Lebesgue Measure on The Real Numbers $\mathbb{R}$

And Lebesgue Theorem on Riemann Integrability

By Ng Tze Beng

In this article, we shall construct the Lebesgue measure on the set of real numbers $\mathbb{R}$. We shall do this via a set function on the collection of all subsets of $\mathbb{R}$. This set function is called the *outer measure* on $\mathbb{R}$. We shall show that the Lebesgue measure is translation invariant and that on interval *I*, it is equal to the length of *I*. We shall characterize Riemann integrability in terms of measure theoretic property.

Definition 1. Let *I* be an interval with end points *a* and *b* with *a* < *b*. The *length* $\lambda(I)$ is defined by $\lambda(I) = b - a$. If *I* is an unbounded interval, then define $\lambda(I) = \infty$.

We want to extend this notion of length to arbitrary subsets of $\mathbb{R}$.

Let Γ be the family of all countable collections of open intervals. Define

$$\lambda^* : \Gamma \to \overline{\mathbb{R}^+},$$

by $\lambda^*(\gamma) = \sum_{I \in \gamma} \lambda(I)$ for any $\gamma \in \Gamma$. Hence, $0 \le \lambda^*(\gamma) \le \infty$. Note that as each $\lambda(I)$ is non-negative, the summation $\sum_{I \in \gamma} \lambda(I)$ is absolutely convergent (including ∞) and does not depend on the order of summation.

Suppose γ is a collection of open intervals and *V* is a subset of $\mathbb{R}$. We say γ is a *covering* for *V* or γ *covers V* if $V \subseteq \bigcup_{I \in \gamma} I$.

Now, let *E* be an arbitrary subset of $\mathbb{R}$. Let $C(E) = \{\gamma \in \Gamma : \gamma \text{ covers } E\}$. Note that $C(E) \neq \varnothing$. Define $\mu^*(E) = \inf\{\lambda^*(\gamma) : \gamma \in C(E)\} \in \overline{\mathbb{R}^+}$. This is called the *Lebesgue outer measure* of *E*. We have thus defined a function from the set of all subsets of $\mathbb{R}$ into $\overline{\mathbb{R}^+}$,

$$\mu^* : \mathscr{P}(\mathbb{R}) = 2^{\mathbb{R}} \to \overline{\mathbb{R}^+}.$$

Then we have:

Proposition 2.

(i) $\mu^*(\varnothing) = 0$.

(ii) $\mu^*(\{x\}) = 0$ for all $x \in \mathbb{R}$.

(iii) For any two subsets A and B of $\mathbb{R}$, $A \subseteq B \Rightarrow \mu^*(A) \le \mu^*(B)$.

Proof.

(i) and (ii)

Take $x \in \mathbb{R}$. Then for any integer $n \ge 1$, the open interval $\left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ covers $\{x\}$.

Therefore, $\mu^*(\{x\}) \le \lambda\left(\left(x - \frac{1}{n}, x + \frac{1}{n}\right)\right) = \frac{2}{n}$. Since $\frac{2}{n} \to 0$, $\mu^*(\{x\}) = 0$.

As $\varnothing \subseteq \{x\}$, $\varnothing \subseteq \{x\} \subseteq \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$, $\mu^*(\varnothing) = 0$.

(iii) Suppose $A \subseteq B$. Then any countable cover of B is also a countable cover of A. Hence, $C(B) \subseteq C(A)$. Therefore,

$$\mu^*(B) = \inf\{\lambda^*(\gamma) : \gamma \in C(B)\} \ge \inf\{\lambda^*(\gamma) : \gamma \in C(A)\} = \mu^*(A).$$

Let $r \in \mathbb{R}$. Let $\tau_r : \mathbb{R} \to \mathbb{R}$ be the translation map given by $\tau_r(x) = x + r$ for $x \in \mathbb{R}$.

The next result gives the desirable property of the Lebesgue outer measure on $\mathbb{R}$. It is translation invariant. Not all outer measures need to be translation invariant but for a generalization of length on subsets of $\mathbb{R}$, translation invariant is expected as the translated interval is still an interval and the length of the interval does not change after the translation.

Proposition 3. For any $r \in \mathbb{R}$, $\mu^*(\tau_r(E)) = \mu^*(E)$ for any subset E of $\mathbb{R}$.

Proof.

If I is an open interval with endpoints $a < b$, then $\tau_r(I)$ is an open interval with end points $a + r$ and $b + r$. Hence, $\lambda(\tau_r(I)) = \lambda(I) = b - a$. For every $\gamma \in \Gamma$, let

$$\tau_r(\gamma) = \{\tau_r(I) : I \in \gamma\} \in \Gamma.$$

Suppose E is a subset of $\mathbb{R}$.

If γ covers E, then plainly, $\tau_r(\gamma)$ covers $\tau_r(E)$. Observe that

$$\lambda^*(\tau_r(\gamma)) = \sum_{I \in \gamma} \lambda(\tau_r(I)) = \sum_{I \in \gamma} \lambda(I) = \lambda^*(\gamma).$$

It follows that $\{\lambda^*(\gamma):\gamma\in C(E)\}=\{\lambda^*(\tau_r(\gamma)):\gamma\in C(E)\}\subseteq\{\lambda^*(\gamma):\gamma\in C(\tau_r(E))\}$.
Hence, $\mu^*(\tau_r(E))=\inf\{\lambda^*(\gamma):\gamma\in C(\tau_r(E))\}\le\inf\{\lambda^*(\gamma):\gamma\in C(E)\}=\mu^*(E)$.

As $\tau_{-r}(\tau_r(E))=E$, by applying the above inequality with $\tau_r(E)$ in place of E and τ_{-r} in place of τ_r, we get $\mu^*(E)=\mu^*(\tau_{-r}(\tau_r(E)))\le\mu^*(\tau_r(E))$. Hence $\mu^*(\tau_r(E))=\mu^*(E)$.

Next, we show that the outer measure does extend the meaning of length of an interval.

Proposition 4. For any interval I, $\mu^*(I)=\lambda(I)$.

Proof. We shall establish the proposition for closed and bounded interval $I=[a,b]$ with $a<b$. Now, $[a,b]\subseteq(a-\varepsilon,b+\varepsilon)$ for any $\varepsilon>0$ and so $\mu^*(I)\le\lambda((a-\varepsilon,b+\varepsilon))=b-a+2\varepsilon$. As ε is arbitrary, we have that $\mu^*(I)\le b-a=\lambda(I)$. We want to show that for every γ in $C(I)$, $\lambda^*(\gamma)\ge b-a$. Since $[a,b]$ is compact, every open covering γ in $C(I)$, has a finite sub-covering, say β, then $\lambda^*(\gamma)\ge\lambda^*(\beta)$. So, we now assume that γ is a finite collection of open intervals that cover I.

Starting with a, since γ covers I, there is an open interval (a_1,b_1) in γ such that $a\in(a_1,b_1)$, i.e., $a_1<a<b_1$. If $b_1>b$, then $\lambda^*(\gamma)\ge\lambda((a_1,b_1))=b_1-a_1>b-a$. If $b_1\le b$, then $b_1\in I$ and there exists an open interval (a_2,b_2) in γ such that $b_1\in(a_2,b_2)$ with $a_2<b_1<b_2$. If $b_2>b$, then
$\lambda((a_1,b_1))+\lambda((a_2,b_2))=b_1-a_1+b_2-a_2=(b_1-a_2)+(b_2-b)+(a-a_1)-a+b>b-a$.

Since γ is a finite collection, this process of covering the end point of the next interval must terminate. Suppose it terminate at the n-th interval (a_n,b_n) such that $b_n>b$ and $n\ge 2$.

Then we have $a_k<b_{k-1}<b_k$, where we have denoted $b_0=a$ for $k=1, 2, \ldots, n-1$. Hence, we have

$$\lambda^*(\gamma)\ge\sum_{i=1}^{n}\lambda((a_i,b_i))=\sum_{i=1}^{n}(b_i-a_i)=\sum_{i=2}^{n}(b_{i-1}-a_i)+b_n-a_1>b_n-a_1>b-a.$$

Therefore, $\mu^*(I)=\inf\{\lambda^*(\gamma):\gamma\in C(I)\}\ge b-a$ and so $\mu^*(I)=b-a=\lambda(I)$.

Now let I be any bounded interval with end points a, b with $a<b$. For any $0<\varepsilon<\dfrac{b-a}{4}$, $[a+\varepsilon,b-\varepsilon]\subseteq I\subseteq[a,b]$. Then by Proposition 2 (iii),

$$\mu^*([a+\varepsilon,b-\varepsilon])\le\mu^*(I)\le\mu^*([a,b]) .$$

It follows that $b-a-2\varepsilon \le \mu^*\left(I\right) \le b-a$. Therefore, as ε is arbitrary, $\mu^*\left(I\right)=b-a=\lambda(I)$.

Finally, let I be an unbounded interval. Then for any real number $K > 0$, the interval I contains a bounded interval H of length $\lambda(H) \ge K$. Therefore, $\mu^*(I) \ge \mu^*(H) \ge K$. It follows that $\mu^*(I)=\infty=\lambda(I)$.

Any non-negative set function β defined on a collection of sets, C, is said to be *countably sub-additive* or *σ sub-additive* if for any countably family Ω of sets in C,

$$\beta\left(\bigcup_{E\in\Omega} E\right) \le \sum_{E\in\Omega} \beta(E) .$$

It is said to be *countably additive* or *σ additive* if for any countable family Ω of pairwise disjoint sets in C,

$$\beta\left(\bigcup_{E\in\Omega} E\right) = \sum_{E\in\Omega} \beta(E) .$$

Next, we show that μ^* is countably sub-additive.

Proposition 5. For any countably family Ω of subsets of $\mathbb{R}$,

$$\mu^*\left(\bigcup_{E\in\Omega} E\right) \le \sum_{E\in\Omega} \mu^*(E) .$$

Proof.

Let $\Omega=\left\{E_n : n\in\mathbb{N}\right\}$. Let $\varepsilon > 0$. By definition of the outer measure μ^*, for each inter $n \ge 1$, there exists a covering γ_n of E_n in $C(E_n)$ such that

$$\lambda^*\left(\gamma_n\right) \le \mu^*(E_n)+\frac{\varepsilon}{2^n} . \text{ ---------------------------- (1)}$$

Let $\gamma=\bigcup_{n=1}^{\infty}\gamma_n$. Then γ is a countable cover of $\bigcup_{E\in\Omega} E$. That is, $\gamma \in C\left(\bigcup_{E\in\Omega} E\right)$.

