The Integrals $\int_0^\infty \frac{\ln(x)}{1+e^x}dx$, $\int_0^\infty \frac{(\ln(x))^2}{1+e^x}dx$, $\int_0^\infty \frac{x\ln(x)}{1+e^x}dx$, $\int_0^\infty \frac{x(\ln(x))^2}{1+e^x}dx$ $\int_0^\infty \frac{x\ln(x)}{e^x-1}dx$, $\int_0^\infty \frac{x}{(1+e^x)^2}dx$, $\int_0^\infty \frac{x^2}{(1+e^x)^2}dx$, $\int_0^\infty \frac{\ln(x)}{(1+e^x)^2}dx$, $\int_0^\infty \frac{x\ln(x)}{(1+e^x)^2}dx$, $\int_0^\infty \frac{x(\ln(x))^2}{(1+e^x)^2}dx$, $\int_0^\infty \frac{x}{(1+e^x)^3}dx$, $\int_0^\infty \frac{\ln(x)}{(1+e^x)^3}dx$ and $\int_0^\infty \frac{x\ln(x)}{(1+e^x)^3}dx$, The Gamma Function and The Eta function

By Ng Tze Beng

Consider the improper integral $\int_0^\infty \frac{x^s}{1+e^x}dx$ for real number $s > -1$. We shall show that this integral is convergent, that is, it has a finite value. Hence, we can define a function $\Lambda(s)$ by $\Lambda(s)=\int_0^\infty \frac{x^s}{1+e^x}dx$ on the interval $(-1,\infty)$. We shall show that, as a real valued function, $\Lambda(s)$ is infinitely differentiable on $(-1,\infty)$ and that it can be differentiated repeatedly by differentiating under the integral sign to give, first of all, the first derivative, $\Lambda'(s)=\int_0^\infty \frac{x^s\ln(x)}{1+e^x}dx$. It follows that the integral $\int_0^\infty \frac{x\ln(x)}{1+e^x}dx$ is the derivative of $\Lambda(s)$ at $s = 1$ and the integral $\int_0^\infty \frac{(\ln(x))^2}{1+e^x}dx$ is the second derivative of $\Lambda(s)$ at $s = 0$. We shall prove a relation relating the function $\Lambda(s)=\int_0^\infty \frac{x^s}{1+e^x}dx$ with the Gamma and Eta function, where the Gamma function $\Gamma(s)$ and the Eta function $\eta(s)$ are defined respectively by $\Gamma(s)=\int_0^\infty e^{-x}x^{s-1}dx$ and $\eta(s)=\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^s}$. More precisely, $\Lambda(s)$ is the product of Gamma and Eta functions. Thus, differentiating the product gives the integral $\int_0^\infty \frac{x^s\ln(x)}{1+e^x}dx=\Lambda'(s)$ in terms of the Gamma and Eta functions and their derivatives. We may evaluate this integral by evaluating the Gamma and Eta functions and their derivatives. We shall show that the Gamma function is well defined, i.e., for each $s > 0$, $\Gamma(s)$ is a convergent improper integral and that it is infinitely differentiable on positive reals by repeatedly differentiating under the integral sign. We prove that $\eta(s)=\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^s}$ for each $s > 0$ is a convergent infinite series and that as a real valued function on $s > 0$, $\eta(s)=\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^s}$ is infinitely differentiable and differentiable repeatedly by term- by-term differentiation.

However, it is not easy to evaluate the Gamma function, the Eta function or the Zeta function, or their derivatives numerically. There are formulas relating the higher order derivatives of Gamma at 1 up to the tenth order in terms of the Riemann zeta function and the Euler constant. Relatively less known are the higher order derivatives of the Zeta function at positive integer greater or equal than 2. Some of the integrals here do evaluate in terms of the first derivative of the Zeta function at the integer 2 and/or the Euler Mascheroni constant γ_0 and γ_1. For some even more complicated integrals, we may get the evaluation in terms of values of Gama, Zeta or Eta functions. Since these constants are not easily computable, it is debatable whether numerical method of evaluation of these complicated integrals with prescribed accuracy is preferred in practice than a formula. A case in point is the evaluation of the Gamma function and its derivatives.

In section 1, we give the pertinent definitions and properties of the Gamma function, the eta function and evaluation of the necessary Gamma and Eta functions and their derivatives at the points $s = 1$ and 2.

1. Definitions and Properties

Definition 1. For each real number $s > 0$, define the *Gamma function* by $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$.

We shall prove that the integral $\int_0^\infty e^{-x} x^{s-1} dx$ is convergent in two parts. We prove that $\int_0^1 e^{-x} x^{s-1} dx$ is convergent if, and only if, $s > 0$ and that $\int_1^\infty e^{-x} x^{s-1} dx$ is convergent for $s > 0$.

Part 1. $\int_0^1 e^{-x} x^{s-1} dx$ is convergent if, and only if $s > 0$.

For $0 < x \le 1$, $e^{-x} x^{s-1} \le x^{s-1}$ for $s > 0$.

Now, for any $0 < \eta < 1$, $\int_\eta^1 e^{-x} x^{s-1} dx \le \int_\eta^1 x^{s-1} dx = \left[\frac{x^s}{s} \right]_\eta^1 = \frac{1-\eta^s}{s} \le \frac{1}{s}$ for $s > 0$.

It follows that $\int_0^1 e^{-x} x^{s-1} dx = \lim_{\eta \to 0^+} \int_\eta^1 e^{-x} x^{s-1} dx \le \frac{1}{s}$ for $s > 0$. That is, $\int_0^1 e^{-x} x^{s-1} dx$ is convergent if $s > 0$.

However, for $0 < x \le 1$, if $s \le 0$, $e^{-x} x^{s-1} \ge \frac{1}{e} \frac{1}{x^{1-s}}$. Since $\int_0^1 \frac{1}{e} \frac{1}{x^{1-s}} dx = \infty$ for $s \le 0$, $\int_0^1 e^{-x} x^{s-1} dx = \infty$ for $s \le 0$. Hence, $\int_0^1 e^{-x} x^{s-1} dx$ is divergent if $s \le 0$.

Part 2. $\int_1^\infty e^{-x} x^{s-1} dx$ is convergent for $s > 0$.

For $s > 1$, $\lim_{x \to \infty} e^{-x/2} x^{s-1} = \lim_{x \to \infty} \frac{x^{s-1}}{e^{x/2}} = \lim_{x \to \infty} \frac{2(s-1) x^{s-2}}{e^{x/2}} = 0$ by repeated use of L'Hôpital's Rule.

For $s \le 1$, $\lim_{x \to \infty} e^{-x/2} x^{s-1} \le \lim_{x \to \infty} e^{-x/2} = 0$. Thus, for $s \ge 0$, $\lim_{x \to \infty} e^{-x/2} x^{s-1} = 0$. Since $e^{-x/2} x^{s-1}$ is continuous on $[1, \infty)$ for each s in $\mathbb{R}$, $e^{-x/2} x^{s-1}$ is bounded above by some K in $[1, \infty)$, that is to say, for $x \ge 1$, $e^{-x} x^{s-1} \le K e^{-x/2}$. Therefore, $\int_1^\infty e^{-x} x^{s-1} dx \le K \int_1^\infty e^{-x/2} dx = K \left[-2e^{-x/2} \right]_1^\infty = 2K \frac{1}{e^{1/2}}$. It follows that $\int_1^\infty e^{-x} x^{s-1} dx$ is convergent for $s \ge 0$.

It follows from part 1 and part 2 that $\int_0^\infty e^{-x}x^{s-1}dx$ is convergent if, and only if, $s > 0$. Hence, since the function is non-negative, it follows that $\int_0^\infty e^{-x}x^{s-1}dx$ is convergent both as improper Riemann integral as well as Lebesgue integral. That is to say, $\Gamma(s)=\int_0^\infty e^{-x}x^{s-1}dx$ is defined for $s > 0$.

Definition 2. For any real numbers $s > -1$, let $\Lambda(s)=\int_0^\infty \frac{x^s}{1+e^x}dx$.

We shall show that $\Lambda(s)$ is well defined, i.e., $\int_0^\infty \frac{x^s}{1+e^x}dx$ is convergent.

Note that for $s > -1$, $\frac{x^s}{1+e^x}=\frac{x^s e^{-x}}{1+e^{-x}}\le x^s e^{-x}$ for all $x \ge 0$. Therefore, as $\int_0^\infty \frac{x^s}{1+e^x}dx \le \int_0^\infty x^s e^{-x}dx=\Gamma(s+1)$ and so $\int_0^\infty \frac{x^s}{1+e^x}dx$ is convergent since $s + 1 > 0$ and $\Gamma(s+1)$ is finite.

Theorem 3. The Gamma function $\Gamma(s)$ is infinitely differentiable for $s > 0$ and $\Gamma^{(n)}(s)=\int_0^\infty e^{-x}x^{s-1}\left(\ln(x)\right)^n dx$ for integer $n \ge 1$.

Proof. Let $f:(0,\infty)\times(0,\infty)\to\mathbb{R}$ be defined by $f(x,s)=x^{s-1}e^{-x}$. Then $\Gamma(s)=\int_0^\infty f(x,s)dx$. Note that $\lim_{x\to 0^+} f(x,s)=\lim_{x\to 0^+} x^{s-1}e^{-x}=0$ for $s > 1$.

Now, $\frac{\partial f}{\partial s}f(x,s)=x^{s-1}e^{-x}\ln(x)$ for (x, s) in $(0,\infty)\times(0,\infty)$ and is continuous on $(0,\infty)\times(0,\infty)$.

Restrict the domain f to $(0,\infty)\times[a,b]$ with $0 < a < b$. We shall show that $\Gamma(s)$ is infinitely differentiable for s in $[a, b]$. Observe that $\frac{\partial^n f}{\partial s^n}(x,s)=x^{s-1}e^{-x}\left(\ln(x)\right)^n$ for (x, s) in $(0,\infty)\times(0,\infty)$ and is continuous on $(0,\infty)\times(0,\infty)$.

Let $g(x,s)=x^{s-1}e^{-x}\left(\ln(x)\right)^n$ for $x > 0$ and $a < s < b$. We shall find a Lebesgue integrable function $h(x)$ such that $g(x,s)\le h(x)$ for all $x > 0$ and $a < s < b$.

We shall construct the function $h(x)$ in two parts. First construct $f(x)$ for x in $[1, \infty)$ and then for x in $[0, 1]$.

Observe that for $x \ge 1$, $0 \le x^{s-1} \le x^b$ and $\lim_{x\to\infty} x^b e^{-x/2}\left(\ln(x)\right)^n=\lim_{x\to\infty}\frac{x^b}{e^{x/4}}\lim_{x\to\infty}\frac{\left(\ln(x)\right)^n}{e^{x/4}}=0\cdot 0=0$.

Therefore, there exists $K > 0$ such that $x^b e^{-x/2}\left(\ln(x)\right)^n \le K$ for all $x \ge 1$. It follows that for all $x \ge 1$ and $a < s < b$,

$$\left|g(x,s)\right|=\left|x^{s-1}e^{-x}\left(\ln(x)\right)^n\right|=\left|x^{s-1}e^{-x/2}\left(\ln(x)\right)^n\right|e^{-x/2}\le\left|x^b e^{-x/2}\left(\ln(x)\right)^n\right|e^{-x/2}\le Ke^{-x/2}.$$

So, we define $h(x) = Ke^{-x/2}$ for $x \ge 1$.

Note that $x^b e^{-x/2}\left(\ln(x)\right)^n$ is not defined at x = 0. Note also that $\lim_{x\to 0^+} x^a \left(\ln(x)\right)^n = 0$ for a > 0. We deduce this as follows.

For n =1, $\lim_{x\to 0^+} x^a \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{e^{-a\ln(x)}} = \lim_{x\to 0^+} \frac{1/(x)}{e^{-a\ln(x)}(-a/x)} = -\lim_{x\to 0^+} \frac{1}{ae^{-a\ln(x)}} = 0$ and for n > 1, apply L'Hôpital's Rule repeatedly.

Note that for $0 < x \le 1$, $\left|g(x,s)\right| = \left|x^{s-1}e^{-x}\left(\ln(x)\right)^n\right| \le x^{a-1}\left|\left(\ln(x)\right)^n\right|$.

Now,

$$\int_\eta^1 x^{a-1}\left(\ln(x)\right)^n dx = \left[\frac{x^a}{a}\left(\ln(x)\right)^n\right]_\eta^1 - n\int_\eta^1 \frac{x^{a-1}}{a}\left(\ln(x)\right)^{n-1} dx = -\frac{\eta^a}{a}\left(\ln(\eta)\right)^n - n\int_\eta^1 \frac{x^{a-1}}{a}\left(\ln(x)\right)^{n-1} dx$$

so that

$$\int_0^1 x^{a-1}\left(\ln(x)\right)^n dx = 0 - \frac{n}{a}\int_0^1 x^{a-1}\left(\ln(x)\right)^{n-1} dx = -\frac{n}{a}\int_0^1 x^{a-1}\left(\ln(x)\right)^{n-1} dx \text{ . ------------- (1)}$$

For n =1, $\int_\eta^1 x^{a-1}\ln(x)dx = \left[\frac{x^a}{a}\ln(x)\right]_\eta^1 - \int_\eta^1 \frac{x^{a-1}}{a}dx = -\frac{\eta^a}{a}\ln(\eta) - \left[\frac{x^a}{a^2}\right]_\eta^1 = -\frac{\eta^a}{a}\ln(\eta) + \frac{\eta^a}{a^2} - \frac{1}{a^2}$.

Therefore, $\int_0^1 x^{a-1}\ln(x)dx = \lim_{\eta\to 0^+}\left(-\frac{\eta^a}{a}\ln(\eta) + \frac{\eta^a}{a^2} - \frac{1}{a^2}\right) = -\frac{1}{a^2}$. Hence, by induction on n using (1), we deduce that $x^{a-1}\left(\ln(x)\right)^n$ is Lebesgue integrable on (0, 1].

Now, define $h(x) = x^{a-1}\left|\ln(x)\right|^n$ for $0 < x \le 1$. Hence, $h(x) = \begin{cases} Ke^{-x/2}, \text{ if } x \ge 1 \\ x^{a-1}\left|\ln(x)\right|^n, \text{ if } 0 < x < 1 \end{cases}$. Since h is Lebesgue integrable on (0,1] and on [1, ∞), it is Lebesgue integrable on (0, ∞).

Hence, we have $g(x,s) \le h(x)$ for all x > 0 and a < s < b.

It follows then by repeated use of Theorem 1 part (ii) of "*Integration Using Differentiation Under The Integral Sign*" that $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$ is infinitely differentiable for a < s < b. Since a and b are arbitrary, it follows that $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$ is infinitely differentiable for s > 0 and $\Gamma^{(n)}(s) = \int_0^\infty e^{-x}x^{s-1}\left(\ln(x)\right)^n dx$ for $n \ge 1$.

Theorem 4. $\Lambda(s) = \int_0^\infty \frac{x^s}{1+e^x}dx$ is infinitely differentiable for s > –1 and

$$\Lambda^{(n)}(s) = \int_0^\infty \frac{x^s\left(\ln(x)\right)^n}{1+e^x}dx.$$

Proof. Let $f(x,s)=\dfrac{x^s}{1+e^x}=\dfrac{x^s e^{-x}}{1+e^{-x}}$ for $s > -1$ and $0 < x < \infty$. Note that

$\dfrac{\partial^n f}{\partial s^n}(x,s)=\dfrac{x^s\left(\ln(x)\right)^n}{1+e^x}=\dfrac{x^s e^{-x}\left(\ln(x)\right)^n}{1+e^{-x}}$ for $n \geq 1$. Observe that

$$\left|\frac{\partial^n f}{\partial s^n}(x,s)\right|=\left|\frac{x^s\left(\ln(x)\right)^n}{1+e^x}\right|=\left|\frac{x^s e^{-x}\left(\ln(x)\right)^n}{1+e^{-x}}\right|\leq x^s e^{-x}\left|\ln(x)\right|^n .$$

Let $g(x,s)=\dfrac{x^s e^{-x}\left(\ln(x)\right)^n}{1+e^{-x}}$ for $x > 0$ and $a < s < b$ and $-1 < a < b$. Note that

$\lim\limits_{x\to\infty} x^b e^{-x/2}\left(\ln(x)\right)^n=0$. Hence $x^b e^{-x/2}\left(\ln(x)\right)^n$ is bounded above by $K > 0$ on $[1, \infty)$. Therefore,

$$\left|g(x,s)\right|\leq\left|x^s e^{-x}\left(\ln(x)\right)^n\right|=\left|x^s e^{-x/2}\left(\ln(x)\right)^n\right|e^{-x/2}\leq\left|x^b e^{-x/2}\left(\ln(x)\right)^n\right|e^{-x/2}\leq Ke^{-x/2} \text{ for } x\geq 1.$$

Since $Ke^{-x/2}$ is integrable on $[1,\infty)$, $g(x,s)=\dfrac{x^s e^{-x}\left(\ln(x)\right)^n}{1+e^{-x}}$ is integrable on $[1,\infty)$ for $-1 < a < s$.

Also, we have as before, for $0 < x \leq 1$, $\left|g(x,s)\right|\leq\left|x^s e^{-x}\left(\ln(x)\right)^n\right|\leq x^a\left|\left(\ln(x)\right)^n\right|$. We have shown in the proof of the previous Theorem that $x^a\left(\ln(x)\right)^n$ is Lebesgue integrable on (0, 1]. Therefore,

$g(x,s)=\dfrac{x^s e^{-x}\left(\ln(x)\right)^n}{1+e^{-x}}$ is Lebesgue integrable on (0, 1]. It follows that $g(x,s)=\dfrac{x^s e^{-x}\left(\ln(x)\right)^n}{1+e^{-x}}$ is Lebesgue integrable on $(0,\infty)$.

We define $h(x)=\begin{cases} Ke^{-x/2}, \text{ if } x\geq 1 \\ x^a\left|\ln(x)\right|^n, \text{ if } 0<x<1 \end{cases}$. Hence, $\left|g(x,s)\right|\leq h(x)$ for $x > 0$ and $a < s < b$ and $-1 < a < b$. We deduce as before that h is Lebesgue integrable on $(0, \infty)$. So, it follows as for the case of Gamma function by repeated use of Theorem 1 part (ii) of "*Integration Using Differentiation Under The Integral Sign*" that $\Lambda(s)=\int_0^\infty \dfrac{x^s}{1+e^x}dx$ is infinitely differentiable at s for $a < s < b$. Since a and b are arbitrary $\Lambda(s)=\int_0^\infty \dfrac{x^s}{1+e^x}dx$ is infinitely differentiable at s for $s > -1$ and

$$\Lambda^{(n)}(s)=\int_0^\infty \frac{x^s\left(\ln(x)\right)^n}{1+e^x}dx \text{ for } n\geq 1.$$

Definition 5. For any $s > 0$, define the *Eta function* $\eta(s)$ by $\eta(s)=\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^s}$. Note that this is well defined since for any $s > 0$, $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^s}$ is convergent by the Alternating Series Test.

Theorem 6. The Eta function $\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$ is infinitely differentiable at *s* for *s* > 0 and

$$\eta^{(p)}(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1+p}\left(\ln(n)\right)^p}{n^s}.$$

Proof.

We note that the series $\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$ converges uniformly with respect to *s* on [*a*, ∞) for any *a* > 0. Write $\eta(s)=\sum_{n=1}^{\infty}\left((-1)^{n+1}\frac{1}{n^{s-\frac{a}{2}}}\bullet\frac{1}{n^{\frac{a}{2}}}\right)$. By the alternating series test, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{\frac{a}{2}}}$ converges and so its sequence of partial sums is bounded and so is uniformly bounded. The function $g_n(s)=\frac{1}{n^{s-\frac{a}{2}}}$ on [*a*, ∞) satisfies $g_{n+1}(s)=\frac{1}{(n+1)^{s-\frac{a}{2}}}\le\frac{1}{n^{s-\frac{a}{2}}}=g_n(s)$ and so the sequence of functions $\left(g_n(s)\right)$ is monotone decreasing. Moreover, $g_n(s)=\frac{1}{n^{s-\frac{a}{2}}}\le\frac{1}{n^{\frac{a}{2}}}$ for all $s\ge a$. It follows that $g_n(s)$ converges uniformly to 0 on [*a*, ∞). Therefore, by the Dirichlet Test, $\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$ converges uniformly on [*a*, ∞).

If we differentiate $\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$ term by term we get $\eta^{(p)}(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1+p}\left(\ln(n)\right)^p}{n^s}$.

