $\alpha \le \eta(y) < \beta$ if, and only if, $g_-(\alpha) \le y < g_-(\beta)$. In particular, if g is also left continuous, we have $g(\alpha) \le y < g(\beta) \Leftrightarrow \alpha \le \eta(y) < \beta$.

Proof.

We shall prove only part (i). Part (ii) is similarly proven.

If $y > g_+(\alpha)$, then $v(y) > \alpha$. Suppose on the contrary that $v(y) \le \alpha$. If $v(y) < \alpha$, then $y \le g(\alpha) \le g_+(\alpha)$, contradicting $y > g_+(\alpha)$. If $v(y) = \alpha$, then for all $z \in I$, $z > v(y) = \alpha$, we have that $g(z) \ge y$ and it follows that $g_+(v(y)) = g_+(\alpha) \ge y$, contradicting $y > g_+(\alpha)$. Hence, we must have $v(y) > \alpha$.

Hence, $v(y) \le \alpha$ implies that $y \le g_+(\alpha)$.

If $y \le g_+(\alpha)$, then $v(y) \le \alpha$. This is because $v(y) \le v(g_+(\alpha)) \le v(g(x)) \le x$ for all $x > \alpha$, since g and v are increasing and $g(x) \ge g_+(\alpha)$. Therefore, $v(y) \le \alpha$. Hence, $y \le g_+(\alpha)$ implies that $v(y) \le \alpha$.

It follows that $v(y) > \alpha$ implies that $y > g_+(\alpha)$.

Therefore, for $\alpha, \beta \in I$ with $\alpha < \beta$, $g_+(\alpha) < y \le g_+(\beta)$ if, and only if $\alpha < v(y) \le \beta$.

We are now ready to state and prove the various versions of change of variable theorem.

Theorem 44. Suppose *I* is an open interval and $g: I \to \mathbb{R}$ is an increasing function. Let $f: I \to [0,\infty)$ be a non-negative Borel function. Let $a, b \in I$ with $a < b$.

(i) Suppose the interval *I* is bounded from below and *g* is right continuous. Then

$$\int_{(a,b]} f(x) d\mu_g(x) = \int_{g(a)}^{g(b)} f(\nu(y))dy ,$$

where μ_g is the Lebesgue Stieltjes measure generated by *g* and ν is the generalised inverse of *g* as defined in Definition 38.

(ii) Suppose the interval *I* is bounded from above and *g* is left continuous. Then

$$\int_{[a,b)} f(x) d\mu_g = \int_{g(a)}^{g(b)} f(\eta(y))dy ,$$

where μ_g is the Lebesgue Stieltjes measure generated by *g* and η is the generalised inverse of *g* as defined in Definition 38.

Proof.

Part (i)

We shall prove the theorem for the case when *f* is the characteristic function of a half-open and half-closed interval, $(\alpha, \beta]$ in *I* where $\alpha < \beta$.

Let $f = \chi_{(\alpha,\beta]}$.

If $(\alpha,\beta] \cap (a,b] = \varnothing$, then $\int_{(a,b]} f(x) d\mu_g(x) = \int_{(a,b]} \chi_{(\alpha,\beta]} d\mu_g(x) = 0$.

By Proposition 43, $g(a) < y \le g(b)$ if, and only if, $a < \nu(y) \le b$. Therefore, $\int_{g(a)}^{g(b)} f(\nu(y))dy = \int_{g(a)}^{g(b)} \chi_{(\alpha,\beta]}(\nu(y))dy = 0$, since $(\alpha,\beta] \cap (a,b] = \varnothing$. Hence the theorem is true for these characteristic functions.

Suppose now $(\alpha,\beta] \cap (a,b] \ne \varnothing$. Then $(\alpha,\beta] \cap (a,b] = (s,t]$, where $s = \max\{\alpha, a\}$ and $t = \min\{\beta, b\}$. Therefore,

$$\int_{(a,b]} f(x) d\mu_g(x) = \int_{(a,b]} \chi_{(\alpha,\beta]} d\mu_g(x) = \int_{(a,b]} \chi_{(s,t]} d\mu_g(x) = g(t) - g(s) .$$

The last equality in the above statement is a consequence of the right continuity of g.

Again, by Proposition 43, $g(a) < y \le g(b)$ if, and only if, $a < \nu(y) \le b$, so that

$$\int_{g(a)}^{g(b)} f(\nu(y))dy = \int_{g(a)}^{g(b)} \chi_{(\alpha,\beta]}(\nu(y))dy = \int_J \chi_{(g(a),g(b)]}\chi_{(g(\alpha),g(\beta)]}dy$$

$$= \min\{g(b), g(\beta)\} - \max\{g(a), g(\alpha)\}$$

$$= g(t) - g(s)\text{, since } g \text{ is increasing.}$$

It follows that $\int_{(a,b]} \chi_{(\alpha,\beta]} d\mu_g(x) = \int_{g(a)}^{g(b)} \chi_{(\alpha,\beta]}(\nu(y))dy$.

Hence, the theorem is true for characteristic functions of a half open and half-closed interval, $(\alpha,\beta]$ in I.

Now we define two positive Radon measures as follows.

For any Borel set B in $\mathcal{B}(I)$, let

$$\mu_1(B) = \int_{(a,b]} \chi_B(x)d\mu_g(x) \quad \text{and} \quad \mu_2(B) = \int_{g(a)}^{g(b)} \chi_B(\nu(y))dy.$$

Then μ_1 and μ_2 are positive Borel measures on $\mathcal{B}(I)$ since I is an open interval, by Corollary 4 of *Product Measure and Fubbini's Theorem*, $\mu_1 = \mu_2$. By Theorem 3, μ_1 and μ_2 are Radon measures.

Therefore, the theorem holds when f is the characteristic function of a Borel set. Hence, the theorem is true for any simple Borel function. By Theorem 16 of *Introduction to Measure Theory*, there exists a monotone sequence of simple Borel function $\{s_n\}$ converging to f pointwise. Therefore, by the Lebesgue Monotone Convergence Theorem (see Theorem 23 of *Introduction to Measure Theory*), $\int_{(a,b]} s_n(x)d\mu_g(x) \to \int_{(a,b]} f(x)d\mu_g(x)$ and $\int_{g(a)}^{g(b)} s_n(\nu(y))dy \to \int_{g(a)}^{g(b)} f(\nu(y))dy$.

Therefore, $\int_{(a,b]} f(x)d\mu_g(x) = \int_{g(a)}^{g(b)} f(\nu(y))dy$. In particular, it follows that

$\int_{(a,b]} f(x)d\mu_g(x) < \infty$ if, and only if, $\int_{g(a)}^{g(b)} f(\nu(y))dy < \infty$.

Part (ii)

The proof is similar to part (i). We proceed analogously as in part (i) to deduce that $\int_{[a,b)} f(x)d\mu_g = \int_{g(a)}^{g(b)} f(\eta(y))dy$ for characteristic function of $[\alpha,\beta)$. The proof then proceeds as in part (i).

This completes the proof of Theorem 44.

Suppose $f : I \to \mathbb{R}$ is a Borel function. Then $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ are Borel functions and $f = f^+ - f^-$. We say *f* is μ_g integrable over (*a*, *b*], if $\int_{(a,b]} f^+(x)d\mu_g(x) - \int_{(a,b]} f^-(x)d\mu_g(x)$ is not of the form $(+\infty) - (+\infty)$. In view of Theorem 44, if *I* is bounded below, *f* is μ_g integrable over (*a*, *b*] if, and only if, $f \circ \nu$ is Lebesgue integrable over $[g(a), g(b)]$, whereas when *I* is bounded above, *f* is μ_g integrable over $[a,b)$ if, and only of, $f \circ \eta$ is Lebesgue integrable over $[g(a), g(b)]$.

We can now extend Theorem 44 to any Borel function $f : I \to \mathbb{R}$.

Theorem 45. Suppose *I* is an open interval and $g : I \to \mathbb{R}$ is an increasing function. Let $f : I \to \mathbb{R}$ be a Borel function. Let $a, b \in I$ with $a < b$.

(i) Suppose the interval *I* is bounded from below and g is right continuous. Then

$$\int_{(a,b]} f(x)d\mu_g(x) = \int_{g(a)}^{g(b)} f(\nu(y))dy ,$$

whenever $\int_{(a,b]} f(x)d\mu_g(x)$ or $\int_{g(a)}^{g(b)} f(\nu(y))dy$ exists finitely or infinitely, where μ_g is the Lebesgue Stieltjes measure generated by *g* and ν is the generalised inverse of *g* as defined in Definition 38.

(ii) Suppose the interval *I* is bounded from above and *g* is left continuous. Then

$$\int_{[a,b)} f(x)d\mu_g(x) = \int_{g(a)}^{g(b)} f(\eta(y))dy ,$$

whenever $\int_{[a,b)} f(x)d\mu_g(x)$ or $\int_{g(a)}^{g(b)} f(\eta(y))dy$ exists finitely or infinitely, where μ_g is the Lebesgue Stieltjes measure generated by *g* and η is the generalised inverse of *g* as defined in Definition 38.

Proof. Immediate from Theorem 44.

The next result is a more familiar version of the change of variable theorem.

Theorem 46. Suppose I is an open interval bounded from below or bounded from above and $g: I \to \mathbb{R}$ is a continuous increasing function. Let $f: \mathbb{R} \to \mathbb{R}$ be a Borel function. Then for $a, b \in I$ with $a < b$,

$$\int_{[a,b]} f \circ g(x) d\mu_g(x) = \int_{g(a)}^{g(b)} f(y)dy ,$$

whenever $\int_{[a,b]} f \circ g(x) d\mu_g(x)$ or $\int_{g(a)}^{g(b)} f(y)dy$ exists finitely or infinitely, where μ_g is the Lebesgue Stieltjes measure generated by g.