Therefore, $\mu^*\left(\bigcup_{E\in\Omega} E\right) \le \lambda^*(\gamma) \le \sum_{n=1}^{\infty}\lambda^*(\gamma_n)$, since any open interval in γ is in some γ_k for some k and $\lambda^*(\gamma_n) \ge 0$ for all $n \ge 1$. It follows then from (1) that

$$\mu^*\left(\bigcup_{E\in\Omega} E\right) \le \sum_{n=1}^{\infty}\mu^*(E_n)+\varepsilon .$$

As this holds for any $\varepsilon > 0$, $\mu^*\left(\bigcup_{E\in\Omega} E\right) \le \sum_{n=1}^{\infty}\mu^*(E_n)$.

Corollary 6. If E is a countable subset of $\mathbb{R}$, then $\mu^*(E) = 0$.

Proof. Suppose E is countable subset of $\mathbb{R}$ and so $E = \bigcup_{n=1}^{\infty} \{x_n\}$. Then

$$\mu^*(E) = \mu^*\left(\bigcup_{n=1}^{\infty} \{x_n\}\right) \le \sum_{i=1}^{\infty} \mu^*(\{x_n\}) = 0 \text{ by Proposition 5 and Proposition 2(ii).}$$

Hence, $\mu^*(E) = 0$.

As a consequence,

Corollary 7. Every interval is not countable.

The Lebesgue outer measure on $\mathbb{R}$ is countably sub-additive on the collection of all subsets of $\mathbb{R}$ but not countably additive. In order to obtain a countably additive function from it, we restrict the domain to a subset of the power set of $\mathbb{R}$. In this procedure, we follow Caratheodory's restriction method, we call the restricted collection, the *Lebesgue measurable subsets* of $\mathbb{R}$ or the *Lebesgue measure* on $\mathbb{R}$.

A subset E of $\mathbb{R}$ is said to be *Lebesgue measurable* if, and only if, for any subset X of $\mathbb{R}$, we have,

$$\mu^*(X) = \mu^*(X \cap E) + \mu^*(X - E).$$

Since $X = (X \cap E) \cup (X - E)$, we have by Proposition 5 that for any subset X of $\mathbb{R}$,

$$\mu^*(X) \le \mu^*(X \cap E) + \mu^*(X - E).$$

We have immediately the following:

Lemma 8. A subset E of $\mathbb{R}$ is Lebesgue measurable if, and only if, for all $X \subseteq \mathbb{R}$,

$$\mu^*(X) \ge \mu^*(X \cap E) + \mu^*(X - E).$$

Proposition 9. If $E \subseteq \mathbb{R}$ is Lebesgue measurable, then its complement $\mathbb{R} - E = E^c$ is also Lebesgue measurable.

Proof. Note that for all $X \subseteq \mathbb{R}$, $X \cap (\mathbb{R} - E) = X - E$ and $X - (\mathbb{R} - E) = X \cap E$. Proposition 9 follows from Lemma 8.

Observe that for any $X \subseteq \mathbb{R}$, $X \cap \varnothing = \varnothing$ and $X - \varnothing = X$. Trivially we have for any $X \subseteq \mathbb{R}$, $\mu^*(X) = \mu^*(X \cap \varnothing) + \mu^*(X - \varnothing) = 0 + \mu^*(X) = \mu^*(X)$. Thus, Ø is Lebesgue measurable and by Proposition 9, $\mathbb{R}$ is Lebesgue measurable. We record our conclusion as:

Proposition 10. $\varnothing$ and $\mathbb{R}$ are Lebesgue measurable.

Proposition 11. If *A* and *B* are Lebesgue measurable subsets of $\mathbb{R}$, then $A \cap B$ is also Lebesgue measurable.

Proof.

Since *B* is Lebesgue measurable, for any $X \subseteq \mathbb{R}$,

$$\mu^*(X \cap A) = \mu^*\big((X \cap A) \cap B\big) + \mu^*\big((X \cap A) - B\big).$$

Now $X - (A \cap B) = (X - A) \cup \big((X \cap A) - B\big)$ and so by Proposition 5,

$$\mu^*\big(X - (A \cap B)\big) \le \mu^*(X - A) + \mu^*\big((X \cap A) - B\big).$$

Therefore,

$$\begin{aligned}\mu^*(X - A) + \mu^*\big((X \cap A) - B\big) &= \mu^*(X - A) + \mu^*(X \cap A) - \mu^*\big((X \cap A) \cap B\big) \\ &\ge \mu^*\big(X - (A \cap B)\big).\end{aligned}$$

Hence, $\mu^*(X - A) + \mu^*(X \cap A) \ge \mu^*\big(X - (A \cap B)\big) - \mu^*\big((X \cap A) \cap B\big)$.

But *A* is Lebesgue measurable and so,

$$\mu^*(X) = \mu^*(X - A) + \mu^*(X \cap A) \ge \mu^*\big(X - (A \cap B)\big) - \mu^*\big(X \cap (A \cap B)\big) \text{ for all } X \subseteq \mathbb{R}.$$

It follows by Lemma 8 that $A \cap B$ is Lebesgue measurable.

Corollary 12. If *A* and *B* are Lebesgue measurable subsets of $\mathbb{R}$, then $A \cup B$ is also Lebesgue measurable.

Proof. Note that $\mathbb{R} - (A \cup B) = (\mathbb{R} - A) \cap (\mathbb{R} - B)$. Since *A* and *B* are Lebesgue measurable, by Proposition 9, $(\mathbb{R} - A)$ and $(\mathbb{R} - B)$ are Lebesgue measurable and consequently, by Proposition 11, $\mathbb{R} - (A \cup B) = (\mathbb{R} - A) \cap (\mathbb{R} - B)$ is Lebesgue measurable. By Proposition 9, $A \cup B$ is Lebesgue measurable.

Let $\mathscr{M}$ be the set of all Lebesgue measurable subsets of $\mathbb{R}$.

Lemma 13. If $E_1, E_2, \ldots, E_n$ are pairwise disjoint Lebesgue measurable sets in $\mathscr{M}$, then for any $X \subseteq \mathbb{R}$,

$$\mu^*\left(X \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} \mu^*\left(X \cap E_i\right).$$

Proof. The lemma is trivially true for $n = 1$.

Let $n > 1$. We shall prove this lemma by induction. Assume the lemma is true for a collection of less than n members of pairwise disjoint Lebesgue measurable sets. Let X be any subset of $\mathbb{R}$.

Since E_n is Lebesgue measurable, for any subset Y of $\mathbb{R}$,

$$\mu^*(Y) = \mu^*\left(Y \cap E_n\right) + \mu^*\left(Y - E_n\right) \text{ . ----------------------- (1)}$$

Take $Y = X \cap \left(\bigcup_{i=1}^{n} E_i\right)$. Now, $Y \cap E_n = X \cap E_n$ and as $\{E_i\}_{i=1}^{n}$ are pairwise disjoint, $Y - E_n = X \cap \bigcup_{i=1}^{n-1} E_i$. It follows from (1) that

$$\mu^*\left(X \cap \bigcup_{i=1}^{n} E_i\right) = \mu^*(Y) = \mu^*\left(X \cap E_n\right) + \mu^*\left(X \cap \bigcup_{i=1}^{n-1} E_i\right)$$

and by the induction hypothesis,

$$\mu^*\left(X \cap \bigcup_{i=1}^{n} E_i\right) = \mu^*\left(X \cap E_n\right) + \sum_{i=1}^{n-1} \mu^*\left(X \cap E_i\right)$$

$$= \sum_{i=1}^{n} \mu^*\left(X \cap E_i\right) \text{ .}$$

This completes the proof.

Next, we have:

Theorem 14. Suppose $\{E_i\}_{i=1}^{\infty}$ is a countable collection of Lebesgue measurable subsets of $\mathbb{R}$, i.e., members of $\mathscr{M}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathscr{M}$.

Proof.

The first thing that we do is to write $\bigcup_{i=1}^{\infty} E_i$ as a disjoint union.

Let $S_1 = E_1$, $S_2 = (E_1 \cup E_2) - E_1 = E_2 - E_1 = E_2 \cap (\mathbb{R} - E_1)$. For integer $n \geq 2$, let

$$S_n = E_n - \bigcup_{i=1}^{n-1} E_i = E_n \cap \left(\mathbb{R} - \bigcup_{i=1}^{n-1} E \right).$$

By Proposition 11, Proposition 12 and Proposition 9, S_n is Lebesgue measurable for integer $n \geq 2$.

Plainly, $S_i \subseteq E_i$ for all integer $i \geq 1$. Therefore, $\bigcup_{i=1}^{\infty} S_i \subseteq \bigcup_{i=1}^{\infty} E_i$. Take $x \in \bigcup_{i=1}^{\infty} E_i$. Then $x \in E_n$ for some integer $n \geq 1$. If $n = 1$, then $x \in E_n = E_1 = S_1$. If $n > 1$, then let k be the least integer such that $x \in E_k$. Then $k \leq n$. If $k = 1$, then $x \in E_k = E_1 = S_1$. If $k > 1$, then $x \notin E_j$ for $j \leq k-1$. Therefore, $x \in E_k - \bigcup_{i=1}^{k-1} E_i = S_k$. It follows that $\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} S_i$. Hence, $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} S_i$.

For $i \neq j$, $S_i \cap S_j = E_i \cap E_j \cap \left(\mathbb{R} - \bigcup_{k=1}^{i-1} E_k \right) \cap \left(\mathbb{R} - \bigcup_{k=1}^{j-1} E_k \right)$

$$= E_i \cap E_j \cap \left(\mathbb{R} - \bigcup_{k=1}^{j-1} E_k \right), \text{ if } i < j,$$

$$= \varnothing.$$

As $i \neq j \Rightarrow$ either $i < j$ or $j < i$. It follows that $S_i \cap S_j = \varnothing$ for $i \neq j$. We conclude that $\bigcup_{i=1}^{\infty} S_i$ is a disjoint union.

Let $D_n = \bigcup_{i=1}^{n} S_i \subseteq E = \bigcup_{i=1}^{\infty} E_i$. Then D_n is Lebesgue measurable by Corollary 12. Therefore, for $X \subseteq \mathbb{R}$,

$$\mu^*(X) = \mu^*(X \cap D_n) + \mu^*(X - D_n) \geq \mu^*(X \cap D_n) + \mu^*(X - E),$$

since $X - E \subseteq X - D_n$. It then follows by Lemma 13 that

$$\mu^*(X) \geq \sum_{i=1}^{n} \mu^*(X \cap S_i) + \mu^*(X - E).$$

Since this holds for any integer $n \geq 1$, we have

$$\mu^*(X) \geq \sum_{i=1}^{\infty} \mu^*(X \cap S_i) + \mu^*(X - E). \quad \text{---------------} \quad (1)$$

But by Proposition 5 (countable sub-additivity of the outer measure),

$$\sum_{i=1}^{\infty} \mu^*\left(X \cap S_i\right) \geq \mu^*\left(\bigcup_{i=1}^{\infty}(X \cap S_i)\right) = \mu^*\left(X \cap \bigcup_{i=1}^{\infty} S_i\right) = \mu^*\left(X \cap E\right).$$

It follows from (1) that $\mu^*(X) \geq \mu^*(X \cap E) + \mu^*\left(X - E\right)$. Hence, by Lemma 8, $E = \bigcup_{i=1}^{\infty} E_i$ is Lebesgue measurable. That is, $E \in \mathscr{M}$.

