**Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem**

# by Ng Tze Beng

We shall begin by examining the properties of the image under a function $f$ of a set in which $f$ has finite derivatives that are bounded by a constant**.** The first property we examine is the relation between the measure of such a set and the measure of its image. We state this property in the next theorem.

This result appears in Saks monograph on the theory of the integral and there are a number of proofs of the result. But I shall present a proof using some finiteness argument, a consequence of compactness and the triangle inequality.

**Theorem 1.** Suppose $f : [a, b] \to \mathbf{R}$ is a function. Suppose $E$ is a subset of $[a, b]$ such that at each point $x$ of $E$, $f$ is differentiable and $|f'(x)| \le K$ for some constant $K \ge 0$. Then if $m$ denotes the Lebesgue outer measure,

$$m(f(E)) \le K\, m(E) \quad \text{----------------------} \quad \mathbf{(A)}$$

**Proof .** Now $E = \{ x \in [a, b] : |f'(x)| \le K \} \subseteq [a, b]$ and so $E$ has finite outer measure. If $E$ is finite or denumerable, then the set $f(E)$ is at most denumerable and so both $m(f(E))$ and $m(E)$ are zero and we have nothing to prove since both sides of the inequality are zero. We shall now assume that $E$ is uncountably infinite. We may assume that neither $a$ nor $b$ is in $E$ since adding any finite number of points to $E$ will not alter the inequality (A).

For any $\varepsilon > 0$, by the definition of outer measure, there exists an open set $U$ *in* $[a, b]$ such that $U \supseteq E$ and $m(U) \le m(E) + \varepsilon$.

Since for each $e$ in $E$, $|f'(x)| \le K$ , for $\varepsilon > o$ there exists a $\delta_e > 0$ such that

$$0 < |x - e| < \delta_e \Rightarrow \left| \left| \frac{f(x) - f(e)}{x - e} \right| - |f'(e)| \right| < \varepsilon .$$

and so

$$0 < |x - e| < \delta_e \Rightarrow \left| \frac{f(x) - f(e)}{x - e} \right| < |f'(e)| + \varepsilon \le K + \varepsilon .$$

Thus we have,

$$|x - e| < \delta_e \Rightarrow |f(x) - f(e)| \le (K + \varepsilon)|x - e| . \quad \text{--------------} \quad (1)$$

Since $U$ is open, we may choose $\delta_e > 0$ such that the open interval $(e - \delta_e , e + \delta_e) \subseteq U$. Denote $(e - \delta_e , e + \delta_e)$ by $I_e$ . Then inequality (1) says that

$$x \in I_e \Rightarrow |f(x) - f(e)| \le (K + \varepsilon)|x - e| \text{-----------------} \quad (2).$$

Then the collection $\mathcal{C} = \{ I_e : e \in E\}$ covers $E$ and the union $W = \cup \{V : V \in \mathcal{C}\} = \cup \{ I_e : e \in E\} \subseteq U$. In particular the union $W$ is open and so is a disjoint union of countable number of open intervals, i.e.,

$$W = \sqcup \{U_i : i \in B\},$$

where $B$ the index set is a subset of the set $\mathbf{N}$ of natural numbers and each $U_i$ is an open interval. We shall show next that for each $i$ in $B$,

$$m(f(U_i \cap E)) \le (K + \varepsilon)\, m(U_i). \text{--------------------------} \quad (3)$$

Note that $U_i = \cup\{ I_e : e \in U_i \cap E \}$. Observe that each $U_i$ is a path component of $W$.

Plainly for $e \in U_i \cap E$, $I_e \cap U_i \neq \varnothing$ and since $I_e \subseteq W$ and $U_i$ is a path component of $W$, $I_e \subseteq U_i$. It follows that $\cup\{ I_e : e \in U_i \cap E \} \subseteq U_i$. For any $x$ in $U_i$, $x \in I_e$ for some $e$ in $E$, since $W = \cup \{ I_e : e \in E\}$ and so $I_e \cap U_i \neq \varnothing$. It follows as in the above argument that $I_e \subseteq U_i$ and so $e \in U_i \cap E$. Thus, $x \in I_e$ for some $e \in U_i \cap E$, that is, $x \in \cup\{ I_e : e \in U_i \cap E \}$ and so $U_i \subseteq \cup\{ I_e : e \in U_i \cap E \}$. This proves that $U_i = \cup\{ I_e : e \in U_i \cap E \}$.

Now take any $x < y$ in $U_i$. Since $U_i$ is an open interval, the closed and bounded interval $[x, y]$ is contained in $U_i$. Now plainly the collection $\mathcal{E} = \{ I_e : e \in U_i \cap E \}$ is an open cover for $[x, y]$. Since $[x, y]$ is compact, there exists a finite subcover say

$$I_1, I_2, \ldots, I_n$$

where $I_i = (e_i - \delta(e_i), e_i + \delta(e_i))$, for some $e_i$ in $E$ and $\delta(e_i)$ is as given in (1). *We* assume that the $e_i$'s are ordered in an increasing order. Hence

$$[x, y] \subseteq I_1 \cup I_2 \cup \ldots \cup I_n$$

and $e_1 < e_2 < \ldots < e_n$.

