Evaluating $\int_0^\infty \frac{x}{1-e^x}\,dx$ without using power series

By Ng Tze Beng

Using power series, we can show that $\Psi(s)=\int_0^\infty \frac{x^s}{e^x-1}dx=\zeta(s+1)\Gamma(s+1)$ for $s > 0$. Thus,

$\int_0^\infty \frac{x}{e^x-1}dx=\zeta(2)\Gamma(2)$ and as $\Gamma(2)=\int_0^\infty ye^{-y}dy=\left[-e^{-y}y\right]_0^\infty+\int_0^\infty e^{-y}dy=0+\left[-e^{-y}\right]_0^\infty=1$,

$\int_0^\infty \frac{x}{e^x-1}dx=\zeta(2)=\frac{\pi^2}{6}$, as it is known that $\zeta(2)=\frac{\pi^2}{6}$. Hence, $\int_0^\infty \frac{x}{1-e^x}dx=-\frac{\pi^2}{6}$.

However, if we do not want to use power series and not using the known value for $\zeta(2)$, we can proceed as follows.

Using a change of variable $u=e^{-x}$, we get

$$\int_0^\infty \frac{x}{1-e^x}dx=\int_0^\infty \frac{xe^{-x}}{e^{-x}-1}dx=\int_1^0 \frac{\ln(u)}{u-1}du=\int_0^1 \frac{\ln(u)}{1-u}du\,.$$

Now we express

$$\int_0^1 \frac{\ln(x)}{1-x}dx=\frac{4}{3}\int_0^1 \frac{\ln(x)}{1-x}dx-\frac{1}{3}\int_0^1 \frac{\ln(x)}{1-x}dx\ . \text{ ------------------------ (1)}$$

We can also express with a change of variable, $x=u^2$,

$$\int_0^1 \frac{\ln(x)}{1-x}dx=\int_0^1 \frac{2\ln(u)}{1-u^2}2udu=4\int_0^1 \frac{u\ln(u)}{1-u^2}du=4\int_0^1 \frac{x\ln(x)}{1-x^2}du\,. \text{ -----------(2)}$$

Hence, from (1),

$$\int_0^1 \frac{\ln(x)}{1-x}dx=\frac{4}{3}\int_0^1 \frac{\ln(x)}{1-x}dx-\frac{4}{3}\int_0^1 \frac{x\ln(x)}{1-x^2}dx=\frac{4}{3}\int_0^1 \frac{\ln(x)}{1-x^2}dx\,. \text{ -------------- (3)}$$

With a change of variable $u=\frac{1}{x}$, we get

$$\int_1^\infty \frac{\ln(x)}{1-x^2}dx=\int_1^0 \frac{-\ln(u)}{1-\frac{1}{u^2}}(-\frac{1}{u^2})du=\int_1^0 \frac{\ln(u)}{u^2-1}du=\int_0^1 \frac{\ln(u)}{1-u^2}du=\int_0^1 \frac{\ln(x)}{1-x^2}dx\,.$$

Therefore, $\int_0^1 \frac{\ln(x)}{1-x^2}dx=\frac{1}{2}\int_0^\infty \frac{\ln(x)}{1-x^2}dx$.

It then follows from (3) that

$$\int_0^1 \frac{\ln(x)}{1-x}dx=\frac{4}{3}\int_0^1 \frac{\ln(x)}{1-x^2}dx=\frac{2}{3}\int_0^\infty \frac{\ln(x)}{1-x^2}dx\,. \text{ ----------------- (4)}$$

We are going to evaluate $\int_0^\infty \frac{\ln(x)}{1-x^2}dx$ by using differentiation under the integral sign.

Let $I(t)=\frac{1}{2}\int_0^\infty \frac{\ln(1-t^2+t^2x^2)}{1-x^2}\,dx$. Note that $I(0)=0$ and $I(1)=\int_0^\infty \frac{\ln(x)}{1-x^2}\,dx$. We shall show later that that $\lim_{t\to 0^+} I(t)=0$ and $\lim_{t\to 1^-} I(t)=\int_0^\infty \frac{\ln(x)}{1-x^2}\,dx$.