We now state our main theorem

Theorem 15. The set $\mathscr{M}$, of all Lebesgue measurable subsets of $\mathbb{R}$, is a σ-algebra and ($\mathbb{R}$, $\mathscr{M}$) is a measure space. The set function on $\mathscr{M}$ given by the restriction of the Lebesgue outer measure to $\mathscr{M}$, $\mu = \mu^*|_{\mathscr{M}} : \mathscr{M} \to \overline{\mathbb{R}^+}$, is a positive measure. Hence, ($\mathbb{R}$, $\mathscr{M}$, μ) is a measure space.

$\mathscr{M}$ is called the *Lebesgue measure* on $\mathbb{R}$ and the set function, $\mu : \mathscr{M} \to \overline{\mathbb{R}^+}$ is called the *Lebesgue measure*.

Proof. By Proposition 9, Proposition 10 and Theorem 14, $\mathscr{M}$ is a σ-algebra and so ($\mathbb{R}$, $\mathscr{M}$) is a measure space. Since $\mu(\varnothing) = \mu^*(\varnothing) = 0$, μ is non-trivial. It remains to show that μ is countably additive on $\mathscr{M}$.

Suppose $\left\{E_i\right\}_{i=1}^{\infty}$ is a countable collection of pairwise disjoint Lebesgue measurable sets in $\mathscr{M}$.

Then for any integer $n \geq 1$, by Lemma 13, with $X = \mathbb{R}$, we have

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \mu^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu^*\left(E_i\right) = \sum_{i=1}^{n} \mu\left(E_i\right) .$$

Since $\bigcup_{i=1}^{n} E_i \subseteq \bigcup_{i=1}^{\infty} E_i$, $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \mu^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu^*\left(E_i\right) = \sum_{i=1}^{n} \mu\left(E_i\right)$ for each $n \geq 1$. Therefore,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} \mu\left(E_i\right).$$

But by Proposition 5 (countable sub-additivity),

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*\left(E_i\right) = \sum_{i=1}^{\infty} \mu\left(E_i\right).$$

It follows that $\mu\left(\bigcup_{i=1}^{\infty} E_i\right)=\sum_{i=1}^{\infty}\mu(E_i)$. Hence, μ is countably additive on $\mathscr{M}$ and so is a positive measure on $\mathscr{M}$.

Proposition 16. Every subset E of $\mathbb{R}$ with $\mu^*(E)=0$ is Lebesgue measurable. Hence, the σ-algebra $\mathscr{M}$ is μ-complete. That is the measure space ($\mathbb{R}$, $\mathscr{M}$, μ) is a complete measure space.

Proof.

Suppose $\mu^*(E)=0$. Take any subset X of $\mathbb{R}$. Since $X\cap E\subseteq E$, by Proposition 2, $0\le\mu^*(X\cap E)\le\mu^*(E)=0$ and so $\mu^*(X\cap E)=0$. Also, as $X\supseteq X-E$, $\mu^*(X)\ge\mu^*(X-E)$. Therefore, $\mu^*(X)\ge\mu^*(X-E)=\mu^*(X-E)+\mu^*(X\cap E)$. Hence, by Lemma 8, E is Lebesgue measurable.

Consider the μ-completion of $\mathscr{M}$,

$$\mathscr{M}^*=\{E\subseteq\mathbb{R} : \text{there exists } A, B\in\mathscr{M}, \text{ such that } A\subseteq E\subseteq B \text{ and } \mu(B-A)=0\}.$$

Note that $\mathscr{M}\subseteq\mathscr{M}^*$. If $A, B\in\mathscr{M}$ is such that $A\subseteq E\subseteq B$ and $\mu(B-A)=0$, then since $E-A\subseteq B-A$, $\mu^*(E-A)\le\mu^*(B-A)=\mu(B-A)=0$ implies that $E-A\in\mathscr{M}$ and so $E=(E-A)\cup A\in\mathscr{M}$. It follows that $\mathscr{M}^*\subseteq\mathscr{M}$. Therefore, $\mathscr{M}^*=\mathscr{M}$ and so $\mathscr{M}$ is μ-complete.

Proposition 17. Every open subset of $\mathbb{R}$ is Lebesgue measurable. Hence the Borel subsets of $\mathbb{R}$, $\mathscr{B}$ is contained in the σ-algebra $\mathscr{M}$ of Lebesgue measurable subsets of $\mathbb{R}$. This means $\mathscr{B}$ is a sub σ-algebra of $\mathscr{M}$.

Proof. Any open subset E of $\mathbb{R}$ is a countable union of open intervals. By Theorem 14, it is sufficient to show that any open interval is Lebesgue measurable.

Since $\{(a,\infty),(-\infty,b): a,b\in\mathbb{R}\}$ is a subbase for the topology on $\mathbb{R}$, by Proposition 11, it is sufficient to show that (a, ∞) and $(-\infty, b)$ for any a and b in $\mathbb{R}$, are Lebesgue measurable. Note that, for any subset X in $\mathbb{R}$, $X-(a,\infty)=X\cap(-\infty,a]$. If we can show that (a, ∞) is Lebesgue measurable, then by Proposition 9, $(-\infty,a]=\mathbb{R}-(a,\infty)$ is Lebesgue measurable. It follows then that for any b in $\mathbb{R}$, by Theorem 14, $(-\infty,b)=\bigcup_{n=1}^{\infty}\left(-\infty,b-\frac{1}{n}\right]$ is Lebesgue measurable. Hence, it is sufficient to prove that (a, ∞) is Lebesgue measurable for any a in $\mathbb{R}$. We shall show that for any subset $X\subseteq\mathbb{R}$,

$$\mu^*(X)\ge\mu^*(X\cap(a,\infty))+\mu^*(X\cap(-\infty,a]).$$

If $\mu^*(X)=\infty$, then we have nothing to prove and so we assume that $\mu^*(X)<\infty$.

Take any $\varepsilon > 0$. Then by the definition of the Lebesgue outer measure μ^*, there exists a countable covering γ of X by open intervals with

$$\lambda^*(\gamma) = \sum_{I\in\gamma} \lambda(I) \le \mu^*(X) + \varepsilon \ .$$

For each open interval $I \in \gamma$, each of the sets $I \cap (a,\infty)$ and $I \cap (-\infty, a]$ is either empty or an interval. Moreover, $I = \left(I \cap (a,\infty)\right) \cup \left(I \cap (-\infty, a]\right)$ is a disjoint union and so

$$\lambda(I) = \lambda\left(I \cap (a,\infty)\right) + \lambda\left(I \cap (-\infty, a]\right) = \mu^*\left(I \cap (a,\infty)\right) + \mu^*\left(I \cap (-\infty, a]\right).$$

As $\left\{I \cap (a,\infty) : I \in \gamma\right\} = \gamma'$ covers $X \cap (a,\infty)$,

$$\mu^*(X \cap (a,\infty)) \le \mu^*\left(\bigcup_{I\in\gamma} \left(I \cap (a,\infty)\right)\right)$$

$$\le \sum_{I\in\gamma} \mu^*\left(I \cap (a,\infty)\right)$$, by Proposition 5 (countable sub-additivity). -----(1)

Similarly, since $X \cap (-\infty, a] \subseteq \bigcup_{I\in\gamma} \left(I \cap (-\infty, a]\right)$,

$$\mu^*(X \cap (-\infty, a]) \le \mu^*\left(\bigcup_{I\in\gamma} \left(I \cap (-\infty, a]\right)\right)$$ by Proposition 2(iii),

$$\le \sum_{I\in\gamma} \mu^*\left(I \cap (-\infty, a]\right)$$, by Proposition 5. ---------------- (2)

It follows from (1) and (2) that

$$\mu^*(X \cap (a,\infty)) + \mu^*(X \cap (-\infty, a])$$

$$\le \sum_{I\in\gamma} \mu^*\left(I \cap (a,\infty)\right) + \sum_{I\in\gamma} \mu^*\left(I \cap (-\infty, a]\right) = \sum_{I\in\gamma} \lambda\left(I\right)$$

$$\le \mu^*(X) + \varepsilon .$$

Since this is true for any $\varepsilon > 0$, $\mu^*(X \cap (a,\infty)) + \mu^*(X \cap (-\infty, a]) \le \mu^*(X)$.

This holds for any subset $X \subseteq \mathbb{R}$, by Lemma 8, (a, ∞) is Lebesgue measurable.

This completes the proof.

In summary, we have

Theorem. $(\mathbb{R}, \mathscr{M}, \mu)$ is a measure space such that $\mathscr{M}$ is μ-complete, μ is non-trivial and $\mathscr{M}$ contains the Borel subsets of $\mathbb{R}$.

Next, we investigate the relation of the Lebesgue integral with the Riemann integral on a bounded interval.

Definition 18. Suppose *E* is a Lebesgue measurable subset of $\mathbb{R}$. We say a real valued Lebesgue measurable function $f : E \to \mathbb{R}$ is *Lebesgue integrable* if

$$\int_E |f| \, d\mu < \infty \ .$$

(See Definition 29, *Introduction to Measure Theory*.)

We recall the following result.

Theorem 19. Suppose *E* is a Lebesgue measurable subset of $\mathbb{R}$ and $\mu(E) < \infty$. Suppose $f : E \to \mathbb{R}$ is bounded. Then f is measurable if, and only if, the lower and upper Lebesgue integral of *f* are the same. The *lower Lebesgue integral* of *f* is defined by $\underline{\int_E} f d\mu = \sup\left\{\int_E \varphi d\mu : \varphi \le f, \varphi \in S(E)\right\}$ and the *upper Lebesgue integral* of f is defined by $\overline{\int_E} f d\mu = \inf\left\{\int_E \varphi d\mu : f \le \varphi, \varphi \in S(E)\right\}$, where *S*(*E*) is the set of real-valued simple measurable functions on *E*.

(This is Theorem 7 in *Positive Borel Measure and Riesz Representation Theorem*.)

Proof.