If g is differentiable everywhere or is absolutely continuous, then

$$\int_{[a,b]} f \circ g(x) \bullet g'(x) d(x) = \int_{g(a)}^{g(b)} f(y)dy$$

Proof. If the interval I is bounded below, by Theorem 45 part (i), we have

$$\int_{(a,b]} f \circ g(x) d\mu_g(x) = \int_{g(a)}^{g(b)} f \circ g(v(y))dy .$$

As g is continuous, $\int_{[a,b]} f \circ g(x) d\mu_g(x) = \int_{(a,b]} f \circ g(x) d\mu_g(x)$.

Since g is continuous, by Proposition 39, $g(v(y)) = y$, for all y in J so that $\int_{g(a)}^{g(b)} f \circ g(v(y))dy = \int_{g(a)}^{g(b)} f(y)dy$.

If the interval I is bounded above, by Theorem 45 part (ii), we have

$$\int_{[a,b)} f \circ g(x) d\mu_g = \int_{g(a)}^{g(b)} f \circ g(\eta(y))dy .$$

Since g is continuous, by Proposition 40, $g(\eta(y)) = y$ and so

$$\int_{[a,b]} f \circ g(x) d\mu_g = \int_{[a,b)} f \circ g(x) d\mu_g = \int_{g(a)}^{g(b)} f(y)dy .$$

If g is differentiable everywhere, then g is a Lusin function by Theorem 12 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation.* By Theorem 15 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*, g is absolutely continuous on the interval $[a, b]$. By Theorem 26, if g is absolutely continuous, then

$$\int_{[a,b]} f \circ g(x) d\mu_g(x) = \int_{[a,b]} f \circ g(x) \bullet g'(x) dx .$$

Hence, $\int_{[a,b]} f \circ g(x) \bullet g'(x) dx = \int_{g(a)}^{g(b)} f(y)dy$.

The next result, with continuity condition on increasing g and differentiability condition on an increasing function ϕ on the codomain of g, has a short and elegant proof.

Theorem 47. Suppose I is an open interval bounded from below and $g : I \to \mathbb{R}$ is a continuous increasing function. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an increasing function, which is differentiable everywhere on $\mathbb{R}$. Let $f : I \to \mathbb{R}$ be a Borel function. Then for any Borel set B in $\mathcal{B}(I)$,

$$\int_B f d\mu_{\phi\circ g} = \int_B f \bullet (\phi' \circ g) d\mu_g ,$$

whenever any side of the equality exists finitely or infinitely,

Proof.

For any interval (a,b) with $a,b \in I$ and $a < b$, since $\phi \circ g$ and ϕ are continuous, by Theorem 7, $\mu_{\phi\circ g}((a,b)) = m(\phi \circ g((a,b))) = m(\phi(g((a,b))) = \mu_\phi(g((a,b)))$. Since ϕ is bounded on $g([a,b])$ and ϕ is differentiable everywhere on $g([a,b])$, by Theorem 15 of *Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*, $\mu_\phi(g((a,b))) = \int_{g((a,b))} \phi' dm$, where m is the Lebesgue measure on $\mathbb{R}$. Note that ϕ' is Borel. Since g is continuous and increasing, by Theorem 46,

$$\int_{g((a,b))} \phi' dm = \int_{(a,b)} \phi' \circ g \; d\mu_g \; .$$

It follows that $\mu_{\phi\circ g}((a,b)) = \int_{(a,b)} \phi' \circ g \; d\mu_g$. Therefore, $\mu_{\phi\circ g}(E) = \int_E \phi' \circ g \; d\mu_g$ for any Borel set E in $\mathcal{B}(I)$. Hence, for a Borel function $f : I \to \mathbb{R}$, by Proposition 28 of *Introduction to Measure Theory*,

$$\int_B f \, d\mu_{\phi\circ g} = \int_B f \cdot (\phi' \circ g) d\mu_g \, .$$

This completes the proof of Theorem 47.

We now consider the Lebesgue Stieltjes measure generated by a composition of two increasing functions, with or without continuity condition. We have a result, which is similar to Theorem 44.

Theorem 48. Suppose I is an open interval and $g : I \to \mathbb{R}$ is an increasing function. Let J be the smallest interval containing the range of g. Let $\phi : J \to \mathbb{R}$

be an increasing function. Let $f: I \to [0,\infty)$ be a non-negative Borel function. Let a, $b \in I$ with $a < b$.

(i) Suppose the interval I is bounded from below, g and ϕ are right continuous.

Then $\int_{(a,b]} f d\mu_{\phi\circ g} = \int_{(g(a),g(b)]} f \circ \nu d\mu_{\phi}$,

where $\mu_{\phi\circ g}$ and μ_{ϕ} are the respective Lebesgue Stieltjes measure generated by $\phi \circ g$ and ϕ respectively and ν is the generalised inverse of g as defined in Definition 38.

(ii) Suppose the interval I is bounded from above, g and ϕ are left continuous.

Then $\int_{[a,b)} f\, d\mu_{\phi\circ g} = \int_{[g(a),g(b))} f \circ \eta d\mu_{\phi}$,

where $\mu_{\phi\circ g}$ and μ_{ϕ} are the respective Lebesgue Stieltjes measure generated by $\phi \circ g$ and ϕ respectively and η is the generalised inverse of g as defined in Definition 38.

Proof.

We shall prove only part (i). The proof of part (ii) is analogous.

We shall prove the theorem for the case when f is the characteristic function of a half-open and half-closed interval, $(\alpha,\beta]$ in I, where $\alpha < \beta$.

Let $f = \chi_{(\alpha,\beta]}$.

If $(\alpha,\beta]\cap(a,b] = \varnothing$, then $\int_{(a,b]} f d\mu_{\phi\circ g} = \int_{(a,b]} \chi_{(\alpha,\beta]} d\mu_{\phi\circ g} = 0$.

By Proposition 43, $g(a) < y \le g(b)$ if, and only if, $a < \nu(y) \le b$. Therefore,

$$\int_{(g(a),g(b)]} f \circ \nu d\mu_{\phi} = \int_{(g(a),g(b)]} \chi_{(\alpha,\beta]} \circ \nu d\mu_{\phi} = 0 \text{ , since } (\alpha,\beta]\cap(a,b] = \varnothing .$$

Hence the theorem is true for these characteristic functions.

Suppose now $(\alpha,\beta]\cap(a,b] \ne \varnothing$. Then $(\alpha,\beta]\cap(a,b] = (s,t]$, where $s = \max\{\alpha, a\}$ and $t = \min\{\beta, b\}$. Therefore,

$$\int_{(a,b]} f d\mu_{\phi\circ g} = \int_{(a,b]} \chi_{(\alpha,\beta]} d\mu_{\phi\circ g} = \int_{(a,b]} \chi_{(s,t]} d\mu_{\phi\circ g} = \phi(g(t)) - \phi(g(s)) ,$$

since $\phi \circ g$ is right continuous.

Again, by Proposition 43, $g(a) < y \le g(b)$ if, and only if, $a < \nu(y) \le b$, so that

$$\int_{(g(a),g(b)]} f\circ v d\mu_\phi = \int_{(g(a),g(b)]} \chi_{(\alpha,\beta]}\circ v d\mu_\phi = \int_{(g(a),g(b)]} \chi_{(g(a),g(b)]}\chi_{(g(\alpha),g(\beta)]} d\mu_\phi$$

$= \int_{(g(a),g(b)]} \chi_{(g(s),g(t)]} d\mu_\phi$, since g is increasing,

$= \phi(g(t)) - \phi(g(s))$, as ϕ is right continuous.

It follows that $\int_{(a,b]} \chi_{(\alpha,\beta]} d\mu_{\phi\circ g} = \int_{(g(a),g(b)]} \chi_{(\alpha,\beta]}\circ v d\mu_\phi$.

Hence, the theorem is true for characteristic functions of a half open and half-closed interval, $(\alpha,\beta]$ in *I*. It follows that $\int_{(a,b]} \chi_B d\mu_{\phi\circ g} = \int_{(g(a),g(b)]} \chi_B \circ v d\mu_\phi$ for any Borel set *B* in $\mathcal{B}$(*I*).

Now we define two positive Radon measures as follows.

For any Borel set in $\mathcal{B}$(*I*), let

$$\mu_1(B) = \int_{(a,b]} \chi_B d\mu_{\phi\circ g} \text{ and } \mu_2(B) = \int_{(g(a),g(b)]} \chi_B \circ v d\mu_\phi .$$

Then μ_1 and μ_2 are positive Borel measures on $\mathcal{B}$(*I*), and since *I* is an open interval, by Corollary 4 of *Product Measure and Fubbini's Theorem*, $\mu_1 = \mu_2$. By Theorem 3, μ_1 and μ_2 are Radon measures.

Therefore, the theorem holds when *f* is the characteristic function of a Borel set.

It follows that the theorem holds for any simple Borel function. By Theorem 16 of *Introduction to Measure Theory*, there exists a monotone sequence of simple Borel function $\{s_n\}$ converging to *f* pointwise. Therefore, by the Lebesgue Monotone Convergence Theorem (see Theorem 23 of *Introduction to Measure Theory*), $\int_{(a,b]} s_n(x) d\mu_{\varphi\circ g}(x) \to \int_{(a,b]} f(x) d\mu_{\varphi\circ g}(x)$ and $\int_{(g(a),g(b)]} s_n \circ v(x) d\mu_\varphi(x) \to \int_{(g(a),(b)]} f\circ v(x) d\mu_\varphi(x)$.