For *p* = 1, we write $\sum_{n=1}^{\infty}\frac{(-1)^{n+2}\ln(n)}{n^s}=\sum_{n=1}^{\infty}\frac{1}{n^{s-\frac{a}{2}}}(-1)^{n+2}\frac{\ln(n)}{n^{\frac{a}{2}}}$ for $s\ge a>0$. Note that the series $\sum_{n=1}^{\infty}(-1)^{n+2}\frac{\ln(n)}{n^{\frac{a}{2}}}$ is convergent since $\frac{\ln(n)}{n^{\frac{a}{2}}}$ is decreasing for $n>e^{\frac{2}{a}}$. Therefore, its sequence of partial sums is uniformly bounded. Hence, $\sum_{n=1}^{\infty}(-1)^{n+2}\frac{\ln(n)}{n^s}$ is uniformly convergent on [*a*, ∞) for any *a* > 0.

Similarly, we can show that $\eta^{(p)}(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1+p}\left(\ln(n)\right)^p}{n^s}$ is uniformly convergent on [*a*, ∞) for any *a* > 0. Therefore, we can conclude that $\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$ can be differentiated infinitely many times term by term on [*a*, ∞). Since *a* is arbitrary, we conclude that $\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$ is differentiable infinitely many times in (0, ∞) and $\eta^{(p)}(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1+p}\left(\ln(n)\right)^p}{n^s}$.

Theorem 7. For $s>-1$, $\Lambda(s)=\eta(s+1)\Gamma(s+1)$.

Proof.

Observe that $\frac{x^s}{1+e^x}=\frac{x^se^{-x}}{1+e^{-x}}=x^se^{-x}\sum_{n=0}^{\infty}(-1)^ne^{-nx}=\sum_{n=0}^{\infty}(-1)^ne^{-x(n+1)}x^s$. Note that each $e^{-x(n+1)}x^s$ is Lebesgue integrable on (0, ∞).

Let $f_n(x)=e^{-x(n+1)}x^s$ for x in $[0,\infty)$. Then $\left(f_n(x)\right)$ is a decreasing sequence of non-negative functions on $[0,\infty)$.

We shall deduce the theorem first for the case when $s > 0$.

Suppose now $s > 0$.

Since, $\frac{d}{dx}f_n(x)=\frac{d}{dx}e^{-x(n+1)}x^s=e^{-x(n+1)}x^{s-1}(s-x(n+1))=0$ if, and only of, $x=\frac{s}{n+1}$, the absolute maximum of $f_n(x)$ on $[0,\infty)$ occurs at $x=\frac{s}{n+1}$. Therefore,

$$\sup_{x\in[0,\infty)} f_n(x)=f_n\left(\frac{s}{n+1}\right)=e^{-s}\left(\frac{s}{n+1}\right)^s=\frac{s^se^{-s}}{(n+1)^s}\to 0 \text{ as } n \text{ tends to infinity.}$$

Therefore, for $s > 0$, $\sum_{n=0}^{\infty}(-1)^ne^{-x(n+1)}x^s=\sum_{n=0}^{\infty}(-1)^n f_n(x)$ converges uniformly to $\frac{x^s}{1+e^x}$ on $(0,\infty)$. Hence, for $s > 0$,

$$\int_0^\infty\frac{x^s}{1+e^x}dx=\int_0^\infty\sum_{n=0}^{\infty}(-1)^ne^{-x(n+1)}x^sdx=\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^\infty e^{-xn}x^sdx$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^\infty e^{-u}\left(\frac{u}{n}\right)^s\frac{1}{n}du=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^{s+1}}\int_0^\infty e^{-u}u^sdu,$$

by a change of variable, $u=nx$,

$$=\eta(s+1)\Gamma(s+1).$$

Suppose now $-1<s\le 0$.

Let k be any real number such that $0<k<\infty$.

For $-1<s\le 0$ and each positive integer n, the function $f_n(x)=e^{-x(n+1)}x^s$ is non-negative and decreasing on (0, ∞), since $\frac{d}{dx}e^{-x(n+1)}x^s=e^{-x(n+1)}x^{s-1}(s-x(n+1))\le 0$ for $s\le 0$. It follows that $f_n(x)$ is a decreasing function of x on $[k,\infty)$ and $\sup_{x\in[k,\infty)} f_n(x)=\sup_{x\in[k,\infty)} e^{-x(n+1)}x^s=f_n(k)=e^{-k(n+1)}k^s\to 0$ as n tends to infinity. Therefore, $\sum_{n=0}^{\infty}(-1)^ne^{-x(n+1)}x^s=\sum_{n=0}^{\infty}(-1)^n f_n(x)$ converges uniformly to $\frac{x^s}{1+e^x}$ on $[k,\infty)$. Hence, for $-1<s\le 0$, $\int_k^\infty\frac{x^s}{1+e^x}dx=\int_k^\infty\sum_{n=0}^{\infty}(-1)^n f_n(x)dx=\sum_{n=1}^{\infty}(-1)^{n-1}\int_k^\infty f_n(x)dx$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \int_k^{\infty} e^{-xn} x^s dx = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{s+1}} \int_{nk}^{\infty} e^{-u} u^s du$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{s+1}} g_n(k) , \qquad \text{-------------------------------- (*)}$$

where $g_n(k) = \int_{nk}^{\infty} e^{-u} u^s du$.

Now, for each $-1 < s \le 0$, the function $g_n(k) = \int_{nk}^{\infty} e^{-u} u^s du$ is a decreasing sequence of functions on $[0, \infty)$ uniformly bounded by $f_n(0) = \int_0^{\infty} e^{-u} u^s du = \Gamma(s+1) < \infty$ and $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{s+1}}$ is uniformly convergent with respect to *k*.

By Abel Test, the series on the right-hand side of (*) converges uniformly with respect to *k* for $-1 < s \le 0$. Therefore, it converges to a continuous function of *k*. Thus, taking limits,

$$\lim_{k \to 0^+} \int_k^{\infty} \frac{x^s}{1+e^x} dx = \lim_{k \to 0^+} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{s+1}} f_n(k) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{s+1}} \lim_{k \to 0^+} f_n(k)$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{s+1}} \Gamma(s+1) = \left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{s+1}} \right) \Gamma(s+1) = \eta(s+1)\Gamma(s+1) .$$

Therefore, for $-1 < s \le 0$, $\Lambda(s) = \int_0^{\infty} \frac{x^s}{1+e^x} dx = \eta(s+1)\Gamma(s+1)$.

It follows that for $s > -1$, $\Lambda(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{s+1}} \Gamma(s+1) = \eta(s+1)\Gamma(s+1)$.

This completes the proof.

Remark 8.

1. Our aim is to determine the integral $\int_0^{\infty} \frac{x^s \ln(x)}{1+e^x} dx$ for *s* = 0 and *s* =1. Note that $\int_0^{\infty} \frac{x^s \ln(x)}{1+e^x} dx = \Lambda'(s)$ for *s* > –1. Hence, for $s > -1$,

$$\int_0^{\infty} \frac{x^s \ln(x)}{1+e^x} dx = \eta'(s+1)\Gamma(s+1) + \eta(s+1)\Gamma'(s+1) .$$

Thus, to evaluate $\Lambda'(s)$, we need to determine $\eta(s+1), \eta'(s+1), \Gamma(s+1)$ and $\Gamma'(s+1)$.

For the evaluation of $\int_0^\infty \frac{\ln(x)}{1+e^x}dx = \Lambda'(0) == \eta'(1)\Gamma(1)+\eta(1)\Gamma'(1)$, we shall determine $\eta'(1)$ and $\Gamma'(1)$. It is easily deduced that $\Gamma(1)=1$ and $\eta(1)=\ln(2)$. We shall prove that $\eta'(1)=\ln(2)\gamma_0 - \frac{(\ln(2))^2}{2}$ and $\Gamma'(1)=-\gamma_0$.

For the evaluation of $\int_0^\infty \frac{x\ln(x)}{1+e^x}dx = \Lambda'(1) = \eta'(2)\Gamma(2)+\eta(2)\Gamma'(2)$, we shall need to determine $\eta'(2)$ and $\Gamma'(2)$. It is easily deduced that $\Gamma(2)=1$ and $\eta(2)=\frac{\pi^2}{12}$.

Theorem 9. $\Gamma'(1)=-\gamma_0$, $\Gamma''(1)=\gamma_0{}^2+\frac{\pi^2}{6}$.

Proof. From Theorem 3, $\Gamma'(1)=\int_0^\infty e^{-x}\ln(x)dx$.

Now, $\left(1-\frac{x}{n}\right)^{n-1} \to e^{-x}$ as n tends to infinity. Let $f_n(x)=\begin{cases}\left(1-\frac{x}{n}\right)^{n-1}, & \text{if } x\le n,\\ 0, & \text{if } x>n\end{cases}$.

Then $f_n(x)\ln(x)$ tends to $e^{-x}\ln(x)$ pointwise on $(0, \infty)$.

Next, we claim that $\left(1-\frac{x}{n}\right)^{n-1} \le e^{-x/2}$ for $n\ge 2$ and $x\le n$. To show this, we note that $1-t\le e^{-t}$, for $t\ge 0$. This is a consequence of the fact that the derivative of $1-t-e^{-t}$ is less than or equal to 0 if $t\ge 0$. Therefore, for $0\le x\le n$ and $n\ge 2$, $1-\frac{x}{n}\le e^{-\frac{x}{n}}$ and so

$$\left(1-\frac{x}{n}\right)^{n-1} \le \left(e^{-\frac{x}{n}}\right)^{n-1} = e^{-x(1-\frac{1}{n})} \le e^{-\frac{x}{2}},$$

as $1-\frac{1}{n}\ge\frac{1}{2}$ for $n\ge 2$.

Therefore, $f_n(x)\le e^{-x/2}$ for $n\ge 2$ and for $x\le n$. Obviously, $f_n(x)\le e^{-x/2}$ for $x>n$. Thus, $f_n(x)\le e^{-x/2}$ for $n\ge 2$ and for $x\ge 0$. It follows that $|f_n(x)\ln(x)|\le e^{-x/2}|\ln(x)|$ for $n\ge 2$ and for $x > 0$.

Now, by a change of variable, $u=\frac{x}{2}$,

$$\int_0^\infty e^{-x/2}\ln(x)dx = 2\int_0^\infty e^{-u}\ln(2u)dx = 2\int_0^\infty e^{-u}\ln(u)dx + 2\int_0^\infty e^{-u}\ln(2)dx.$$

We have already shown that $e^{-x}\ln(x)$ is Lebesgue integrable on (0, ∞) and since e^{-x} is also integrable on (0, ∞), it follows that $e^{-x/2}\ln(x)$ is Lebesgue integrable on (0, ∞). Thus, $e^{-x/2}\left|\ln(x)\right| = \left|e^{-x/2}\ln(x)\right|$ is Lebesgue integrable on (0, ∞). Therefore, by the Lebesgue Dominated Convergence Theorem, $\int_0^\infty f_n(x)\ln(x)dx \to \int_0^\infty e^{-x}\ln(x)dx$.

Now, $\int_0^\infty f_n(x)\ln(x)dx = \int_0^n \left(1-\frac{x}{n}\right)^{n-1}\ln(x)dx = n\int_0^1 u^{n-1}\ln(n(1-u))du$,

by change of variable, $u = 1-\frac{x}{n}$,

$$= n\int_0^1 u^{n-1}\ln(n)du + n\int_0^1 u^{n-1}\ln(1-u))du$$

$$= \ln(n) + n\int_0^1 u^{n-1}\ln(1-u))du .$$

If n =1,

$$\int_0^1 u^{n-1}\ln(1-u))du = \int_0^1 \ln(1-u))du = \lim_{s\to 1^-}\int_0^s \ln(1-u))du = \lim_{s\to 1^-}\left[-u-\ln(1-u)(1-u)\right]_0^s = -1 .$$

For $n > 1$, $\int_0^s u^{n-1}\ln(1-u))du = \left[\frac{1}{n}(u^n-1)\ln(1-u)\right]_0^s + \int_0^s \frac{1}{n}(u^n-1)\frac{1}{1-u}du$

$$= \frac{1}{n}(s^n-1)\ln(1-s) - \int_0^s \frac{1}{n}\sum_{k=1}^n u^{k-1}du = \frac{1}{n}(s^n-1)\ln(1-s) - \frac{1}{n}\sum_{k=1}^n \frac{s^k}{k} .$$

Therefore, $\int_0^1 u^{n-1}\ln(1-u))du = \lim_{s\to 1^-}\int_0^s u^{n-1}\ln(1-u))du =$

$$= \lim_{s\to 1^-}\left(\frac{1}{n}(s^n-1)\ln(1-s) - \frac{1}{n}\sum_{k=1}^n \frac{s^k}{k}\right) = 0 - \frac{1}{n}\sum_{k=1}^n \frac{1}{k} = -\frac{H_n}{n} , \text{ ------------------- } (1)$$

where, $H_n = \sum_{k=1}^n \frac{1}{k}$ is the harmonic sum.

Therefore, $\int_0^\infty f_n(x)\ln(x)dx = \ln(n) + n\int_0^1 u^{n-1}\ln(1-u))du = \ln(n) - H_n$.

Thus, $\lim_{n\to\infty}\int_0^\infty f_n(x)\ln(x)dx = \lim_{n\to\infty}\left(\ln(n) - H_n\right) = -\gamma_0$.

Now, we shall evaluate $\Gamma''(1)$.

From Theorem 3, $\Gamma''(1) = \int_0^\infty e^{-x}\left(\ln(x)\right)^2 dx$. As for the case of $\Gamma'(1)$, we can show that

$$\int_0^\infty f_n(x)(\ln(x))^2 dx \to \int_0^\infty e^{-x}(\ln(x))^2 dx .$$

Now, as above,

$$\int_0^\infty f_n(x)\left(\ln(x)\right)^2 dx = \int_0^n \left(1-\frac{x}{n}\right)^{n-1} \left(\ln(x)\right)^2 dx = n\int_0^1 u^{n-1}\left(\ln(n(1-u))\right)^2 du ,$$

by change of variable, $u = 1-\dfrac{x}{n}$,

$$= n\int_0^1 u^{n-1}\left(\ln(n)\right)^2 du + n\int_0^1 u^{n-1}\left(\ln(1-u)\right)^2 dx + 2n\int_0^1 u^{n-1}\ln(n)\ln(1-u)dx$$

$$= \left(\ln(n)\right)^2 - 2n\ln(n)\frac{H_n}{n} + n\int_0^1 u^{n-1}\left(\ln(1-u)\right)^2 dx \text{ , by applying identity (1).}$$

$$= \left(\ln(n)\right)^2 - 2\ln(n)H_n + n\int_0^1 u^{n-1}\left(\ln(1-u)\right)^2 dx . \text{ ------------------------------ (2)}$$

If n =1,

$$\int_0^1 u^{n-1}\ln(1-u))du = \int_0^1 \ln(1-u))du = \lim_{s\to 1^-}\int_0^s \ln(1-u))du = \lim_{s\to 1^-}\left[-u-\ln(1-u)(1-u)\right]_0^s = -1 .$$

For $n > 1$,

$$\int_0^1 u^{n-1}\left(\ln(1-u)\right)^2 du = \left[\frac{1}{n}(u^n-1)\left(\ln(1-u)\right)^2\right]_0^1 + \int_0^s \frac{2}{n}(u^n-1)\frac{\ln(1-u)}{1-u}du$$

$$= 0 - \int_0^1 \frac{2}{n}\sum_{k=1}^{n} u^{k-1}\ln(1-u)du = -\frac{2}{n}\sum_{k=1}^{n}\int_0^1 u^{k-1}\ln(1-u)du$$

$$= -\frac{2}{n}\sum_{k=1}^{n}\left(-\frac{H_k}{k}\right) = \frac{2}{n}\sum_{k=1}^{n}\frac{H_k}{k} \text{ , ----------------------------(3)}$$

by applying identity (1).

Now, $$H_n{}^2 = \left(\sum_{k=1}^{n}\frac{1}{k}\right)^2 = \sum_{i=1}^{n}\frac{1}{i}\left(\sum_{j=1}^{n}\frac{1}{j}\right) = \sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{n}\frac{1}{j}\right) = \sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i}\frac{1}{j} + \frac{1}{i}\sum_{j=i}^{n}\frac{1}{j}\right) - \sum_{i=1}^{n}\frac{1}{i^2}$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{i}\frac{1}{ij}\right) + \sum_{i=1}^{n}\left(\sum_{j=i}^{n}\frac{1}{ij}\right) - \sum_{i=1}^{n}\frac{1}{i^2} = 2\sum_{i=1}^{n}\left(\sum_{j=1}^{i}\frac{1}{ij}\right) - \sum_{i=1}^{n}\frac{1}{i^2} \text{ , by symmetry,}$$

$$= 2\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i}\frac{1}{j}\right) - \sum_{i=1}^{n}\frac{1}{i^2} = 2\sum_{i=1}^{n}\left(\frac{H_i}{i}\right) - \sum_{i=1}^{n}\frac{1}{i^2} = 2\sum_{i=1}^{n}\left(\frac{H_i}{i}\right) - Z_n ,$$

where $Z_n = \sum_{i=1}^{n}\frac{1}{i^2}$.

Hence, $\sum_{i=1}^{n}\left(\frac{H_i}{i}\right) = \frac{1}{2}\left(H_n{}^2 + Z_n\right)$.

Therefore, $\int_0^1 u^{n-1}\left(\ln(1-u)\right)^2 du = \frac{2}{n}\sum_{k=1}^{n}\frac{H_k}{k} = \frac{1}{n}\left(H_n{}^2 + Z_n\right)$. -------------------- (4)

Hence, $\int_0^\infty f_n(x)\left(\ln(x)\right)^2 dx = \left(\ln(n)\right)^2 - 2\ln(n)H_n + H_n{}^2 + Z_n = (H_n - \ln(n))^2 + Z_n$.

Therefore, $\Gamma''(1) = \int_0^\infty e^{-x}\left(\ln(x)\right)^2 dx = \lim_{n\to\infty}\int_0^\infty f_n(x)\left(\ln(x)\right)^2 dx = \gamma_0{}^2 + \zeta(2) = \gamma_0{}^2 + \frac{\pi^2}{6}$, since $\lim_{n\to\infty} Z_n = \zeta(2) = \frac{\pi^2}{6}$.

Theorem 10. $\eta(1) = \ln(2)$, $\eta'(1) = \sum_{n=1}^{\infty}\frac{(-1)^{n+2}\ln(n)}{n} = \ln(2)\gamma_0 - \frac{\left(\ln(2)\right)^2}{2}$.

Before we embark on the proof of Theorem 10, we shall define a series of Stieltjes constants γ_j for $j \geq 0$. $\gamma_0 = \gamma$ is the Euler Mascheroni constant. The γ_j's are also known as the Euler Mascheroni constants.

$$\gamma_0 = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \int_1^n \frac{1}{x}dx\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right),$$

$$\gamma_1 = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\ln(k)}{k} - \int_1^n \frac{\ln(x)}{x}dx\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\ln(k)}{k} - \frac{1}{2}\left(\ln(n)\right)^2\right),$$

$$\gamma_j = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\left(\ln(k)\right)^j}{k} - \int_1^n \frac{\left(\ln(x)\right)^j}{x}dx\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\left(\ln(k)\right)^j}{k} - \frac{1}{j+1}\left(\ln(n)\right)^{j+1}\right).$$

We shall make use of γ_1 to evaluate the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+2}\ln(n)}{n}$.

We shall show that γ_1 is defined, that is, it is finite. γ_0 can be shown in a similar way to be finite. The proof for finiteness of γ_1 is exemplary for γ_j, $j > 1$.

Theorem 11. $\gamma_1 = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\ln(k)}{k} - \int_1^n \frac{\ln(x)}{x}dx\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\ln(k)}{k} - \frac{1}{2}\left(\ln(n)\right)^2\right)$ is finite.

Proof. Let $f:[1,\infty)\to\mathbb{R}$ be defined by $f(x) = \frac{\ln(x)}{x}$ for $x \geq 1$. For each integer $n \geq 1$, let $s_n = \sum_{k=1}^{n}\frac{\ln(k)}{k}$. Let $d_n = s_n - \int_1^n f(x)dx$. Note that $f(x)$ is decreasing on $[e, \infty)$. Therefore,

For $n \geq 3$, $\int_n^{n+1} f(x)dx \geq \int_n^{n+1} f(n+1)dx = \frac{\ln(n+1)}{n+1}$ and $\int_n^{n+1} f(x)dx \leq \int_n^{n+1} f(n)dx = \frac{\ln(n)}{n}$.

That is, for $n \geq 3$,

$$\frac{\ln(n+1)}{n+1} \leq \int_n^{n+1} f(x)dx \leq \frac{\ln(n)}{n} . \text{ ----------------------------- } (1)$$

Hence, for $n \geq 3$,

$$\sum_{k=3}^{n-1} \frac{\ln(k+1)}{k+1} \leq \sum_{k=3}^{n-1} \int_n^{n+1} f(x)dx = \int_3^n f(x)dx \leq \sum_{k=3}^{n-1} \frac{\ln(k)}{k} , \text{ ------------------ } (2)$$

Let $t_n = \sum_{k=3}^{n} \frac{\ln(k)}{k}$ and $H_n = \int_3^n f(x)dx$ for $n \geq 3$.