Suppose *f* is measurable. As *f* is bounded, we assume $\alpha \le f < \beta$. Let $\delta_n = \dfrac{\beta - \alpha}{n}$, for integer $n \ge 1$. Define $E_{n.i} = f^{-1}[\alpha + (i-1)\delta_n, \alpha + i\delta_n)$ for $1 \le i \le n$, $n = 1, 2, \ldots$.
Then $E_{n,i}$ are measurable and for each integer $n \ge 1$,

$$\mu(E) = \mu\left(\bigcup_{i=1}^{n} E_{n,i}\right) \text{ , where } \bigcup_{i=1}^{n} E_{n,i} \text{ is a disjoint union,}$$

$$= \sum_{i=1}^{n} \mu\left(E_{n,i}\right) < \infty .$$

Let $\phi_n = \sum_{i=1}^{n} \left(\alpha + (i-1)\delta_n\right)\chi_{E_{n,i}}$ and $\psi_n = \sum_{i=1}^{n} \left(\alpha + i\,\delta_n\right)\chi_{E_{n,i}}$ for each integer $n \ge 1$. Thus, ϕ_n, ψ_n are simple measurable functions on *E* such that

$$\phi_n(x) \le f(x) < \psi_n(x) \quad \text{for all } x \text{ in } E.$$

Therefore,

$$\underline{\int}_E f\,d\mu \geq \int_E \phi_n\,d\mu = \sum_{i=1}^{n}\left(\alpha+(i-1)\delta_n\right)\mu(E_{n,i})$$

and

$$\overline{\int}_E f\,d\mu \leq \int_E \psi_n d\mu = \sum_{i=1}^{n}\left(\alpha+i\delta_n\right)\mu\left(E_{n,i}\right) \quad .$$

Hence, $\overline{\int}_E f\,d\mu - \underline{\int}_E f\,d\mu \leq \sum_{i=1}^{n}\delta_n\mu(E_{n,i}) = \delta_n\mu(E)$.

But $\delta_n \to 0$ as $n\to\infty$ and so $\overline{\int}_E f\,d\mu \leq \underline{\int}_E f\,d\mu$. Since $\underline{\int}_E f\,d\mu \leq \overline{\int}_E f\,d\mu$, it follows that $\underline{\int}_E f\,d\mu = \overline{\int}_E f\,d\mu$

Conversely, suppose $\underline{\int}_E f\,d\mu = \overline{\int}_E f\,d\mu$. We shall show that f is measurable or μ-measurable.

Let $L(f)=\{\phi\in S(E):\phi\leq f\}$ and $U(f)=\{\psi\in S(E): f\leq\psi\}$. Then $\underline{\int}_E f\,d\mu = \sup\left\{\int_E \phi d\mu : \phi\in L(f)\right\}$ and $\overline{\int}_E f\,d\mu = \inf\left\{\int_E \psi d\mu : \psi\in U(f)\right\}$. Since f is bounded and $\mu(E)<\infty$, $\underline{\int}_E f\,d\mu = \overline{\int}_E f\,d\mu < \infty$. Thus, for any integer $n \geq 1$, there exists $\phi_n\in L(f)$ and $\psi_n\in U(f)$ such that

$$\int_E \phi_n d\mu > \underline{\int}_E f\,d\mu - \frac{1}{n} \quad \text{and} \quad \int_E \psi_n d\mu < \overline{\int}_E f\,d\mu + \frac{1}{n}.$$

Hence, $\int_E(\psi_n-\phi_n)d\mu = \int_E \psi_n d\mu - \int_E \phi_n d\mu \leq \frac{2}{n}$. This holds for all integer $n \geq 1$.

Define $\phi,\psi: E\to\mathbb{R}$, by $\phi=\sup\{\phi_n\}_{n=1}^{\infty}$ and $\psi=\inf\{\psi_n\}_{n=1}^{\infty}$. Then both ϕ and ψ are measurable since each ϕ_n and ψ_n are measurable for all integer $n \geq 1$.

Plainly, $\phi_n\leq\phi\leq f\leq\psi\leq\psi_n$.

Let $D_k=\left\{x\in E:\psi(x)-\phi(x)>\frac{1}{k}\right\}$.

Obviously, $D_k\subseteq\left\{x\in E:\psi_n(x)-\phi_n(x)>\frac{1}{k}\right\}=D_{k,n}$ for all integer $n \geq 1$.

Hence, $\frac{1}{k}\chi_{D_{k,n}}\leq\psi_n-\phi_n$ and so $\int_E\frac{1}{k}\chi_{D_{k,n}}d\mu\leq\int_E(\psi_n-\phi_n)d\mu$. It follows that

$$\frac{1}{k}\mu\left(D_{k,n}\right) \le \int_E (\psi_n - \phi_n)\, d\mu \le \frac{2}{n}.$$

Therefore, $\mu\left(D_k\right) \le \mu\left(D_{k,n}\right) \le \frac{2k}{n}$ for all integer $n \ge 1$. It follows that $\mu\left(D_k\right) = 0$.

Let $D = \{x \in E : \psi(x) - \phi(x) > 0\}$. Then $D = \bigcup_{k=1}^{\infty} D_k$ and $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_k \subseteq \cdots \subseteq D$. Therefore, $\mu(D) = \lim_{k\to\infty} \mu\left(D_k\right) = 0$. This means $\psi = \phi$ almost everywhere with respect to the Lebesgue measure μ. As $\phi \le f \le \psi$, $\phi = f = \psi$ on $E - D$ and $f = \psi$ almost everywhere with respect to the Lebesgue measure μ. Since E is measurable and the Lebesgue measure is complete, $E - D$ is measurable. Therefore, f is measurable on $E - D$ since ψ is measurable. Hence, f is measurable.

Theorem 20. Suppose E is a Lebesgue measurable subset of $\mathbb{R}$ and $\mu(E) < \infty$. Suppose $f : E \to \mathbb{R}$ is a bounded measurable function. Then f is Lebesgue integrable and

$$\int_E f\, d\mu = \int_E f^+ d\mu - \int_E f^- d\mu = \underline{\int_E} f\, d\mu = \overline{\int_E} f\, d\mu \quad .$$

(This is Theorem 8 in *Positive Borel Measure and Riesz Representation Theorem*.)

Proof.

Since $f : E \to \mathbb{R}$ is measurable, $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$ are measurable. Thus, $f = f^+ - f^-$ and $|f| = f^+ + f^-$ is measurable. Note that f^+, f^- and $|f|$ are bounded non-negative functions. Then by definition,

$$\int_E f^+ d\mu = \sup\left\{\int_E s\, d\mu : 0 \le s \le f^+, s \in S(E)\right\}$$

and $$\int_E f^- d\mu = \sup\left\{\int_E s\, d\mu : 0 \le s \le f^-, s \in S(E)\right\}.$$

Since both $\left\{\int_E s\, d\mu : 0 \le s \le f^+, s \in S(E)\right\}$ and $\left\{\int_E s\, d\mu : 0 \le s \le f^-, s \in S(E)\right\}$ are bounded above by $K\mu(E)$ for some constant K such that $|f(x)| < K$ for all x in E, $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ exist and are finite and so $\int_E |f|\, d\mu = \int_E f^+ d\mu + \int_E f^- d\mu < \infty$. Thus, by definition, f is Lebesgue integrable on E and

$$\int_E f\, d\mu = \int_E f^+ d\mu - \int_E f^- d\mu \quad .$$

Since f^+ is measurable,

$$\underline{\int_E} f^+ d\mu = \sup\left\{\int_E \phi d\mu : \phi \in L(f^+)\right\} = \sup\left\{\int_E \phi d\mu : \phi \in S(E) : \phi \le f^+\right\}$$

$$= \sup\left\{\int_E \phi d\mu : \phi \in S(E) : 0 \le \phi \le f^+\right\} = \int_E f^+ d\mu .$$

Similarly, we have $\underline{\int_E} f^- d\mu = \int_E f^- d\mu$. By Theorem 19, $\underline{\int_E} f^- d\mu = \int_E f^- d\mu = \overline{\int_E} f^- d\mu$ and $\underline{\int_E} f^+ d\mu = \int_E f^+ d\mu = \overline{\int_E} f^+ d\mu$.

$$\int_E f^+ d\mu - \int_E f^- d\mu = \overline{\int_E} f^+ d\mu - \underline{\int_E} f^- d\mu$$

$$= \inf\left\{\int_E \psi d\mu : f^+ \le \psi, \psi \in S(E)\right\} - \sup\left\{\int_E \phi d\mu : \phi \in S(E), \phi \le f^-\right\}$$

$$= \inf\left\{\int_E \psi d\mu : f^+ \le \psi, \psi \in S(E)\right\} + \inf\left\{-\int_E \phi d\mu : \phi \in S(E), \phi \le f^-\right\}$$

$$= \inf\left\{\int_E \psi d\mu : f^+ \le \psi, \psi \in S(E)\right\} + \inf\left\{\int_E -\phi d\mu : -\phi \in S(E), -\phi \ge -f^-\right\}$$

$$= \inf\left\{\int_E \psi d\mu : f^+ \le \psi, \psi \in S(E)\right\} + \inf\left\{\int_E \phi d\mu : -f^- \le \phi, \phi \in S(E),\right\}$$

$$= \inf\left\{\int_E \psi d\mu + \int_E \phi d\mu : f^+ \le \psi, -f^- \le \phi, \psi, \phi \in S(E)\right\}$$

$$= \inf\left\{\int_E (\psi + \phi) d\mu : f^+ \le \psi, -f^- \le \phi, \psi, \phi \in S(E)\right\}$$

$$\ge \inf\left\{\int_E \psi d\mu : f = f^+ - f^- \le \psi, \psi \in S(E)\right\} = \overline{\int_E} f d\mu .$$

Similarly,

$$\int_E f^+ d\mu - \int_E f^- d\mu = \underline{\int_E} f^+ d\mu - \overline{\int_E} f^- d\mu$$

$$= \sup\left\{\int_E \psi d\mu : \psi \le f^+, \psi \in S(E)\right\} - \inf\left\{\int_E \phi d\mu : \phi \in S(E), f^- \le \phi\right\}$$

$$= \sup\left\{\int_E \psi d\mu : \psi \le f^+, \psi \in S(E)\right\} + \sup\left\{-\int_E \phi d\mu : \phi \in S(E), f^- \le \phi\right\}$$

$$= \sup\left\{\int_E \psi d\mu : \psi \le f^+, \psi \in S(E)\right\} + \sup\left\{\int_E -\phi d\mu : -\phi \in S(E), -f^- \ge -\phi\right\}$$

$$= \sup\left\{\int_E \psi d\mu : \psi \le f^+, \psi \in S(E)\right\} + \sup\left\{\int_E \phi d\mu : \phi \in S(E), \phi \le -f^-\right\}$$

$$= \sup\left\{\int_E \psi d\mu + \int_E \phi d\mu : \psi \le f^+, \phi \le -f^-, \psi, \phi \in S(E)\right\}$$

$$= \sup\left\{\int_E (\psi + \phi) d\mu : \psi \le f^+, \phi \le -f^-, \psi, \phi \in S(E)\right\}$$

$$\le \sup\left\{\int_E \psi d\mu : \psi \le f^+ - f^- = f, \psi \in S(E)\right\} = \underline{\int_E} f d\mu .$$