Therefore, $\int_{(a,b]} f(x) d\mu_{\varphi\circ g}(x) = \int_{(g(a),(b)]} f\circ v(x) d\mu_\varphi(x)$. In particular, it follows that $\int_{(a,b]} f(x) d\mu_{\varphi\circ g}(x) < \infty$ if, and only if, $\int_{(g(a),(b)]} f\circ v(x) d\mu_\varphi(x) < \infty$.

Corollary 49. Suppose *I* is an open interval and $g: I \to \mathbb{R}$ is an increasing function. Let *J* be the smallest interval containing the range of *g*. Let $\phi: J \to \mathbb{R}$ be an increasing function. Let $f: I \to \mathbb{R}$ be a Borel function. Let *a*, *b* $\in$ *I* with *a* < *b*.

(i) Suppose the interval I is bounded from below, g and ϕ are right continuous. Then $\int_{(a,b]} f\,d\mu_{\phi\circ g} = \int_{(g(a),g(b)]} f\circ v\,d\mu_\phi$.

(ii) Suppose the interval I is bounded from above, g and ϕ are left continuous. Then $\int_{[a,b)} f\,d\mu_{\phi\circ g} = \int_{[g(a),g(b))} f\circ\eta\,d\mu_\phi$,

Proof. This follows from Theorem 48.

Corollary 50. Suppose I is an open interval and $g: I \to \mathbb{R}$ is a continuous increasing function. Let J be the smallest interval containing the range of g. Let $\phi: J \to \mathbb{R}$ be an increasing function. Let $f: \mathbb{R} \to \mathbb{R}$ or $f: J \to \mathbb{R}$ be a Borel function. Let $a, b \in I$ with $a < b$.

(i) Suppose the interval I is bounded from below and ϕ is right continuous. Then

$$\int_{(a,b]} f\circ g\,d\mu_{\phi\circ g} = \int_{(g(a),g(b)]} f\,d\mu_\phi .$$

(ii) Suppose the interval I is bounded from above and ϕ is left continuous. Then

$$\int_{[a,b)} f\circ g\,d\mu_{\phi\circ g} = \int_{[g(a),g(b))} f\,d\mu_\phi .$$

Proof.

We prove part (i) only. Part (ii) is similarly proven.

By Corollary 49, $\int_{(a,b]} f\circ g\,d\mu_{\phi\circ g} = \int_{(g(a),g(b)]} f\circ g\circ v\,d\mu_\phi$. Since g is continuous, $\int_{(g(a),g(b)]} f\circ g\circ v\,d\mu_\phi = \int_{(g(a),g(b)]} f\,d\mu_\phi$. Therefore, $\int_{(a,b]} f\circ g\,d\mu_{\phi\circ g} = \int_{(g(a),g(b)]} f\,d\mu_\phi$.

We now introduce the idea of using a measurable function and a measure space to define a measure by taking the preimage of a measurable set.

Definition 51.

Suppose $(X, \mathcal{M}, \mu)$ is a measure space. That is, $\mathcal{M}$ is a σ-algebra of subsets of X and $\mu: \mathcal{M} \to [0, \infty]$ is a positive measure. Suppose $f: X \to \mathbb{R}$ is a $\mathcal{M}$-measurable function. Define a collection $\mathcal{S}$ of subsets of $\mathbb{R}$ by

$$\mathcal{S} = \{\ E \subseteq \mathbb{R} : f^{-1}(E) \in \mathcal{M}\}.$$

Plainly, $\mathcal{S}$ is a σ-algebra and contains every Borel set of $\mathbb{R}$. Define the measure μf^{-1} on $\mathcal{S}$ by $\mu f^{-1}(E) = \mu(f^{-1}(E))$. Obviously, every Borel set of $\mathbb{R}$ is $\mathcal{S}$ -measurable or μf^{-1}-measurable, if we specify the measure $\mu : \mathcal{M} \to [0, \infty]$. on X.

Theorem 52. Suppose $f : X \to \mathbb{R}$ is a $\mathcal{M}$-measurable function, where $(X, \mathcal{M}, \mu)$ is a measure space. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function on $\mathbb{R}$.

Then $\int_{\mathbb{R}} gd\left(\mu f^{-1}\right) = \int_X g \circ f d\mu$, whenever any side of the equality exists.

Moreover, if B is a Borel set in $\mathbb{R}$, then $\int_B gd\left(\mu f^{-1}\right) = \int_{f^{-1}(B)} g \circ f d\mu$

Proof. It is enough to prove the theorem when g is a non-negative Borel measurable function. We note that for any Borel set E in $\mathbb{R}$,

$$\int_{\mathbb{R}} \chi_E d\left(\mu f^{-1}\right) = \mu\left(f^{-1}(E)\right) = \int_X \chi_{f^{-1}(E)} d\mu = \int_X \chi_E \circ f \, d\mu \ .$$

It follows that the theorem is true for any simple Borel measurable function. Since g is a non-negative function, there exists a monotone sequence of Borel simple function converging pointwise to g. Therefore, by the Lebesgue Monotone Convergence Theorem, the theorem holds for a non-negative Borel measurable function. In general, by writing $g = g^+ - g^-$, we see that whenever $\int_{\mathbb{R}} g^+ d\left(\mu f^{-1}\right) - \int_{\mathbb{R}} g^+ d\left(\mu f^{-1}\right)$ is not of the form $(+\infty) - (+\infty)$, the right hand side $\int_X g^+ \circ f d\mu - \int_X g^- \circ f d\mu$ is also not of the form $(+\infty) - (+\infty)$ and vice versa. Therefore,

$$\int_B gd\left(\mu f^{-1}\right) = \int_{\mathbb{R}} \chi_B gd\left(\mu f^{-1}\right) = \int_X (g\chi_B) \circ f \, d\mu$$

$$= \int_X (g \circ f)(\chi_B \circ f) d\mu = \int_X (g \circ f)\left(\chi_{f^{-1}(B)}\right) d\mu = \int_{f^{-1}(B)} g \circ f d\mu .$$

We shall apply Theorem 52 to the next result.

Theorem 53. Suppose I is an open interval and $g : I \to \mathbb{R}$ is an increasing function. Let J be the smallest interval containing the range of g. Let $\phi : J \to \mathbb{R}$ be an increasing function and $f : I \to \mathbb{R}$ be a Borel function. Let B be any Borel set in I.

(i) Suppose I is bounded from below and ϕ is right continuous. Then

$$\int_B f d\mu_{\phi \circ g} = \int_{v^{-1}(B)} f \circ v d\mu_\phi .$$

In particular, for $a, b \in I$ and $a < b$, $\int_{(a,b]} f d\mu_{\phi\circ g} = \int_{(g_+(a),g_+(b)]} f \circ v d\mu_\phi$.

(ii) Suppose I is bounded from above and ϕ is left continuous. Then

$$\int_B f d\mu_{\phi\circ g} = \int_{\eta^{-1}(B)} f \circ \eta d\mu_\phi .$$

In particular, for $a, b \in I$ and $a < b$, $\int_{[a,b)} f d\mu_{\phi\circ g} = \int_{[g_-(a),g_-(b))} f \circ \eta d\mu_\phi$.

Proof.

We shall prove only part (i). Part (ii) is analogously proven.

Let $a, b \in I$ and $a < b$. Then

$$\mu_{\phi\circ g}((a,b]) = (\phi\circ g)_+(b) - (\phi\circ g)_+(a) = \phi(g_+(b)) - \phi(g_+(a)),$$

since ϕ is right continuous,

$$= \mu_\phi\big((g_+(a), g_+(b)]\big) = \mu_\phi\big(v^{-1}((a,b])\big),$$ by Proposition 43.

Since half-open and half-closed intervals generate the Borel algebra,

$$\mu_{\phi\circ g}(E) = \mu_\phi\big(v^{-1}(E)\big),$$

for all Borel set E in $\mathcal{B}(I)$. Therefore, by Theorem 52, for any Borel set B in $\mathcal{B}(I)$,

$$\int_B f d\mu_{\phi\circ g} = \int_{v^{-1}(B)} f \circ v d\mu_\phi .$$

Hence, for $a, b \in I$ and $a < b$, $\int_{(a,b]} f d\mu_{\phi\circ g} = \int_{v^{-1}((a,b])} f \circ v d\mu_\phi = \int_{(g_+(a),g_+(b)]} f \circ v d\mu_\phi$.

However, for strictly increasing function g and any increasing function ϕ, the conclusion of Theorem 53 holds without any continuity condition.

Theorem 54. Suppose I is an open interval and $g : I \to \mathbb{R}$ is a strictly increasing function. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an increasing function and $f : I \to \mathbb{R}$ be a Borel function. Let B be any Borel set in $\mathcal{B}(I)$.

(i) If the interval I is bounded from below,

$$\int_B f d\mu_{\phi\circ g} = \int_{v^{-1}(B)} f \circ v d\mu_\phi ,$$

whenever any side of the equality exists finitely or infinitely. In particular, for a, $b \in I$ and $a < b$, $\int_{(a,b]} f d\mu_{\phi\circ g} = \int_{(g_+(a),g_+(b)]} f\circ v d\mu_\phi$. Moreover, if g is a strictly increasing continuous function, then $\int_B f d\mu_{\phi\circ g} = \int_{g(B)} f\circ v d\mu_\phi$.

(ii) If the interval I is bounded from above,

$$\int_B f d\mu_{\phi\circ g} = \int_{\eta^{-1}(B)} f\circ\eta d\mu_\phi$$

whenever any side of the equality exists finitely or infinitely. In particular, for a, $b \in I$ and $a < b$, $\int_{[a,b)} f d\mu_{\phi\circ g} = \int_{[g_-(a),g_-(b))} f\circ\eta d\mu_\phi$. Moreover, if g is a strictly increasing continuous function, then for any Borel set in $\mathcal{B}(I)$, $\int_B f d\mu_{\phi\circ g} = \int_{g(B)} f\circ\eta d\mu_\phi$.