We may assume that $x \in I_1$. This is seen as follows. If $x \notin I_1$ $x$ must belong to $I_j$ for some $1 < j \leq n$ and $x \notin I_i$ for for $1 \leq i < j$. Then $[x, y] \cap I_i = \varnothing$ for $1 \leq i < j$. It follows that $[x, y] \subseteq I_j \cup I_{j+1} \cup \ldots \cup I_n$ and so we can rename if need be $I_j$ to be $I_1$, $I_{j+1}$ to be $I_2$ and so on. By a similar argument we may assume that $y \in I_n$. We may also assume that $I_i \cap I_{i+1} \neq \varnothing$ for $1 \leq i \leq n-1$ and that $I_i \not\subset I_j$ for $j \neq i$. We can deduce this as follows. If $I_i \subseteq I_j$, then the collection of the $I_k$'s without $I_i$ still covers $[x, y]$ and so we can discard $I_i$ and rename the $I_j$'s. Then starting with $I_1$, suppose $I_1 \cap I_2 = \varnothing$. Then since $[x, y]$ is path connected, $I_1 \cap \cup\{ I_j : 1< j \leq n\} \neq \varnothing$ implies for some $2 < j \leq n$, $I_1 \cap I_j \neq \varnothing$. Then $e_j - \delta(e_j) < e_2 - \delta(e_2)$ implies that $\delta(e_j) > e_j - e_2 + \delta(e_2) > \delta(e_2)$ and so $e_j + \delta(e_j) > e_2 + \delta(e_2)$ and so $I_2 \subseteq I_j$. This contradicts that $I_2 \not\subset I_j$. We can repeat the same argument to show that $I_i \cap I_{i+1} \neq \varnothing$ for $i > 1$.

Thus, in this way we may assume that we have a sequence of points $x_1, x_2, \ldots, x_{n-1}$ such that

$$e_1 < x_1 < e_2 < x_2 < \ldots < e_{n-1} < x_{n-1} < e_n$$

and $x_i \in I_i \cap I_{i+1}$ for $1 \leq i \leq n-1$. Therefore, by (2) and the triangle inequality.

$$\begin{aligned}
&|f(x)-f(y)| \leq |f(x)-f(e_1)| + |f(e_1)-f(x_1)| + |f(x_1)-f(e_2)| + |f(e_2)-f(x_2)| \\
&+\cdots+ |f(x_{n-2})-f(e_{n-1})| + |f(e_{n-1})-f(x_{n-1})| + |f(x_{n-1})-f(e_n)| + |f(e_n)-f(y)| \\
&\leq (K+\varepsilon)\{|x-e_1| + |e_1-x_1| + |x_1-e_2| + |e_2-x_2| + \cdots \\
&\qquad + |x_{n-2}-e_{n-}| + |e_{n-1}-x_{n-1}| + |x_{n-1}-e_n| + |e_n-y|\} \\
&\leq (K+\varepsilon)\{|x-e_1| + |e_1-e_n| + |e_n-y|\} \\
&\leq (K+\varepsilon)\, m(I_1 \cup I_2 \cup \ldots \cup I_n) \leq (K+\varepsilon)\, m(U_i).
\end{aligned}$$

Hence the diameter of $f(U_i) \leq (K+\varepsilon)\, m(U_i)$. It follows that $m(f(U_i \cap E)) \leq (K+\varepsilon)\, m(U_i)$. This proves (3).

Then using (3), we see that

$$\begin{aligned}
&m(f(E)) = m(\cup\{f(U_i \cap E) : i \in B\}) \leq \sum_{i\in B} m(f(U_i \cap E)) \\
&\leq \sum_{i\in B} (K+\varepsilon) m(U_i) = (K+\varepsilon) m(W) \leq (K+\varepsilon) m(U) \\
&\leq (K+\varepsilon)(m(E)+\varepsilon).
\end{aligned}$$

Since ε is arbitrary, we conclude that $m(f(E)) \leq Km(E)$.

**Theorem 2**. Suppose $f : [a, b] \to \mathbf{R}$ is a measurable function. Suppose $E$ is any measurable subset such that $f'(x)$ exists finitely for every $x$ in $E$. Then

$$m(f(E)) \le \int_E |f'|,$$

where $m$ is the Lebesgue outer measure.

**Proof.** Since $f$ is measurable and finite on $[a, b]$, its Dini derivatives are measurable. (Banach Theorem). Consequently, $f'$ is measurable on $E$ and so $|f'|$ is measurable on $E$. Suppose now g = $|f'|$ is bounded on $E$, by a positive integer $K$, i.e., $|f'(x)| < K$ for each $x$ in $E$. For any positive integer $n$ and integer $i$ =1,2, …,$2^n K$, let $E_{n,i} = g^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) \cap E$. Define $g_n = \sum_{i=1}^{2^n K} \frac{i-1}{2^n}\chi_{E_{n,i}}$ for each positive integer $n$. Then ( $g_n$ ) is a sequence of simple functions converging pointwise to g on $E$. In particular,

$$\int_E g_n \to \int_E g.$$

By Theorem 1, $m(f(E_{n,i})) \le \frac{i}{2^n} m(E_{n,i})$ for integer $i$ =1,2, …,$2^n K$, Thus,

$$\begin{aligned} m(f(E)) = m(f(\bigcup_{i=1}^{2^nK} E_{n,i})) &\le \sum_{i=1}^{2^nK} \frac{i}{2^n} m(E_{n,i}) \\ &= \sum_{i=1}^{2^nK} \frac{i-1}{2^n} m(E_{n,i}) + \frac{1}{2^n}\sum_{i=1}^{2^nK} m(E_{n,i}) \\ &= \int_E g_n + \frac{1}{2^n} m(E). \quad \text{------------------------ (1)} \end{aligned}$$