Thus, we have $\overline{\int_E} f d\mu \le \int_E f^+ d\mu - \int_E f^- d\mu \le \underline{\int_E} f d\mu$. Now, as f is measurable and $\mu(E) < \infty$, by Theorem 19, $\overline{\int_E} f d\mu = \underline{\int_E} f d\mu$ and so it follows that

$$\overline{\int_E} f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu = \underline{\int_E} f d\mu .$$

Suppose $f : [a,b] \to \mathbb{R}$ is bounded. A *step function s* on [a, b] is a function that assumes finite constant values on the open subintervals of [a, b] defined by some partition of [a, b]. More precisely, there is a partition $x_0 = a < x_1 < x_2 < \cdots < x_n = b$ and a set of constants, $\xi_1, \xi_2, \cdots, \xi_n$ such that $s(x) = \xi_i$ for $x_{i-1} < x < x_i$ for $1 \le i \le n$. Plainly, a step function is a simple μ-measurable function. We define the *Riemann integral on step function s* by the expression,

$$\int_a^b s = R\int_a^b s = \sum_{i=1}^{n} \xi_i (x_i - x_{i-1}) .$$

This is the usual definition of Riemann integral on step function. Observe that as a step function s is bounded and measurable, s is Lebesgue integrable and

$$\int_{[a,b]} s\, d\mu = \int_{[a,b]-\{x_i\}_{i=0}^n} s\, d\mu = \sum_{i=1}^{n} \xi_i \mu\left((x_{i-1}, x_i)\right) = \sum_{i=1}^{n} \xi_i (x_i - x_{i-1}) .$$

Thus, the Riemann integral of a step function and the Lebesgue integral of a step function are the same. This is the main lead to showing that Riemann integrals are Lebesgue integrals.

Let $S^*([a,b])$ be the set of all step functions on [a, b]. The *lower Riemann integral* of f is defined to be

$$R\underline{\int_a^b} f = \sup\left\{\int_a^b \varphi : \varphi \le f, \varphi \in S^*([a,b])\right\}$$

and the *upper Riemann integral* of f is defined to be

$$R\overline{\int_a^b} f = \inf\left\{\int_a^b \varphi : f \le \varphi, \varphi \in S^*([a,b])\right\}.$$

As *f* is bounded, the upper and lower Riemann integrals exist. The bounded function *f* is said to be *Riemann integra*ble if

$$R\underline{\int_a^b} f = R\overline{\int_a^b} f .$$

Now a step function in $S^*([a,b])$ is a linear combination of characteristic functions of subintervals plus a finite number of linear combination of characteristic functions of singleton sets. So, by Proposition 4, for $\varphi \in S^*([a,b])$, $\int_a^b \varphi = \int_{[a,b]} \varphi d\mu$. Now let $S([a,b])$ be the set of real-valued measurable simple functions on [*a*, *b*]. Then $S^*([a,b]) \subseteq S([a,b])$.
Moreover,

$$R\overline{\int_a^b} f = \inf\left\{\int_a^b \varphi : f \le \varphi, \varphi \in S^*([a,b])\right\} = \inf\left\{\int_{[a,b]} \varphi d\mu : f \le \varphi, \varphi \in S^*([a,b])\right\}$$

$$\ge \inf\left\{\int_a^b \varphi d\mu : f \le \varphi, \varphi \in S([a,b])\right\}.$$

and $$R\underline{\int_a^b} f = \sup\left\{\int_a^b \varphi : \varphi \le f, \varphi \in S^*([a,b])\right\} = \sup\left\{\int_{[a,b]} \varphi d\mu : \varphi \le f, \varphi \in S^*([a,b])\right\}$$

$$\le \sup\left\{\int_{[a,b]} \varphi d\mu : \varphi \le f, \varphi \in S([a,b])\right\}.$$

The *lower Lebesgue integral* of f is

$$\underline{\int_{[a,b]}} f d\mu = \sup\left\{\int_{[a,b]} \varphi d\mu : \varphi \le f, \varphi \in S([a,b])\right\}$$

and the *upper Lebesgue integral* of f is

$$\overline{\int_{[a,b]}} f d\mu = \inf\left\{\int_a^b \varphi d\mu : f \le \varphi, \varphi \in S([a,b])\right\}$$

Thus, we have

$$R\underline{\int_a^b} f \le \underline{\int_{[a,b]}} f d\mu \le \overline{\int_{[a,b]}} f d\mu \le R\overline{\int_a^b} f .$$

So, if f is Riemann integrable on [*a*, *b*], then $\underline{\int_{[a,b]}} f d\mu = \overline{\int_{[a,b]}} f d\mu$. By Theorem 19, if $\underline{\int_{[a,b]}} f = \overline{\int_{[a,b]}} f$, then $f : [a,b] \to \mathbb{R}$ is μ-measurable. Therefore, *f* is bounded and Lebesgue measurable and so *f* is Lebesgue integrable and the Lebesgue integral,

$\int_{[a,b]} f\,d\mu = \underline{\int_{[a,b]}} f\,d\mu = \overline{\int_{[a,b]}} f\,d\mu = R\underline{\int_a^b} f = R\overline{\int_a^b} f$. Thus, if f is Riemann integrable, then f is Lebesgue integrable and the Riemann integral and the Lebesgue integral are the same.

Hence, we have proved the following theorem.

Theorem 21. Suppose $f:[a,b]\to\mathbb{R}$ is a bounded and Riemann integrable function. Then f is Lebesgue integrable (therefore, μ-measurable) and

$$\int_{[a,b]} f\,d\mu = R\int_a^b f \text{ , the Riemann integral of } f \text{ on } [a, b].$$

Example. A bounded function, not Riemann integrable but Lebesgue integrable, the Dirichlet function.

Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}1, & \text{if } x \text{ is rational,}\\ 0, & \text{if } x \text{ is irrational}\end{cases}$. It is easily seen that $R\overline{\int_0^1} f = 1$ and $R\underline{\int_0^1} f = 0$ so that f is not Riemann integrable. Note that since the rational numbers in [0, 1] is a set of Lebesgue measure zero, $f = 0$ except on a set of Lebesgue measure zero. Therefore, by Proposition 39 in *Introduction To Measure Theory*, f is Lebesgue measurable. Since f is bounded and the interval [0, 1] is of finite Lebesgue measure, by Theorem 20, f is Lebesgue integrable and $\int_{[a,b]} f\,d\mu = 0$.

We next characterize Riemann integrable function in terms of Lebesgue measure.

Theorem 22. A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable if, and only if, it is continuous *a.e.* [μ] on [a, b].

Before we prove Theorem 22. We show that the lower and upper Riemann integrals of a bounded function as defined above via step functions are the same as the usual ones using Darboux sums.

Suppose now f: [a, b] $\to$ **R** is a bounded function.

Let P : $a = x_0 < x_1 < \ldots < x_n = b$ be a partition for [a, b].
The *upper Darboux sum* with respect to the partition P is defined by

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$. Note that since f is bounded on [a, b], f is bounded on each $[x_{i-1}, x_i]$ and so the supremum M_i exists for each i. Likewise for each i, $m_i = \inf\{f$

$(x) : x \in [x_{i-1}, x_i]\}$ exists since f is bounded on each $[x_{i-1}, x_i]$. We define the *lower Darboux sum* with respect to the partition P by

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

Because for each integer i such that $1 \le i \le n$, $m_i \le M_i$, $L(f, P) \le U(f, P)$.

Since f is bounded, there exist real numbers m and M such that $m \le f(x) \le M$ for all x in $[a, b]$. Hence $m \le M_i \le M$ and $m \le m_i \le M$ for $i = 1, 2, ..., n$. Therefore, for any partition P the upper Darboux sum

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \ge m \sum_{i=1}^{n}(x_i - x_{i-1}) = m(b-a).$$

Hence the set of all upper Darboux sums (over all partitions of $[a, b]$) is *bounded below* by $m(b - a)$. Likewise, the lower Darboux sum

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le M \sum_{i=1}^{n}(x_i - x_{i-1}) = M(b-a).$$

We conclude that the set of all lower Darboux sums (over all partitions of $[a, b]$) is *bounded above* by $M(b-a)$. We may now make the following definition following Darboux.

Definition 23. Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. Then the *upper Darboux integral* or *upper integral* is defined to be

$$U\int_a^b f = \inf\{U(f,P) : P \text{ a partition of } [a,b]\}.$$

The *lower Darboux integral* or *lower integral* is defined to be

$$L\int_a^b f = \sup\{L(f,P) : P \text{ a partition of } [a,b]\}.$$

We say f is *Darboux integrable* if $U\int_a^b f = L\int_a^b f$.

Note that by the completeness property of the real numbers, the upper integral exists, because the set of all upper Darboux sum is bounded below and the lower integral exists because the set of all lower Darboux sum is bounded above.

We shall show that $\underline{R\int_a^b} f = L\int_a^b f$ and $\overline{R\int_a^b} f = U\int_a^b f$.

For a partition P : $a = x_0 < x_1 < ... < x_n = b$ of $[a, b]$, we can associate a step function, g, to the lower Darboux sum $L(f, P)$, defined by

$$g(x) = m_i \text{ for } x \text{ in } (x_{i-1}, x_i),\ 1 \le i \le n,\ \ g(x_i) = f(x_i) \text{ for } 0 \le i \le n.$$

Then $\int_a^b g = L(f,P)$. It follows that

$$\{L(f,P) : P \text{ a partition of } [a,b]\} \subseteq \left\{\int_a^b \varphi : \varphi \le f, \varphi \in S^*([a,b])\right\}.$$

Therefore,

$$L\int_a^b f = \sup\{L(f,P) : P \text{ a partition of } [a,b]\} \le \underline{R\int_a^b} f = \sup\left\{\int_a^b \varphi : \varphi \le f, \varphi \in S^*([a,b])\right\}.$$

Next, we show that for any step function $s \le f$, $\int_a^b s \le L\int_a^b f$.

Suppose the step function $s \le f$ is given by the partition *P*: $x_0 = a < x_1 < x_2 < \cdots < x_n = b$ and a set of constants, $\xi_1, \xi_2, \cdots, \xi_n$ such that $s(x) = \xi_i$ for $x_{i-1} < x < x_i$ for $1 \le i \le n$. As $s \le f$, $\xi_i = s(x) \le f(x)$ for $x_{i-1} < x < x_i$ and so $\xi_i \le \inf\{f(x) : x \in (x_{i-1}, x_i)\}$.

Let $K = \min_{1\le i\le n} |x_i - x_{i-1}|$. Then there exists an integer *N* such that $k \ge N \Rightarrow \frac{1}{k} < \frac{K}{4}$.