Proof. We shall prove only part (i). Part (ii) is similarly proven.

Let a, $b \in I$ and $a < b$. Since g is strictly increasing and ϕ is increasing,

$$\mu_{\phi\circ g}((a,b]) = (\phi\circ g)_+(b) - (\phi\circ g)_+(a) = \phi_+(g_+(b)) - \phi_+(g_+(a)) ,$$

since g is strictly increasing,

$$= \mu_\phi((g_+(a), g_+(b)]) = \mu_\phi\left(v^{-1}((a,b])\right), \text{ by Proposition 43.}$$

Since half-open and half-closed intervals generate the Borel algebra,

$$\mu_{\phi\circ g}(E) = \mu_\phi\left(v^{-1}(E)\right),$$

for all Borel set E in $\mathcal{B}(I)$. Therefore, by Theorem 52, for any Borel set B in $\mathcal{B}(I)$,

$$\int_B f d\mu_{\varphi\circ g} = \int_B f d\left(\mu_\phi v^{-1}\right) = \int_{v^{-1}(B)} f\circ v d\mu_\phi .$$

In particular, $\int_{(a,b]} f d\mu_{\phi\circ g} = \int_{v^{-1}((a,b])} f\circ v d\mu_\phi = \int_{(g_+(a),g_+(b)]} f\circ v d\mu_\phi$.

If g is a continuous, strictly increasing function, then v is both left and right inverse function of g and so $v^{-1}(B) = g(B)$.

Theorem 55. Suppose I is an open interval and $g: I \to \mathbb{R}$ is a strictly increasing continuous function. Let $\phi: \mathbb{R} \to \mathbb{R}$ be an increasing function. Let J be the smallest interval containing the range of g. Let $f: \mathbb{R} \to \mathbb{R}$ or $f: J \to \mathbb{R}$ be a Borel function. Let a, $b \in I$ with $a < b$. If the interval I is bounded from below, then

$\int_{(a,b]} f \circ g d\mu_{\phi\circ g} = \int_{(g(a),g(b)]} f d\mu_\phi$. If the interval *I* is bounded from above, then $\int_{[a,b)} f \circ g d\mu_{\phi\circ g} = \int_{[g(a),g(b))} f d\mu_\phi$.

Moreover, $\int_{[a,b]} f \circ g d\mu_{\phi\circ g} = \int_{[g(a),g(b)]} f d\mu_\phi$.

Proof. Suppose the interval *I* is bounded from below.

By Theorem 54 part (i),

$$\int_{(a,b]} f \circ g d\mu_{\phi\circ g} = \int_{(g(a),g(b)]} (f \circ g) \circ \nu d\mu_\phi = \int_{(g(a),g(b)]} f \circ (g \circ \nu) d\mu_\phi = \int_{(g(a),g(b)]} f d\mu_\phi ,$$

since *g* is continuous so that $(g \circ \nu)(y) = g(\nu(y)) = y$ for all *y* in *J*. Note that

$$f(g(a))\mu_{\phi\circ g}\{a\} = f(g(a))\big((\phi \circ g)_+(a) - (\phi \circ g)_-(a)\big)$$

$$= f(g(a))\big(\phi_+(g(a)) - \phi_-(g(a))\big),$$

since g is strictly increasing and continuous.

Now $f(g(a))\mu_\phi\{g(a)\} = f(g(a))\big(\phi_+(g(a)) - \phi_-(g(a))\big)$ and so $f(g(a))\mu_{\phi\circ g}\{a\} = f(g(a))\mu_\phi\{g(a)\}$. Hence adding the end points to the integral gives $\int_{[a,b]} f \circ g d\mu_{\phi\circ g} = \int_{[g(a),g(b)]} f d\mu_\phi$.

If the interval *I* is bounded from above, similar conclusion is reached by using Theorem 54 part (ii).

Remark.

If the interval *I* is both bounded above and bounded below, then the conclusion of Theorem 55 holds when *g* is just only continuous and increasing. For this relaxation of strictly increasing condition, we shall examine the contribution of the points of discontinuity in the domain of the function ϕ, where its inverse image under *g* contains more than one point. See Corollary 61 below.

We now discuss a situation when we can dispense with the continuity condition of *g*.

The result of Theorem 44 is also true when the integrand function *f* is increasing with no continuity condition.

We shall need the following properties of the generalized inverse function of *g*.

Proposition 56. Suppose *I* is an open interval bounded below and $g: I \to \mathbb{R}$ is an increasing function. Let *J* be the smallest interval containing the range of *g*. Let $v: J \to \mathbb{R}$ be the generalised left continuous inverse of *g* as defined in Definition 38.

Suppose $\alpha, \beta \in I$ are such that $\alpha < \beta$. Then

(i) $\alpha < v(y) \le \beta$ if, and only if, $g_+(\alpha) < y \le g_+(\beta)$;

(ii) $v(y) < \beta \Rightarrow y \le g_-(\beta)$, $y < g_-(\beta) \Rightarrow v(y) < \beta$;

(iii) $v(y) \ge \alpha \Rightarrow y \ge g_-(\alpha)$, $y > g_-(\alpha) \Rightarrow v(y) \ge \alpha$;

(iv) $\alpha < v(y) < \beta \Rightarrow g_+(\alpha) < y \le g_-(\beta)$, $g_+(\alpha) < y < g_-(\beta) \Rightarrow \alpha < v(y) < \beta$;

(v) $\alpha \le v(y) < \beta \Rightarrow g_-(\alpha) \le y \le g_-(\beta)$, $g_-(\alpha) < y < g_-(\beta) \Rightarrow \alpha \le v(y) < \beta$;

(vi) $\alpha \le v(y) \le \beta \Rightarrow g_-(\alpha) \le y \le g_+(\beta)$, $g_-(\alpha) < y \le g_+(\beta) \Rightarrow \alpha \le v(y) \le \beta$.

(vii) If $v(y) = z$, then $g_-(z) \le y \le g_+(z)$.

Proof.

(i) This follows from Proposition 43.

(ii) Suppose $y < g_-(\beta)$. If $v(y) = \beta$, then for all $z < \beta$, $g(z) < y$. Hence, $g_-(\beta) \le y$ contradicting $y < g_-(\beta)$ and so $v(y)$ cannot be equal to β. If $v(y) > \beta$, then $y > g(\beta) \ge g_-(\beta)$, contradicting $y < g_-(\beta)$. Therefore, $y < g_-(\beta) \Rightarrow v(y) < \beta$.

Suppose $v(y) < \beta$. Then by definition of *v*, $y \le g(\beta)$. By the proof of Proposition 39 part (iv), for $g_-(\beta) < y \le g(\beta)$, $v(y) = \beta$, if $g_-(\beta) < g(\beta)$. Therefore, $y \le g_-(\beta)$. Hence, $v(y) < \beta \Rightarrow y \le g_-(\beta)$.

(iii). Part (iii) is equivalent to Part (ii).

(iv), (v), (vi).

Parts (iv), (v) and (vi) follow from parts (i), (ii) and (iii).

(vii). If $v(y) = z$, then by definition of *v*, $y \le g(x)$ for all $x > z$. Hence, $y \le g_+(z)$.

For $x < v(y) = z$, $g(x) < y$ for if on the contrary $g(x) \ge y$, then $x \ge v(y)$, contradicting $x < v(y) = z$. It follows that $g_-(z) \le y$.

Theorem 57. Suppose I is an open interval bounded from below and $g: I \to \mathbb{R}$ is an increasing function. Let $f: I \to [0,\infty)$ be a non-negative increasing function. Then for $a,b \in I$ with $a < b$,

$$\int_{[a,b]} f\,d\mu_g = \int_{\nu^{-1}([a,b])} f \circ \nu(y)\,dy = \int_{[g_-(a),g_+(b)]} f \circ \nu(y)\,dy\,,$$

where μ_g is the Lebesgue Stieltjes measure generated by g and ν is the generalised inverse of g as defined in Definition 38.

Proof.

Since f is a Borel function, there exists a monotone increasing sequence of (non-negative) Borel measurable simple functions (s_n) converging pointwise to f. (See Theorem 16, *Introduction To Measure Theory*.)

For simplicity we take $I = [a,b]$ and assumed that $g_-(a) = g(a)$.

The sequence is constructed as follows as in *Introduction To Measure Theory*.

For each integer $n \geq 1$, divide the interval $[0, n]$ into $n \times 2^n$ sub-intervals of length $\frac{1}{2^n}$.

Let $E_{n,i} = f^{-1}\left(\left[\frac{i-1}{2^n},\frac{i}{2^n}\right)\right)$, $i = 1,2,\cdots,n2^n$, $F_n = f^{-1}([n,\infty))$ and

$$s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n}\chi_{E_{n,i}} + n\chi_{F_n} \quad .$$

Since f is increasing, the sets $E_{n,i}$ are bounded intervals and F_n is an interval, which may not be bounded.

Note that $E_{n,i} = E_{n+1,j} \cup E_{n+1,j+1}$, where $\frac{j-1}{2^{n+1}} = \frac{i-1}{2^n}$ or $j = 2i-1$. On the set $E_{n,i}$, $s_{n+1}(x)$ takes on the value $\frac{j-1}{2^{n+1}} = \frac{i-1}{2^n}$ when x is in $E_{n+1,j}$ and the value $\frac{j}{2^{n+1}} > \frac{i-1}{2^n}$ when x is in $E_{n+1,j+1}$. Observe also that

$$F_n = f^{-1}([n,\infty)) = f^{-1}([n+1,\infty)) \cup f^{-1}\big([n,n+1)\big) = F_{n+1} \cup f^{-1}\big([n,n+1)\big)$$

and $f^{-1}\big([n,n+1)\big) = \bigcup\left\{E_{n+1,i} : i = n2^{n+1}+1 \text{ to } (n+1)2^{n+1}\right\}$.