Then for $n \geq 3$, $t_n - H_n = \sum_{k=3}^{n} \frac{\ln(k)}{k} - H_n = \frac{\ln(3)}{3} + \sum_{k=3}^{n-1} \frac{\ln(k+1)}{k+1} - H_n \leq \frac{\ln(3)}{3} < 1$, by inequality (2),

and $t_n - H_n = \sum_{k=3}^{n} \frac{\ln(k)}{k} - H_n = \frac{\ln(n)}{n} + \sum_{k=3}^{n-1} \frac{\ln(k)}{k} - H_n \geq \frac{\ln(n)}{n} > 0$, by inequality (2).

For $n \geq 3$, let $c_n = t_n - H_n$.

Then,

$$c_{n+1} - c_n = t_{n+1} - H_{n+1} - (t_n - H_n) = t_{n+1} - t_n - (H_{n+1} - H_n) = \frac{\ln(n+1)}{n+1} - \int_n^{n+1} f(x)dx$$

< 0, by inequality (1).

It follows that the sequence (c_n) for $n \geq 3$ is a decreasing sequence. Moreover, $0 < c_n < 1$. Therefore, by the Monotone Convergence Theorem, the sequence (c_n) is convergent, since the sequence (c_n) is bounded below. Let $\lim_{n\to\infty} c_n = C$.

For $n \geq 3$, $s_n = \sum_{k=1}^{2} \frac{\ln(k)}{k} + \sum_{k=3}^{n} \frac{\ln(k)}{k} = \frac{\ln(2)}{2} + t_n$.

Therefore, $d_n = s_n - \int_1^n f(x)dx = \frac{\ln(2)}{2} - \int_1^3 f(x)dx + t_n - \int_3^n f(x)dx = \frac{\ln(2)}{2} - \left[\frac{(\ln(x))^2}{2} \right]_1^3 + c_n$.

Hence, $d_n \to \frac{\ln(2)}{2} - \frac{(\ln(3))^2}{2} + C$. That is to say, $\gamma_1 = \frac{\ln(2)}{2} - \frac{(\ln(3))^2}{2} + C$.

For the function $f_j(x) = \frac{(\ln(x))^j}{x}$, we note that it is decreasing for $x \geq e^j$. Let n_j be the first integer greater than e^j. Let $t_n{}^j = \sum_{k=n_j}^{n} \frac{(\ln(k))^j}{k}$. For $n > n_j$, we can show similarly that

$t_n^j - \int_{n_j}^{n} \frac{(\ln(x))^j}{x} dx$ tends to a constant C_j. Note that the antiderivative of $f_j(x) = \frac{(\ln(x))^j}{x}$ is $\frac{(\ln(x))^{j+1}}{j+1}$. We can show similarly that

$$\sum_{k=1}^{n} \frac{(\ln(k))^j}{k} - \int_1^n \frac{(\ln(x))^j}{x} dx = \sum_{k=1}^{n_j-1} \frac{(\ln(k))^j}{k} - \int_1^{n_j} \frac{(\ln(x))^j}{x} dx + \sum_{k=n_j}^{n} \frac{(\ln(k))^j}{k} - \int_{n_j}^{n} \frac{(\ln(x))^j}{x} dx$$

$$= \sum_{k=1}^{n_j-1} \frac{(\ln(k))^j}{k} - \frac{(\ln(n_j))^j}{j+1} + \sum_{k=n_j}^{n} \frac{(\ln(k))^j}{k} - \int_{n_j}^{n} \frac{(\ln(x))^j}{x} dx \to \sum_{k=1}^{n_j-1} \frac{(\ln(k))^j}{k} - \frac{(\ln(n_j))^j}{j+1} + C_j .$$

Hence, $\gamma_j = \sum_{k=1}^{n_j-1} \frac{(\ln(k))^j}{k} - \frac{(\ln(n_j))^j}{j+1} + C_j$.

Proof of Theorem 10. $\eta(1) = \ln(2)$, $\eta'(1) = \sum_{k=2}^{\infty} (-1)^k \frac{\ln(k)}{k} = \ln(2)\gamma_0 - \frac{(\ln(2))^2}{2}$.

(i) $\eta(1) = \ln(2)$.

Since $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$, $\eta(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

By the alternating series test, $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ is convergent. Let $s_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}$ and $d_n = s_n - \int_1^n \frac{1}{x} dx = s_n - \ln(n)$. As remarked in Theorem 11, d_n is convergent and tends to the Euler Mascheroni constant γ_0. Consider $s_{2n} = \sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k}$. Then s_{2n} tends to $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$.

We shall rewrite s_{2n} in the following way:

$$s_{2n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} + \cdots - \frac{1}{2n}$$

$$= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n} - \ln(2n) \right) - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n} \right) + \ln(2n)$$

$$= d_{2n} - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \right) + \ln(n) + \ln(2) = d_{2n} - d_n + \ln(2)$$

Therefore, $s_{2n} \to \gamma_0 - \gamma_0 + \ln(2) = \ln(2)$ as n tends to infinity. Hence, $\eta(1) = \ln(2)$.

(ii) $\eta'(1)=\sum_{k=2}^{\infty}(-1)^k\frac{\ln(k)}{k}=\ln(2)\gamma_0-\frac{\left(\ln(2)\right)^2}{2}$**.** Differentiating $\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$ formally term by term we get $\eta'(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+2}\ln(n)}{n^s}=\sum_{n=1}^{\infty}\frac{(-1)^{n}\ln(n)}{n^s}$.

By the alternating series test, $\sum_{k=2}^{\infty}(-1)^k\frac{\ln(k)}{k}$ is convergent. Let $s_{2n}=\sum_{k=1}^{2n}(-1)^k\frac{\ln(k)}{k}$ and since $\sum_{k=2}^{\infty}(-1)^k\frac{\ln(k)}{k}$ is convergent, the limit of s_{2n} tends to $\sum_{k=2}^{\infty}(-1)^k\frac{\ln(k)}{k}$**.**

We shall rewrite s_{2n} in the following way:

$$s_{2n}=-\frac{\ln(1)}{1}+\frac{\ln(2)}{2}-\frac{\ln(3)}{3}+\cdots+\frac{\ln(2n)}{2n}$$

$$=-\left(\frac{\ln(1)}{1}+\frac{\ln(2)}{2}+\frac{\ln(3)}{3}+\cdots+\frac{\ln(2n)}{2n}-\frac{\left(\ln(2n)\right)^2}{2}\right)+2\left(\frac{\ln(2)}{2}+\frac{\ln(4)}{4}+\frac{\ln(6)}{6}+\cdots+\frac{\ln(2n)}{2n}\right)-\frac{\left(\ln(2n)\right)^2}{2}$$

$$=-d_{2n}+2\left(\frac{\ln(2)}{4}+\frac{\ln(3)}{6}+\cdots+\frac{\ln(n)}{2n}\right)+2\left(\frac{\ln(2)}{2}+\frac{\ln(2)}{4}+\cdots+\frac{\ln(2)}{2n}\right)-\frac{\left(\ln(2n)\right)^2}{2}$$

$$=-d_{2n}+\left(\frac{\ln(2)}{2}+\frac{\ln(3)}{3}+\cdots+\frac{\ln(n)}{n}\right)+\ln(2)\left(1+\frac{1}{2}+\frac{1}{4}\cdots+\frac{2}{n}\right)-\frac{\left(\ln(2n)\right)^2}{2}$$

$$=-d_{2n}+s_n+\ln(2)\left(1+\frac{1}{2}+\frac{1}{4}\cdots+\frac{1}{n}\right)-\frac{\left(\ln(n)\right)^2}{2}-\frac{\left(\ln(2)\right)^2}{2}-\ln(2)\ln(n)$$

$$=-d_{2n}+s_n-\frac{\left(\ln(n)\right)^2}{2}+\ln(2)\left(1+\frac{1}{2}+\frac{1}{4}\cdots+\frac{1}{n}-\ln(n)\right)-\frac{\left(\ln(2)\right)^2}{2}$$

$$=-d_{2n}+d_n+\ln(2)\left(1+\frac{1}{2}+\frac{1}{4}\cdots+\frac{1}{n}-\ln(n)\right)-\frac{\left(\ln(2)\right)^2}{2}$$

Therefore, $s_{2n}\to-\gamma_1+\gamma_1+\ln(2)\gamma_0-\frac{\left(\ln(2)\right)^2}{2}=\ln(2)\gamma_0-\frac{\left(\ln(2)\right)^2}{2}$ as *n* tends to infinity.

It follows that $\sum_{k=2}^{\infty}(-1)^k\frac{\ln(k)}{k}=\ln(2)\gamma_0-\frac{\left(\ln(2)\right)^2}{2}$**.**

Theorem 12. Recall $\eta^{(p)}(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1+p}\left(\ln(n)\right)^p}{n^s}$. For $p\geq 2$,

$$\eta^{(p)}(1)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1+p}\left(\ln(n)\right)^p}{n^s}=(-1)^{1+p}\sum_{k=2}^{\infty}(-1)^k\frac{\left(\ln(k)\right)^p}{k}$$ and

$$\sum_{k=2}^{\infty}(-1)^k \frac{\left(\ln(k)\right)^p}{k}$$

$$= -\frac{1}{p+1}\left(\ln(2)\right)^{p+1} + \left(\ln(2)\right)^p \gamma_0 + \binom{p}{1}\ln(2)\gamma_{p-1} + \binom{p}{2}\left(\ln(2)\right)^2 \gamma_{p-2} + \cdots + \binom{p}{p-1}\left(\ln(2)\right)^{p-1}\gamma_1$$

Proof. Let $f(x) = \frac{\left(\ln(x)\right)^p}{x}$. For integers $n, p \geq 1$, let $s_n^p = \sum_{k=1}^{n}(-1)^k \frac{\left(\ln(k)\right)^p}{k}$. Note that

$\int_1^n f(x)dx = \int_1^n \frac{\left(\ln(x)\right)^p}{x} dx = \frac{1}{p+1}\left(\ln(n)\right)^{p+1}$. Let

$$d_n^p = s_n^p - \int_1^n \frac{\left(\ln(x)\right)^p}{x} dx = \sum_{k=1}^{n} \frac{\left(\ln(k)\right)^p}{k} - \frac{1}{p+1}\left(\ln(n)\right)^{p+1}.$$

Then

$$s_{2n}^p = \sum_{k=1}^{2n}(-1)^k \frac{\left(\ln(k)\right)^p}{k} = -\sum_{k=1}^{2n}\frac{\left(\ln(k)\right)^p}{k} + 2\sum_{k=1}^{n}\frac{\left(\ln(2k)\right)^p}{2k}$$

$$= -\left(\sum_{k=1}^{2n}\frac{\left(\ln(k)\right)^p}{k} - \frac{1}{p+1}\left(\ln(2n)\right)^{p+1}\right) + \sum_{k=1}^{n}\frac{\left(\ln(2k)\right)^p}{k} - \frac{1}{p+1}\left(\ln(2n)\right)^{p+1}$$

$$= -d_{2n}^p + \sum_{k=1}^{n}\frac{\left(\ln(2k)\right)^p}{k} - \frac{1}{p+1}\left(\ln(2n)\right)^{p+1}. \quad \text{------------------------- (1)}$$

Now,

$$\left(\ln(2k)\right)^p = \left(\ln(2) + \ln(k)\right)^p$$

$$= \left(\ln(2)\right)^p + \left(\ln(k)\right)^p + \binom{p}{1}\ln(2)(\ln(k))^{p-1} + \binom{p}{2}(\ln(2))^2(\ln(k))^{p-1} + \cdots + \binom{p}{p-1}(\ln(2))^{p-1}(\ln(k))$$

------------------------- (2)

and

$$\left(\ln(2n)\right)^{p+1} = \left(\ln(2)\right)^{p+1} + \left(\ln(n)\right)^{p+1} + \binom{p+1}{1}\ln(2)(\ln(n))^p$$

$$+\binom{p+1}{2}(\ln(2))^2(\ln(n))^{p-1} + \cdots + \binom{p+1}{p}(\ln(2))^p \ln(n). \quad \text{------------------------ (3)}$$

Hence,

$$s_{2n}^p = -d_{2n}^p + \sum_{k=1}^{n}\frac{\left(\ln(2)\right)^p}{k} + \sum_{k=1}^{n}\frac{\left(\ln(k)\right)^p}{k}$$

$$+\binom{p}{1}\sum_{k=1}^{n}\ln(2)\frac{(\ln(k))^{p-1}}{k}+\binom{p}{2}\sum_{k=1}^{n}(\ln(2))^{2}\frac{(\ln(k))^{p-2}}{k}+\cdots+\binom{p}{p-1}\sum_{k=1}^{n}(\ln(2))^{p-1}\frac{\ln(k)}{k}$$

$$-\frac{1}{p+1}\left(\ln(2)\right)^{p+1}-\frac{1}{p+1}\left(\ln(n)\right)^{p+1}$$

$$-\binom{p+1}{1}\frac{1}{p+1}\ln(2)(\ln(n))^{p}-\binom{p+1}{2}\frac{1}{p+1}(\ln(2))^{2}(\ln(n))^{p-1}-\cdots-\binom{p+1}{p}\frac{1}{p+1}(\ln(2))^{p}(\ln(n))$$

$$s_{2n}^{p}=-d_{2n}^{p}+\left(\sum_{k=1}^{n}\frac{\left(\ln(k)\right)^{p}}{k}-\frac{1}{p+1}\left(\ln(n)\right)^{p+1}\right)+\left(\sum_{k=1}^{n}\frac{\left(\ln(2)\right)^{p}}{k}-\frac{1}{p+1}\left(\ln(2)\right)^{p+1}\right)-\binom{p+1}{p}\frac{1}{P+1}(\ln(2))^{p}\ln(n)$$

$$+\ln(2)\left(\binom{p}{1}\sum_{k=1}^{n}\frac{(\ln(k))^{p-1}}{k}-\binom{p+1}{1}\frac{1}{p+1}(\ln(n))^{p}\right)$$

$$+(\ln(2))^{2}\left(\binom{p}{2}\sum_{k=1}^{n}\frac{(\ln(k))^{p-2}}{k}\binom{p+1}{2}\frac{1}{p+1}(\ln(n))^{p-1}\right)+\cdots+$$

$$+(\ln(2))^{p-1}\left(\binom{p}{p-1}\sum_{k=1}^{n}\frac{\ln(k)}{k}-\binom{p+1}{p-1}\frac{1}{p+1}(\ln(n))^{2}\right).$$

Thus,

$$s_{2n}^{p}=-d_{2n}^{p}+d_{n}^{p}+\left(\ln(2)\right)^{p}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln(n)\right)-\frac{1}{p+1}\left(\ln(2)\right)^{p+1}$$

$$+\ln(2)\binom{p}{1}\left(\sum_{k=1}^{n}\frac{(\ln(k))^{p-1}}{k}-\frac{\binom{p+1}{1}}{\binom{p}{1}(p+1)}(\ln(n))^{p}\right)$$

$$+(\ln(2))^{2}\binom{p}{2}\left(\sum_{k=1}^{n}\frac{(\ln(k))^{p-2}}{k}-\frac{\binom{p+1}{2}}{\binom{p}{2}(p+1)}(\ln(n))^{p-1}\right)+\cdots+$$

$$+(\ln(2))^{p-1}\binom{p}{p-1}\left(\sum_{k=1}^{n}\frac{\ln(k)}{k}-\frac{\binom{p+1}{p-1}}{\binom{p}{p-1}(p+1)}(\ln(n))^{2}\right)$$

$$=-d_{2n}^{p}+d_{n}^{p}+\left(\ln(2)\right)^{p}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln(n)\right)-\frac{1}{p+1}\left(\ln(2)\right)^{p+1}$$

$$+\ln(2)\binom{p}{1}\left(\sum_{k=1}^{n}\frac{(\ln(k))^{p-1}}{k}-\frac{1}{p}(\ln(n))^{p}\right)+(\ln(2))^{2}\binom{p}{2}\left(\sum_{k=1}^{n}\frac{(\ln(k))^{p-2}}{k}-\frac{1}{p-1}(\ln(n))^{p-1}\right)+\cdots+$$

$$+(\ln(2))^{p-1}\binom{p}{p-1}\left(\sum_{k=1}^{n}\frac{\ln(k)}{k}-\frac{1}{2}(\ln(n))^{2}\right),$$

since $\dfrac{\binom{p+1}{j}}{\binom{p}{j}(p+1)}=\dfrac{1}{p+1-j}$.

Therefore,

$$s_{2n}^{p}=-d_{2n}^{p}+d_{n}^{p}+\left(\ln(2)\right)^{p}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln(n)\right)-\frac{1}{p+1}\left(\ln(2)\right)^{p+1}$$

$$+\ln(2)\binom{p}{1}d_{n}^{p-1}+(\ln(2))^{2}\binom{p}{2}d_{n}^{p-2}+\cdots+(\ln(2))^{p-1}\binom{p}{p-1}d_{n}^{1}$$

Thus,

$$\lim_{n\to\infty}s_{2n}^{p}=-\gamma_{p}+\gamma_{p}+\left(\ln(2)\right)^{p}\gamma_{0}-\frac{\left(\ln(2)\right)^{p+1}}{p+1}+\ln(2)\binom{p}{1}\gamma_{p-1}+(\ln(2))^{2}\binom{p}{2}\gamma_{p-2}+\cdots+(\ln(2))^{p-1}\binom{p}{p-1}\gamma_{1}$$

$$=\left(\ln(2)\right)^{p}\gamma_{0}-\frac{\left(\ln(2)\right)^{p+1}}{p+1}+\binom{p}{1}\ln(2)\gamma_{p-1}+\binom{p}{2}(\ln(2))^{2}\gamma_{p-2}+\cdots+\binom{p}{p-1}(\ln(2))^{p-1}\gamma_{1}.$$

Hence,

$$\sum_{k=2}^{\infty}(-1)^{k}\frac{\left(\ln(k)\right)^{p}}{k}$$

$$=\left(\ln(2)\right)^{p}\gamma_{0}-\frac{\left(\ln(2)\right)^{p+1}}{p+1}+\binom{p}{1}\ln(2)\gamma_{p-1}+\binom{p}{2}(\ln(2))^{2}\gamma_{p-2}+\cdots+\binom{p}{p-1}(\ln(2))^{p-1}\gamma_{1}\ .$$

Theorem 13. $\eta'(2)=\sum_{n=1}^{\infty}\dfrac{(-1)^{n}\ln(n)}{n^{2}}=\dfrac{1}{2}\zeta'(2)+\dfrac{\pi^{2}}{12}\ln(2)$, where $\zeta(s)$ is the Riemann Zeta function.

Proof. From Theorem 12, $\eta'(2)=\sum_{n=1}^{\infty}\dfrac{(-1)^{n}\ln(n)}{n^{2}}$. Let $f(x)=\dfrac{\ln(x)}{x^{2}}$ for $x > 0$. Then $f'(x)=\dfrac{1}{x^{3}}(1-2\ln(x))$ for $x > 0$ and so $f'(x)<0$ if $x>\sqrt{e}$. Hence, $f(x)$ is non-negative and strictly decreasing on $[2, \infty)$. Note that $\int_{1}^{n}\dfrac{\ln(x)}{x^{2}}dx=\left[-\dfrac{\ln(x)+1}{x}\right]_{1}^{n}=1-\dfrac{\ln(n)+1}{n}$ and so

$\int_1^\infty \frac{\ln(x)}{x^2}\,dx = 1$. It follows that $\sum_{n=1}^{\infty}\frac{\ln(n)}{n^2}$ is convergent. Note that this is just the negative of the derivative of the Riemann Zeta function $\zeta(s)$ at $s = 2$. That is, $\zeta'(2) = -\sum_{n=1}^{\infty}\frac{\ln(n)}{n^2}$. Let $s_{2n} = \sum_{k=2}^{2n}(-1)^k\frac{\ln(k)}{k^2}$. Then since $\sum_{n=1}^{\infty}\frac{(-1)^n\ln(n)}{n^2}$ is convergent, $\lim_{n\to\infty} s_{2n} = \sum_{n=1}^{\infty}\frac{(-1)^n\ln(n)}{n^2}$.

$$s_{2n} = -\frac{\ln(1)}{1^2}+\frac{\ln(2)}{2^2}-\frac{\ln(3)}{3^2}+\frac{\ln(4)}{4^2}+\cdots+\frac{\ln(2n)}{(2n)^2}$$

$$= -\left(\frac{\ln(1)}{1^2}+\frac{\ln(2)}{2^2}+\frac{\ln(3)}{3^2}+\frac{\ln(4)}{4^2}+\cdots+\frac{\ln(2n)}{(2n)^2}\right)+2\left(\frac{\ln(2)}{2^2}+\frac{\ln(4)}{4^2}+\cdots+\frac{\ln(2n)}{(2n)^2}\right)$$

$$= -\left(\frac{\ln(1)}{1^2}+\frac{\ln(2)}{2^2}+\frac{\ln(3)}{3^2}+\frac{\ln(4)}{4^2}+\cdots+\frac{\ln(2n)}{(2n)^2}\right)+\frac{2}{4}\left(\frac{\ln(2)}{1}+\frac{\ln(4)}{2^2}+\cdots+\frac{\ln(2n)}{n^2}\right)$$

$$= -\left(\frac{\ln(1)}{1^2}+\frac{\ln(2)}{2^2}+\cdots+\frac{\ln(2n)}{(2n)^2}\right)+\frac{\ln(2)}{2}\left(\frac{1}{1}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)+\frac{1}{2}\left(\frac{\ln(2)}{2^2}+\cdots+\frac{\ln(n)}{n^2}\right).$$

Hence, $\lim_{n\to\infty} s_{2n} = -(-\zeta'(2))+\frac{\ln(2)}{2}\zeta(2)-\frac{1}{2}\zeta'(2) = \frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)$, as $\zeta(2)=\frac{\pi^2}{6}$.