Using $k \ge N$, we introduce more partition points into *P* to give Q_k :

$$x_0 = a < b_0 < a_1 < x_1 < b_1 < a_2 < x_2 < b_2 < \cdots < a_i < x_i < b_i < \cdots a_{n-1} < x_{n-1} < b_{n-1} < a_n < x_n = n,$$

where $b_i = x_i + \frac{1}{k}$ for $0 \le i \le n-1$ and $a_i = x_i - \frac{1}{k}$ for $1 \le i \le n$.

Note that $\inf\{f(x) \in [b_{i-1}, a_i]\} \ge \xi_i$ for $1 \le i \le n$. Note that $a_i - b_{i-1} = x_i - x_{i-1} - \frac{2}{k} > 0$ for $1 \le i \le n$.

$$L(f, Q_k)$$

$$= \sum_{i=1}^{n} \inf\{f(x) \in [b_{i-1}, a_i]\}(a_i - b_{i-1}) + \sum_{i=0}^{n-1} \inf\{f(x) \in [x_i, b_i]\}(b_i - x_i) + \sum_{i=1}^{n} \inf\{f(x) \in [a_i, x_i]\}(x_i - a_i)$$

$$\ge \sum_{i=1}^{n} \xi_i (x_i - x_{i-1} - \frac{2}{k}) + \sum_{i=0}^{n-1} \inf\{f(x) \in [x_i, b_i]\}\frac{1}{k} + \sum_{i=1}^{n} \inf\{f(x) \in [a_i, x_i]\}\frac{1}{k} .$$

Hence, $\lim_{k\to\infty} L(f, Q_k) \ge \sum_{i=1}^{n} \xi_i (x_i - x_{i-1}) = \int_a^b s$. It follows that

$$\int_a^b s \le \sup\{L(f,P) : P \text{ a partition of } [a,b]\} = L\int_a^b f .$$

Therefore, $R\underline{\int_a^b} f \le L\int_a^b f$ and so $R\underline{\int_a^b} f = L\int_a^b f$.

Similarly, we can show that $R\overline{\int_a^b} f = U\int_a^b f$.

We have thus proved:

Proposition 24. Suppose f: $[a, b] \to \mathbf{R}$ is a bounded function. Then the lower Darboux integral of f is equal to the lower Riemann integral of f, the upper Darboux integral of f is equal to the upper Riemann integral of f. The function f is Riemann integrable if, and only if, f is Darboux integrable.

The next result is a technical result that helps to find a sequence of step functions whose integrals converge to the lower Riemann integral and also a sequence of step functions whose integrals converge to the upper Riemann integral.

Lemma 25. The Refinement Lemma.
Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. Suppose Q and P are partitions of $[a, b]$ such that Q is a refinement of P, that is, every partition point of P is also a partition point of Q. Then

$$L(f, P) \le L(f, Q) \text{ and } U(f, Q) \le U(f, P).$$

Proof.
This is a well known result. We prove the result first when Q has just one additional point than P. Then proceed to the general case by induction.
Suppose Q contains just one additional point y than P. Let P be denoted by $P: \ a = x_0 < x_1 < \ldots < x_n = b$. Suppose $y \in (x_{j-1}, x_j)$ for some j between 1 and n. Then Q is the partition $Q: \ a = x_0 < x_1 < \ldots < x_{j-1} < y < x_j < \ldots < x_n = b$. Let $m_j' = \inf\{f(x) : x \in [x_{j-1}, y]\}$, $m_j'' = \inf\{f(x) : x \in [y, x_j]\}$. Then

$$m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\} \le m_j', m_j''.$$

Therefore,

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{j-1} m_i(x_i - x_{i-1}) + m_j(y - x_{j-1}) + m_j(x_j - y) + \sum_{i=j+1}^{n} m_i(x_i - x_{i-1})$$
$$\le \sum_{i=1}^{j-1} m_i(x_i - x_{i-1}) + m_j'(y - x_{j-1}) + m_j''(x_j - y) + \sum_{i=j+1}^{n} m_i(x_i - x_{i-1}) = L(f,Q).$$

Let $M_j' = \sup\{f(x) : x \in [x_{j-1}, y]\}$, $M_j'' = \sup\{f(x) : x \in [y, x_j]\}$. Then

$$M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\} \ge M_j', M_j''.$$

Therefore,

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$
$$= \sum_{i=1}^{j-1} M_i(x_i - x_{i-1}) + M_j(y - x_{j-1}) + M_j(x_j - y) + \sum_{i=j+1}^{n} M_i(x_i - x_{i-1})$$
$$\ge \sum_{i=1}^{j-1} M_i(x_i - x_{i-1}) + M_j'(y - x_{j-1}) + M_j''(x_j - y) + \sum_{i=j+1}^{n} M_i(x_i - x_{i-1}) = U(f,Q).$$

This proves the lemma for the case when Q has just one additional partition point than P.
For the general case, if Q contains k points not in P, then there is a sequence of partitions, $P = P_0, P_1, P_2, \ldots, P_k = Q$ where Q is obtained by adding one point at a time. That is P_{i+1}, is obtained by adding one point in Q not in P_i to P_i. Thus by the special case,

$$L(f, P) = L(f, P_0) \le L(f, P_1) \le L(f, P_2) \le \ldots \le (f, P_k) = L(f, Q)$$

and

$$U(f, P) = U(f, P_0) \ge U(f, P_1) \ge U(f, P_2) \ge \ldots \ge U(f, P_k) = U(f, Q).$$

This completes the proof.

Proposition 26. Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. Then there exists a sequence of partitions (P_k) of $[a, b]$ such that $P_k \subseteq P_{k+1}$, $\lim_{k\to\infty} \|P_k\| = 0$,

$$L(f,P_k)\to L\int_a^b f \text{ and } U(f,P_k)\to U\int_a^b f \ .$$

Proof. By definition of the lower and upper Datboux integral, there exist partitions P_1' and P_1'' of $[a, b]$ such that

$$L\int_a^b f-1< L(f,P_1')\le L\int_a^b f \quad\text{and}\quad U\int_a^b f\le U(f,P_1'')<U\int_a^b f+1.$$

Let P_1 be a common refinement of P_1' and P_1'' for which $\|P_1\| < 1$. Then by the Refinement Lemma (Lemma 25),

$$L\int_a^b f-1< L(f,P_1)\le L\int_a^b f \quad\text{and}\quad U\int_a^b f\le U(f,P_1)<U\int_a^b f+1.$$

Similarly, there exist partitions P_2' and P_2'' of $[a, b]$ such that

$$L\int_a^b f-\frac{1}{2}< L(f,P_2')\le L\int_a^b f \quad\text{and}\quad U\int_a^b f\le U(f,P_2'')<U\int_a^b f+\frac{1}{2} \ .$$

Let P_2 be a common refinement of P_1 , P_2' and P_2'' for which $\|P_2\| < 1/2$. Then

$$L\int_a^b f-\frac{1}{2}< L(f,P_2)\le L\int_a^b f \quad\text{and}\quad U\int_a^b f\le U(f,P_2)<U\int_a^b f+\frac{1}{2} \ .$$

We now define the sequence $\{P_k\}$ by repeating the above process. Suppose we have defined partition P_k such that $P_{k-1} \subseteq P_k$ and $\|P_k\| < 1/k$. By the definition of the lower and upper Darboux integrals, there exist partitions P_{k+1}' and P_{k+1}'' of $[a, b]$ such that

$$L\int_a^b f-\frac{1}{k+1}< L(f,P_{k+1}')\le L\int_a^b f \text{ and } U\int_a^b f\le U(f,P_{k+1}'')<U\int_a^b f+\frac{1}{k+1} \ .$$

Let P_{k+1} be a common refinement of P_k , P_{k+1}' and P_{k+1}'' for which $\|P_{k+1}\| < 1/(k+1)$. Then by the Refinement Lemma, we have

$$L\int_a^b f-\frac{1}{k+1}< L(f,P_{k+1})\le L\int_a^b f\ ,\quad U\int_a^b f\le U(f,P_{k+1})<U\int_a^b f+\frac{1}{k+1}$$

and that $P_k \subseteq P_{k+1}$.

In this way we obtain the sequence of partitions $\{P_k\}$ of $[a, b]$ such that $P_k \subseteq P_{k+1}$ and $\lim_{k\to\infty}\|P_k\|=0$. In particular, by the definition of convergence of sequence or by the Comparison Test, $L(f,P_k)\to L\int_a^b f$ and $U(f,P_k)\to U\int_a^b f$. This completes the proof.

Proof of Theorem 22.

Suppose f is Riemann integrable.

By Proposition 26, there exists a sequence of partitions (P_k) of [a, b] such that $P_k \subseteq P_{k+1}$, $\lim_{k\to\infty}\|P_k\| = 0$,

$$L(f,P_k) \to L\int_a^b f \text{ and } U(f,P_k) \to U\int_a^b f .$$

For each partition P_k : $a = x_0 < x_1 < ... < x_n = b$. We can associate a step function φ_k to the lower Darboux sum $L(f, P_k)$ and a step function ψ_k to the upper Darboux sum $U(f, P_k)$ defined by

$$\varphi_k(x) = m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} \text{ for } x \text{ in } (x_{i-1}, x_i),\ 1 \le i \le n,$$

$$\varphi_k(x_i) = f(x_i) \text{ for } 0 \le i \le n$$

and

$$\psi_k(x) = M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} \text{ for } x \text{ in } (x_{i-1}, x_i),\ 1 \le i \le n,$$

$$\varphi_k(x_i) = f(x_i) \text{ for } 0 \le i \le n\ \psi_k(x_i) = f(x_i).$$

Then $\int_a^b \varphi_k = L(f,P_k)$ and $\int_a^b \psi_k = U(f,P_k)$. It is easily seen that (φ_k) is a monotonic increasing sequence of step functions and (ψ_k) is a monotonic decreasing sequence.
Note that since φ_k and ψ_k are step functions, they are Lebesgue measurable and $\varphi_k \le f \le \psi_k$ for all integer $k \ge 1$. Since f is bounded, the sequence (φ_k) is uniformly bounded and so converges pointwise in [a, b] to a function φ on [a, b]. Similarly, (ψ_k) converges pointwise on [a, b] to a function ψ on [a, b]. By Corollary 14 of *Introduction To Measure Theory*, φ and ψ are Lebesgue measurable. Observe that $\phi \le f \le \psi$.
Now, f is Riemann integrable implies, by Theorem 21, that f is Lebesgue integrable and $\int_{[a,b]} f\, d\mu = R\int_a^b f$.