Thus, on the set F_{n+1}, $s_{n+1}(x)$ takes on the value $n+1$ when x is in $E_{n+1,j}$ and on the set $f^{-1}([n,n+1))$, $s_{n+1}(x)$ takes on values $\geq n$, when $s_n(x)$ is defined and is equal to n. Therefore, $s_{n+1} \geq s_n$.

Since $f(x) < \infty$, take an integer N such that $N > f(x)$, then for all $n \geq N$, $s_{n+1}(x) \leq N$ as $\chi_{F_n}(x) = 0$ and so the sequence is pointwise convergence.

Since f is increasing, we may modify the sequence by dropping the final term.

We redefine $s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}}$ Thus,

$$\int_I s_n d\mu_g = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \mu_g(E_{n,i}) \text{. ---------------------------------- (1)}$$

Note that each $E_{n,i}$ is an interval. We examine the intervals with respect to the Lebesgue Stieltjes measure. If $E_{n,i} = [c,d)$ or $[c,d]$ or $(c,d]$ or (c,d) , then

$$\mu_g(E_{n,i}) = g_-(d) - g_-(c) \text{ or } g_+(d) - g_-(c) \text{ or } g_+(d) - g_+(c) \text{ or } g_-(d) - g_+(c)$$

respectively. It follows that
$\mu_g(E_{n,i}) = m([g_-(c), g_-(d)))$ or $m((g_-(c), g_+(d)])$

or $m((g_+(c), g_+(d)])$ or $m((g_+(c), g_-(d)))$.

Let $E_{n,i}{}^g = [g_-(c), g_-(d)]$ or $[g_-(c), g_+(d)]$ or $[g_+(c), g_+(d)]$ or $[g_+(c), g_-(d)]$,

respectively. We note that by Proposition 56,

$v^{-1}((c,d]) = (g_+(c), g_+(d)]$, $(g_+(c), g_-(d)) \subseteq v^{-1}((c,d)) \subseteq (g_+(c), g_-(d)]$,

$(g_-(c), g_-(d)) \subseteq v^{-1}([c,d)) \subseteq [g_-(c), g_-(d)]$, $(g_-(c), g_+(d)] \subseteq v^{-1}([c,d]) \subseteq [g_-(c), g_+(d)]$.

Therefore, if $E_{n,i}{}^g = [c,d]$, then $(c,d) \subseteq v^{-1}(E_{n,i}) \subseteq [c,d] = E_{n,i}{}^g$.

Thus, $m(E_{n,i}{}^g) = m(v^{-1}(E_{n,i}))$, if $E_{n,i}$ is not a singleton set. If $E_{n,i}$ is a singleton set, say $\{c\}$ with $c \in I$. Then $d\mu_g(\{c\}) = g_+(c) - g_-(c)$ and $E_{n,i}{}^g = [g_-(c), g_+(c)]$.

By Proposition 56 part (vi) $v^{-1}(\{c\}) \subseteq [g_-(c), g_+(c)]$. On the other hand, if $g_-(c) < g_+(c)$, then for all $y \in (g_-(c), g_+(c)]$, $v(y) = c$. Hence,

$$(g_-(c), g_+(c)] \subseteq \nu^{-1}(\{c\}) \subseteq [g_-(c), g_+(c)]\,.$$

Therefore, $m(E_{n,i}{}^g) = m([g_-(c), g_+(c)]) = m\left(\nu^{-1}(\{c\})\right) = \mu_g\left(\{c\}\right) = \mu_g\left(E_{n,i}\right)$.

It follows that $\mu_g\left(E_{n,i}\right) = m\left(\nu^{-1}\left(E_{n,i}\right)\right)$. Therefore, taking *I* to be $[a, b]$, we have

$$\int_I s_n d\mu_g = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \mu_g\left(E_{n,i}\right) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} m\left(\nu^{-1}(E_{n,i})\right) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} m\left(\nu^{-1}\left(f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right)\right)\right)$$

$$= \sum_{i=1}^{n2^n} \frac{i-1}{2^n} m\left((f \circ \nu)^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right)\right) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \int_J \chi_{(f\circ\nu)^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right)} dy\,.$$

Evidently, $\sum_{i=1}^{n2^n} \frac{i-1}{2^n} \int_J \chi_{(f\circ\nu)^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right)} dy$ tends to $\int_{\nu^{-1}(I)} f \circ \nu(y) dy$.

Therefore, $\int_{[a,b]} f\, d\mu_g = \int_{\nu^{-1}[a,b]} f \circ \nu(y)\, dy = \int_{[g(a), g_+(b)]} f \circ \nu(y)\, dy$.

In general, if $[a,b] \subseteq I$ and *I* is an open interval, then

$$\int_I s_n \chi_{[a,b]} d\mu_g = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \mu_g\left(E_{n,i} \cap [a,b]\right) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} m\left(\nu^{-1}(E_{n,i} \cap [a,b])\right)$$

$$= \sum_{i=1}^{n2^n} \frac{i-1}{2^n} m\left(\nu^{-1}\left(f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) \cap [a,b]\right)\right)$$

$$= \sum_{i=1}^{n2^n} \frac{i-1}{2^n} m\left(\nu^{-1}\left(f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right)\right) \cap \nu^{-1}([a,b])\right)$$

$$= \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \int_J \chi_{(f\circ\nu)^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right)} \chi_{\nu^{-1}([a,b])} dy\,.$$

Note that $\sum_{i=1}^{n2^n} \frac{i-1}{2^n} \int_J \chi_{(f\circ\nu)^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right)} \chi_{\nu^{-1}([a,b])} dy \to \int_J (f \circ \nu) \chi_{\nu^{-1}([a,b])} dy = \int_{\nu^{-1}([a,b])} (f \circ \nu) dy$. It follows that $\int_{[a,b]} f\, d\mu_g = \int_{\nu^{-1}([a,b])} f \circ \nu(y)\, dy = \int_{[g_-(a), g_+ b)]} f \circ \nu(y)\, dy$.

Instead of requiring the function *g* to be strictly increasing, we can impose some restrictive continuity condition so that the conclusion of Theorem 54 holds. This is just to make sure the limit of composition function behaves well.

Theorem 58. Suppose *I* is an open interval bounded from below and $g: I \to \mathbb{R}$ is an increasing function. Let *J* be the smallest interval containing *g*(*I*). Let $G = \{y \in J : g^{-1}(y) \text{ contains more than one point.}\}$. Suppose $\phi : J \to \mathbb{R}$ is an increasing function with the property that ϕ is right continuous at every point *y* in *G*. Let $f : I \to \mathbb{R}$ be a Borel function. Then for any Borel set *B* in $\mathcal{B}(I)$,

$$\int_B f d\mu_{\phi\circ g} = \int_{v^{-1}(B)} f \circ v d\mu_\phi ,$$

whenever any side of the equality exists finitely or infinitely. In particular, for *a*, *b* ∈ *I* and *a* < *b*, $\int_{(a,b]} f d\mu_{\phi\circ g} = \int_{(g_+(a), g_+(b)]} f \circ v d\mu_\phi$. If $I = [a,b]$ with *a* < *b*, then $\int_{[a,b]} f d\mu_{\phi\circ g} = \int_{[g(a), g(b)]} f \circ v d\mu_\phi$.

Proof.

The key to the proof is a proper handling of the limit of composition functions. Note that the collection $\{g^{-1}(\{y\}) : y \in G\}$ is a collection of disjoint intervals. Since any collection of disjoint non-trivial intervals is countable, *G* is countable. Note that by associating each $g^{-1}(\{y\})$ with a rational number, we deduce that the collection of disjoint non-trivial intervals is countable. Note that if $x \notin \bigcup \left\{ g^{-1}(\{y\}) : y \in G \right\}$, then for all *z* in *I* such that *x* < *z*, $g(z) \neq g(x)$, consequently $(\phi \circ g)_+(x) = \lim_{z \searrow x} (\phi \circ g)(z) = \lim_{y \searrow g_+(x)} (\phi(y)) = \phi_+(g_+(x))$. If $x \in \bigcup \left\{ g^{-1}(\{y\}) : y \in G \right\}$, then there exists $x_0 \in I$ such that $x_0 \neq x$ and $g(x_0) = g(x)$. If $x < x_0$, then $(\phi \circ g)_+(x) = (\phi \circ g)(x) = \phi(g(x)) = \phi_+(g(x)) = \phi_+(g_+(x))$, since ϕ is right continuous at *g*(*x*). If *x* is such that for all *z* > *x*, $g(z) > g(x)$, i.e., $x \in \partial\left(g^{-1}(g(x))\right)$ $(\phi \circ g)_+(x) = \phi_+(g_+(x))$. This means if $g^{-1}(\{y\}) = (c,d)$ or $(c,d]$ or $[c,d]$ or (c,d), $(\phi \circ g)_+(z) = \phi_+(g_+(z))$ for $z \in (c,d)$. If $c \in g^{-1}(\{y\})$, then $(\phi \circ g)_+(z) = \phi_+(g_+(z))$. If $d \in g^{-1}(\{y\})$, then for all $z > d$, $g(z) > g(d)$ and so $(\phi \circ g)_+(z) = \phi_+(g_+(z))$. It follows that for all $x \in I$, $(\phi \circ g)_+(x) = \phi_+(g_+(x))$. Therefore, for any $\alpha, \beta \in I$, with $\alpha < \beta$,

$$\mu_{\phi\circ g}\left((\alpha,\beta]\right) = (\phi \circ g)_+(\beta) - (\phi \circ g)_+(\alpha) = \phi_+(g_+(\beta)) - \phi_+(g_+(\alpha))$$

$$= \mu_\phi\left((g_+(\alpha), g_+(\beta)]\right) = \mu_\phi v^{-1}\left((\alpha,\beta]\right).$$

Since half-open and half-closed intervals of the form $(\alpha, \beta]$ generates the Borel σ-algebra on *I*, for all Borel set, *B*, in *I*, $\mu_{\phi\circ g}(B) = \mu_\phi v^{-1}(B)$. Therefore, by Theorem 52,

$$\int_B f d\mu_{\phi\circ g} = \int_B f d\left(\mu_\phi v^{-1}\right) = \int_{v^{-1}(B)} f \circ v d\mu_\phi \ .$$

If the interval $I = [a, b]$, with $a < b$, then $J = [g(a), g(b)]$ and $v^{-1}\left([a,b]\right) = [g(a), g(b)]$. This is because by Proposition 56, $v^{-1}\left((a,b]\right) = (g_+(a), g_+(b)] = (g_+(a), g(b)]$ and as $v(g(a)) = a$ and $v(g_+(a)) = \inf\left\{z : g(z) \geq g_+(a)\right\} = a$, $v^{-1}\left([a,b]\right) = [g(a), g(b)]$. Thus, under the condition that $I = [a,b]$ and $J = [g(a), g(b)]$, $\int_{[a,b]} f d\mu_{\phi\circ g} = \int_{[g(a),g(b)]} f \circ v d\mu_\phi$. This completes the proof of Theorem 58.