Thus, $\eta'(2) = \sum_{n=1}^{\infty}\frac{(-1)^n\ln(n)}{n^2} = \frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)$.

2. Evaluation of some integrals

(1) The integral $\int_0^\infty \frac{\ln(x)}{1+e^x}\,dx = \Lambda'(0) = -\frac{(\ln(2))^2}{2}$.

Since $\Lambda^{(n)}(s) = \int_0^\infty \frac{x^s(\ln(x))^n}{1+e^x}\,dx$, $\int_0^\infty \frac{x^s\ln(x)}{1+e^x}\,dx = \Lambda'(s)$. It follows that $\int_0^\infty \frac{\ln(x)}{1+e^x}\,dx = \Lambda'(0)$.

By Theorem 7, for $s > -1$, $\Lambda(s) = \eta(s+1)\Gamma(s+1)$ and so

$$\Lambda'(s) = \eta'(s+1)\Gamma(s+1)+\eta(s+1)\Gamma'(s+1) .$$

Hence, $\Lambda'(0) = \eta'(1)\Gamma(1)+\eta(1)\Gamma'(1)$. By Theorem 10, $\eta'(1) = \sum_{k=2}^{\infty}(-1)^k\frac{\ln(k)}{k} = \ln(2)\gamma_0 - \frac{(\ln(2))^2}{2}$.
By Theorem 9, $\Gamma'(1) = -\gamma_0$. Now, $\eta(1) = \ln(2)$ and $\Gamma(1) = 1$.

Therefore, $\int_0^\infty \frac{(\ln(x))}{1+e^x}\,dx = \ln(2)\gamma_0 - \frac{(\ln(2))^2}{2} - \ln(2)\gamma_0 = -\frac{(\ln(2))^2}{2}$.

(2) The integral $\int_0^\infty \frac{\ln(x)}{1+e^{2x}}\,dx = -\frac{3}{4}(\ln(2))^2$.

$$\int_0^\infty \frac{\ln(x)}{1+e^{2x}}\,dx=\frac{1}{2}\int_0^\infty \frac{\ln(\frac{u}{2})}{1+e^u}\,dx$$, by a change of variable $u=2x$,

$$=-\frac{1}{2}\int_0^\infty \frac{\ln(2)}{1+e^u}\,du+\frac{1}{2}\int_0^\infty \frac{\ln(u)}{1+e^u}\,du=-\frac{1}{2}\left(\ln(2)\right)^2-\frac{1}{4}\left(\ln(2)\right)^2=-\frac{3}{4}\left(\ln(2)\right)^2 .$$

(3) The integral $\int_0^\infty \frac{x\ln(x)}{\cosh^2(x)}\,dx=\ln(2)-\frac{3}{2}\left(\ln(2)\right)^2$.

$$\int_0^\infty \frac{x\ln(x)}{\cosh^2(x)}\,dx=4\int_0^\infty \frac{x\ln(x)e^{2x}}{(1+e^{2x})^2}\,dx=4\left[-\frac{x\ln(x)}{2(1+e^{2x})}\right]_0^\infty+4\int_0^\infty \frac{\ln(x)+1}{2(1+e^{2x})}\,dx$$

$$=4\times 0+2\int_0^\infty \frac{\ln(x)}{1+e^{2x}}\,dx+2\int_0^\infty \frac{1}{1+e^{2x}}\,dx=2\left(-\frac{3}{4}\left(\ln(2)\right)^2\right)+2\left[\frac{\ln(1+e^{-2x})}{-2}\right]_0^\infty$$

$$=-\frac{3}{2}\left(\ln(2)\right)^2+\ln(2) .$$

Before we evaluate the subsequent integrals, we state the following formulae or properties concerning the Gamma function. We list also the approximate values of $\zeta'(2),\zeta''(2),\gamma_0$ and γ_1.

(i) $\Gamma(1)=1,\ \ \Gamma(2)=1$.

$$\Gamma(1)=\int_0^\infty e^{-x}dx=\left[-e^{-x}\right]_0^\infty=1. \quad \Gamma(2)=\int_0^\infty e^{-x}xdx=\left[-e^{-x}x\right]_0^\infty+\int_0^\infty e^{-x}dx=0+1=1.$$

(ii) $\Gamma(s+1)=s\Gamma(s)$ for $s > 0$.

$$\Gamma(s+1)=\int_0^\infty e^{-x}x^s dx=\left[-e^{-x}x^s\right]_0^\infty+s\int_0^\infty e^{-x}x^{s-1}dx=0+s\Gamma(s)=s\Gamma(s) .$$

(iii) $\Gamma(n)=(n-1)!$ for integer $n\geq 1$.

This follows from (ii) above.

(iv) $\Gamma'(1)=-\gamma_0$.

(v) $\Gamma''(1)=\gamma_0{}^2+\frac{\pi^2}{6}$.

(vi) $\Gamma'(n)=-\Gamma(n)\left(\frac{1}{n}+\gamma_0-\sum_{k=1}^{n}\frac{1}{k}\right)$.

(vii) $\Gamma^{(n+1)}(1)=-\gamma_0\Gamma^{(n)}(1)+n!\sum_{k=1}^{n}\frac{(-1)^{k+1}}{(n-k)!}\zeta(k+1)\Gamma^{(n-k)}(1)$ **,** for integer $n\geq 0$.

(viii) $(1-2^{1-s})\zeta(s)=\eta(s)$ for $s > 1$.

(ix) $\zeta(s)\Gamma(s)=\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx$**,** for $s > 1$. (See Theorem 18.)

(x) $\zeta'(2) \approx -0.93754825431584375370257409456786497789786028861482...$

(xi) $\zeta''(2) \approx 1.98928023429890102342085868742151638149 44...$

(xii) $\gamma_0 \approx 0.57721566490153286060651209008240243104 2...$

(xiii) $\gamma_1 \approx -0.07281584548367672486058637587 49...$

(xiv) $\zeta(3) \approx 1.2020569031595942853...$

(xv) $\zeta'(3) \approx -0.19812624288563685333...$

(4) The integral $\int_0^\infty \frac{(\ln(x))^2}{1+e^x}dx$,

$$\int_0^\infty \frac{(\ln(x))^2}{1+e^x}dx = \Lambda''(0) = \frac{1}{3}(\ln(2))^3 - 2\ln(2)\gamma_1 - \ln(2)\gamma_0^{\,2} + \frac{\pi^2}{6}\ln(2)$$

Using $\Lambda(s) = \eta(s+1)\Gamma(s+1)$ and differentiating twice, we get for $s > -1$,

$$\Lambda''(s) = \eta''(s+1)\Gamma(s+1) + \eta(s+1)\Gamma''(s+1) + 2\eta'(s+1)\Gamma'(s+1).$$

Therefore, $\int_0^\infty \frac{(\ln(x))^2}{1+e^x}dx = \Lambda''(0) = \eta''(1)\Gamma(1) + 2\eta'(1)\Gamma'(1) + \eta(1)\Gamma''(1)$.

By Theorem 12, $\eta''(1) = \frac{1}{3}(\ln(2))^3 - (\ln(2))^2\gamma_0 - 2\ln(2)\gamma_1$. By Theorem 10,

$\eta'(1) = \sum_{k=2}^{\infty}(-1)^k \frac{\ln(k)}{k} = \ln(2)\gamma_0 - \frac{(\ln(2))^2}{2}$ and $\eta(1) = \ln(2)$. $\Gamma(1) = 1$, by Theorem 9,

$\Gamma'(1) = -\gamma_0$, $\Gamma''(1) = \gamma_0^{\,2} + \frac{\pi^2}{6}$.

Thus, $\Lambda''(0) = \eta''(1)\Gamma(1) + 2\eta'(1)\Gamma'(1) + \eta(1)\Gamma''(1)$

$$= \left(\frac{1}{3}(\ln(2))^3 - (\ln(2))^2\gamma_0 - 2\ln(2)\gamma_1\right)\Gamma(1) + 2\left(\ln(2)\gamma_0 - \frac{(\ln(2))^2}{2}\right)(-\gamma_0) + \eta(1)\Gamma''(1),$$

$$= \frac{1}{3}(\ln(2))^3 - (\ln(2))^2\gamma_0 - 2\ln(2)\gamma_1 - 2\ln(2)\gamma_0^{\,2} + (\ln(2))^2\gamma_0 + \ln(2)\Gamma''(1)$$

$$= \frac{1}{3}(\ln(2))^3 - 2\ln(2)\gamma_1 - 2\ln(2)\gamma_0^{\,2} + \ln(2)\left(\gamma_0^{\,2} + \frac{\pi^2}{6}\right),$$

$$= \frac{1}{3}(\ln(2))^3 - 2\ln(2)\gamma_1 - \ln(2)\gamma_0^{\,2} + \frac{\pi^2}{6}\ln(2).$$

(5) The integral $\int_0^\infty \frac{x\ln(x)}{1+e^x}dx = \Lambda'(1) = \frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}(\ln(2) + 1 - \gamma_0)$.

Proof. Since $\Lambda'(s) = \eta'(s+1)\Gamma(s+1) + \eta(s+1)\Gamma'(s+1)$ for $s > -1$,

$$\Lambda'(1) = \eta'(2)\Gamma(2) + \eta(2)\Gamma'(2).$$

Now, $\eta'(2) = \sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n^2} = \frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\ln(2)$ by Theorem 13, $\Gamma(2) = 1$, $\eta(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ and $\Gamma'(2) = 1 + \Gamma'(1) = 1 - \gamma_0$. We may deduce the last identity as follows.

$$\Gamma'(1) = \int_0^{\infty} e^{-x}\ln(x)dx = \left[\left(x\ln(x) - x\right)e^{-x}\right]_0^{\infty} + \int_0^{\infty}\left(x\ln(x) - x\right)e^{-x}dx = 0 + \int_0^{\infty}\left(x\ln(x) - x\right)e^{-x}dx$$

$= \int_0^{\infty} x\ln(x)e^{-x}dx - \int_0^{\infty} xe^{-x}dx = \Gamma'(2) - \Gamma(2) = \Gamma'(2) - 1$. Therefore, $\Gamma'(2) = 1 + \Gamma'(1) = 1 - \gamma_0$.

Thus, $\Lambda'(1) = \eta'(2)\Gamma(2) + \eta(2)\Gamma'(2) = \left(\frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\ln(2)\right)\times 1 + \frac{\pi^2}{12}(1 - \gamma_0)$

$$= \frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\left(\ln(2) + 1 - \gamma_0\right).$$

It follows that $\int_0^{\infty} \frac{x\left(\ln(x)\right)}{1+e^x}dx = \frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\left(\ln(2) + 1 - \gamma_0\right)$.

Remark. Given that $\zeta(2) = \frac{\pi^2}{6}$, we can deduce easily that $\eta(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$. There are many methods of evaluating $\zeta(2)$. $\zeta'(2)$ can be expressed in terms of less familiar or computable constants, $\zeta'(2) = \frac{\pi^2}{6}(\gamma_0 + \ln(2\pi) - 12\ln(A))$, where A is the Glaisher-Kinkelin constant, which is approximately equal to 1.282427129100622636 87... . $\zeta'(2) \approx -0.9375482543158437537 0...$ It was shown by using a series representation of the Riemann Zeta function of Helmut Hasse that

$$\ln(A) = \frac{1}{8} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{n+1}\sum_{k=0}^{n}(-1)^k \binom{n}{k}(k+1)^2\ln(k+1).$$

(6) The integral $\int_0^{\infty} \frac{x\left(\ln(x)\right)^2}{1+e^x}dx = \Lambda''(1)$.

We have $\Lambda''(1) = \eta''(2)\Gamma(2) + \eta(2)\Gamma''(2) + 2\eta'(2)\Gamma'(2)$.

We have $\Gamma(2) = 1$, $\Gamma'(2) = 1 - \gamma_0$, $\eta(2) = \frac{\pi^2}{12}$ and $\eta'(2) = \frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\ln(2)$. It remains to determine $\Gamma''(2)$ and $\eta''(2)$. Starting with $\Gamma''(1) = \gamma_0^{\,2} + \frac{\pi^2}{6}$, we have

$$\Gamma''(1)=\int_0^{\infty} e^{-x}\left(\ln(x)\right)^2 dx=\left[e^{-x}(x\left(\ln(x)\right)^2-2x\ln(x)+2x)\right]_0^{\infty}+\int_0^{\infty} e^{-x}(x\left(\ln(x)\right)^2-2x\ln(x)+2x)dx$$

$$=0+\int_0^{\infty} e^{-x}(x\left(\ln(x)\right)^2-2x\ln(x)+2x)dx$$

$$=\int_0^{\infty} e^{-x}x\left(\ln(x)\right)^2 dx-2\int_0^{\infty} e^{-x}x\ln(x)dx+2\int_0^{\infty} e^{-x}xdx$$

$$=\Gamma''(2)-2\Gamma'(2)+2\Gamma(2)=\Gamma''(2)-2(1-\gamma_0)+2=\Gamma''(2)+2\gamma_0 .$$

Therefore, $\Gamma''(2)=\Gamma''(1)-2\gamma_0=\gamma_0{}^2+\frac{\pi^2}{6}-2\gamma_0$.

Now, since $\eta(s)=(1-2^{1-s})\zeta(s)$ for $s > 1$, $\eta'(s)=2^{1-s}\ln(2)\zeta(s)+(1-2^{1-s})\zeta'(s)$ for $s > 1$.
Differentiating again, we get for $s > 1$,

$$\eta''(s)=-2^{1-s}\left(\ln(2)\right)^2\zeta(s)+2^{1-s}\ln(2)\zeta'(s)+2^{1-s}\ln(2)\zeta'(s)+(1-2^{1-s})\zeta''(s)$$

$$=-2^{1-s}\left(\ln(2)\right)^2\zeta(s)+2^{2-s}\ln(2)\zeta'(s)+(1-2^{1-s})\zeta''(s) .$$

Thus, $\eta''(2)=-\frac{1}{2}\left(\ln(2)\right)^2\zeta(2)+\ln(2)\zeta'(2)+\frac{1}{2}\zeta''(2)$

$$=-\frac{\pi^2}{12}\left(\ln(2)\right)^2+\ln(2)\zeta'(2)+\frac{1}{2}\zeta''(2) .$$

Therefore,

$$\Lambda''(1)=-\frac{\pi^2}{12}\left(\ln(2)\right)^2+\ln(2)\zeta'(2)+\frac{1}{2}\zeta''(2)+\frac{\pi^2}{12}\Gamma''(2)+2\eta'(2)\Gamma'(2)$$

$$=-\frac{\pi^2}{12}\left(\ln(2)\right)^2+\ln(2)\zeta'(2)+\frac{1}{2}\zeta''(2)+\frac{\pi^2}{12}(\gamma_0{}^2+\frac{\pi^2}{6}-2\gamma_0)+2(\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2))\Gamma'(2)$$

$$=\frac{\pi^2}{12}\left(-\left(\ln(2)\right)^2+\gamma_0{}^2-2\gamma_0+\frac{\pi^2}{6}\right)+\ln(2)\zeta'(2)+\frac{1}{2}\zeta''(2)+\left(\zeta'(2)+\frac{\pi^2}{6}\ln(2)\right)\Gamma'(2)$$

$$=\frac{\pi^2}{12}\left(-\left(\ln(2)\right)^2+\gamma_0{}^2-2\gamma_0+\frac{\pi^2}{6}\right)+\ln(2)\zeta'(2)+\frac{1}{2}\zeta''(2)+\left(\zeta'(2)+\frac{\pi^2}{6}\ln(2)\right)\left(1-\gamma_0\right)$$

$$=\frac{\pi^2}{12}\left(-\left(\ln(2)\right)^2+\gamma_0{}^2-2\gamma_0+\frac{\pi^2}{6}\right)+\zeta'(2)\left(\ln(2)+1-\gamma_0\right)+\frac{1}{2}\zeta''(2)+\left(\frac{\pi^2}{6}\ln(2)\right)\left(1-\gamma_0\right)$$

$$=\frac{\pi^2}{12}\left(2\ln(2)-\left(\ln(2)\right)^2-2\ln(2)\gamma_0+\gamma_0{}^2-2\gamma_0+\frac{\pi^2}{6}\right)+\zeta'(2)\left(\ln(2)+1-\gamma_0\right)+\frac{1}{2}\zeta''(2) .$$

$$=\frac{\pi^2}{12}\left(2\ln(2)-\left(\ln(2)\right)^2-2(\ln(2)+1)\gamma_0+\gamma_0{}^2+\frac{\pi^2}{6}\right)+\zeta'(2)\left(\ln(2)+1-\gamma_0\right)+\frac{1}{2}\zeta''(2) .$$

(7) The integral $\int_0^\infty \frac{x\ln(x)}{e^x-1}dx = \zeta'(2)+\frac{\pi^2}{6}(1-\gamma_0)$.

Property (ix) states that $\zeta(s)\Gamma(s)=\int_0^\infty \frac{x^{s-1}}{e^x-1}dx$, for s > 1. We may show in the same fashion as for the relation for $\Lambda(s)$ that this relation holds for s > 1. Differentiating the relation with respect to s gives $\int_0^\infty \frac{x^{s-1}\ln(x)}{e^x-1}dx = \zeta'(s)\Gamma(s)+\zeta(s)\Gamma'(s)$ for s > 1.

Therefore,

$$\int_0^\infty \frac{x\ln(x)}{e^x-1}dx = \zeta'(2)\Gamma(2)+\zeta(2)\Gamma'(2)$$

$$=\zeta'(2)\cdot 1+\frac{\pi^2}{6}\Gamma'(2)=\zeta'(2)\cdot 1+\frac{\pi^2}{6}(1-\gamma_0)=\zeta'(2)+\frac{\pi^2}{6}(1-\gamma_0).$$

(8) The integrals, $\int_0^\infty \frac{x}{(1+e^x)^2}dx$, $\int_0^\infty \frac{x^2}{(1+e^x)^2}dx$ $\int_0^\infty \frac{\ln(x)}{(1+e^x)^2}dx$, $\int_0^\infty \frac{x\ln(x)}{(1+e^x)^2}dx$ and $\int_0^\infty \frac{x\left(\ln(x)\right)^2}{(1+e^x)^2}dx$.

For this set of integrals, we shall make use of the integral $\int_0^\infty \frac{x^s}{(1+e^x)^2}dx$. Since $\int_0^\infty \frac{x^s}{(1+e^x)^2}dx \le \int_0^\infty \frac{x^s}{1+e^x}dx$ and $\int_0^\infty \frac{x^s}{1+e^x}dx$ is convergent for all $s > -1$, $\int_0^\infty \frac{x^s}{(1+e^x)^2}dx$ is convergent for all $s > -1$. We can prove that $\int_0^\infty \frac{x^s}{(1+e^x)^2}dx$ is infinitely differentiable with respect to s for all $s > -1$ in the same way as proving the same for the function $\Lambda(x)$.

Theorem 14. The function $G(s)=\int_0^\infty \frac{x^s}{(1+e^x)^2}dx$ is finite for all $s > -1$. $G(s)$ is infinitely differentiable on $(-1,\infty)$. The derivatives $G^{(n)}(s)$ is given successively by differentiating under the integration sign and is given by

$$G^{(p)}(s)=\int_0^\infty \frac{x^s\left(\ln(x)\right)^p}{(1+e^x)^2}dx.$$

The proof of Theorem 14 is almost exactly the same as for the function $\Lambda(x)$ in Theorem 4 and is omitted.

Thus, $\int_0^\infty \frac{\ln(x)}{(1+e^x)^2}dx$ is the derivative of $G(s)$ at s = 0 and is equal to $G'(0)$, $\int_0^\infty \frac{x\ln(x)}{(1+e^x)^2}dx$ is the derivative of $G(s)$ at s =1, i.e., $\int_0^\infty \frac{x\ln(x)}{(1+e^x)^2}dx = G'(1)$ and $\int_0^\infty \frac{x\left(\ln(x)\right)^2}{(1+e^x)^2}dx = G''(1)$.