Therefore, $m(f(E)) \le \lim_{n\to\infty} (\int_E g_n + \frac{1}{2^n} m(E))$. Since, $\int_E g_n \to \int_E g$ and $\frac{1}{2^n} m(E) \to 0$, we conclude that

$$m(f(E)) \le \int_E g.$$

We now consider the case when g is unbounded. For each integer $k > 1$, let

$$E_k = g^{-1}([k-1, k)) \cap E.$$

Then it is obvious that $E$ is a disjoint union of the $E_k$'s. Note that on $E_k$, g is bounded by $k$. Hence, by what we have just shown, for each integer $k > 0$, $m(f(E_k)) \le \int_{E_k} g$. Therefore,

$$m(f(E)) \le \sum_{k=1}^{\infty} m(f(E_k)) \le \sum_{k=1}^{\infty} \int_{E_k} g = \int_E g = \int_E |f'|.$$

This completes the proof of Theorem 2.

We have some easy consequences of the above theorems.

**Theorem 3.** Suppose $f$ is defined and finite on $[a, b]$. Suppose $E = \{x \in [a, b]:\ f$ is differentiable at $x$ and $f'(x) = 0\}$. Then $m(f(E)) = 0$.

**Proof.** By Theorem 1 $m(f(E)) \le 1/n\ m(E)$ for any positive integer $n$. Therefore, $m(f(E)) = 0$.

Recall a set is called a *null set* if its measure is zero.

**Theorem 4.** Suppose $f : [a, b] \to \mathbf{R}$ has a finite derivative at every point of $[a, b]$. Then $f$ maps null sets onto null sets.

**Proof.** Suppose $E$ is a null set in $[a, b]$. Then by Theorem 2,

$$m(f(E)) \le \int_E |f'| = 0.$$

Hence $m(f(E)) = 0$. This proves the theorem.

**Theorem 5.** Suppose $f : [a, b] \to \mathbf{R}$ has a finite derivative at every point of $[a, b]$ and $f'$ is Lebesgue integrable on $[a, b]$. Then for every closed and bounded interval $[c, d]$ in $[a, b]$,

$$\int_c^d |f'| \geq |f(d) - f(c)|.$$

**Proof.** Since $f$ is continuous on $[a, b]$, $|f(d) - f(c)| \leq m(f([c, d]))$. Since $f$ is differentiable at every point of $[c, d]$, by Theorem 2,

$$m(f([c,d])) \leq \int_{[c,d]} |f'| = \int_c^d |f'|.$$

It follows that $|f(d) - f(c)| \leq \int_c^d |f'|$.

We can apply Theorem 5 to the next result.

**Theorem 6.** Suppose $f : [a, b] \to \mathbf{R}$ has a finite derivative at every point of $[a, b]$ and $f'$ is Lebesgue integrable on $[a, b]$. Then $f$ is absolutely continuous.

**Proof.** Since $f'$ is Lebesgue integrable, $|f'|$ is also Lebesgue integrable on $[a, b]$. For each positive integer $n$, let $g_n = \min(|f'|, n)$. Then each $g_n$ is Lebesgue integrable on $[a, b]$ and the sequence $(g_n)$ converges pointwise to $|f'|$. In particular, for each $n$, $|g_n| = g_n \leq |f'|$ and so by the Lebesgue Dominated Convergence Theorem,

$$\int_a^b g_n \to \int_a^b |f'|.$$

Hence, given any $\varepsilon > 0$, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow \left| \int_a^b |f'| - \int_a^b g_n \right| < \frac{\varepsilon}{2}.$$

It follows that

$$n \geq N \Rightarrow 0 \leq \int_a^b (|f'| - g_n) < \frac{\varepsilon}{2}. \qquad \text{-------------------------- (1)}$$

Now take $\delta = \frac{\varepsilon}{2N}$. Suppose $[a_i, b_i]$, $i = 1, 2, \ldots, k$ are non-overlapping intervals in $[a, b]$. If $\sum_{i=1}^{k} |b_i - a_i| < \delta$, then

$$\begin{aligned}
\sum_{i=1}^{k} |f(b_i) - f(a_i)| &\leq \sum_{i=1}^{k} \int_{a_i}^{b_i} |f'| \qquad \text{by Theorem 5,} \\
&= \sum_{i=1}^{k} \int_{a_i}^{b_i} (|f'| - g_N) + \sum_{i=1}^{k} \int_{a_i}^{b_i} g_N \\
&\leq \int_a^b (|f'| - g_N) + \sum_{i=1}^{k} \int_{a_i}^{b_i} N \\
&= \int_a^b (|f'| - g_N) + N \sum_{i=1}^{k} |b_i - a_i| \\
&< \frac{\varepsilon}{2} + N\delta = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{by (1),}
\end{aligned}$$

This shows that $f$ is absolutely continuous on $[a, b]$.

**Remark.** Theorem 6 is Theorem 8.21 in Rudin's *Real and Complex Analysis* in an equivalent formulation.

More generally we may relax the requirement of everywhere differentiability but we need to impose the requirement that $f$ maps null sets to null sets. This is a necessary condition for absolute continuity.