In general, if we relax the condition of right continuity on the set G, we may obtain a sort of change of variable with a correction term coming from the set G. The correction term is a contribution from the discontinuity of the function ϕ in the set G.

Now we state the corresponding result to Theorem 58 using left continuity.

Theorem 59. Suppose I is an open interval bounded from above and $g : I \to \mathbb{R}$ is an increasing function. Let J be the smallest interval containing $g(I)$. Let $G = \{y \in J : g^{-1}(y) \text{ contains more than one point.}\}$. Suppose $\phi : J \to \mathbb{R}$ is an increasing function with the property that ϕ is left continuous at every point y in G. Let $f : I \to \mathbb{R}$ be a Borel function. Then for any Borel set B in $\mathcal{B}(I)$,

$$\int_B f d\mu_{\phi\circ g} = \int_{\eta^{-1}(B)} f \circ \eta d\mu_\phi \ ,$$

whenever any side of the equality exists finitely or infinitely. In particular, for $a, b \in I$ and $a < b$, $\int_{[a,b)} f d\mu_{\phi\circ g} = \int_{[g_-(a),g_-(b))} f \circ \eta d\mu_\phi$. If $I = [a,b]$ with $a < b$, then $\int_{[a,b]} f d\mu_{\phi\circ g} = \int_{[g(a),g(b)]} f \circ \eta d\mu_\phi$.

The proof of Theorem 59 is similar to the proof of Theorem 58 and is omitted.

We now discuss how we can dispense with the right continuity of the function ϕ on the set G in Theorem 58. We can make use of the Saltus function of ϕ to write it as the sum of an increasing continuous function and an increasing Saltus function. We can further write the Saltus function into a sum of an increasing right continuous and an increasing left continuous function. We can then apply the previous theorems to each of the functions. For now, we shall use this idea on the set G.

Suppose I is bounded from above and below. Suppose $g : I \to \mathbb{R}$ is an increasing function. Let $G = \{y \in J : g^{-1}(y) \text{ contains more than one point.}\}$. Let J be the smallest interval containing the range of g. Suppose $\phi : J \to \mathbb{R}$ is an increasing bounded function.

We now define the left and right jump function at the points of G as follows. For simplicity we let $I = [a,b]$ with $a < b$ and so the range of g is the closed interval $[g(a), g(b)]$.

For each y in G, let $\phi_{ls}^{y} = (\phi(y) - \phi_{-}(y))\chi_{[y,g(b)]}$. This gives the value of the left jump at y for all $x \in J$ and $x \geq y$. Define $\Phi_{ls} = \sum_{y \in G} \phi_{ls}^{y}$. Note that since ϕ is increasing on $J = [g(a), g(b)]$ and each term $\phi(y) - \phi_{-}(y)$ is non-negative, $\sum_{y \in G} (\phi(y) - \phi_{-}(y))$ is non negative and is less than the total Saltus of ϕ, which is finite. Therefore, $\sum_{y \in G} (\phi(y) - \phi_{-}(y))$ is well defined and so Φ_{ls} is uniformly convergent. Plainly, it is an increasing function and right continuous. Now, define for each y in G, $\phi_{rs}^{y} = (\phi_{+}(y) - \phi(y))\chi_{(y,g(b)]}$ and define $\Phi_{rs} = \sum_{y \in G} \phi_{rs}^{y}$. Note that $\phi_{rs}^{y} = (\phi_{+}(y) - \phi(y))\chi_{(y,g(b)]}$ gives the value of the right jump at y after y. Obviously, it is an increasing function. Note that if $x \notin G$, then $\Phi_{rs}(x) = \sum_{y<x} \phi_{rs}^{y}$ and obviously, $\lim_{z \nearrow x} \Phi_{rs}(z) = \Phi_{rs}(x)$. If $x \in G$, $\Phi_{rs}(x) = \sum_{y<x} \phi_{rs}^{y}$, since at y in G, $\chi_{(y,g(b)]}(y) = 0$. Thus, $\Phi_{rs}(x)$ is left continuous. Observe that $\phi - \Phi_{ls} - \Phi_{rs}$ is continuous at each y in G.

Note that

$$\Phi_{rs}(x) = \sum_{y \in G} \phi_{rs}^{y}(x) = \sum_{y \in G} (\phi_{+}(y) - \phi(y))\chi_{(y,g(b)]}(x) = \begin{cases} \sum_{y \in G, y<x} (\phi_{+}(y) - \phi(y)), \text{ if } x \notin G, \\ \sum_{y \in G, y<x} (\phi_{+}(y) - \phi(y)), \text{ if } x \in G \end{cases} \quad \text{and}$$

$$\Phi_{ls}(x) = \sum_{y \in G} \phi_{ls}^{y}(x) = \sum_{y \in G} (\phi(y) - \phi_{-}(y))\chi_{[y,g(b)]}(x) = \begin{cases} \sum_{y \in G, y<x} (\phi(y) - \phi_{-}(y)), \text{ if } x \notin G, \\ \sum_{y \in G, y \leq x} (\phi(y) - \phi_{-}(y)), \text{ if } x \in G \end{cases}.$$

If $x \in G$, $\left(\Phi_{rs}\right)_{-}(x) = \Phi_{rs}(x) = \sum_{y \in G, y<x} (\phi_{+}(y) - \phi(y))$,

$$\left(\Phi_{rs}\right)_{+}(x) = \Phi_{rs}(x) = \sum_{y \in G, y \leq x} (\phi_{+}(y) - \phi(y)),$$

$$\left(\Phi_{ls}\right)_+(x)=\Phi_{ls}(x)=\sum_{y\in G,y\le x}(\phi(y)-\phi_-(y)) \text{ and}$$

$$\left(\Phi_{ls}\right)_-(x)=\Phi_{ls}(x)=\sum_{y\in G,y<x}(\phi(y)-\phi_-(y)).$$

If $y\in G$, $\left(\phi-\Phi_{ls}-\Phi_{rs}\right)(y)=\phi(y)-\left(\Phi_{ls}\right)(y)-\left(\Phi_{rs}\right)(y)$

$$=\phi(y)-\sum_{z\in G,z<y}(\phi_+(z)-\phi(z))-\sum_{z\in G,z\le y}(\phi(z)-\phi_-(z))$$

$$=\phi_-(y)-\sum_{z\in G,z<y}(\phi_+(z)-\phi(z))-\sum_{z\in G,z<y}(\phi(z)-\phi_-(z))$$

$$=\phi_-(y)-\sum_{z\in G,z<y}(\phi_+(z)-\phi_-(z)),$$

$$\left(\phi-\Phi_{ls}-\Phi_{rs}\right)_-(y)=\phi_-(y)-\left(\Phi_{ls}\right)_-(y)-\left(\Phi_{rs}\right)_-(y)=\phi_-(y)-\Phi_{rs}(y)-\sum_{z\in G,z<y}(\phi(z)-\phi_-(z))$$

$$=\phi_-(y)-\sum_{z\in G,z<y}(\phi_+(z)-\phi(z))-\sum_{z\in G,z<y}(\phi(z)-\phi_-(z))$$

$$=\phi_-(y)-\sum_{z\in G,z<y}(\phi_+(z)-\phi_-(z)).$$

Therefore, $\left(\phi-\Phi_{ls}-\Phi_{rs}\right)_-(y)=\left(\phi-\Phi_{ls}-\Phi_{rs}\right)(y)$.

If $y\in G$, $\left(\phi-\Phi_{ls}-\Phi_{rs}\right)_+(y)=\phi_+(y)-\left(\Phi_{ls}\right)_+(y)-\left(\Phi_{rs}\right)_+(y)$

$$=\phi_+(y)-\Phi_{ls}(y)-\left(\Phi_{rs}\right)_+(y)$$

$$=\phi_+(y)-\sum_{z\in G,z\le y}(\phi(z)-\phi_-(z))-\sum_{z\in G,z\le y}(\phi_+(z)-\phi(z))$$

$$=\phi_-(y)-\sum_{z\in G,z<y}(\phi_+(z)-\phi_-(z)).$$

Therefore, $\left(\phi-\Phi_{ls}-\Phi_{rs}\right)_+(y)=\left(\phi-\Phi_{ls}-\Phi_{rs}\right)(y)$. Hence, $\phi-\Phi_{ls}-\Phi_{rs}$ is continuous at every point in *G*.