By Proposition 26,

$$\int_{[a,b]} \varphi_k d\mu = \int_a^b \varphi_k = L(f,P_k) \to L\int_d^b f \text{ and } \int_{[a,b]} \psi_k d\mu = \int_a^b \psi_k = U(f,P_k) \to U\int_d^b f .$$

By the Lebesgue Dominated Convergence Theorem, since both (φ_k) and (ψ_k) are bounded by constant function,

$$\int_{[a,b]} \varphi_k d\mu \to \int_{[a,b]} \varphi d\mu \text{ and } \int_{[a,b]} \psi_k d\mu \to \int_{[a,b]} \psi d\mu .$$

Therefore, $\int_{[a,b]} \varphi d\mu = L\int_d^b f = U\int_d^b f = \int_{[a,b]} \psi d\mu$. Hence, $\int_{[a,b]} (\psi - \varphi)\, d\mu = 0$. As $\psi - \varphi \ge 0$, by Proposition 38 part (1) of *Introduction to Measure Theory*, $\psi - \varphi = 0$ *a.e.* [μ] on [a, b].

Hence, there is a set $E \subseteq [a,b]$ such that $\mu(E) = 0$ and $\varphi(x) = \psi(x)$ for $x \in [a,b] - E$. Let L_k be the set of partition points of P_k for each integer $k \geq 1$. Then $L = \bigcup_{k=1}^{\infty} L_k$ is countable and so $\mu(L) = 0$. Let $H = E \cup L$ and $\mu(H) = 0$ since H is the union of two sets of Lebesgue measure zero. Now we claim that f is continuous at each point $x \in [a,b] - H$. Take $x_0 \in [a,b] - H$ and so $\varphi(x_0) = f(x_0) = \psi(x_0)$. As $\varphi_k(x_0) \nearrow \varphi(x_0)$, given $\varepsilon > 0$, there exists an integer $N \geq 1$ such that $k \geq N \Rightarrow \varphi(x_0) - \varepsilon < \varphi_k(x_0) \leq \varphi(x_0)$. Likewise, as $\psi_k(x_0) \searrow \psi(x_0)$, there exists an integer $M \geq 1$ such that $k \geq M \Rightarrow \psi(x_0) \leq \psi_k(x_0) < \psi(x_0) + \varepsilon$. Let $J \geq \max\{N, M\}$. Let L_J be the partition points of the partition P_J associated with the lower and upper sum $L(f, P_J)$ and $U(f, P_J)$. Hence $x_0 \notin L_J$ and so x_0 is in some open interval, say *I*, defined by the partition points of P_J. Thus, there exists a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq I$. Observe that $\varphi_J(x) = \varphi_J(x_0) \leq f(x_0) \leq \psi_J(x_0) = \psi_J(x)$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Therefore, for $x \in (x_0 - \delta, x_0 + \delta)$,

$$f(x_0) - \varepsilon = \varphi(x_0) - \varepsilon < \varphi_J(x_0) = \varphi_J(x) \leq f(x) \leq \psi_J(x) = \psi_J(x_0) < \psi(x_0) + \varepsilon = f(x_0) + \varepsilon .$$

It follows that $|f(x) - f(x_0)| < \varepsilon$ for $x \in (x_0 - \delta, x_0 + \delta)$. This means that the function f is continuous at $x_0 \in [a,b] - H$. Hence, f is continuous *a.e.* $[\mu]$ on $[a, b]$.

Suppose f is bounded and continuous *a.e.* $[\mu]$ on $[a, b]$. By Proposition 40 in *Introduction To Measure Theory*, f is Lebesgue measurable. By Theorem 19, f is Lebesgue integrable. The gist of the proof is to prove that both the lower and upper Darboux integrals of f are equal to the Lebesgue integral of f.

Recall that (φ_k) and (ψ_k) converge pointwise respectively to Lebesgue integrable functions φ and ψ. Moreover,

$$\int_{[a,b]} \varphi_k \, d\mu \nearrow \int_{[a,b]} \varphi \, d\mu = L\int_a^b f \text{ and } \int_{[a,b]} \psi_k \, d\mu \searrow \int_{[a,b]} \psi \, d\mu = U\int_a^b f \; .$$

We shall show that $\varphi = f = \psi$ *a.e.* $[\mu]$ on $[a, b]$.

Let $F \subseteq [a,b]$ be such that $\mu(F) = 0$ and f is continuous at x for all x in $[a,b] - F$.

Let $G = F \cup \bigcup_{k=1}^{\infty} L_k$. We shall show that $\varphi = f = \psi$ on $[a,b] - G$.

Take $x \in [a,b] - G$. Then x is not a partition point of any P_k and f is continuous at x. Therefore, given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all y in $[a, b]$,

$$|y-x|<\delta \Rightarrow |f(y)-f(x)|<\frac{\varepsilon}{2} .$$

By definition of the partitions $\{P_i\}$, $\|P_i\| \to 0$. Therefore, there exists an integer N such that $k \geq N \Rightarrow \|P_k\| < \delta$. Take any integer $k \geq N$. Suppose the partition P_k is given by $a = x_0 < x_1 < \ldots < x_n = b$. Then $x \in (x_{i-1}, x_i)$ for some $1 \leq i \leq n$. Since $|x_i - x_{i-1}| \leq \|P_k\| < \delta$,

$$|f(y)-f(x)|<\frac{\varepsilon}{2} \text{ for all } y \in [x_{i-1}, x_i]. \text{ ------------------------ (1)}$$

Now, $\psi_k(x) = \sup\{f(y) : y \in [x_{i-1}, x_i]\}$. Therefore, there exists $y_0 \in [x_{i-1}, x_i]$ such that

$$\psi_k(x) - \frac{\varepsilon}{2} < f(y_0) \leq \psi_k(x).$$

It follows that

$$|\psi_k(x) - f(y_0)| = \psi_k(x) - f(y_0) < \frac{\varepsilon}{2} . \text{ ------------------------- (2)}$$

Therefore, with this value of y_0, it follows from (1) and (2) that

$$|\psi_k(x) - f(x)| \leq |\psi_k(x) - f(y_0)| + |f(y_0) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Since this is true for any $k \geq N$, $|\psi(x) - f(x)| = \lim_{k\to\infty} |\psi_k(x) - f(x)| \leq \varepsilon$. As this is true for all $\varepsilon > 0$, $\psi(x) = f(x)$. Hence, $\psi = f$ on $[a,b] - G$.

We have $\varphi_k(x) = \inf\{f(y) : y \in [x_{i-1}, x_i]\}$. Therefore, there exists $y_0 \in [x_{i-1}, x_i]$ such that

$$\varphi_k(x) \leq f(y_0) < \varphi_k(x) + \frac{\varepsilon}{2}.$$

It follows that

$$|\varphi_k(x) - f(y_0)| = f(y_0) - \varphi_k(x) < \frac{\varepsilon}{2} . \text{ ------------------------- (3)}$$

Therefore, with this value of y_0, it follows from (1) and (3) that

$$|\varphi_k(x) - f(x)| \leq |\varphi_k(x) - f(y_0)| + |f(y_0) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Hence, taking limit, we get $|\varphi(x) - f(x)| \leq \varepsilon$. As ε is arbitrary, $\varphi(x) = f(x)$. Therefore, $\varphi = f$ on $[a,b] - G$.

Therefore,

$$L\int_a^b f = \int_{[a,b]} \varphi d\mu = \int_{[a,b]} f\, d\mu \text{ and } U\int_a^b f = \int_{[a,b]} \psi d\mu = \int_{[a,b]} f\, d\mu .$$

It follows that $L\int_a^b f = \int_{[a,b]} f\, d\mu = U\int_a^b f$ and consequently f is Riemann integrable.

Non Lebesgue measurable set

We have stated that the Lebesgue outer measure is *not* countably additive on the collection of all subsets of $\mathbb{R}$. This presupposes that a non-Lebesgue measurable subset exist. The restriction of the Lebesgue outer measure on the σ-algebra of all Lebesgue measurable subsets is countably additive. If every subset is Lebesgue measurable, then the Lebesgue outer measure would be countably additive. The question is: Does non-Lebesgue measurable set exist?

The answer depends on our system of set theory. If we admit the Axiom of Choice, then it does. If we don't admit the Axiom of Choice, then every set is Lebesgue measurable, that is, if we replace the Axiom of Choice by Solovay Axiom (Axiom of Dependent Choice and every subset of $\mathbb{R}$ is measurable). The two systems of axioms for set theory (Zermelo-Fraenkel plus Axiom of Choice or Zermelo-Fraenkel plus Axiom of Solovay) are mutually incompatible although they are both consistent. The following is thus of interest to those ardent supporters of the Zermelo-Fraenkel plus Axiom of Choice.

We shall use the Axiom of Choice to define a non-Lebesgue measurable subset of [0,1]. Indeed, we shall also define a non-Lebesgue measurable subset of $\mathbb{R}$ and use this subset to show the existence of a non-Lebesgue measurable subset of any set with positive Lebesgue outer measure.

Define an equivalence relation R on [0, 1] by $x\, R\, y$ if and only if $x - y$ is a rational number. This then partitions [0, 1] into disjoint equivalence classes. By the Axiom of Choice, we can choose a point from each of these equivalent classes to form a subset E of [0, 1]. That is, E intersects each equivalence class in exactly one point. Then E is not Lebesgue measurable. We shall prove this by contradiction.
Suppose E is Lebesgue measurable.
To see this, consider the set $[0, 1] \cup -[0, 1] = [-1, 1]$. The set of rational numbers in [−1, 1] is countable. Let $\{a_n : n = 1, \ldots\}$ be an enumeration of the set of rational numbers in [−1, 1]. Then for each integer $n \geq 1$, $E_n = \{a_n + x : x \in E\} = E + a_n$ is Lebesgue measurable if E is. Obviously, $E_n \cap E_m = \varnothing$, if $n \neq m$.
We deduce this as follows.
If $x \in E_n \cap E_m$, then $x = z + a_n$ for some z in E and $x = z' + a_m$ for some z' in E .
Therefore, $0 = z - z' - (a_m - a_n)$ and so $z - z' = (a_m - a_n)$ and so $z\, R\, z'$. But if $z \neq z'$, then

z and z' would come from different equivalence classes and so $z\,R\,z'$ cannot hold. Thus $z = z'$ and so $a_m = a_n$, contradicting $a_m \neq a_n$.
For x in [0, 1] either x is in E or $x\,R\,y$ for some y in E. Note that $E = E_\alpha = E + a_\alpha$, when $a_\alpha = 0$. If $x\,R\,y$ for some y in E, then $x - y$ is a rational number in $[-1, 1]$. Thus $x - y = a_j$ for some j. Hence, $x = y + a_j \in E + \{a_j\} = E_j$. We have thus shown that $[0, 1] \subseteq \bigcup_{i=1}^{\infty} E_i$.