**Theorem 7.** Suppose $f : [a, b] \to \mathbf{R}$ is continuous and $f'$ exists almost everywhere and is Lebesgue integrable on $[a, b]$. Suppose $f$ maps null sets to null sets. Then $f$ is absolutely continuous.

**Proof.** Let $E \subseteq [a, b]$ be the subset where $f'$ exists at each point so that the measure of $[a, b] - E$ is zero. Then $|f'| = g$ almost everywhere, where $g(x) = |f'(x)|$ for $x$ in $E$ and $g(x) = 0$ for $x$ outside $E$. Then there exists an increasing sequence of simple functions $(g_n)$ converging pointwise to g almost everywhere and

$$\int_a^b g_n \to \int_a^b |f'| = \int_a^b g.$$

Thus, given any $\varepsilon > 0$, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow \left|\int_a^b g - \int_a^b g_n\right| = \int_a^b (g - g_n) < \frac{\varepsilon}{2}. \qquad \text{----------------} \ (1)$$

Suppose $[a_i , b_i]$, $i = 1, 2, \ldots, k$ are non-overlapping intervals in $[a, b]$. Let $E_i = \{x \in [a_i , b_i]: f'(x) \text{ exists.}\}$. Then since $f$ maps null sets to null sets and $m([a_i , b_i]-E_i) = 0$, $m(f([a_i , b_i]) = m(f(E_i))$. By Theorem 2, $m(f(E_i)) \leq \int_{E_i} |f'|$ and so for each $i$,

$$m(f([a_i, b_i])) \leq \int_{E_i} |f'|. \qquad \text{------------------------} \ (2)$$

Since $f$ is continuous, $f$ is also continuous on $[a_i , b_i]$ and so by continuity,

$$|f(b_i) - f(a_i)| \leq m(f([a_i , b_i])) \text{ for each } i = 1, 2, \ldots, k.$$

Therefore, by (2) we have

$$\sum_{i=1}^{k} |f(b_i) - f(a_i)| \leq \sum_{i=1}^{k} \int_{E_i} |f'| = \sum_{i=1}^{k} \int_{E_i} g$$

$$= \sum_{i=1}^{k} \int_{a_i}^{b_i} g \quad \text{since } m([a_i , b_i]-E_i) = 0$$

$$= \sum_{i=1}^{k} \int_{a_i}^{b_i} (g - g_N) + \sum_{i=1}^{k} \int_{a_i}^{b_i} g_N$$

$$\leq \int_a^b (g - g_N) + \sum_{i=1}^{k} \int_{a_i}^{b_i} K,$$

where $K > 0$ is an upper bound for $g_N$..

$$< \frac{\varepsilon}{2} + K \sum_{i=1}^{k} |b_i - a_i|$$

------------------------------- (3).

Take $\delta = \frac{\varepsilon}{2K}$, It follows from (3) that if $\sum_{i=1}^{k} |b_i - a_i| < \delta$, then

$$\sum_{i=1}^{k} |f(b_i) - f(a_i)| < \frac{\varepsilon}{2} + K\frac{\varepsilon}{2K} = \varepsilon.$$

This shows that $f$ is absolutely continuous.

As a corollary we have the Banach Zarecki Theorem.

**Theorem 8 (Banach Zarecki)** . Suppose $f : [a, b] \to \mathbf{R}$ is continuous and is a function of bounded variation. Suppose $f$ maps null sets to null sets. Then $f$ is absolutely continuous.

**Proof.** Since $f$ is of bounded variation, $f$ is differentiable almost everywhere and $f'$ is Lebesgue integrable. Therefore, by Theorem 7, $f$ is absolutely continuous.

**Remark.** It is easy to see that if $f$ is absolutely continuous on $[a, b]$, then $f$ is continuous and of bounded variation on $[a, b]$. Any function of bounded variation on $[a, b]$ is the difference of two increasing functions (see for instance Theorem 13 of "*Monotone functions, function of bounded variation , fundamental theorem of Calculus*"). Since any increasing function on $[a, b]$ is differentiable almost everywhere on $[a, b]$ and its derived function is Lebesgue integrable on $[a, b]$, any function of bounded variation is therefore, differentiable almost everywhere on $[a, b]$ and its derivative is Lebesgue integrable on $[a, b]$. So if $f$ is absolutely continuous on $[a, b]$, then $f$ is differentiable almost everywhere on $[a, b]$ and $f'$ is Lebesgue integrable on $[a, b]$. If $f$ is absolutely continuous on $[a, b]$, then $f$ maps null sets in $[a, b]$ to null sets (see for instance Proposition 9 of my article "*Change of variable or substitution in Riemann and Lebesgue Integration*"). Thus the converse of Theorem 7 and Theorem 8 are also true.

With a little thought we shall derive the following theorem.

**Theorem 9.** Suppose $f : [a, b] \to \mathbf{R}$ is absolutely continuous and $f'(x) = 0$ almost everywhere on $[a, b]$. Then $f$ is a constant function.