Let $\Phi=\phi-\Phi_{ls}-\Phi_{rs}$. Then Φ is continuous at every point in *G*. We shall show that Φ is an increasing function.

$$\Phi_s(x)=\left(\Phi_{ls}\right)(x)+\left(\Phi_{rs}\right)(x)=\begin{cases}\sum_{z\in G,z\le x}(\phi(z)-\phi_-(z))+\sum_{z\in G,z<x}(\phi_+(z)-\phi(z)), & \text{if } x\in G,\\ \sum_{z\in G,z<x}(\phi(z)-\phi_-(z))+\sum_{z\in G,z<x}(\phi_+(z)-\phi(z)), & \text{if } x\notin G\end{cases}$$

$$= \begin{cases} \sum_{z\in G, z<x} (\phi_+(z)-\phi_-(z)) + \phi(x)-\phi_-(x), \text{ if } x\in G \\ \sum_{z\in G, z<x} (\phi_+(z)-\phi_-(z)), \text{ if } x\notin G \end{cases}.$$

Suppose $x < y$. Then

$$\Phi_s(y)-\Phi_s(x) = \begin{cases} \sum_{z\in G, x<z<y} (\phi_+(z)-\phi_-(z)) + \phi(y)-\phi_-(y), \text{ if } y\in G, x\notin G, \\ \sum_{z\in G, x<z<y} (\phi_+(z)-\phi_-(z)), \text{ if } y\notin G, x\notin G, \\ \sum_{z\in G, x<z<y} (\phi_+(z)-\phi_-(z)) + \phi_+(x)-\phi(x), \text{ if } y\notin G, x\in G, \\ \sum_{z\in G, x<z<y} (\phi_+(z)-\phi_-(z)) + \phi(y)-\phi_-(y)+\phi_+(x)-\phi(x), \text{ if } y\in G, x\in G \end{cases}.$$

Since ϕ is increasing, for $y > x$,

$$\phi(y)-\phi(x) \geq \begin{cases} \sum_{z\in G, x\leq z<y} (\phi_+(z)-\phi_-(z)) + \phi(y)-\phi_-(y), \text{if } y\in G, x\notin G, \\ \sum_{z\in G, x\leq z<y} (\phi_+(z)-\phi_-(z))), \text{if } y\notin G, x\notin G, \\ \sum_{z\in G, x\leq z<y} (\phi_+(z)-\phi_-(z)) + \phi(y)-\phi_-(y)+\phi_+(x)-\phi(x), \text{if } y\in G, x\in G, \\ \sum_{z\in G, x\leq z<y} (\phi_+(z)-\phi_-(z)) + \phi_+(x)-\phi(x), \text{ if } y\notin G, x\in G \end{cases}$$

$$= \Phi_s(y)-\Phi_s(x).$$

This is just a consequence of the fact that ϕ is increasing and so $\phi(y)-\phi(x)$ is greater than the sum of all the saltus between x and y plus the right jump at x and the left jump at y. Thus, $\Phi = \phi - \Phi_{rs} - \Phi_{ls} = \phi - \Phi_s$ is an increasing function, continuous at every point in G, Φ_{ls} is right continuous at every point in G and Φ_{rs} is left continuous at every point in G.

We can now formulate our next result.

Theorem 60. Suppose $I = [a,b]$, with $a < b$, is a closed and bounded interval and $g: I \to \mathbb{R}$ is an increasing function. Let $J = [g(a), g(b)]$ and $G = \{y \in J : g^{-1}(y) \text{ contains more than one point.}\}$. Suppose $\phi: J \to \mathbb{R}$ is an increasing function. Let $f:[a,b] \to \mathbb{R}$ be a Borel function. Then

$$\int_{[a,b]} f d\mu_{\phi\circ g} = \int_{[g(a),g(b)]} f\circ v\, d\mu_\Phi + \sum_{y\in G} f(v(y))\big(\phi(y)-\phi_-(y)\big) + \sum_{y\in G} f(\eta(y))\big(\phi_+(y)-\phi(y)\big).$$

For any Borel set B in $\mathcal{B}([a,b])$,

$$\int_B f d\mu_{\phi\circ g}$$

$$= \int_{\nu^{-1}(B)} f\circ \nu d\mu_\Phi + \sum_{y\in G\cap \nu^{-1}(B)} f(\nu(y))\big(\phi(y)-\phi_-(y)\big) + \sum_{y\in G\cap \eta^{-1}(B)} f(\eta(y))\big(\phi_+(y)-\phi(y)\big).$$

Moreover, $\int_{[g(a),g(b)]} f\circ \nu d\mu_\Phi = \int_{[g(a),g(b)]} f\circ \eta d\mu_\Phi$ and $\int_{\nu^{-1}(B)} f\circ \nu d\mu_\Phi = \int_{\eta^{-1}(B)} f\circ \eta d\mu_\Phi$.

Proof.

$$\int_{[a,b]} f d\mu_{\phi\circ g} = \int_{[a,b]} f d\mu_{\Phi\circ g+\Phi_{ls}\circ g+\Phi_{rs}\circ g} = \int_{[a,b]} f d\mu_{\Phi\circ g} + \int_{[a,b]} f d\mu_{\Phi_{ls}\circ g} + \int_{[a,b]} f d\mu_{\Phi_{rs}\circ g}.$$

By Theorem 58, $\int_{[a,b]} f d\mu_{\Phi\circ g} = \int_{[g(a),g(b)]} f\circ \nu d\mu_\Phi$. -------------------- (1)

Since Φ_{ls} is right continuous, by Theorem 58, $\int_{[a,b]} f d\mu_{\Phi_{ls}\circ g} = \int_{[g(a),g(b)]} f\circ \nu d\mu_{\Phi_{ls}}$.
Note that since Φ_{ls} is a constant function except for countable number of points in *G*,

$$\int_{[g(a),g(b)]} f\circ \nu d\mu_{\Phi_{ls}} = \sum_{y\in G}\int_{\{y\}} f\circ \nu d\mu_{\Phi_{ls}} = \sum_{y\in G} f(\nu(y))\mu_{\Phi_{ls}}(\{y\}).$$

Now for each *y* in *G*, $\mu_{\Phi_{ls}}(\{y\}) = (\Phi_{ls})_+(y) - (\Phi_{ls})_-(y)$

$$= (\Phi_{ls})(y) - (\Phi_{ls})_-(y)$$

$$= \sum_{z\in G, z\le y} (\phi(z)-\phi_-(z)) - \sum_{z\in G, z<y} (\phi(z)-\phi_-(z)) = \phi(y)-\phi_-(y).$$

Hence, $\int_{[g(a),g(b)]} f\circ \nu d\mu_{\Phi_{ls}} = \sum_{y\in G} f(\nu(y))\big(\phi(y)-\phi_-(y)\big)$. ------------------- (2)

By Theorem 59, $\int_{[a,b]} f d\mu_{\Phi_{rs}\circ g} = \int_{[g(a),g(b)]} f\circ \eta d\mu_{\Phi_{rs}} = \sum_{y\in G} f(\eta(y))\mu_{\Phi_{rs}}(\{y\})$.

Observe that for each *y* in *G*, $\mu_{\Phi_{rs}}(\{y\}) = (\Phi_{rs})_+(y) - (\Phi_{rs})_-(y)$

$$= (\Phi_{rs})_+(y) - (\Phi_{rs})(y)$$

$$= \sum_{z\in G, z\le y} (\phi_+(z)-\phi(z)) - \sum_{z\in G, z<y} (\phi_+(z)-\phi(z)) = \phi_+(y)-\phi(y).$$

Hence, $\int_{[a,b]} f d\mu_{\Phi_{rs}\circ g} = \sum_{y\in G} f(\eta(y))\big(\phi_+(y)-\phi(y)\big)$. ---------------------- (3)

Putting (1), (2) and (3) together we have that

$$\int_{[a,b]} f d\mu_{\phi\circ g} = \int_{[g(a),g(b)]} f\circ \nu d\mu_\Phi + \sum_{y\in G} f(\nu(y))\big(\phi(y)-\phi_-(y)\big) + \sum_{y\in G} f(\eta(y))\big(\phi_+(y)-\phi(y)\big).$$

In general, for a Borel set *B* in $\mathcal{B}([a,b])$,

$$\int_B fd\mu_{\phi\circ g} = \int_B fd\mu_{\Phi\circ g+\Phi_{ls}\circ g+\Phi_{rs}\circ g} = \int_B fd\mu_{\Phi\circ g} + \int_B fd\mu_{\Phi_{ls}\circ g} + \int_B fd\mu_{\Phi_{rs}\circ g}$$

$$= \int_{\nu^{-1}(B)} f\circ\nu d\mu_{\Phi} + \int_{\nu^{-1}(B)} f\circ\nu d\mu_{\Phi_{ls}} + \int_{\eta^{-1}(B)} f\circ\eta d\mu_{\Phi_{rs}},$$

by Theorem 58 and Theorem 59.

The last statement is a consequence of Theorem 58 and Theorem 59.

Corollary 61. Suppose $I = [a,b]$, with $a < b$, is a closed and bounded interval, and $g: I \to \mathbb{R}$ is an increasing continuous function. Let $J = [g(a), g(b)]$ and suppose $\phi: J \to \mathbb{R}$ is an increasing function. Let $f: J \to \mathbb{R}$ be a Borel function. Then $\int_{[a,b]} f\circ g d\mu_{\phi\circ g} = \int_{[g(a),g(b)]} fd\mu_{\phi}$.