We seek to find a relation of $G(s)$ with the Gamma and Eta functions similar to that of $\Lambda(s)$.

Now, $\frac{x^s}{(1+e^x)^2} = \frac{x^s e^{-2x}}{(1+e^{-x})^2} = x^s e^{-x}\frac{e^{-x}}{(1+e^{-x})^2}$. We want to express $\frac{e^{-x}}{(1+e^{-x})^2}$ as a series.

Differentiating $\frac{1}{1+e^{-x}}$, we get $\frac{d}{dx}\frac{1}{1+e^{-x}} = \frac{e^{-x}}{(1+e^{-x})^2}$. Now, for $x > 0$,

$\frac{1}{1+e^{-x}} = \sum_{n=0}^{\infty}(-e^{-x})^n = \sum_{n=0}^{\infty}(-1)^n e^{-nx}$. Note that $\sum_{n=0}^{\infty}(-1)^n e^{-nx}$ converges uniformly on $[k, \infty)$ for any $k > 0$. And for $k > 0$, the differentiated series $\sum_{n=0}^{\infty}(-1)^{n+1} ne^{-nx}$ is uniformly convergent on $[k, \infty)$ since $\left|(-1)^{n+1} ne^{-nx}\right| \le ne^{-nk}$ and $\sum_{n=0}^{\infty} ne^{-nk}$ is convergent by the Weierstrass M-test. Therefore, we can differentiate $\frac{1}{1+e^{-x}} = \sum_{n=0}^{\infty}(-e^{-x})^n = \sum_{n=0}^{\infty}(-1)^n e^{-nx}$ term by term for x in $[k, \infty)$. Hence, for all $x > k$, $\frac{e^{-x}}{(1+e^{-x})^2} = \sum_{n=0}^{\infty}(-1)^{n+1} ne^{-nx}$. It follows that for all $x > k$,

$\frac{x^s}{(1+e^x)^2} = \frac{x^s e^{-2x}}{(1+e^{-x})^2} = x^s e^{-x}\sum_{n=0}^{\infty}(-1)^{n+1} ne^{-nx} = \sum_{n=0}^{\infty}(-1)^{n+1} x^s ne^{-(n+1)x} = \sum_{n=1}^{\infty}(-1)^{n+1} nx^s e^{-(n+1)x}$. Since this is true for any $k > 0$, $\sum_{n=1}^{\infty}(-1)^{n+1} nx^s e^{-(n+1)x}$ converges pointwise to $\frac{x^s}{(1+e^x)^2}$ for $x > 0$ and for any $s > -1$.

Now $nx^s e^{-(n+1)x}$ is Lebesgue integrable on $(0, \infty)$ and

$$\int_0^\infty nx^s e^{-(n+1)x}dx = \int_0^\infty n\left(\frac{y}{n+1}\right)^s e^{-y}\frac{1}{n+1}dy = \frac{n}{(n+1)^{s+1}}\int_0^\infty y^s e^{-y}dy = \left(\frac{1}{(n+1)^s} - \frac{1}{(n+1)^{s+1}}\right)\Gamma(s+1)$$

and $\Gamma(s+1)$ is finite for $s > -1$.

Note that $\int_0^\infty x^s e^{-x}dx = \Gamma(s+1) < \infty$ for $s > -1$, $\int_0^\infty x^{2s} e^{-x}dx = \Gamma(2s+1) < \infty$ for $s > -\frac{1}{2}$ and

$\int_0^\infty x^{2s} e^{-2x}dx \le \int_0^\infty x^{2s} e^{-x}dx = \Gamma(2s+1) < \infty$ for $s > -\frac{1}{2}$.

Therefore, $\int_0^\infty \frac{x^{2s}}{(1+e^x)^2}dx \le \int_0^\infty \frac{x^{2s}}{1+e^x}dx \le \int_0^\infty x^{2s} e^{-x}dx = \Gamma(2s+1) < \infty$ for $s > -\frac{1}{2}$. Similarly,

$\int_0^\infty \frac{x^{2s}}{(1+e^x)^4}dx \le \int_0^\infty \frac{x^{2s}}{1+e^x}dx \le \int_0^\infty x^{2s} e^{-x}dx = \Gamma(2s+1) < \infty$ for $s > -\frac{1}{2}$.

Recall previously we have shown that $\frac{x^s}{1+e^x}=\sum_{n=0}^{\infty}(-1)^n e^{-x(n+1)}x^s = x^s e^{-x}+\sum_{n=1}^{\infty}(-1)^n e^{-x(n+1)}x^s$.

Recall that $\frac{x^s}{(1+e^x)^2}=\sum_{n=1}^{\infty}(-1)^{n+1}nx^s e^{-(n+1)x}$. Note that each $nx^s e^{-x(n+1)}$ is Lebesgue integrable on (0, ∞). Let the p-th partial sum of $\sum_{n=1}^{\infty}(-1)^{n+1}nx^s e^{-(n+1)x}$ be given by $g_p(x)=\sum_{n=1}^{p}(-1)^n nx^s e^{-x(n+1)}$.

Then $g_p(x)$ converges pointwise to $\frac{x^s}{(1+e^x)^2}$.

$$\frac{x^s}{(1+e^x)^2}-g_p(x)=\sum_{n=p+1}^{\infty}(-1)^n x^s ne^{-x(n+1)}=(-1)^p e^{-xp}\sum_{n=1}^{\infty}(-1)^{n+1}(n+p)x^s e^{-x(n+1)}$$

$$=(-1)^p e^{-xp}\sum_{n=1}^{\infty}(-1)^{n+1}x^s ne^{-x(n+1)}+p(-1)^p e^{-xp}\sum_{n=1}^{\infty}(-1)^{n+1}x^s e^{-x(n+1)}$$

$$=(-1)^p e^{-xp}\sum_{n=1}^{\infty}(-1)^{n+1}x^s ne^{-x(n+1)}+p(-1)^p e^{-xp}\sum_{n=0}^{\infty}(-1)^{n+1}x^s e^{-x(n+1)}+p(-1)^p e^{-xp}x^s e^{-x}$$

Therefore, for $k > 0$ and for $s>-\frac{1}{2}$

$$\left|\int_k^{\infty}\left(\frac{x^s}{(1+e^x)^2}-g_p(x)\right)dx\right|\le\left|\int_k^{\infty}\left((-1)^p e^{-xp}\sum_{n=1}^{\infty}(-1)^{n+1}x^s ne^{-x(n+1)}\right)dx\right|+\left|\int_k^{\infty}\left(p(-1)^p e^{-xp}\sum_{n=0}^{\infty}(-1)^{n+1}x^s ne^{-x(n+1)}\right)dx\right|$$

$$+p\left|\int_k^{\infty}e^{-xp}x^s e^{-x}dx\right|$$

$$\le\sqrt{\int_k^{\infty}\left|e^{-2xp}\right|dx}\sqrt{\int_k^{\infty}\left|\left(\frac{x^s}{(1+e^x)^2}\right)\right|^2dx}+p\sqrt{\int_k^{\infty}\left|e^{-2xp}\right|dx}\sqrt{\int_k^{\infty}\left|\sum_{n=0}^{\infty}(-1)^{n+1}x^s e^{-x(n+1)}\right|^2dx}$$

$$+p\sqrt{\int_k^{\infty}e^{-2xp}dx}\sqrt{\int_k^{\infty}x^{2s}e^{-2x}dx}\text{, by the Holder inequality,}$$

$$\le\sqrt{\int_k^{\infty}e^{-2xp}dx}\sqrt{\int_k^{\infty}\frac{x^{2s}}{(1+e^x)^4}dx}+p\sqrt{\int_k^{\infty}e^{-2xp}dx}\sqrt{\int_k^{\infty}\left|\frac{x^s}{1+e^x}\right|^2dx}+p\sqrt{\int_k^{\infty}e^{-2xp}dx}\sqrt{\int_k^{\infty}\left|x^{2s}e^{-2x}\right|dx}$$

$$\le\sqrt{\frac{e^{-2kp}}{2p}}\sqrt{\int_k^{\infty}\frac{x^{2s}}{(1+e^x)^4}dx}+p\sqrt{\frac{e^{-2kp}}{2p}}\sqrt{\int_k^{\infty}\frac{x^{2s}}{(1+e^x)^2}dx}+p\sqrt{\frac{e^{-2kp}}{2p}}\sqrt{\int_k^{\infty}x^{2s}e^{-2x}dx}$$

$$\le\frac{e^{-2kp}}{\sqrt{2}\sqrt{p}}\sqrt{\int_k^{\infty}\frac{x^{2s}}{(1+e^x)^4}dx}+\frac{1}{\sqrt{2}}\sqrt{p}e^{-2kp}\sqrt{\int_k^{\infty}\frac{x^{2s}}{(1+e^x)^2}dx}+\frac{1}{\sqrt{2}}\sqrt{p}e^{-2kp}\sqrt{\int_k^{\infty}x^{2s}e^{-2x}dx}$$

$$\le\frac{e^{-2kp}}{\sqrt{2}\sqrt{p}}\sqrt{\int_0^{\infty}\frac{x^{2s}}{(1+e^x)^4}dx}+\frac{1}{\sqrt{2}}\sqrt{p}e^{-2kp}\sqrt{\int_0^{\infty}\frac{x^{2s}}{(1+e^x)^2}dx}+\frac{1}{\sqrt{2}}\sqrt{p}e^{-2kp}\sqrt{\int_0^{\infty}x^{2s}e^{-2x}dx}.$$

Note that $\int_0^\infty \left(\frac{x^{2s}}{(1+e^x)^4} \right) dx$, $\int_0^\infty \left(\frac{x^{2s}}{(1+e^x)^2} \right) dx$ and $\int_0^\infty x^{2s} e^{-2x} dx$ are finite for $s > -\frac{1}{2}$.

Since for any $k > 0$, $\lim_{p\to\infty} \frac{e^{-2kp}}{\sqrt{p}} = \lim_{p\to\infty} \sqrt{p} e^{-2kp} = 0$, $\lim_{p\to\infty} \left| \int_k^\infty \left(\frac{x^s}{(1+e^x)^2} - g_p(x) \right) dx \right| = 0$ for $s > -\frac{1}{2}$.

Therefore, for $s > -\frac{1}{2}$, $\int_k^\infty g_p(x) dx = \sum_{n=1}^{p} (-1)^n \int_k^\infty n x^s e^{-x(n+1)} dx \to \int_k^\infty \frac{x^s}{(1+e^x)^2} dx$.

That is, for $s > -\frac{1}{2}$ and $k > 0$,

$$\int_k^\infty \frac{x^s}{(1+e^x)^2} dx = \sum_{n=1}^{\infty} (-1)^n \int_k^\infty n x^s e^{-x(n+1)} dx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{(n+1)^{s+1}} \int_{(n+1)k}^\infty y^s e^{-y} dy ,$$

since $\int_k^\infty n x^s e^{-(n+1)x} dx = \int_{(n+1)k}^\infty n \left(\frac{y}{n+1} \right)^s e^{-y} \frac{1}{n+1} dy$.

Hence, for $s > -\frac{1}{2}$ and $k > 0$,

$$\int_k^\infty \frac{x^s}{(1+e^x)^2} dx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{(n+1)^{s+1}} \int_{(n+1)k}^\infty y^s e^{-y} dy . \quad \text{------------------------ (1)}$$

Now, for each $s > 0$, the function $f_n(k) = \int_{(n+1)k}^\infty y^s e^{-y} dy$ is a decreasing sequence of functions on $[0, \infty)$ uniformly bounded by $f_n(0) = \int_0^\infty y^s e^{-y} dy$ and

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{(n+1)^{s+1}} = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{(n+1)^s} - \frac{1}{(n+1)^{s+1}} \right)$ is uniformly convergent with respect to *k*.

By Abel Test, the series on the right-hand side of (1) converges uniformly with respect to *k* for $s > 0$. Therefore, it converges to a continuous function of *k*. Thus, taking limits,

$$\lim_{k\to 0} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{(n+1)^s} \int_{nk}^\infty y^s e^{-y} dy = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{(n+1)^{s+1}} \lim_{k\to 0} \int_{nk}^\infty y^s e^{-y} dy = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{(n+1)^{s+1}} \int_0^\infty y^s e^{-y} dy$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{(n+1)^s} - \frac{1}{(n+1)^{s+1}} \right) \Gamma(s+1) = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^s} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^{s+1}} \right) \Gamma(s+1)$$

$$= \left(-\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} + 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{s+1}} - 1 \right) \Gamma(s+1) = -\big(\eta(s) - \eta(s+1) \big) \Gamma(s+1) .$$

That is, $G(s) = \int_0^\infty \frac{x^s}{(1+e^x)^2} dx = \big(\eta(s+1) - \eta(s) \big) \Gamma(s+1)$ for $s > 0$. ------------------- (1)

Since $\eta(s)$ as a complex function can be analytically continued over the whole complex plane to an entire function and $\Gamma(s)$ can be extended over the complex plane with a countable number of poles at $\{0,-1,-2,-3,\cdots\}$, we can extend the relation (1) to $s>-1$. For the extended $\eta(s)$, we have $\eta(0)=\lim_{s\to 0^+}\eta(s)$. This means (1) is also valid for s = 0 if we define $\eta(0)$ to be its right hand limit at 0.

We shall show later that $\eta(0)=\lim_{s\to 0^+}\eta(s)=\frac{1}{2}$.

Hence, we have proved:

Theorem 15. For $s \geq 0$,

$$G(s)=\int_0^\infty \frac{x^s}{(1+e^x)^2}\,dx=\left(\eta(s+1)-\eta(s)\right)\Gamma(s+1)\,. \text{ --------------------- (2)}$$

Since $\eta(s)$ is infinitely differentiable for $s \geq 0$ and $\Gamma(s)$ is infinitely differentiable for $s > 0$, both sides of (2) are infinitely differentiable for $s \geq 0$.

Differentiating (2) for $s \geq 0$, we get,

$$G'(s)=\int_0^\infty \frac{x^s\ln(x)}{(1+e^x)^2}\,dx=\left(\eta'(s+1)-\eta'(s)\right)\Gamma(s+1)+\left(\eta(s+1)-\eta(s)\right)\Gamma'(s+1)\,. \text{ ------- (3)}$$

It follows that, $\int_0^\infty \frac{\ln(x)}{(1+e^x)^2}\,dx=G'(0)=\left(\eta'(1)-\eta'(0)\right)\Gamma(1)+\left(\eta(1)-\eta(0)\right)\Gamma'(1)$. It remains to determine $\eta(0)$ and $\eta'(0)$.

We claim that $\eta(0)=\frac{1}{2}$.

It is well known that $\eta(0)$ is the Abel sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1}$, which is $\frac{1}{2}$. We shall derive this as follows.

For $0<x\leq 1$, $(1+x)\sum_{n=1}^{\infty}(-x)^{n-1}\frac{1}{n^s}=1+\sum_{n=2}^{\infty}(-x)^{n-1}\left(\frac{1}{n^s}-\frac{1}{(n-1)^s}\right)$.

Then, $(1+x)^2\sum_{n=1}^{\infty}(-x)^{n-1}\frac{1}{n^s}=1+2x-\frac{x}{2^s}+\sum_{n=3}^{\infty}(-x)^{n-1}\left(\frac{1}{n^s}-\frac{2}{(n-1)^s}+\frac{1}{(n-2)^s}\right)$

$$=1+2x-\frac{x}{2^s}+\sum_{n=3}^{\infty}x^n(-1)^{n-1}\left(\frac{1}{n^s}-\frac{2}{(n-1)^s}+\frac{1}{(n-2)^s}\right)$$

This series converges uniformly for $0<x\leq 1$ and $s\in[0,1]$ by Abel's Test since $\sum_{n=3}^{\infty}(-1)^{n-1}\left(\frac{1}{n^s}-\frac{2}{(n-1)^s}+\frac{1}{(n-2)^s}\right)$ is convergent (uniformly with respect to x) and x^n is

monotone decreasing for all $0 < x \le 1$ and uniformly bounded. Therefore, we can interchange the order of the limits.

Thus, $4\eta(0) = \lim_{s\to 0^+}\lim_{x\to 1^-}(1+x)^2\sum_{n=1}^{\infty}(-x)^{n-1}\frac{1}{n^s} = \lim_{x\to 1^-}\lim_{s\to 0^{+-}}(1+x)^2\sum_{n=1}^{\infty}(-x)^{n-1}\frac{1}{n^s}$

$= \lim_{x\to 1^-}(1+x) = 2$.

Therefore, $\eta(0) = \frac{1}{2}$.

Lemma 16. $\eta'(0) = \frac{1}{2}\ln\left(\frac{\pi}{2}\right)$.

Proof. To determine $\eta'(0)$, we shall use the following series representation for $\eta(s)$

$$\eta(s) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^S} = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right).$$

Let $g_n(s) = \frac{1}{n^s} - \frac{1}{(n+1)^s}$ for integer $n \ge 1$ and for $-1 < s$. Note that the derivative of $\frac{1}{y^s} - \frac{1}{(y+1)^s}$ is $x^{-(s+1)}(x+1)^{-(s+1)}s(x^{s+1} - (x+1)^{s+1})$ and is negative for $s > 0$ and $x > 0$. It is positive for $-1 < s < 0$. We can thus conclude that $g_n(s) = \frac{1}{n^s} - \frac{1}{(n+1)^s}$ is a decreasing sequence of non-negative function for $s > 0$, i.e., $g_{n+1}(s) \le g_n(s)$ converging to the zero function. Moreover, $g_n(0) = 0$ for all $n \ge 1$. Thus, $g_n(s)$ is a decreasing sequence of non-negative function for $s \ge 0$. However, for $-1 < s < 0$, $g_n(s) = \frac{1}{n^s} - \frac{1}{(n+1)^s}$ is an increasing sequence of non-positive function. For $s > 0$, and $n \ge 2$, $g_n{}'(s) = \frac{\ln(n+1)}{(n+1)^s} - \frac{\ln(n)}{n^s} = 0 \Leftrightarrow s = \ln\left(\frac{\ln(n)}{\ln(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)$. Thus, for each integer $n \ge 2$, $\sup\{g_n(s) : s \ge 0\} = g_n\left(\ln\left(\frac{\ln(n)}{\ln(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)\right)$ and is positive and occurs at the point $a_n = \ln\left(\frac{\ln(n)}{\ln(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)$. Note that $a_n = \ln\left(\frac{\ln(n)}{\ln(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right) \to 0$, $\lim_{n\to\infty}\sup\{g_n(s) : s \ge 0\} = \lim_{n\to\infty} g_n(a_n)$. Now,

$$\lim_{n\to\infty} g_n(a_n) = \lim_{n\to\infty}\left(\frac{1}{n^{\ln\left(\frac{\ln(n)}{\ln(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)}} - \frac{1}{(n+1)^{\ln\left(\frac{\ln(n)}{\ln(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)}}\right) = e^{-1} - e^{-1} = 0, \text{ since}$$

$$\lim_{n\to\infty}\frac{1}{n^{\ln\left(\frac{\ln(n)}{\ln(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)}} = e^{-1} = \lim_{n\to\infty}\frac{1}{(n+1)^{\ln\left(\frac{\ln(n)}{\ln(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)}}$$ by applying L'Hôpital's Rule.

Hence, $\lim_{n\to\infty}\sup\{g_n(s): s\geq 0\}=0$.

Therefore, $\eta(s)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^S}=\frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)$ is uniformly convergent on $[0,\infty)$. Therefore, we may interchange the limit and the summation and

$$\lim_{s\to 0^+}\eta(s)=\frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\lim_{s\to 0^+}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)=\frac{1}{2}.$$

The term by term differentiated series of $\eta(s)=\frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)$ is

$$\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\left(-\frac{\ln(n)}{n^s}+\frac{\ln(n+1)}{(n+1)^s}\right)=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{\ln(n)}{n^s}-\frac{\ln(n+1)}{(n+1)^s}\right).$$

We shall show that $\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{\ln(n)}{n^s}-\frac{\ln(n+1)}{(n+1)^s}\right)$ converges uniformly on $[0,\infty)$. Let $h_n(s)=\frac{\ln(n)}{n^s}-\frac{\ln(n+1)}{(n+1)^s}$. Note that $h_n(s)$ is not a nonnegative function on $[0,\infty)$. Now, for $n\geq 2$, $h_n'(s)=\frac{(\ln(n+1))^2}{(n+1)^s}-\frac{(\ln(n))^2}{n^s}=0$ for $s>0$ if, and only if, $s=2\ln\left(\frac{\ln(n)}{\ln(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)$.
Therefore, $\inf_{s\geq 0}h_n(s)=\lim_{s\to 0^+}h(s)=\ln(n)-\ln(n+1)<0$ and $h_n(s)>0$ for $s>\ln\left(\frac{\ln(n)}{\ln(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)$ for $n\geq 2$. Thus, for $n\geq 2$ and for $s\geq 0$,

$f_n(s)=h_n(s)-\ln(n)+\ln(n+1)=\frac{\ln(n)}{n^s}-\frac{\ln(n+1)}{(n+1)^s}+\ln\left(\frac{1+n}{n}\right)$ is non-negative. We shall show that $f_n(s)$ is a monotonic decreasing sequence of functions for $n\geq 3$ on $s\geq 0$.