**Proof.** It is enough to show that the range of $f$ has measure zero. Let $E = \{x \in [a, b] : f'(x) = 0\}$. Then $m([a, b] - E) = 0$. By Theorem 3, $m(f(E)) = 0$. Since $f$ is absolutely continuous, it maps null sets to null sets (see Proposition 9 of my article "*Change of variable or substitution in Riemann and Lebesgue Integration*"). It follows that $m(f([a, b] - E)) = 0$. Therefore, $m(f([a, b])) \leq m(f(E)) + m(f([a, b] - E)) = 0$. It follows that $m(f([a, b])) = 0$. Since $f$ is continuous and $[a, b]$ is compact and connected, $f([a, b]))$ is compact and connected and so is either a non-trivial closed and bounded interval or a single point. Since a non-trivial closed and bounded interval has non-zero measure, $f([a, b]))$ must be a single point, consequently $f$ is a constant function.

The next result is a consequence of a function having the property of being a continuous $N$ function. In particular the result applies to an absolutely continuous function on $[a, b]$.

**Theorem 10.** Suppose $f : [a, b] \to \mathbf{R}$ is continuous and maps null sets to null sets, i.e., $f$ is a continuous $N$ function. Then $f$ maps measurable sets to measurable sets.

**Proof.** Since the Lebesgue measure is a regular measure, for any measurable set $E$ there is a subset, a $F_\sigma$ set, $K$ in $[a, b]$ such that $K \subseteq E$ and $m(E - K) = 0$. By a $F_\sigma$ set $K$, we mean $K$ is a countable union of closed sets in $[a, b]$. Thus

$$K = \bigcup_{n=1} K_n ,$$

where each $K_n$ is a closed subset in $[a, b]$.
Each $K_n$ is closed and bounded and so by the Heine Borel Theorem, is compact. Since $f$ is continuous, each $f(K_n)$ is compact and so is closed and bounded by the Heine Borel Theorem. Since $f(K_n)$ is closed, it is measurable.

Therefore,

$$f(K) = \bigcup_{n=1} f(K_n),$$

being a countable union of measurable sets, is measurable.
Since $f$ maps null sets to null sets, $m(f(E-K)) = 0$. It then follows that $f(E-K)$ is measurable. Hence,

$$f(E) = f(K) \cup f(E-K))$$

is a union of two measurable sets and so is measurable.

**Corollary 11.** Suppose $f : [a, b] \to \mathbf{R}$ is absolutely continuous. Then $f$ maps measurable sets to measurable sets.

**Proof.** Since $f$ is absolutely continuous on $[a, b]$, $f$ maps null sets in $[a, b]$ to null sets (see for instance Proposition 9 of my article "*Change of variable or substitution in Riemann and Lebesgue Integration*"). Thus $f$ is a continuous $N$ function and so by Theorem 10, $f$ maps measurable sets to measurable sets.

For functions that are strictly increasing (or strictly decreasing) we have the following useful result for absolute continuity.

**Theorem 12 (Zarecki)**. Suppose $f : [a, b] \to \mathbf{R}$ is strictly increasing and continuous.
(a) $f$ is absolutely continuous if and only if $m(f(\{x \in [a, b]: f'(x) = \infty\})) = 0$.
(b) The inverse function $f^{-1}$ is absolutely continuous if and only if
$m(\{x \in [a, b]: f'(x) = 0\}) = 0$

**Proof.**
(a) By Theorem 8, $f$ is absolutely continuous if and only if $f$ maps null sets to null sets. Since $f$ is increasing, $f$ is differentiable (finitely) almost everywhere on $[a, b]$. Hence $m(\{x \in [a, b]: f'(x) = \infty\}) = 0$. If $f$ maps null sets to null sets, then $m(f(\{x \in [a, b]: f'(x) = \infty\})) = 0$.
Conversely, suppose $m(f(\{x \in [a, b]: f'(x) = \infty\})) = 0$. Let $E$ be a set of measure 0 in $[a, b]$. Let $A = \{x \in [a, b]: f'(x) = \infty\}$, $B = \{x \in [a, b]: f'(x)$ does not exists and $f'(x) \neq \infty\}$. By the Theorem of De La Vallee Poussin, $m(f(B)) = 0$. Write $E = (E \cap A) \cup (E \cap B) \cup (E - (A \cup B))$. Then $m(E) = 0$ implies that $m(E - (A \cup B)) = 0$.
By the Theorem of De La Vallee Poussin we may assume that $f'(x)$ exists finitely on $E - (A \cup B)$. Therefore, by Theorem 2,

$$m(f(E-(A\cup B))) \leq \int_{E-(A\cup B)} |f'| = 0.$$

Hence $m(f(E-(A\cup B))) = 0$. Since $f(E \cap B) \subseteq f(B)$ and $m(f(B)) = 0$, $m(f(E \cap B)) = 0$. Since $E \cap A \subseteq A$ and $m(f(A)) = 0$, $m(f(E \cap A)) = 0$. Thus,

$$m(f(E)) \leq m(f(E-(A\cup B))) + m(f(E\cap A)) + m(f(E\cap B) = 0.$$

It follows that $m(f(E) = 0$. This means $f$ maps null sets to null sets and it follows that $f$ is absolutely continuous.

(b) Suppose $f^{-1}$ is absolutely continuous. Let $C = \{x \in [a, b]: f'(x) = 0\}$. Then by Theorem 3, $m(f(C)) = 0$. Then since $f^{-1}$ is absolutely continuous,

$$m(C) = m(f^{-1}(f(C))) = 0.$$

As in part (a), note that $f^{-1}$ is absolutely continuous if and only if $f$ maps null sets to null sets.
Now assume that $m(C) = 0$.