Let $f(x,t)=\frac{1}{2}\frac{\ln(1-t^2+t^2x^2)}{1-x^2}$ for $(x,t)\in(0,\infty)\times[a,b]$, where $0<a<b<1$ Then

$$\frac{\partial f}{\partial t}(x,t)=\frac{1}{2}\left(\frac{2tx^2-2t}{1-t^2+t^2x^2}\right)\frac{1}{(1-x^2)}=-\frac{t}{1-t^2+t^2x^2} \text{ for } (x,t)\in(0,\infty)\times[a,b].$$

Now, $\left|\frac{\partial f}{\partial t}(x,t)\right|=\left|\frac{t}{1-t^2+t^2x^2}\right|\le\frac{b}{1-b^2+a^2x^2}$ for $(x,t)\in(0,\infty)\times[a,b]$ and $\frac{b}{1-b^2+a^2x^2}$ is Lebesgue integrable on $(0,\infty)$ and so by Theorem 59 part (ii) of Chapter 14, *Mathematical Analysis, An Introduction* in My Calculus Web, we can differentiate under the integral sign. That is, for $t\in[a,b]$ with $0<a<b<1$,

$$I'(t)=-\int_0^\infty \frac{t}{1-t^2+t^2x^2}\,dx=-\int_0^\infty \frac{\frac{1}{t}}{\frac{1}{t^2}-1+x^2}\,dx=-\frac{1}{t}\left[\frac{1}{\sqrt{\frac{1}{t^2}-1}}\tan^{-1}\left(\frac{x}{\sqrt{\frac{1}{t^2}-1}}\right)\right]_0^\infty=-\frac{1}{\sqrt{1-t^2}}\frac{\pi}{2}.$$

Since *a* and *b* are arbitrarily chosen, we have that

$$I'(t)=-\frac{1}{\sqrt{1-t^2}}\frac{\pi}{2} \text{ for } t\in(0,1).$$

Therefore, $I(t)=-\frac{\pi}{2}\sin^{-1}(t)+C$ for $t\in(0,1)$.

Note that $f(x,t)=\frac{1}{2}\frac{\ln(1-t^2+t^2x^2)}{1-x^2}\le 0$ for $0\le t\le 1$ and $0\le x<\infty$ with $x\ne 1$. This is because $1-t^2+t^2x^2>1$ if $x>1$ and $1-t^2+t^2x^2<1$ if $0\le x<1$ so that $\ln(1-t^2+t^2x^2)\ge 0$ if $x>1$ and $\ln(1-t^2+t^2x^2)\le 0$ if $0\le x<1$. Therefore,

$-f(x,t)=-\frac{1}{2}\frac{\ln(1-t^2+t^2x^2)}{1-x^2}\ge 0$. Since $-\frac{\partial f}{\partial t}(x,t)=\frac{t}{1-t^2+t^2x^2}>0$ for $0<t\le 1$ and $0\le x<\infty$ with $x\ne 1$, by the Lebesgue Monotone Convergence Theorem,

$-\lim_{t\to 0^+} I(t)=\frac{1}{2}\lim_{t\to 0^+}\int_0^\infty\left(-\frac{\ln(1-t^2+t^2x^2)}{1-x^2}\right)dx=0$ since $\frac{\ln(1-t^2+t^2x^2)}{1-x^2}$ converges pointwise to the zero function as *t* tends to 0 on the right. Also, we have that

$-\lim_{t\to 1^-} I(t)=\frac{1}{2}\lim_{t\to 1^-}\int_0^\infty\left(-\frac{\ln(1-t^2+t^2x^2)}{1-x^2}\right)dx=-\frac{1}{2}\int_0^\infty \frac{\ln(x^2)}{1-x^2}\,dx=-\int_0^\infty \frac{\ln(x)}{1-x^2}\,dx$ since

$\frac{\ln(1-t^2+t^2x^2)}{1-x^2}$ tends to $\frac{\ln(x^2)}{1-x^2}$ as t tends to 1 on the left. It follows that $\lim_{t\to 0^+} I(t)=0$ and $\lim_{t\to 1^-} I(t)=\int_0^\infty \frac{\ln(x)}{1-x^2}\,dx$.

Thus, $0=\lim_{t\to 0^+} I(t)=-\lim_{t\to 0^+}\left(\frac{\pi}{2}\sin^{-1}(t)\right)+C=C$. Hence, $C=0$.

Therefore, $\int_0^\infty \frac{\ln(x)}{1-x^2}\,dx=\lim_{t\to 1^-} I(t)=-\lim_{t\to 1^-}\left(\frac{\pi}{2}\sin^{-1}(t)\right)=-\frac{\pi^2}{4}$, and so,

$$\int_0^1 \frac{\ln(x)}{1-x}\,dx=\frac{2}{3}\int_0^\infty \frac{\ln(x)}{1-x^2}\,dx=-\frac{\pi^2}{6}.$$