Proof. With the notation as in Theorem 60, by Theorem 60,

$$\int_{[a,b]} f\circ g d\mu_{\phi\circ g} = \int_{[g(a),g(b)]} f\circ g\circ\nu d\mu_{\Phi} + \sum_{y\in G} f(g(\nu(y)))\big(\phi(y)-\phi_-(y)\big) + \sum_{y\in G} f(g(\eta(y)))\big(\phi_+(y)-\phi(y)\big)$$

$$= \int_{[g(a),g(b)]} fd\mu_{\Phi} + \sum_{y\in G} f(y)\big(\phi(y)-\phi_-(y)\big) + \sum_{y\in G} f(y)\big(\phi_+(y)-\phi(y)\big),$$ since g is continuous,

$$= \int_{[g(a),g(b)]} fd\mu_{\Phi} + \int_{[g(a),g(b)]} fd\mu_{\Phi_{ls}} + \int_{[g(a),g(b)]} fd\mu_{\Phi_{rs}}$$

$$= \int_{[g(a),g(b)]} fd\mu_{\Phi+\Phi_{ls}+\Phi_{rs}} = \int_{[g(a),g(b)]} fd\mu_{\phi}.$$

If ϕ is an increasing function on a closed and bounded interval $[c, d]$, then it is a function of bounded variation. By Theorem 19, ϕ can be decomposed as a sum of three functions, $\phi = \Phi_{ac} + \Phi_c + \Phi_s$, where Φ_{ac} is an increasing absolutely continuous function with $\Phi_{ac}{}' = \phi'$ almost everywhere, Φ_c is a continuous increasing function with $\Phi_c{}' = 0$ almost everywhere and Φ_s is the Saltus function of ϕ. Evidently, Φ_s is an increasing singular function. As explained in the proof of Theorem 19, $G(x) = \phi - \Phi_s$ is a continuous increasing function. Moreover, we note that ϕ is increasing and bounded so that ϕ is differentiable almost everywhere and ϕ' is finite and greater or equal to zero almost everywhere. In particular ϕ' is Lebesgue integrable and we may define $\Phi_{ac}(x) = \int_c^x \phi' dm = \int_c^x G' dm$, where m is the Lebesgue measure. It follows that Φ_{ac} is increasing. Note that Φ_{ac} is absolutely continuous. Let $\Phi_c = \phi - \Phi_s - \Phi_{ac}$. Then

Φ_c is continuous and in particular, $\Phi_c{}' = 0$ almost everywhere. Let $c \le x < y \le d$. Then

$$\Phi_c(x) - \Phi_c(y) = G(x) - \Phi_{ac}(x) - \left(G(y) - \Phi_{ac}(y)\right)$$

$$= G(x) - G(y) + \int_x^y \phi' dm = G(x) - G(y) + \int_x^y H' dm$$

$$\le G(x) - G(y) + G(y) - G(x) = 0 \ ,$$

by Theorem 6 *of Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*. Therefore, Φ_c is an increasing function. Observe that if we define Φ_{ls} and Φ_{rs} for ϕ with the set *D* of discontinuity of ϕ in place of *G*, as we have done in the construction preceding Theorem 60, we can show that Φ_s as given in the proof of Theorem 18 is equal to $\Phi_{ls} + \Phi_{rs}$.

We now make use of the decomposition of ϕ into a sum of absolute continuous increasing function, a continuous increasing singular function and an increasing saltus function. Using the same argument as in Theorem 60, we obtain:

Theorem 62. Suppose $I = [a,b]$ is a closed and bounded interval, with $a < b$ and $g : I \to \mathbb{R}$ is an increasing function. Let $J = [g(a), g(b)]$ and suppose $\phi : J \to \mathbb{R}$ is an increasing function. Let $D = \{y \in J : \phi \text{ is discontinuous at } y\}$ be the set of discontinuity of ϕ. Let $f : [a,b] \to \mathbb{R}$ be a Borel function. Then

$$\int_{[a,b]} f d\mu_{\phi\circ g} = \int_{[g(a),g(b)]} \left(f \circ \nu\right) \cdot \phi' dy$$

$$+ \int_{[g(a),g(b)]} \left(f \circ \nu\right) d\mu_{\Phi_c} + \sum_{y \in D} f(\nu(y))\left(\phi(y) - \phi_-(y)\right) + \sum_{y \in D} f(\eta(y))\left(\phi_+(y) - \phi(y)\right).$$

For any Borel set *B* in $\mathcal{B}$([*a*,*b*]),

$$\int_B f d\mu_{\phi\circ g} = \int_{\nu^{-1}(B)} \left(f \circ \nu\right) \cdot \phi' dy$$

$$+ \int_{\nu^{-1}(B)} f \circ \nu d\mu_{\Phi_c} + \sum_{y \in D \cap \nu^{-1}(B)} f(\nu(y))\left(\phi(y) - \phi_-(y)\right) + \sum_{y \in D \cap \eta^{-1}(B)} f(\eta(y))\left(\phi_+(y) - \phi(y)\right),$$

where Φ_c is the continuous increasing singular function in the decomposition, $\phi = \Phi_{ac} + \Phi_c + \Phi_s$.

Moreover, $\int_{[g(a),g(b)]} \left(f \circ \nu\right) \cdot \phi' dy = \int_{[g(a),g(b)]} \left(f \circ \eta\right) \cdot \phi' dy$,
$\int_{[g(a),g(b)]} \left(f \circ \nu\right) d\mu_{\Phi_c} = \int_{[g(a),g(b)]} \left(f \circ \eta\right) d\mu_{\Phi_c}$, $\int_{\nu^{-1}(B)} \left(f \circ \nu\right) \cdot \phi' dy = \int_{\eta^{-1}(B)} \left(f \circ \eta\right) \cdot \phi' dy$ and
$\int_{\nu^{-1}(B)} f \circ \nu d\mu_{\Phi_c} = \int_{\eta^{-1}(B)} f \circ \eta d\mu_{\Phi_c}$.

Proof. Take a decomposition of ϕ , $\phi = \Phi_{ac} + \Phi_c + \Phi_{ls} + \Phi_{rs}$, where Φ_{ac} is an absolutely continuous increasing function with $\Phi_{ac}'(x) = \phi'(x)$ almost everywhere on *J*, Φ_c is a continuous increasing singular function, i.e., $\Phi_c'(x) = 0$ almost everywhere, Φ_{ls} and Φ_{rs} are defined as before with the set *D* of discontinuity in place of *G*. Then

$$\int_{[a,b]} f d\mu_{\phi\circ g} = \int_{[a,b]} f d\mu_{\Phi_{ac}\circ g + \Phi_c\circ g + \Phi_{ls}\circ g + \Phi_{rs}\circ g}$$

$$= \int_{[a,b]} f d\mu_{\Phi_{ac}\circ g} + \int_{[a,b]} f d\mu_{\Phi_c\circ g} + \int_{[a,b]} f d\mu_{\Phi_{ls}\circ g} + \int_{[a,b]} f d\mu_{\Phi_{rs}\circ g} .$$

By Theorem 58 and Theorem 59,

$$\int_{[a,b]} f d\mu_{\phi\circ g}$$

$$= \int_{[g(a),g(b)]} f\circ\nu d\mu_{\Phi_{ac}} + \int_{[g(a),g(b)]} f\circ\nu d\mu_{\Phi_c} + \int_{[g(a),g(b)]} f\circ\nu d\mu_{\Phi_{ls}} + \int_{[g(a),g(b)]} f\circ\eta d\mu_{\Phi_{rs}}$$

$$= \int_{[g(a),g(b)]} \left(f\circ\nu\right)\cdot\phi' dy + \int_{[g(a),g(b)]} f\circ\nu d\mu_{\Phi_c} + \int_{[g(a),g(b)]} f\circ\nu d\mu_{\Phi_{ls}} + \int_{[g(a),g(b)]} f\circ\eta d\mu_{\Phi_{rs}} ,$$

since Φ_{ac} is absolutely continuous and $\Phi_{ac}' = \phi'$ almost everywhere,

$$= \int_{[g(a),g(b)]} \left(f\circ\nu\right)\cdot\phi' dy + \int_{[g(a),g(b)]} f\circ\nu d\mu_{\Phi_c}$$

$$+ \sum_{y\in D} f(\nu(y))\left(\phi(y) - \phi_-(y)\right) + \sum_{y\in D} f(\eta(y))\left(\phi_+(y) - \phi(y)\right),$$

where the last two terms are derived as in the proof of Theorem 60.

The last assertion follows from Theorem 58 and Theorem 59 as follows:

For a Borel set *B* in *I*,

$$\int_B f d\mu_{\phi\circ g} = \int_B f d\mu_{\Phi_{ac}\circ g + \Phi_c\circ g + \Phi_{ls}\circ g + \Phi_{rs}\circ g}$$

$$= \int_B f d\mu_{\Phi_{ac}\circ g} + \int_B f d\mu_{\Phi_c\circ g} + \int_B f d\mu_{\Phi_{ls}\circ g} + \int_B f d\mu_{\Phi_{rs}\circ g}$$

$$= \int_{\nu^{-1}(B)} f\circ\nu d\mu_{\Phi_{ac}} + \int_{\nu^{-1}(B)} f\circ\nu d\mu_{\Phi_c} + \int_{\nu^{-1}(B)} f\circ\nu d\mu_{\Phi_{ls}} + \int_{\eta^{-1}(B)} f\circ\eta d\mu_{\Phi_{rs}} ,$$

$$= \int_{\nu^{-1}(B)} f\circ\nu\cdot\phi' dy + \int_{\nu^{-1}(B)} f\circ\nu d\mu_{\Phi_c}$$

$$+ \sum_{y\in D\cap\nu^{-1}(B)} f(\nu(y))\left(\phi(y) - \phi_-(y)\right) + \sum_{y\in D\cap\eta^{-1}(B)} f(\eta(y))\left(\phi_+(y) - \phi(y)\right).$$

The assertions in the last statement are a consequence of the continuity of Φ_{ac} and Φ_c and follows from Theorem 58 and Theorem 59.

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