Let $E$ be a subset of $f([a, b]) = [c, d]$ of measure 0. Then $E = f(A)$, where $A = f^{-1}(E)$. We shall show that $m(A) = 0$.
By Theorem 15 of "*Functions of Bounded Variation and Johnson's Indicatrix*", $f' = 0$ almost everywhere on $A$.
Write $A = (A\cap C) \cup (A - C)$. Since $f' = 0$ almost everywhere on $A$, $m(A - C) = 0$.
But $A\cap C \subseteq C$ and $m(C) = 0$ and so $m(A\cap C) = 0$. Hence $m(A) = m(f^{-1}(E)) = 0$.
This completes the proof.

The proof of Theorem 12 (a) word for word with minor modification changing "increasing" to "of bounded variation" and "$\infty$" to "$\pm\infty$" gives the following theorem.

**Theorem 13.** Suppose $f: [a, b] \to \mathbf{R}$ is continuous and of bounded variation. Then $f$ is absolutely continuous if and only if $m(f(\{x \in [a, b]: f'(x) = \pm\infty\})) = 0$.

We shall now give a proof of the Theorem of De La Vallée Poussin.

**Theorem 14 (De La Vallée Poussin).** Suppose $f: [a, b] \to \mathbf{R}$ is a function of bounded variation. Then there is a subset $N$ of $[a, b]$ such that

$$m(v_f(N)) = m(f(N)) = m(N) = 0,$$

where $v_f$ is the total variation function of $f$, and for each $x$ in $[a, b] - N$, $f'(x)$ and $v_f'(x)$ exist (finite or infinite) and that

$$v_f'(x) = |f'(x)|.$$

The following elementary proof is due to F. S. Cater.

The following technical lemma is the key to the proof.

**Lemma 15**. Suppose $f: [a, b] \to \mathbf{R}$ is a function of bounded variation. Let $h$ and $k$ be positive numbers such that $h < k$. Suppose $E = \{ x \in [a, b]$: there is a derived number of $v_f$ at $x$ greater than $k$ and a derived number of $f$ at $x$, whose absolute value is less than $h$.$\}$. Suppose $S = \{ x \in [a, b]$: there is a positive derived number and a negative derived number of $f$ at $x\}$.
Then

$$m(v_f(E \cup S)) = m(f(E \cup S)) = m(E \cup S) = 0.$$

**Proof.** We assume that $E \cup S$ is non-denumerable, otherwise trivially all three sets have measure zero.
The first step is to choose some anchor partition for $[a, b]$ to approximate the total variation of $f$. Recall the definition of a function of bounded variation,

$$v_f(b) = \sup\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| : (P: a = x_0 < x_1 < \cdots < x_n = b) \text{ is a partition for } [a, b]\}.$$

Then given any $\varepsilon > 0$, there exists a partition $P: a = u_0 < u_1 < \cdots < u_n = b$ such that

$$v_f(b) - \varepsilon < \sum_{i=1}^{n} |f(u_i) - f(u_{i-1})| \le v_f(b).$$

i.e.,

$$v_f(b) < \sum_{i=1}^{n} |f(u_i) - f(u_{i-1})| + \varepsilon. \quad \text{------------------------------} \quad (A)$$

Then for any partition $Q: a = z_0 < z_1 < \cdots < z_t = b$ containing all the points of the partition $P$,

$$v_f(b) = \sum_{i=1}^{t} (v_f(z_i) - v_f(z_{i-1}) < \sum_{i=1}^{t} |f(z_i) - f(z_{i-1})| + \varepsilon \text{. ----------- (1)}$$

Let $P$ denote also the set of points of the partition. $P : a = u_0 < u_1 < \cdots < u_n = b$.
We may assume also that $f$ is continuous at every point of $E \cup S$. Then $v_f$ is also continuous at every point of $E \cup S$. Since $f$ is of bounded variation the set of discontinuity of $f$ is denumerable and so we may remove these points of discontinuity from $E \cup S$ without affecting the conclusion of the lemma.
Let $U$ be an open set containing the image $v_f(E)$ such that $m(U) < m(v_f(E)) + \varepsilon$. Since $U$ is open and $v_f$ is continuous at $e$ for each $e$ in $E$, there exists an $\zeta > 0$ so that $(v_f(e) - \zeta, v_f(e) + \zeta) \subseteq U$ and corresponding to this $\zeta > 0$ there exists $\delta > 0$ such that

$$x \in (e - \delta, e + \delta) \Rightarrow v_f(x) \in (v_f(e) - \zeta, v_f(e) + \zeta).$$

Thus we can find arbitrary small non trivial intervals $[x, y]$ with $x \le e \le y$ such that $v_f(e) \in [v_f(x), v_f(y)] \subseteq (v_f(e) - \zeta, v_f(e) + \zeta)$. In particular, since $v_f$ has a positive derived number $> k$ at $e$ we can find arbitrary such small intervals $[x, y]$ such that

$$\frac{v_f(y) - v_f(x)}{y - x} > k.$$

(Note that since $v_f$ has a positive derived number at $e$, the interval $[v_f(x), v_f(y)]$ is never degenerate.) Thus we can cover $v_f(E)$ by arbitrary such small closed intervals. Therefore, by the Vitali Covering Theorem, we can cover $v_f(E)$ almost every where by countable mutually disjoint closed interval

$$\{[v_f(a_i), v_f(b_i)]\}$$

such that $[v_f(a_i), v_f(b_i)] \subseteq U$ and $v_f(b_i) - v_f(a_i) > k(b_i - a_i)$ for each $i$.
Therefore, the intervals $\{[a_i, b_i]\}$ are also mutually disjoint and

$$m(v_f(E)) + \varepsilon > m(U) \ge \sum_i (v_f(b_i) - v_f(a_i)) > k \sum_i (b_i - a_i)$$

and so

$$\sum_i (b_i - a_i) < (m(v_f(E)) + \varepsilon)/k \text{. ----------------------------- (2)}$$