Note that each E_n is a subset of $[-1, 2]$ and so $\bigcup_{i=1}^{\infty} E_i \subseteq [-1, 2]$.
Now, for each i, $\mu(E_i) = \mu(E + a_i) = \mu(E)$ as μ is translation invariant. Since each E_i is measurable and $\{E_i\}_{i=1}^{\infty}$ are pairwise disjoint, by the countable additivity of μ,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) = \lim_{n\to\infty} n\mu(E) \leq \mu([-1,2]) = 3.$$

This is only possible if $\mu(E) = 0$. Therefore, $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = 0$. But then we have, because $[0, 1] \subseteq \bigcup_{i=1}^{\infty} E_i$, $1 = \mu[0,1] \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = 0$, which is absurd. Therefore, E is not Lebesgue measurable.

Note that $1 = \mu^*([0,1]) \leq \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \mu^*([-1,2]) = 3$.

If $\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu^*(E_i) = \lim_{n\to\infty} n\mu^*(E)$, then it is only possible if $\mu^*(E) = 0$ and so by Proposition 16, E is Lebesgue measurable and we get a contradiction as before. This means $\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) < \sum_{i=1}^{\infty} \mu^*(E_i)$, $\mu^*(E) > 0$ and $1 \leq \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) < \mu^*([-1,2]) = 3$. Hence, we can conclude that the Lebesgue outer measure is not countably additive. It also follows that there exists an integer n such that $\mu^*\left(\bigcup_{i=1}^{n} E_i\right) < \sum_{i=1}^{n} \mu^*(E_i) = n\mu^*(E)$. This means that the Lebesgue outer measure is also not finitely additive.

The following is a general way of getting a non-measurable set.
The device we would be using is the algebraic difference of two sets. This time we shall obtain a non-measurable subset of $\mathbb{R}$. Define the same relation *as before but now denote by* $\sim$, on $\mathbb{R}$ as follows. $x \sim y$ if, and only if, $x - y$ is a rational number. Plainly, this is an equivalence relation on $\mathbb{R}$. Denote the set of equivalence classes by $\mathbb{R}/\sim$. Each equivalence class has the form

$$\{x + r : r \in \mathbb{Q}\}.$$

Thus, the collection of rational numbers constitutes one equivalence class, $\left\{\sqrt{2}+r : r \in \mathbb{Q}\right\}$ is another equivalence class and $\{\pi + r : r \in \mathbb{Q}\}$ is yet another. Obviously, each equivalence class is countable and so since the set of real numbers $\mathbb{R}$ is uncountable, the number of distinct equivalence classes is uncountable. This is because if the number of equivalence classes were countable then $\mathbb{R}$ being the union of countable number of equivalence classes, each of which is countable, would be countable and thus contradicts the fact that $\mathbb{R}$ is uncountable. By the Axiom of Choice, we can choose a point from each equivalence class to form an uncountable set F. We claim that this set is non-measurable. This is because the set of algebraic difference

$$F - F = \{ x - y : x, y \in F \}$$

cannot contain an interval. Because any two distinct points of F must differ by an irrational number and since F contains only one rational number, $F - F$ contains exactly one rational number namely 0. If $F - F$ were to contain an interval, it would contain rational number different from zero which is not possible. Hence by the following lemma, either F is not Lebesgue measurable *or* $\mu^*(F) = 0$ and F is Lebesgue measurable.

Lemma 27. If E is a Lebesgue measurable subset of $\mathbb{R}$ with positive measure, i.e., $\mu(E) > 0$, then $E - E$ contains a non-trivial interval centred at the origin.

We shall prove this lemma later. Enumerate the set of rational numbers as $\{a_n : n = 1, \ldots\}$. Now define $F_n = \{a_n + x : x \in F\} = F + a_n$. Then by the definition of F, we have

$$\mathbb{R} = \bigcup_{n=1}^{\infty} F_n \text{ . ---------------- (A)}$$

Observe that by the definition of F, $F_n \cap F_m = \varnothing$ for $n \neq m$ so that $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$ is a countable disjoint union.

If $\mu^*(F) = 0$, then F is Lebesgue measurable by Proposition 16 and since μ^* is translation invariant, $\mu^*(F_n) = \mu^*(F + a_n) = \mu^*(F) = 0$. Thus, by the countable sub-additivity of the Lebesgue outer measure, $\mu^*(\mathbb{R}) = \mu^*\left(\bigcup_{n=1}^{\infty} F_n\right) \leq \sum_{n=1}^{\infty} \mu^*(F_n) = 0$ implying that $\mu^*(\mathbb{R}) = 0$, which is not true. Hence $\mu^*(F) \neq 0$. Therefore, by Lemma 27, F is not Lebesgue measurable as $F - F$ does not contain *an interval*.

We have thus produce two non-measurable subsets, one in [0, 1] and one in $\mathbb{R}$. We shall make use of F to produce other non-measurable set.

Now for the proof of Lemma 27.

Proof of Lemma 27.

Suppose E is a Lebesgue measurable subset of $\mathbb{R}$ with positive measure, i.e., $\mu(E) > 0$, Firstly, we take a special open set G containing E such that

$$\mu(G) < (1 + \varepsilon)\, \mu(E).$$

How can we obtain G? Recall that $\mu(E) = \mu^*(E) = \inf\{\lambda^*(\gamma) : \gamma \in C(E)\}$.

Now for any $\varepsilon > 0$, $(1 + \varepsilon)\, \mu(E) > \mu(E)$. Therefore, by the definition of infimum, there exists a countable cover γ of E, i.e., $\gamma \in C(E)$, by open intervals such that

$$(1 + \varepsilon)\, \mu(E) > \lambda^*(\gamma) \geq \mu(E).$$

Let $G = \bigcup_{I \in \gamma} I$. Then G is an open set in $\mathbb{R}$ and so is Lebesgue measurable. Since $\mathbb{R}$ is locally path connected, any open set of $\mathbb{R}$ is also locally path connected and so G is locally path connected. Hence, the path components are open sets and therefore, are open intervals. As $\mathbb{R}$ has a countable basis for its topology, these components are disjoint open cover for G and can at most be countable. Thus, we conclude that G is a countable union of pairwise disjoint open intervals. Let this collection of pairwise disjoint open intervals be denoted by β. Then $G = \bigcup_{I \in \beta} I = \bigcup_{I \in \gamma} I \supseteq E$. By the countable additivity of Lebesgue measure, μ,

$$\mu^*(G) = \mu(G) = \mu\left(\bigcup_{I \in \beta} I\right) = \sum_{I \in \beta} \mu(I) = \lambda^*(\beta)$$

$$= \mu^*\left(\bigcup_{I \in \gamma} I\right) \leq \sum_{I \in \gamma} \mu^*(I) = \lambda^*(\gamma) < (1+\varepsilon)\mu(E),$$

by the countable sub-additivity of μ^*.

Thus, we have

$$\mu(G) = \lambda^*(\beta) < (1+\varepsilon)\mu(E).$$

Let $\{I_n : n = 1, \ldots\}$ be an enumeration of the open intervals in β. Then $G = \bigcup_{I \in \beta} I = \bigcup_{n=1}^{\infty} I_n$. Let $E_n = E \cap I_n$. We have,

$$E = E \cap G = E \cap \left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} (E \cap I_n) = \bigcup_{n=1}^{\infty} E_n.$$

Since I_n is Lebesgue measurable and E is Lebesgue measurable, E_n is Lebesgue measurable for integer $n \geq 1$. As $\beta = \{I_n : n = 1, \ldots\}$ is a countable collection of pairwise disjoint sets, $\{E_n : n = 1, \ldots\}$ is a countable collection of pairwise disjoint measurable subsets. Therefore, by the countable additivity of the Lebesgue measure μ,

$$\mu(E) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Note that $\mu(G) = \mu\left(\bigcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} \mu(I_n)$. Since $\mu(G) < (1+\varepsilon)\mu(E)$, we must have that for some integer $j \geq 1$,

$$\mu(I_j) < (1+\varepsilon)\mu(E_j). \qquad \text{-------------------} \quad (1)$$

Let $I = I_j$ and $J = E_j$. Then $J = E_j = E \cap I_j \subseteq I_j = I$.

Take $\varepsilon = 1/3$. Then by (1), $\mu(I) < (4/3)\, \mu(J)$. That is,

$$\mu(J) > (3/4)\mu(I). \qquad \text{-------------------------} \quad (2)$$

We now claim that if J is translated by any number d with $|d| < (1/2)\mu(I)$, the translated set J_d has some points in common with J. If this is not the case, i.e., $J \cap J_d = \varnothing$, then since $J \cup J_d \subseteq I \cup I_d$,

$$2\mu(J) = \mu(J) + \mu(J_d) = \mu(J \cup J_d) \le \mu(I \cup I_d) \le \mu(I) + |d| < \mu(I) + (1/2)\mu(I) = (3/2)\mu(I).$$

We would get $\mu(J) < (3/4)\mu(I)$ contradicting (2).
Hence, $J \cap J_d \neq \varnothing$,
This means that for some $y = x + d$ in J_d, where x is in J, y is also in J. Therefore, $d = y - x$ is in $J - J$. This is true for any d with $|d| < (1/2)\mu(I)$ and so, the open interval

$$(-(1/2)\mu(I), (1/2)\mu(I)) \subseteq J - J \subseteq E - E.$$

This completes the proof of lemma 27.

Theorem 28. For any subset A of $\mathbb{R}$ with positive outer measure, i.e., $\mu^*(A) > 0$, there is a non-measurable subset $B \subseteq A$.
Proof. Suppose $\mu^*(A) > 0$. Take F_n as defined before using the non-Lebesgue measurable subset F of $\mathbb{R}$ By (A), $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$ and so

$$A = A \cap \mathbb{R} = A \cap \left(\bigcup_{n=1}^{\infty} F_n \right) = \bigcup_{n=1}^{\infty} \left(A \cap F_n \right).$$

Therefore,

$$\mu^*(A) = \mu^*\left(\bigcup_{n=1}^{\infty} \left(A \cap F_n \right) \right) \le \sum_{n=1}^{\infty} \mu^*\left(A \cap F_n \right). \qquad \text{----------------} \quad (1)$$

If $A \cap F_n$ is measurable, then since $A \cap F_n - A \cap F_n$ does not contain a non-trivial interval (because $F_n - F_n$ dose not contain a non-trivial interval, a consequence of the fact that $F - F$ dose not contain a non-trivial interval), by Lemma 27, $\mu(A \cap F_n) = 0$. Therefore, since $\mu^*(A) > 0$, $A \cap F_n$ cannot be measurable for all integer n. This is because if $A \cap F_n$ were measurable for all integer n, then by (1), $\mu^*(A) \le 0$ contradicting $\mu^*(A) > 0$. Hence, for some integer j, $A \cap F_j$ is not Lebesgue measurable. Take $B = A \cap F_j$.
This completes the proof of Theorem 28.

With this theorem proven, we conclude this modest introduction to the Lebesgue measure on the real numbers $\mathbb{R}$.