Observe that for $n\geq 2$, $h_n(s)=\frac{\ln(n)}{n^s}-\frac{\ln(n+1)}{(n+1)^s}=0$ if, and only if, $s=\ln\left(\frac{\ln(n)}{\ln(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)$.

We shall show that $\sup_{s\geq 0}h_n(s)\searrow 0$.

$h_n'(s)=\frac{(\ln(n+1))^2}{(n+1)^s}-\frac{(\ln(n))^2}{n^s}=0$ if, and only if, $s=2\ln\left(\frac{\ln(n)}{\ln(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)$ and the absolute maximum of h_n occurs at $b_n=\ln\left(\frac{\ln^2(n)}{\ln^2(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)$. We shall show that $h_n(b_n)\searrow 0$.

$$h_n(b_n)=\frac{\ln(n)}{n^{\ln\left(\frac{\ln^2(n)}{\ln^2(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)s}}-\frac{\ln(n+1)}{(n+1)^{\ln\left(\frac{\ln^2(n)}{\ln^2(n+1)}\right)\Big/\ln\left(\frac{n}{n+1}\right)}}$$

$$=\frac{\ln(n)\left(\frac{\ln^2(n+1)}{\ln^2(n)}\right)^{\ln(n)/(\ln(n)-\ln(n+1)}\left(\ln(n+1)-\ln(n)\right)}{\ln(n+1)}.$$

Since $\lim_{n\to\infty}\left(\frac{\ln^2(n+1)}{\ln^2(n)}\right)^{\ln(n)/(\ln(n)-\ln(n+1)}=e^{-2}$ by applying L'Hôpital's rule, $\lim_{n\to\infty}h_n(b_n)=0$.

That is, $\sup_{s\geq 0}h_n(s)\to 0$.

Then it follows that $f_n(s)=h_n(s)-\ln(n)+\ln(n+1)=\frac{\ln(n)}{n^s}-\frac{\ln(n+1)}{(n+1)^s}+\ln\left(\frac{1+n}{n}\right)$ tends to the zero function uniformly. Next, we claim that $f_n(s)$ is monotonic decreasing for $n>2$.

Consider the function, $g(x)=\frac{\ln(x)}{x^s}-\frac{\ln(x+1)}{(x+1)^s}+\ln\left(\frac{1+x}{x}\right)$.

Then $g'(x)=-\left(\frac{s\ln(x)}{x^{s+1}}-\frac{1}{x^{s+1}}+\frac{1}{x(x+1)}+\frac{1}{(x+1)^{s+1}}-s\frac{\ln(x+1)}{(x+1)^{s+1}}\right)$

$$=-\left(\left(\frac{s\ln(x)}{x^{s+1}}-\frac{1}{x^{s+1}}+\frac{1}{x}\right)-\left(s\frac{\ln(x+1)}{(x+1)^{s+1}}-\frac{1}{(x+1)^{s+1}}+\frac{1}{x+1}\right)\right)$$

We show that $\frac{s\ln(x)}{x^{s+1}}-\frac{1}{x^{s+1}}+\frac{1}{x}$ is decreasing with respect to x.

Observe that $\frac{1}{dx}\left(\frac{s\ln(x)}{x^{s+1}}-\frac{1}{x^{s+1}}+\frac{1}{x}\right)=-x^{-s-2}\left(x^s+s(s+1)\ln(x)-2s-1\right)$ and note that

for $x\geq 3$, $x^s+s(s+1)\ln(x)>2s+1$. Hence, $\frac{1}{dx}\left(\frac{s\ln(x)}{x^{s+1}}-\frac{1}{x^{s+1}}+\frac{1}{x}\right)<0$ for $x\geq 3$. Therefore, $\frac{s\ln(x)}{x^{s+1}}-\frac{1}{x^{s+1}}+\frac{1}{x}$ is decreasing for $x\geq 3$. Hence, $g'(x)<0$ for $x\geq 3$. Thus, $g(x)$ is a decreasing function for $x\geq 3$. It follows that $f_n(s)$ is monotonic decreasing for $n>2$. Note that $\sup_{s\geq 0}f_n(s)=\sup_{s\geq 0}\left(h_n(s)-\ln(n)+\ln(n+1)\right)=\sup_{s\geq 0}h_n(s)+\ln\left(\frac{1+n}{n}\right)\to 0$ as n tends to infinity.

Therefore, $\sum_{n=1}^{\infty}\left((-1)^n\left(h_n(s)+\ln\left(\frac{1+n}{n}\right)\right)\right)$ converges uniformly for $s\geq 0$. Hence, $\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n f_n(s)$ converges uniformly on $[0,\infty)$. Since $\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\ln\left(\frac{n+1}{n}\right)$ converges uniformly on $[0,\infty)$. It follows that $\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n h_n(s) = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\ln(n)}{n^s} - \frac{\ln(n+1)}{(n+1)^s}\right)$ converges uniformly on $[0,\infty)$.

We may thus differentiate $\eta(s)$ term by term to give:

$$\eta'(s) = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\left(-\frac{\ln(n)}{n^s} + \frac{\ln(n+1)}{(n+1)^s}\right) = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\left(-\ln(n^{n^{-s}}) + \ln\left((n+1)^{(n+1)^{-s}}\right)\right) \text{ ----- (4)}$$

$$= \frac{1}{2}\left[-\ln(1^{1^{-s}}) + \ln(2^{2^{-s}}) + \ln(2^{2^{-s}}) - \ln(3^{3^{-s}}) - \ln(3^{3^{-s}}) + \ln(4^{4^{-s}}) + \ln(4^{4^{-s}}) + \cdots\right]$$

$$= \frac{1}{2}\ln\left[\frac{2^{2^{-s}}}{1^{1^{-s}}}\frac{2^{2^{-s}}}{3^{3^{-s}}}\frac{4^{4^{-s}}}{3^{3^{-s}}}\frac{4^{4^{-s}}}{5^{5^{-s}}}\frac{6^{6^{-s}}}{5^{5^{-s}}}\frac{6^{6^{-s}}}{7^{7^{-s}}}\frac{8^{8^{-s}}}{7^{7^{-s}}}\frac{8^{8^{-s}}}{9^{9^{-s}}}\cdots\right] \quad \text{for any } s > 0.$$

Thus, $\eta'(0) = \lim_{s\to 0^+}\eta'(s) = \frac{1}{2}\ln\left[\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9}\cdots\right] = \frac{1}{2}\ln\left(\frac{\pi}{2}\right)$, by Wallis's product formula, since we may interchange the limit operation with the summation as the series on the right of (4) is uniformly convergent.

We list the values of the relevant eta function, gamma function and their derivatives.

$$\Gamma(1) = 1,\ \Gamma(2) = 1,\ \Gamma'(1) = -\gamma_0,\ \Gamma''(1) = \gamma_0^{\,2} + \frac{\pi^2}{6},\ \Gamma'(2) = 1 - \gamma_0,\ \Gamma''(2) = \gamma_0^{\,2} + \frac{\pi^2}{6} - 2\gamma_0,$$

$$\eta(0) = \frac{1}{2},\ \eta'(0) = \frac{1}{2}\ln\left(\frac{\pi}{2}\right),\ \eta(1) = \ln(2),\ \eta(2) = \frac{\pi^2}{12},\ \eta'(1) = \ln(2)\gamma_0 - \frac{(\ln(2))^2}{2},$$

$$\eta''(1) = \frac{1}{3}(\ln(2))^3 - (\ln(2))^2\gamma_0 - 2\ln(2)\gamma_1,\ \eta'(2) = \sum_{n=1}^{\infty}\frac{(-1)^n\ln(n)}{n^2} = \frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\ln(2),$$

$$\eta''(2) = -\frac{\pi^2}{12}(\ln(2))^2 + \ln(2)\zeta'(2) + \frac{1}{2}\zeta''(2).$$

Recall that for $s \ge 0$, $G(s) = \int_0^\infty \frac{x^s}{(1+e^x)^2}dx = \left(\eta(s+1) - \eta(s)\right)\Gamma(s+1)$ and

$$G'(s) = \int_0^\infty \frac{x^s\ln(x)}{(1+e^x)^2}dx = \left(\eta'(s+1) - \eta'(s)\right)\Gamma(s+1) + \left(\eta(s+1) - \eta(s)\right)\Gamma'(s+1).$$

Differentiating again gives,

$$G''(s) = \int_0^\infty \frac{x^s\ln^2(x)}{(1+e^x)^2}dx = \left(\eta''(s+1) - \eta''(s)\right)\Gamma(s+1) + \left(\eta'(s+1) - \eta'(s)\right)\Gamma'(s+1)$$

$$+\left(\eta'(s+1) - \eta'(s)\right)\Gamma'(s+1) + \left(\eta(s+1) - \eta(s)\right)\Gamma''(s+1)$$

$$= \left(\eta''(s+1) - \eta''(s)\right)\Gamma(s+1) + 2\left(\eta'(s+1) - \eta'(s)\right)\Gamma'(s+1) + \left(\eta(s+1) - \eta(s)\right)\Gamma''(s+1).$$

Therefore,

$$\int_0^\infty \frac{x}{(1+e^x)^2}\,dx = G(1) = \left(\eta(2)-\eta(1)\right)\Gamma(2) = \left(\frac{\pi^2}{12} - \ln(2)\right)\times 1 = \frac{\pi^2}{12} - \ln(2).$$

$$\int_0^\infty \frac{x^2}{(1+e^x)^2}\,dx = G(2) = (\eta(3)-\eta(2))\Gamma(3) = \left(\eta(3) - \frac{\pi^2}{12}\right)\times 2 = 2\eta(3) - \frac{\pi^2}{6}.$$

Now, by formula **(x)**, $\eta(3) = (1-2^{-2})\zeta(3) = \frac{3}{4}\zeta(3)$, and so $\int_0^\infty \frac{x^2}{(1+e^x)^2}\,dx = \frac{3}{2}\zeta(3) - \frac{\pi^2}{6}$.

$$\int_0^\infty \frac{\ln(x)}{(1+e^x)^2}\,dx = G'(0) = \left(\eta'(1)-\eta'(0)\right)\Gamma(1) + \left(\eta(1)-\eta(0)\right)\Gamma'(1)$$

$$= \left(\ln(2)\gamma_0 - \frac{\left(\ln(2)\right)^2}{2} - \frac{1}{2}\ln\left(\frac{\pi}{2}\right)\right)\times 1 + \left(\ln(2) - \frac{1}{2}\right)\left(-\gamma_0\right)$$

$$= \left(-\frac{1}{2}\ln\left(\frac{\pi}{2}\right) - \frac{\left(\ln(2)\right)^2}{2}\right)\times 1 + \frac{1}{2}\gamma_0$$

$$= \frac{1}{2}\left(\gamma_0 - \ln\left(\frac{\pi}{2}\right) - \left(\ln(2)\right)^2\right).$$

$$\int_0^\infty \frac{x\ln(x)}{(1+e^x)^2}\,dx = G'(1) = \left(\eta'(2)-\eta'(1)\right)\Gamma(2) + \left(\eta(2)-\eta(1)\right)\Gamma'(2)$$

$$= \left(\left(\frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\ln(2)\right) - \left(\ln(2)\gamma_0 - \frac{\left(\ln(2)\right)^2}{2}\right)\right)\times 1 + \left(\frac{\pi^2}{12} - \ln(2))\right)\left(1-\gamma_0\right)$$

$$= \left(\left(\frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\ln(2)\right) + \frac{\left(\ln(2)\right)^2}{2}\right) + \frac{\pi^2}{12}\left(1-\gamma_0\right) - \ln(2)$$

$$= \frac{1}{2}\left(\zeta'(2) + \left(\ln(2)\right)^2 - 2\ln(2) + \frac{\pi^2}{6}\left(\ln(2) + 1 - \gamma_0\right)\right).$$

$$\int_0^\infty \frac{x\ln^2(x)}{(1+e^x)^2}\,dx = G''(1) = \left(\eta''(2)-\eta''(1)\right)\Gamma(2) + 2\left(\eta'(2)-\eta'(1)\right)\Gamma'(2) + \left(\eta(2)-\eta(1)\right)\Gamma''(2)$$

$$= \left(\left(-\frac{\pi^2}{12}\left(\ln(2)\right)^2 + \ln(2)\zeta'(2) + \frac{1}{2}\zeta''(2)\right) - \left(\frac{1}{3}\left(\ln(2)\right)^3 - \left(\ln(2)\right)^2\gamma_0 - 2\ln(2)\gamma_1\right)\right)\times 1$$

$$+2\left(\left(\frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\ln(2)\right) - \left(\ln(2)\gamma_0 - \frac{\left(\ln(2)\right)^2}{2}\right)\right)\left(1-\gamma_0\right)$$

$$+\left(\frac{\pi^2}{12}-\ln(2)\right)\left(\gamma_0{}^2+\frac{\pi^2}{6}-2\gamma_0\right)$$

$$=\ln(2)\zeta'(2)+\frac{1}{2}\zeta''(2)+\zeta'(2)\left(1-\gamma_0\right)+\frac{\pi^2}{12}\left(2\left(1-\gamma_0\right)\ln(2)-\left(\ln(2)\right)^2\right)$$

$$-\left(\frac{1}{3}\left(\ln(2)\right)^3-\left(\ln(2)\right)^2\gamma_0-2\ln(2)\gamma_1\right)-2\left(\ln(2)\gamma_0-\frac{\left(\ln(2)\right)^2}{2}\right)(1-\gamma_0)$$

$$+\frac{\pi^2}{12}\left(\gamma_0{}^2+\frac{\pi^2}{6}-2\gamma_0\right)-\ln(2)\left(\gamma_0{}^2-2\gamma_0\right)-\frac{\pi^2}{6}\ln(2)$$

$$=\frac{1}{2}\zeta''(2)+\zeta'(2)\left(1-\gamma_0+\ln(2)\right)+\frac{\pi^2}{12}\left(2\left(1-\gamma_0\right)\ln(2)-\left(\ln(2)\right)^2+\gamma_0{}^2+\frac{\pi^2}{6}-2\gamma_0-2\ln(2)\right)$$

$$-\frac{1}{3}\left(\ln(2)\right)^3+\left(\ln(2)\right)^2\gamma_0+2\ln(2)\gamma_1+\left(1-\gamma_0\right)\left(\ln(2)\right)^2-2\left(\ln(2)\gamma_0\right)\left(1-\gamma_0\right)-\ln(2)\left(\gamma_0{}^2-2\gamma_0\right)$$

$$=\frac{1}{2}\zeta''(2)+\zeta'(2)\left(1-\gamma_0+\ln(2)\right)+\frac{\pi^2}{12}\left(-2\ln(2)\gamma_0-\left(\ln(2)\right)^2+\gamma_0{}^2+\frac{\pi^2}{6}-2\gamma_0\right)$$

$$-\frac{1}{3}\left(\ln(2)\right)^3+2\ln(2)\gamma_1+\left(\ln(2)\right)^2+\ln(2)\gamma_0{}^2$$

$$=\frac{1}{2}\zeta''(2)+\zeta'(2)\left(1-\gamma_0+\ln(2)\right)+\frac{\pi^2}{12}\left(\frac{\pi^2}{6}-\left(\ln(2)\right)^2+\gamma_0{}^2-2\gamma_0-2\ln(2)\gamma_0\right)$$

$$+\left(\ln(2)\right)^2-\frac{1}{3}\left(\ln(2)\right)^3+\ln(2)\gamma_0{}^2+2\ln(2)\gamma_1 .$$

The integrals $\int_0^\infty \frac{x^s}{(1+e^x)^3}\,dx$ and $\int_0^\infty \frac{x^s\ln(x)}{(1+e^x)^3}\,dx$ for $s\geq 0$.

We can show as for the case of $\int_0^\infty \frac{x^s}{(1+e^x)^2}\,dx$, that $\int_0^\infty \frac{x^s}{(1+e^x)^3}\,dx$ is convergent for all $s>-1$ and that as a function of *s*, it is infinitely differentiable for $s>-1$ by repeated us of differentiation under the integral sign. Let $H(s)=\int_0^\infty \frac{x^s}{(1+e^x)^3}\,dx$ for $s>-1$.

For $s>0$,

$$\int_0^\infty \frac{x^s}{(1+e^x)^3}\,dx=\int_0^\infty \frac{x^s e^{-3x}}{(1+e^{-x})^3}\,dx=\left[x^s e^{-2x}\frac{1}{2(1+e^{-x})^2}\right]_0^\infty-\int_0^\infty \frac{1}{2(1+e^{-x})^2}(sx^{s-1}e^{-2x}-2x^s e^{-2x})dx$$

$$=0-\int_0^\infty \frac{sx^{s-1}e^{-2x}}{2(1+e^{-x})^2}\,dx+\int_0^\infty \frac{x^s e^{-2x}}{(1+e^{-x})^2}\,dx=-\frac{s}{2}\int_0^\infty \frac{x^{s-1}e^{-2x}}{(1+e^{-x})^2}\,dx+\int_0^\infty \frac{x^s e^{-2x}}{(1+e^{-x})^2}\,dx$$

$$= -\frac{s}{2}\int_0^\infty \frac{x^{s-1}}{(1+e)^2}\,dx + \int_0^\infty \frac{x^s}{(1+e^x)^2}\,dx = -\frac{s}{2}G(s-1)+G(s)$$

$$= -\frac{s}{2}\big(\eta(s)-\eta(s-1)\big)\Gamma(s) + \big(\eta(s+1)-\eta(s)\big)\Gamma(s+1)\text{, if } s\geq 1$$

$$= \frac{s}{2}\big(\eta(s-1)-\eta(s)\big)\Gamma(s) + \big(\eta(s+1)-\eta(s)\big)\Gamma(s+1)\ .$$

Thus, for $s\geq 1$,

$$H(s) = \frac{1}{2}\big(\eta(s-1)-\eta(s)\big)\Gamma(s+1) + \big(\eta(s+1)-\eta(s)\big)\Gamma(s+1)\,.$$

$$= \frac{1}{2}\big(\eta(s-1)+2\eta(s+1)-3\eta(s)\big)\Gamma(s+1)\,.$$

By the analyticity of the Eta function, we can extend the above relation to $s>-1$.

Therefore, for $s>-1$,

$$H'(s) = \int_0^\infty \frac{x^s\ln(x)}{(1+e^x)^3}\,dx$$

$$= \frac{1}{2}\big(\eta'(s-1)+2\eta'(s+1)-3\eta'(s)\big)\Gamma(s+1) + \frac{1}{2}\big(\eta(s-1)+2\eta(s+1)-3\eta(s)\big)\Gamma'(s+1)\,.$$

$$\int_0^\infty \frac{x}{(1+e^x)^3}\,dx = H(1) = \frac{1}{2}\big(\eta(0)+2\eta(2)-3\eta(1)\big)\Gamma(2)$$

$$= \frac{1}{2}\left(\frac{1}{2}+\frac{\pi^2}{6}-3\ln(2)\right)\times 1 = \frac{1}{2}\left(\frac{1}{2}-3\ln(2)\right)+\frac{\pi^2}{12}\ .$$

Now,

$$H'(1) = \int_0^\infty \frac{x\ln(x)}{(1+e^x)^3}\,dx == \frac{1}{2}\big(\eta'(0)+2\eta'(2)-3\eta'(1)\big)\Gamma(2) + \frac{1}{2}\big(\eta(0)+2\eta(2)-3\eta(1)\big)\Gamma'(2)$$

$$= \frac{1}{2}\left(\frac{1}{2}\ln\left(\frac{\pi}{2}\right)+2\left(\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)\right)-3\left(\gamma_0\ln(2)-\frac{\big(\ln(2)\big)^2}{2}\right)\right)\times 1$$

$$+\frac{1}{2}\left(\frac{1}{2}+\frac{\pi^2}{6}-3\ln(2)\right)(1-\gamma_0)$$

$$= \frac{1}{4}\left(1-7\ln(2)+3\big(\ln(2)\big)^2+\ln(\pi)+\frac{\pi^2}{3}(1+\ln(2)-\gamma_0)-\gamma_0+2\zeta'(2)\right).$$

$$\int_0^\infty \frac{\ln(x)}{(1+e^x)^3}\,dx = H'(0) = \frac{1}{2}\big(\eta'(-1)+2\eta'(1)-3\eta'(0)\big)\Gamma(1) + \frac{1}{2}\big(\eta(-1)+2\eta(1)-3\eta(0)\big)\Gamma'(1)$$

$$=\frac{1}{2}\left(3\ln(A)-\frac{1}{4}-\frac{\ln(2)}{3}+2\left(\gamma_0\ln(2)-\frac{(\ln(2))^2}{2}\right)-\frac{3}{2}\ln\left(\frac{\pi}{2}\right)\right)$$

$$+\frac{1}{2}\left(\frac{1}{4}+2\ln(2)-3\frac{1}{2}\right)(-\gamma_0),$$

since $\eta'(1)=\gamma_0\ln(2)-\frac{(\ln(2))^2}{2}$, $\eta'(0)=\frac{1}{2}\ln\left(\frac{\pi}{2}\right)$, $\eta'(-1)=3\ln(A)-\frac{1}{4}-\frac{\ln(2)}{3}$,

$\eta(-1)=\frac{1}{4}$, $\eta(0)=\frac{1}{2}$ and $\eta(1)=\ln(2)$,

$$=\frac{3\ln(A)}{2}-\frac{1}{8}-\frac{\ln(2)}{6}+\gamma_0\ln(2)-\frac{(\ln(2))^2}{2}-\frac{3}{4}\ln\left(\frac{\pi}{2}\right)+\left(-\frac{1}{8}-\ln(2)+\frac{3}{4}\right)\gamma_0$$

$$=\frac{3\ln(A)}{2}-\frac{1}{8}-\frac{\ln(2)}{6}-\frac{(\ln(2))^2}{2}-\frac{3}{4}\ln\left(\frac{\pi}{2}\right)+\left(\frac{5}{8}\right)\gamma_0.$$

3. Integrals Connected with the Riemann Zeta Function

For $s > 0$, define $\Psi(s)=\int_0^\infty \frac{x^s}{e^x-1}dx$. We shall prove a relation with the Riemann Zeta function and the Gamma function. We note that it is finite and infinitely differentiable for $s > 0$.