Without loss of generality we may assume that the set of points of the partition . $P : a = u_0 < u_1 < \cdots < u_n = b$ does not contain any points of $E$. If $P$ contains a point in $E$ we may just remove this point from $E$. We may thus remove all the points in $P$ that are in $E$ from $E$ without affecting the conclusion of the lemma as only a finite number of points is removed from $E$. We may take $\varepsilon = 1/N$ then by passing to the limit as $N$ tends to infinity only at most a denumerable number of points are removed from $E$. Consequently as the measure of a set of denumerable number of points and its image under $f$ or $v_f$ is of measure zero, the conclusion of the lemma remains valid.
Now since at each point $e$ of $E - \{a_i, b_i : i = 1,2, \ldots\}$, there is a derived number of $f$ whose absolute value is less than $h$, we may pick arbitrary small interval $[c, d]$ such that $e$ is either one of the end points of the interval,

$$\frac{|f(d) - f(c)|}{d - c} < h,$$

$[c, d] \subseteq \bigcup_i [a_i, b_i]$ and that $P \cap [c, d] = \varnothing$. Note that $\{a_i, b_i : i = 1,2, \ldots\}$ is countable and so its image under $v_f$ is also countable and so is of measure zero. Hence again by the Vitali Covering Theorem we can cover $v_f(E)$ almost every where with countable mutually disjoint closed intervals $\{[v_f(c_i), v_f(d_i)]\}$ such that $P \cap [c_i, d_i] = \varnothing$, $[c_i, d_i] \subseteq \bigcup_i [a_i, b_i]$,

$$| f(d_i) - f(c_i)| < h(d_i - c_i)$$

for each $i$ and

$$m(v_f(E)) \le \sum_i (v_f(d_i) - v_f(c_i)) \quad . \qquad \text{-------------------- (3)}$$

But using (1) and the fact that for any $x < y$, $|f(y) - f(x)| \le v_f(y) - v_f(x)$, we can show that

$$\sum_i (v_f(d_i) - v_f(c_i)) \le \sum_i |f(d_i) - f(c_i)| + \varepsilon \quad . \qquad \text{---------------- (4)}$$

(Show this for finite number of the intervals $[c_i, d_i]$ and pass to the limit.)

Since $\bigcup_i [c_i, d_i] \subseteq \bigcup_i [a_i, b_i]$,

$$\sum_i (d_i - c_i) \le \sum_i (b_i - a_i) \quad . \qquad \text{------------------------- (5)}$$

Then from (3) and (4), we arrive at

$$m(v_f(E)) \le \sum_i (v_f(d_i) - v_f(c_i)) \le \sum_i |f(d_i) - f(c_i)| + \varepsilon$$

$$\le h\sum_i (d_i - c_i) + \varepsilon \le h\sum_i (b_i - a_i) + \varepsilon$$

Thus,

$$(m(v_f(E)) - \varepsilon)/h \le \sum_i (b_i - a_i)$$

and using (2) we get

$$(m(v_f(E)) - \varepsilon)/h < (m(v_f(E)) + \varepsilon)/k.$$

Since ε = $1/N$ is arbitrary by passing $N$ to infinity we deduce that, $m(v_f(E))/h \le m(v_f(E))/k$. But since $h < k$. This is only possible if $m(v_f(E)) = 0$.

Now we proceed to show that $m(v_f(S)) = 0$. Using the fact that at each $e$ in $S$, there is a positive derived number of $f$, and as before, we may pick arbitrary small interval $[r, s]$ such that $e$ is either one of the end points of the interval,

$$\frac{f(s) - f(r)}{s - r} > 0$$

and $P \cap [r, s] = \varnothing$. Hence we may cover $v_f(e)$ by arbitrary small intervals $[v_f(r), v_f(s)]$. Therefore, by the Vitali Covering Theorem we may cover $v_f(S)$ almost everywhere by countable mutually disjoint closed intervals $\{[v_f(r_i), v_f(s_i)]\}$ such that $P \cap [r_i, s_i] = \varnothing$, $f(s_i) - f(r_i) > 0$ for each $i$ and

$$m(v_f(S)) \le \sum_i (v_f(s_i) - v_f(r_i)) \le \sum_i f(s_i) - f(r_i) + \varepsilon \quad , \qquad \text{----------------- (6)}$$

where the last inequality is deduced using (1).