Theorem 17. $\Psi(s)=\int_0^\infty \frac{x^s}{e^x-1}dx$ is infinitely differentiable for $s > 0$ and

$$\Psi^{(n)}(s)=\int_0^\infty \frac{x^s(\ln(x))^n}{e^x-1}dx.$$

Proof. That $\Psi(s)=\int_0^\infty \frac{x^s}{e^x-1}dx$ is finite is a consequence of the Lebesgue Monotone Convergence Theorem as asserted in the next theorem.

Let $f(x,s)=\frac{x^s}{e^x-1}=\frac{x^se^{-x}}{1-e^{-x}}$ for $s > 0$ and $0 < x < \infty$. Note that

$\frac{\partial^n f}{\partial s^n}(x,s)=\frac{x^s(\ln(x))^n}{e^x-1}=\frac{x^se^{-x}(\ln(x))^n}{1-e^{-x}}$ for $n\geq 1$. Observe that

$$\left|\frac{\partial^n f}{\partial s^n}(x,s)\right|=\left|\frac{x^s(\ln(x))^n}{e^x-1}\right|=\left|\frac{x^se^{-x}(\ln(x))^n}{1-e^{-x}}\right|\leq\frac{e}{e-1}x^se^{-x}\left|\ln(x)\right|^n \text{ for } x\geq 1.$$

Let $g(x,s)=\frac{x^s(\ln(x))^n}{e^x-1}=\frac{x^se^{-x}(\ln(x))^n}{1-e^{-x}}$ for $x > 0$ and $a < s < b$ and $0 < a < b$. Note that

$\lim_{x\to\infty} x^b e^{-x/2}\left(\ln(x)\right)^n = 0$. Hence $x^b e^{-x/2}\left(\ln(x)\right)^n$ is bounded above by $K > 0$ on $[1, \infty)$. Therefore,

$$|g(x,s)| \le \left|\frac{e}{e-1} x^s e^{-x}\left(\ln(x)\right)^n\right| = \frac{e}{e-1}\left|x^s e^{-x/2}\left(\ln(x)\right)^n\right| e^{-x/2} \le \frac{e}{e-1}\left|x^b e^{-x/2}\left(\ln(x)\right)^n\right| e^{-x/2} \le K\frac{e}{e-1}e^{-x/2}$$

for $x \ge 1$. Since $\frac{Ke}{e-1}e^{-x/2}$ is integrable on $[1,\infty)$, $g(x,s) = \frac{x^s e^{-x}\left(\ln(x)\right)^n}{1-e^{-x}}$ is integrable on $[1,\infty)$ for $0 < a < s$.

For $0 < x \le 1$,

$$|g(x,s)| \le \left|\frac{x^s e^{-x}\left(\ln(x)\right)^n}{1-e^{-x}}\right| \le \left|\frac{x^a e^{-x}\left(\ln(x)\right)^n}{1-e^{-x}}\right| \le \frac{xe^{-x}}{1-e^{-x}} x^{a-1}\left|\left(\ln(x)\right)^n\right|.$$

Note that $\frac{xe^{-x}}{1-e^{-x}}$ is continuous on $(0,1]$. Moreover,

$$\lim_{x\to 0^+}\frac{xe^{-x}}{1-e^{-x}} = \lim_{x\to 0^+}\frac{x}{e^x-1} = \lim_{x\to 0^+}\frac{1}{e^x} = 1.$$

Note that $\frac{x}{e^x-1}$ is differentiable on $(0,1]$ and its derivative, $\frac{1}{dx}\frac{xe^{-x}}{1-e^{-x}} = -\frac{e^x(x-1)+1}{(e^x-1)^2}$.

Since $e^x(x-1)+1$ cannot have zero in (0, 1]. It follows that $\frac{x}{e^x-1}$ has no stationary point in $(0,1]$.

Since $\frac{x}{e^x-1}$ is non-negative, its supremum in (0, 1] must be 1 as $\lim_{x\to 0^+}\frac{x}{e^x-1} = \lim_{x\to 0^+}\frac{1}{e^x} = 1 > \frac{1}{e-1}$.

That is, supremum of $\frac{x}{e^x-1}$ on $(0,1]$ is 1. Thus, $\frac{x}{e^x-1} \le 1$ for x in $(0,1]$. In particular, $\frac{xe^{-x}}{1-e^{-x}}$ is a decreasing function on $(0,1]$.

$$|g(x,s)| \le \left|\frac{x^s e^{-x}\left(\ln(x)\right)^n}{1-e^{-x}}\right| \le \left|\frac{x^a e^{-x}\left(\ln(x)\right)^n}{1-e^{-x}}\right| \le \frac{xe^{-x}}{1-e^{-x}} x^{a-1}\left|\left(\ln(x)\right)^n\right| \le x^{a-1}\left|\left(\ln(x)\right)^n\right|.$$

We have shown in the proof of Theorem 3 that $x^{a-1}\left(\ln(x)\right)^n$ is Lebesgue integrable on (0, 1].

Therefore, $g(x,s) = \frac{x^s e^{-x}\left(\ln(x)\right)^n}{1-e^{-x}}$ is Lebesgue integrable on (0, 1]. It follows that

$g(x,s) = \frac{x^s e^{-x}\left(\ln(x)\right)^n}{1+e^{-x}}$ is Lebesgue integrable on $(0,\infty)$.

We define $h(x) = \begin{cases} \frac{Ke}{e-1}e^{-x/2}, & \text{if } x \ge 1 \\ x^{a-1}\left|\left(\ln(x)\right)^n\right|, & \text{if } 0 < x < 1 \end{cases}$. Hence, $|g(x,s)| \le h(x)$ for $x > 0$ and $a < s < b$ and $0 < a < b$. Note that h is Lebesgue integrable on $(0, \infty)$. So, it follows as for the case of Gamma function by repeated use of Theorem 1 part (ii) of "*Integration Using Differentiation Under The Integral Sign*" that $\int_0^\infty \frac{x^s}{e^x-1}dx$ is infinitely differentiable at *s* for $a < s < b$. Since *a* and *b* are arbitrary $\int_0^\infty \frac{x^s}{e^x-1}dx$ is infinitely differentiable at *s* for $s > 0$ and

$$\Psi^{(n)}(s) = \int_0^\infty \frac{x^s\left(\ln(x)\right)^n}{e^x-1}dx \text{ for } n \geq 1.$$

Theorem 18. For $s > 0$, $\Psi(s) = \int_0^\infty \frac{x^s}{e^x-1}dx = \zeta(s+1)\Gamma(s+1)$.

Proof. Now $\frac{x^s}{e^x-1} = \frac{x^s e^{-x}}{1-e^{-x}} = x^s e^{-x}\sum_{n=0}^{\infty} e^{-nx} = \sum_{n=1}^{\infty} x^s e^{-(n+1)x}$.

For $s > 0$, $\int_0^\infty \frac{x^s}{e^x-1}dx = \int_0^\infty \frac{x^s e^{-x}}{1-e^{-x}}dx = \sum_{n=0}^{\infty}\int_0^\infty x^s e^{-(n+1)x} = \sum_{n=0}^{\infty}\int_0^\infty \frac{1}{n+1}\left(\frac{y}{n+1}\right)^s e^{-y}dy$

$$= \sum_{n=0}^{\infty}\int_0^\infty \frac{1}{(n+1)^{s+1}} y^s e^{-y}dy = \left(\sum_{n=0}^{\infty}\frac{1}{(n+1)^{s+1}}\right)\int_0^\infty y^s e^{-y}dy = \zeta(s+1)\Gamma(s+1),$$

by the Lebesgue Monotone Convergence Theorem.

Now we turn our attention to the integral $\frac{x^s}{(e^x-1)^2}$ for $s > 1$.

Theorem 19. For $s > 1$,

(1) $\int_0^\infty \frac{x^s}{(e^x-1)^2}dx = \Gamma(s+1)\left(\zeta(s)-\zeta(s+1)\right)$, (2) $\int_0^\infty \frac{x^s e^x}{(e^x-1)^2}dx = \Gamma(s+1)\zeta(s)$.

Proof.

Now, $\frac{x^s}{(e^x-1)^2} = \frac{x^s e^{-2x}}{(1-e^{-x})^2} = x^s\frac{e^{-2x}}{(1-e^{-x})^2}$. We want to express $\frac{e^{-2x}}{(1-e^{-x})^2}$ as a series.

Differentiating $\frac{1}{1-e^{-x}}$, we get $\frac{d}{dx}\frac{1}{1-e^{-x}} = -\frac{e^{-x}}{(1-e^{-x})^2}$. Now, for $x > 0$, $\frac{1}{1-e^{-x}} = \sum_{n=0}^{\infty} e^{-nx}$.

Note that $\sum_{n=0}^{\infty} e^{-nx}$ converges uniformly on $[k, \infty)$ for any $k > 0$. And for $k > 0$, the differentiated series $\sum_{n=0}^{\infty} -ne^{-nx}$ is uniformly convergent on $[k, \infty)$ since $\left|ne^{-nx}\right| \leq ne^{-nk}$ and $\sum_{n=0}^{\infty} ne^{-nk}$ is convergent by the Weierstrass M-test. Therefore, we can differentiate $\frac{1}{1-e^{-x}} = \sum_{n=0}^{\infty} e^{-nx}$ term by term for *x* in $[k, \infty)$.

That is, for *x* in $[k, \infty)$, $\frac{d}{dx}\frac{1}{1-e^{-x}} = -\frac{e^{-x}}{(1-e^{-x})^2} = -\sum_{n=0}^{\infty} ne^{-nx}$.

It follows that for all $x > k$,

$\dfrac{x^s}{(e^x-1)^2} = \dfrac{x^s e^{-2x}}{(1-e^{-x})^2} = x^s e^{-x}\sum_{n=0}^{\infty} n e^{-nx} = \sum_{n=1}^{\infty} n x^s e^{-(n+1)x}$. Since this is true for any $k > 0$,

$\sum_{n=1}^{\infty} n x^s e^{-(n+1)x}$ converges pointwise to $\dfrac{x^s}{(e^x-1)^2}$ for $x > 0$ and for any $s > 1$.

Now, $\int_0^{\infty} n x^s e^{-(n+1)x} dx = \int_0^{\infty} n\left(\dfrac{y}{n+1}\right)^s e^{-y}\dfrac{1}{n+1}dy = \dfrac{n}{(n+1)^{s+1}}\int_0^{\infty} y^s e^{-y} dy = \dfrac{n}{(n+1)^{s+1}}\Gamma(s+1)$.

Therefore, for $s > 1$, by the Lebesgue Monotone Convergence Theorem,

$$\int_0^{\infty}\frac{x^s}{(e^x-1)^2}dx = \sum_{n=1}^{\infty}\int_0^{\infty} n x^s e^{-(n+1)x}dx = \sum_{n=0}^{\infty}\frac{n}{(n+1)^{s+1}}\Gamma(s+1) = \left(\sum_{n=0}^{\infty}\frac{n}{(n+1)^{s+1}}\right)\Gamma(s+1)$$

$$= \Gamma(s+1)\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)^s} - \frac{1}{(n+1)^{s+1}}\right)$$

$$= \Gamma(s+1)\big(\zeta(s) - \zeta(s+1)\big).$$

In a similar way, we have,

$$\int_0^{\infty}\frac{x^s e^x}{(e^x-1)^2}dx = \sum_{n=1}^{\infty}\int_0^{\infty} n x^s e^{-nx}dx = \sum_{n=1}^{\infty}\frac{1}{n^s}\Gamma(s+1) = \Gamma(s+1)\sum_{n=1}^{\infty}\frac{1}{n^s} = \Gamma(s+1)\zeta(s) .$$

Example of application.

1. $$\int_0^{\infty}\frac{x^s}{\cosh(x)-1}dx = \int_0^{\infty}\frac{x^s}{2\sinh^2(\left(\frac{x}{2}\right))}dx = \frac{1}{2}\int_0^{\infty}\frac{x^s}{\left(\frac{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}{2}\right)^2}dx$$

$$= 2\int_0^{\infty}\frac{x^s}{\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)^2}dx = 2\int_0^{\infty}\frac{x^s e^x}{\left(e^x-1\right)^2}dx = 2\Gamma(s+1)\zeta(s) .$$

2. $\int_0^{\infty}\dfrac{x^2}{(e^x-1)^2}dx = \Gamma(3)\big(\zeta(2)-\zeta(3)\big) = 2\left(\dfrac{\pi^2}{6}-\zeta(3)\right)$,

3. $\int_0^{\infty}\dfrac{x^3}{(e^x-1)^2}dx = \Gamma(4)\big(\zeta(3)-\zeta(4)\big) = 6\big(\zeta(3)-\zeta(4)\big)$

4. For $s > 0$, $\Psi(s) = \int_0^{\infty}\dfrac{x^s}{e^x-1}dx = \zeta(s+1)\Gamma(s+1)$,

$$\Psi'(s) = \int_0^{\infty}\frac{x^s\ln(x)}{e^x-1}dx = \zeta'(s+1)\Gamma(s+1) + \zeta(s+1)\Gamma'(s+1) .$$

Differentiate again gives,

$$\int_0^\infty \frac{x^s(\ln(x))^2}{e^x-1}dx = \Psi''(s) = \zeta''(s+1)\Gamma(s+1)+\zeta'(s+1)\Gamma'(s+1)+\zeta'(s+1)\Gamma'(s+1)+\zeta(s+1)\Gamma''(s+1)$$

$$= \zeta''(s+1)\Gamma(s+1)+2\zeta'(s+1)\Gamma'(s+1)+\zeta(s+1)\Gamma''(s+1).$$

Therefore, $\int_0^\infty \frac{x\ln(x)}{e^x-1}dx = \Psi'(1) = \zeta'(2)\Gamma(2)+\zeta(2)\Gamma'(2) = \zeta'(2)+\frac{\pi^2}{6}(1-\gamma_0)$ and

$$\int_0^\infty \frac{x(\ln(x))^2}{e^x-1}dx = \Psi''(1) = \zeta''(2)\Gamma(2)+2\zeta'(2)\Gamma'(2)+\zeta(2)\Gamma''(2)$$

$$= \zeta''(2)+2\zeta'(2)(1-\gamma_0)+\frac{\pi^2}{6}\left(\gamma_0{}^2+\frac{\pi^2}{6}-2\gamma_0\right).$$

4. More Integrals of the type $\int_0^\infty \frac{x^s e^x}{(1+e^x)^2}dx$, $\int_0^\infty \frac{x^s e^x\ln(x)}{(1+e^x)^2}dx$, $\int_0^\infty \frac{x^s\ln^2(x)}{(1+e^x)^2}dx$ $\int_0^\infty \frac{x^s e^{-kx}}{(1+e^x)^2}dx$, $\int_0^\infty \frac{x^s e^{-kx}\ln(x)}{(1+e^x)^2}dx$, $\int_0^\infty \frac{x^s e^{-kx}\left(\ln(x)\right)^2}{(1+e^x)^2}dx$ $k\geq 0$.

We can show that the integrals $\int_0^\infty \frac{x^s e^x}{(1+e^x)^2}dx$ and $\int_0^\infty \frac{x^s e^{-kx}}{(1+e^x)^2}dx$ for $k\geq 0$ and $s>-1$ are finite and infinite differentiable with respect to s in the same way that we show in Theorem 4 that $\Lambda(s)=\int_0^\infty \frac{x^s}{1+e^x}dx$ is infinitely differentiable for $s > -1$ and that we can apply differentiation under the integration sign repeatedly.

Using the expansion for $\frac{x^s}{(1+e^x)^2}$, $\frac{x^s}{(1+e^x)^2} = \sum_{n=1}^{\infty}(-1)^{n+1}nx^s e^{-(n+1)x}$.

In the proof of Theorem 14, we get for $s>0$ and $k\geq 0$,

$$\frac{x^s e^{-kx}}{(1+e^x)^2} = \sum_{n=1}^{\infty}(-1)^{n+1}nx^s e^{-(n+k+1)x}.$$

Note that $\int_0^\infty nx^s e^{-(n+k+1)x}dx = \int_0^\infty n\left(\frac{y}{n+k+1}\right)^s e^{-y}\frac{1}{n+k+1}dy = \frac{n}{(n+k+1)^{s+1}}\int_0^\infty y^s e^{-y}dy$

$$= \frac{n}{(n+k+1)^{s+1}}\Gamma(s+1) .$$

As in the proof of Theorem 14, we can integrate $\frac{x^s e^{-kx}}{(1+e^x)^2}$ term by term.

$$\int_0^\infty \frac{x^s e^{-kx}}{(1+e^x)^2}dx = \left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(n+k+1)^{s+1}}\right)\Gamma(s+1) \quad \text{for } s>0$$

$$=\left(\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{(n+k+1)^s}-\frac{k+1}{(n+k+1)^{s+1}}\right)\right)\Gamma(s+1)$$

$$=\left(\sum_{n=1}^{\infty}(-1)^{n+k}\left(\frac{1}{n^s}-(k+1)\frac{1}{n^{s+1}}\right)-\sum_{n=1}^{k+1}(-1)^{n+k}\left(\frac{1}{n^s}-(k+1)\frac{1}{n^{s+1}}\right)\right)\Gamma(s+1)$$

$$=(-1)^{k+1}\left(\eta(s)-(k+1)\eta(s+1)\right)\Gamma(s+1)-\sum_{n=1}^{k+1}(-1)^{n+k}\left(\frac{1}{n^s}-(k+1)\frac{1}{n^{s+1}}\right)\Gamma(s+1)$$

$$=(-1)^{k+1}\left(\eta(s)-(k+1)\eta(s+1)\right)\Gamma(s+1)-(-1)^{k+1}\sum_{n=1}^{k+1}(-1)^{n+1}\left(\frac{1}{n^s}-(k+1)\frac{1}{n^{s+1}}\right)\Gamma(s+1).$$

With $k = 0$, we recover Theorem 15, for $s > 0$,

$$\int_0^{\infty}\frac{x^s}{(1+e^x)^2}\,dx=\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(n+1)^{s+1}}\right)\Gamma(s+1)$$

$$=(-1)\left(\eta(s)-\eta(s+1)\right)\Gamma(s+1)=\left(\eta(s+1)-\eta(s)\right)\Gamma(s+1).$$

By continuity of η at $s=0$, we can extend this relation to $s = 0$.

Hence, for $s\geq 0$,

$$\int_0^{\infty}\frac{x^s}{(1+e^x)^2}\,dx=\left(\eta(s+1)-\eta(s)\right)\Gamma(s+1). \quad \text{-------------------------------} \quad (1)$$

With $k = 1$, we get for $s > 0$,

$$\int_0^{\infty}\frac{x^s e^{-x}}{(1+e^x)^2}\,dx=\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(n+2)^{s+1}}\right)\Gamma(s+1)$$

$$=\left(\eta(s)-2\eta(s+1)\right)\Gamma(s+1)-\left(\sum_{n=1}^{2}(-1)^{n+1}\left(\frac{1}{n^s}-2\frac{1}{n^{s+1}}\right)\right)\Gamma(s+1)$$

$$=\left(\eta(s)-2\eta(s+1)\right)\Gamma(s+1)-\left(1-2-\frac{1}{2^s}+2\frac{1}{2^{S+1}}\right)\Gamma(s+1)$$

$$=\left(\eta(s)-2\eta(s+1)+1\right)\Gamma(s+1).$$

Again, by continuity, we have for $s\geq 0$,

$$\int_0^{\infty}\frac{x^s e^{-x}}{(1+e^x)^2}\,dx=\left(\eta(s)-2\eta(s+1)+1\right)\Gamma(s+1). \quad \text{---------------------------} \quad (2)$$

With $k =2$, we get for $s>0$,

$$\int_0^{\infty}\frac{x^s e^{-2x}}{(1+e^x)^2}\,dx=\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(n+3)^{s+1}}\right)\Gamma(s+1)$$

$$= -\left(\eta(s) - 3\eta(s+1)\right)\Gamma(s+1) + \sum_{n=1}^{3}(-1)^{n+1}\left(\frac{1}{n^s} - 3\frac{1}{n^{s+1}}\right)\Gamma(s+1)$$

$$= \left(3\eta(s+1) - \eta(s)\right)\Gamma(s+1) + (1-3) - \left(\frac{1}{2^s} - 3\frac{1}{2^{s+1}} + \frac{1}{3^s} - 3\frac{1}{3^{s+1}}\right)\Gamma(s+1)$$

$$= \left(3\eta(s+1) - \eta(s) - 2 + \frac{1}{2^{s+1}}\right)\Gamma(s+1).$$

By continuity, we have, for $s \geq 0$,

$$\int_0^\infty \frac{x^s e^{-2x}}{(1+e^x)^2}dx = \left(3\eta(s+1) - \eta(s) - 2 + \frac{1}{2^{s+1}}\right)\Gamma(s+1). \text{ ------------------ (3)}$$

Now, $\frac{x^s e^x}{(1+e^x)^2} = \sum_{n=1}^{\infty}(-1)^{n+1} n x^s e^{-nx}$ and for $s > 0$, we get

$$\int_0^\infty \frac{x^s e^x}{(1+e^x)^2}dx = \sum_{n=1}^{\infty}(-1)^{n+1}\int_0^\infty n x^s e^{-nx}dx = \sum_{n=1}^{\infty}(-1)^{n+1}\int_0^\infty n\left(\frac{y}{n}\right)^s e^{-y}\frac{1}{n}dy$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\int_0^\infty \frac{1}{n^s} y^s e^{-y}dy = \left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^s}\right)\int_0^\infty e^{-y}y^s dy = \eta(s)\Gamma(s+1).$$

Again, by continuity, we get for $s \geq 0$,

$$\int_0^\infty \frac{x^s e^x}{(1+e^x)^2}dx = \eta(s)\Gamma(s+1). \text{ -- } \mathbf{(4)}$$

Note that for $s \geq 0$ and $k \geq 2$, $\int_0^\infty \frac{x^s e^{kx}}{(1+e^x)^2}dx$ is divergent.

Therefore, for $s \geq 0$,

$$\int_0^\infty \frac{x^s e^x \ln(x)}{(1+e^x)^2}dx = \eta'(s)\Gamma(s+1) + \eta(s)\Gamma'(s+1), \text{ ------------------ } \mathbf{(5)}$$

$$\int_0^\infty \frac{x^s e^{-x}\ln(x)}{(1+e^x)^2}dx = \left(\eta'(s) - 2\eta'(s+1)\right)\Gamma(s+1) + \left(\eta(s) - 2\eta(s+1) + 1\right)\Gamma'(s+1). \text{ ------- } \mathbf{(6)}$$

$$\int_0^\infty \frac{x^s e^{-x}\left(\ln(x)\right)^2}{(1+e^x)^2}dx = \left(\eta''(s) - 2\eta''(s+1)\right)\Gamma(s+1) + \left(\eta'(s) - 2\eta'(s+1)\right)\Gamma'(s+1)$$

$$+\left(\eta'(s) - 2\eta'(s+1)\right)\Gamma'(s+1) + \left(\eta(s) - 2\eta(s+1) + 1\right)\Gamma''(s+1)$$

$$= \left(\eta''(s) - 2\eta''(s+1)\right)\Gamma(s+1) + 2\left(\eta'(s) - 2\eta'(s+1)\right)\Gamma'(s+1) + \left(\eta(s) - 2\eta(s+1) + 1\right)\Gamma''(s+1).$$

------------------ (7)

Application.

Using (2),

$$\int_0^\infty \frac{e^{-x}}{(1+e^x)^2}\,dx == \left(\eta(0)-2\eta(1)+1\right)\Gamma(1)=\left(\frac{1}{2}-2\ln(2)+1\right)\bullet 1=\frac{3}{2}-2\ln(2)\,,$$

$$\int_0^\infty \frac{xe^{-x}}{(1+e^x)^2}\,dx == \left(\eta(1)-2\eta(2)+1\right)\Gamma(2)=\left(\ln(2)-\frac{\pi^2}{6}+1\right)\bullet 1=\ln(2)-\frac{\pi^2}{6}+1\,.$$

Using (3),

$$\int_0^\infty \frac{e^{-2x}}{(1+e^x)^2}\,dx = \left(3\eta(1)-\eta(0)-2+\frac{1}{2}\right)\Gamma(1)=3\ln(2)-\frac{1}{2}-2+\frac{1}{2}=3\ln(2)-2\,,$$

$$\int_0^\infty \frac{xe^{-2x}}{(1+e^x)^2}\,dx = \left(3\eta(2)-\eta(1)-2+\frac{1}{2^2}\right)\Gamma(2)=3\frac{\pi^2}{12}-\ln(2)-\frac{7}{4}=\frac{\pi^2}{4}-\ln(2)-\frac{7}{4}\,,$$

$$\int_0^\infty \frac{x^2e^{-2x}}{(1+e^x)^2}\,dx = \left(3\eta(3)-\eta(2)-2+\frac{1}{2^3}\right)\Gamma(3)=2\left(3(1-2^{-2})\zeta(3)-\frac{\pi^2}{12}-\frac{15}{8}\right)$$

$$=2\left(3(1-2^{-2})\zeta(3)-\frac{\pi^2}{12}-\frac{15}{8}\right)=\frac{9}{2}\zeta(3)-\frac{\pi^2}{6}-\frac{15}{4}\quad .$$

From (6),

$$\int_0^\infty \frac{e^{-x}\ln(x)}{(1+e^x)^2}\,dx = \left(\eta'(0)-2\eta'(1)\right)\Gamma(1)+\left(\eta(0)-2\eta(1)+1\right)\Gamma'(1)$$

$$\int_0^\infty \frac{e^{-x}\ln(x)}{(1+e^x)^2}\,dx = \left(\frac{1}{2}\ln\left(\frac{\pi}{2}\right)-2\left(\ln(2)\gamma_0-\frac{\left(\ln(2)\right)^2}{2}\right)\right)+\left(\frac{1}{2}-2\ln(2)+1\right)\left(-\gamma_0\right)$$

$$=\frac{1}{2}\ln\left(\frac{\pi}{2}\right)-2\ln(2)\gamma_0+\left(\ln(2)\right)^2+\left(2\ln(2)-\frac{3}{2}\right)\gamma_0$$

$$=\frac{1}{2}\ln\left(\frac{\pi}{2}\right)+\left(\ln(2)\right)^2-\frac{3}{2}\gamma_0\,.$$

$$\int_0^\infty \frac{xe^{-x}\ln(x)}{(1+e^x)^2}\,dx = \left(\eta'(1)-2\eta'(2)\right)\Gamma(2)+\left(\eta(1)-2\eta(2)+1\right)\Gamma'(2)$$

$$=\left(\eta'(1)-2\eta'(2)\right)\Gamma(2)+\left(\eta(1)-2\eta(2)+1\right)\Gamma'(2)$$

$$=\left(\left(\ln(2)\gamma_0-\frac{\left(\ln(2)\right)^2}{2}\right)-2\left(\frac{1}{2}\zeta'(2)+\frac{\pi^2}{12}\ln(2)\right)\right)+\left(\ln(2)-2\frac{\pi^2}{12}+1\right)\left(1-\gamma_0\right)$$

$$= \ln(2)\gamma_0 - \frac{(\ln(2))^2}{2} - \zeta'(2) - \frac{\pi^2}{6}\ln(2) + \ln(2) - \frac{\pi^2}{6} + 1 - \left(\ln(2) - \frac{\pi^2}{6} + 1\right)\gamma_0$$

$$= \ln(2) - \frac{(\ln(2))^2}{2} - \frac{\pi^2}{6}\ln(2) - \frac{\pi^2}{6} + 1 - \zeta'(2) + \left(\frac{\pi^2}{6} - 1\right)\gamma_0 .$$

Using (4),

$$\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx = \eta(0)\Gamma(1) = \frac{1}{2},$$

$$\int_0^\infty \frac{xe^x}{(1+e^x)^2}\,dx = \eta(1)\Gamma(2) = \ln(2),$$

$$\int_0^\infty \frac{x^2e^x}{(1+e^x)^2}\,dx = \eta(2)\Gamma(3) = \frac{\pi^2}{12}\bullet 2 = \frac{\pi^2}{6},$$

$$\int_0^\infty \frac{x^3e^x}{(1+e^x)^2}\,dx = \eta(3)\Gamma(4) = 6(1-2^{-2})\zeta(3) = \frac{9}{2}\zeta(3).$$

From (5),

$$\int_0^\infty \frac{e^x\ln(x)}{(1+e^x)^2}\,dx = \eta'(0)\Gamma(1) + \eta(0)\Gamma'(1) = \frac{1}{2}\ln\left(\frac{\pi}{2}\right) + \frac{1}{2}(-\gamma_0) = \frac{1}{2}\left(\ln\left(\frac{\pi}{2}\right) - \gamma_0\right)$$

$$\int_0^\infty \frac{xe^x\ln(x)}{(1+e^x)^2}\,dx = \eta'(1)\Gamma(2) + \eta(1)\Gamma'(2) = \left(\ln(2)\gamma_0 - \frac{(\ln(2))^2}{2}\right) + \ln(2)(1-\gamma_0)$$

$$= \ln(2) - \frac{(\ln(2))^2}{2},$$

$$\int_0^\infty \frac{x^2e^x\ln(x)}{(1+e^x)^2}\,dx = \eta'(2)\Gamma(3) + \eta(2)\Gamma'(3) = \left(\frac{1}{2}\zeta'(2) + \frac{\pi^2}{12}\ln(2)\right)\bullet 2 + \frac{\pi^2}{12}\Gamma'(3)$$

$$= \zeta'(2) + \frac{\pi^2}{6}\ln(2) + \frac{\pi^2}{12}(3-2\gamma_0) = \zeta'(2) + \frac{\pi^2}{6}\ln(2) + \frac{\pi^2}{4} - \frac{\pi^2}{6}\gamma_0 ,$$

since $\Gamma'(3) = -\Gamma(3)\left(\frac{1}{3} + \gamma_0 - \sum_{k=1}^{3}\frac{1}{k}\right) = -\Gamma(3)\left(\frac{1}{3} + \gamma_0 - 1 - \frac{1}{2} - \frac{1}{3}\right) = 2\left(\frac{3}{2} - \gamma_0\right) = 3 - 2\gamma_0$.

We can differentiate (4) and (5) repeatedly to obtain relation of the derivatives,

$$\int_0^\infty \frac{x^s e^x (\ln(x))^n}{(1+e^x)^2}\,dx, \quad \int_0^\infty \frac{x^s e^{-x} (\ln(x))^n}{(1+e^x)^2}\,dx$$

for $n \geq 2$.

For instance:

Recall from (7),

$$\int_0^\infty \frac{x^s e^{-x}\left(\ln(x)\right)^2}{(1+e^x)^2}\,dx$$

$$=\left(\eta''(s)-2\eta''(s+1)\right)\Gamma(s+1)+2\left(\eta'(s)-2\eta'(s+1)\right)\Gamma'(s+1)+\left(\eta(s)-2\eta(s+1)+1\right)\Gamma''(s+1).$$

Thus,

$$\int_0^\infty \frac{e^{-x}\left(\ln(x)\right)^2}{(1+e^x)^2}\,dx=\left(\eta''(0)-2\eta''(1)\right)\Gamma(1)+2\left(\eta'(0)-2\eta'(1)\right)\Gamma'(1)+\left(\eta(0)-2\eta(1)+1\right)\Gamma''(1).$$

We shall need to determine $\eta''(0)$. From the identity $\eta(s)=(1-2^{1-s})\zeta(s)$, we can express $\eta''(0)$ in terms of the Zeta function.

By analyticity for $s<1$, we can differentiate the above identity twice to give,

$$\eta'(s)=2^{1-s}\ln(2)\zeta(s)+(1-2^{1-s})\zeta'(s)$$

and $\eta''(s)=-2^{1-s}\left(\ln(2)\right)^2\zeta(s)+2^{2-s}\ln(2)\zeta'(s)+(1-2^{1-s})\zeta''(s)$.

Therefore,

$$\eta''(0)=-2\left(\ln(2)\right)^2\zeta(0)+2^2\ln(2)\zeta'(0)-\zeta''(0)$$

$$=\left(\ln(2)\right)^2+2^2\ln(2)\left(-\frac{1}{2}\ln(2\pi)\right)-\zeta''(0),\ \text{since}\ \zeta(0)=-\frac{1}{2}\ \text{and}\ \zeta'(0)=-\frac{1}{2}\ln(2\pi),$$

$$=-\left(\ln(2)\right)^2-2\ln(2)\ln(\pi)-\zeta''(0).$$

By Apostol’s formula,

$$\zeta''(0)=-\frac{1}{2}\left(\ln(2\pi)\right)^2+\frac{\pi^2}{24}-\frac{1}{2}\zeta(2)+\frac{1}{2}\gamma_0^{\,2}+\gamma_1$$

$$=-\frac{1}{2}\left(\ln(2\pi)\right)^2-\frac{\pi^2}{24}+\frac{1}{2}\gamma_0^{\,2}+\gamma_1. \qquad \text{----------------------------------- (8)}$$

Therefore,

$$\eta''(0)=-\left(\ln(2)\right)^2-2\ln(2)\ln(\pi)-\left(-\frac{1}{2}\left(\ln(2\pi)\right)^2-\frac{\pi^2}{24}+\frac{1}{2}\gamma_0^{\,2}+\gamma_1\right)$$

$$=-\left(\ln(2)\right)^2-2\ln(2)\ln(\pi)+\frac{1}{2}\left(\ln(2)\right)^2+\frac{1}{2}\left(\ln(\pi)\right)^2+\ln(2)\ln(\pi)+\frac{\pi^2}{24}-\frac{1}{2}\gamma_0^{\,2}-\gamma_1$$

$$=-\frac{1}{2}\left(\ln(2)\right)^2+\frac{1}{2}\left(\ln(\pi)\right)^2-\ln(2)\ln(\pi)+\frac{\pi^2}{24}-\frac{1}{2}\gamma_0^{\,2}-\gamma_1, \qquad \text{----------------------- (9)}$$

where γ_1 is the Euler Mascheroni or Stieltjes constant.

Thus,

$$\int_0^\infty \frac{e^{-x}\left(\ln(x)\right)^2}{(1+e^x)^2}dx = \left(\eta''(0)-2\eta''(1)\right)\Gamma(1)+2\left(\eta'(0)-2\eta'(1)\right)\Gamma'(1)+\left(\eta(0)-2\eta(1)+1\right)\Gamma''(1)$$

$$= -\frac{1}{2}\left(\ln(2)\right)^2+\frac{1}{2}\left(\ln(\pi)\right)^2-\ln(2)\ln(\pi)+\frac{\pi^2}{24}-\frac{1}{2}\gamma_0{}^2-\gamma_1$$

$$-2\left(\frac{1}{3}\left(\ln(2)\right)^3-\left(\ln(2)\right)^2\gamma_0-2\ln(2)\gamma_1\right)$$

$$-2\gamma_0\left(\frac{1}{2}\ln\left(\frac{\pi}{2}\right)-2\left(\ln(2)\gamma_0-\frac{1}{2}\left(\ln(2)\right)^2\right)\right)$$

$$+\left(\frac{1}{2}-2\ln(2)+1\right)\left(\gamma_0{}^2+\frac{\pi^2}{6}\right)$$

$$= -\frac{1}{2}\left(\ln(2)\right)^2+\frac{1}{2}\left(\ln(\pi)\right)^2-\ln(2)\ln(\pi)+\frac{\pi^2}{24}-\frac{1}{2}\gamma_0{}^2-\gamma_1$$

$$-\frac{2}{3}\left(\ln(2)\right)^3+4\ln(2)\gamma_1-\ln\left(\frac{\pi}{2}\right)\gamma_0+2\ln(2)\gamma_0{}^2+\frac{3}{2}\gamma_0{}^2+\frac{\pi^2}{4}-\frac{\pi^2}{3}\ln(2)$$

$$= -\frac{1}{2}\left(\ln(2)\right)^2+\frac{1}{2}\left(\ln(\pi)\right)^2-\frac{2}{3}\left(\ln(2)\right)^3-\ln(2)\ln(\pi)-\frac{\pi^2}{3}\ln(2)+\frac{7\pi^2}{24}+\gamma_0{}^2-\gamma_1$$

$$+4\ln(2)\gamma_1-\ln\left(\frac{\pi}{2}\right)\gamma_0+2\ln(2)\gamma_0{}^2.$$

Similarly, we can evaluate $\int_0^\infty \frac{\left(\ln(x)\right)^2}{(1+e^x)^2}dx$ as follows.

Recall that $\int_0^\infty \frac{x^s\ln^2(x)}{(1+e^x)^2}dx$

$$=\left(\eta''(s+1)-\eta''(s)\right)\Gamma(s+1)+2\left(\eta'(s+1)-\eta'(s)\right)\Gamma'(s+1)+\left(\eta(s+1)-\eta(s)\right)\Gamma''(s+1).$$

Therefore,

$$\int_0^\infty \frac{\ln^2(x)}{(1+e^x)^2}dx = \left(\eta''(1)-\eta''(0)\right)\Gamma(1)+2\left(\eta'(1)-\eta'(0)\right)\Gamma'(1)+\left(\eta(1)-\eta(0)\right)\Gamma''(1)$$

$$=\eta''(1)-\eta''(0)-2\gamma_0\left(\ln(2)\gamma_0-\frac{1}{2}\left(\ln(2)\right)^2-\frac{1}{2}\ln\left(\frac{\pi}{2}\right)\right)+\left(\ln(2)-\frac{1}{2}\right)\left(\gamma_0{}^2+\frac{\pi^2}{6}\right)$$

$$=\frac{1}{3}\left(\ln(2)\right)^3-\left(\ln(2)\right)^2\gamma_0-2\ln(2)\gamma_1$$

$$-\left(-\frac{1}{2}\left(\ln(2)\right)^2+\frac{1}{2}\left(\ln(\pi)\right)^2-\ln(2)\ln(\pi)+\frac{\pi^2}{24}-\frac{1}{2}\gamma_0{}^2-\gamma_1\right)$$

$$-2\ln(2)\gamma_0{}^2+\left(\ln(2)\right)^2\gamma_0+\ln\left(\frac{\pi}{2}\right)\gamma_0+\ln(2)\gamma_0{}^2-\frac{1}{2}\gamma_0{}^2+\frac{\pi^2}{6}\ln(2)-\frac{\pi^2}{12}$$

$$=\frac{1}{3}\left(\ln(2)\right)^3-\left(\ln(2)\right)^2\gamma_0-2\ln(2)\gamma_1$$

$$+\frac{1}{2}\left(\ln(2)\right)^2-\frac{1}{2}\left(\ln(\pi)\right)^2+\ln(2)\ln(\pi)-\frac{\pi^2}{24}+\frac{1}{2}\gamma_0{}^2+\gamma_1$$

$$-2\ln(2)\gamma_0{}^2+\left(\ln(2)\right)^2\gamma_0+\ln\left(\frac{\pi}{2}\right)\gamma_0+\ln(2)\gamma_0{}^2-\frac{1}{2}\gamma_0{}^2+\frac{\pi^2}{6}\ln(2)-\frac{\pi^2}{12}$$

$$=\frac{1}{2}\left(\ln(2)\right)^2+\frac{1}{3}\left(\ln(2)\right)^3-\frac{1}{2}\left(\ln(\pi)\right)^2+\ln(2)\ln(\pi)-\frac{\pi^2}{24}+\frac{\pi^2}{6}\ln(2)-\frac{\pi^2}{12}+\gamma_1-2\ln(2)\gamma_1$$

$$-\ln(2)\gamma_0{}^2+\ln\left(\frac{\pi}{2}\right)\gamma_0.$$

Revised 27/2/2024