Similarly as before using the negative derived number of $f$ at each of the point $e$ of $S$, we may cover $v_f(E)$ almost every where with countable mutually disjoint closed intervals $\{[v_f(p_i), v_f(q_i)]\}$ such that $P \cap [p_i, q_i] = \varnothing$, $[p_i, q_i] \subseteq \bigcup_i [r_i, s_i]$, $f(p_i) > f(q_i)$ for each $i$ and

$$m(v_f(S)) \le \sum_i (v_f(q_i) - v_f(p_i)) \le \sum_i f(p_i) - f(q_i) + \varepsilon \qquad \text{-------------------- (7)}$$

Since , $\bigcup_i [p_i, q_i] \subseteq \bigcup_i [r_i, s_i]$,

$$\sum_i f(p_i) - f(q_i) \le \sum_i (N_f(s_i) - N_f(r_i)),$$

$$\sum_i f(s_i) - f(r_i) \le \sum_i (P_f(s_i) - P_f(r_i)),$$

where $N_f$ and $P_f$ are the negative and positive variations of $f$. Therefore, because $v_f = N_f + P_f$,

$$\sum_i f(p_i) - f(q_i) + \sum_i f(s_i) - f(r_i) \le \sum_i (v_f(s_i) - v_f(r_i)).$$

This combining with (6) and (7) yields,

$$\sum_i (v_f(s_i) - v_f(r_i)) - \varepsilon + \sum_i (v_f(q_i) - v_f(p_i)) - \varepsilon \le \sum_i (v_f(s_i) - v_f(r_i))$$

and so

$$\sum_i (v_f(q_i) - v_f(p_i)) \le 2\varepsilon.$$

Hence, $m(v_f(S)) \le 2\varepsilon$. Since ε = 1/$N$ by passing to the limit as $N$ tends to infinity, $m(v_f(S)) = 0$.
Therefore,

$$m(v_f(S \cup E)) \le m(v_f(E)) + m(v_f(S)) = 0$$

and so $m(v_f(S \cup E)) = 0$.
Now for any ε > 0, take an open set $U$ such that $E \cup S \subseteq U$ and $m(U) \le \varepsilon$. Since U is open, $U$ is a countable union of mutually disjoint non-trivial intervals $I_i$. Then the collection $\{ v_f^{-1}(I_i) \}$ covers $E \cup S$. Therefore,

$$m(f(S \cup E)) \le m(f(v_f^{-1}(\bigcup I_i))) = \sum_i m(f(v_f^{-1}(I_i))) \le \sum_i m(I_i) = m(U) \le \varepsilon.$$

We have used the fact that $m(f(v_f^{-1}(I_i))) \le m(I_i)$ for each $i$. We deduce this as follows. For any point $x$, $y$ in $v_f^{-1}(I_i)$, $|f(x) - f(y)| \le |v_f(x) - v_f(y)| \le diameter(I_i)$. Therefore, the diameter of $f(v_f^{-1}(I_i)) \le$ diameter of $I_i$ = length of $I_i$ = m($I_i$) That means $m(f(v_f^{-1}(I_i))) \le m(I_i)$. Since ε was arbitrary, $m(f(E \cup S)) = 0$.
It remains now to show that $m(E \cup S) = 0$.
Since $f$ is of bounded variation, $f$ is differentiable almost everywhere. So we may assume that $f$ has finite derivative at every point of $E \cup S$. $f$ is obviously not differentiable at every point of $S$ since each point of $S$ has a positive and negative derived numbers. Note that since $|f'| = v_f'$ almost every where, we may look only at points $x$ in $E$ where the derived number for $f$ at $x$ has the same absolute value as the only derived number of $v_f$ at $x$. So since points in $E$ do not have this property, $E$ must have measure 0. It follows that $m(E \cup S) = 0$. We may alternatively prove directly that $m(E \cup S) = 0$ by using a Vitali covering argument.

## 16. Proof of de La Vallée Poussin Theorem (Theorem 14)

Let $E_{h,k} = \{ x \in [a, b]$: there is a derived number of $v_f$ at $x$ greater than $k$ and a derived number of $f$ at $x$, whose absolute value is less than $h$, $h < k.\}$.
Let $E = \bigcup \{E_{h,k} : h, k \text{ rational and } h < k\}$. Let $N = E \cup S$. We have already shown in the proof of Lemma 15 that $m(S) = m(f(S)) = m(v_f(S)) = 0$..
By Lemma 15, $m(E_{h,k}) = 0$ for each pair $(h, k)$, $h < k$. Thus $E$ is a countable union of sets of measure zero and so $m(N) = m(E \cup S) = 0$. Note that

$$m(f(E)) \le \sum_{0 < h < k,\ h \text{ and } k \text{ rational}} m(f(E_{h,k})) = 0$$

since the set $f(E) = \bigcup \{ f(E_{h,k}) : h, k \text{ rational and } h < k.\}$ is a countable union of sets $f(E_{h,k})$ each of measure zero by Lemma 15. Thus $m(f(E)) = 0$. It follows that $m(f(N)) = 0$. Similarly, we show that $m(v_f(N)) = 0$.
We now prove the property of $N$ as stated in the theorem. Take any $x$ in $[a, b] - N$. Then $x$ is not in $S$ and not in any $E_{h,k}$. Hence $f$ does not have a positive and a

negative derived numbers at $x$. Moreover for any finite derived number $DV$ of $v_f$ at $x$,

$$DV \le |Df| \text{ for any derived number } Df \text{ of } f \text{ at } x.$$

Therefore, for any derived number $DV$ of $v_f$ at $x$, we have

$$DV \le \inf\{ |Df| : Df \text{ is a derived number of } f \text{ at } x.\}.$$

Note that if $DV$ is a derived number of $v_f$ at $x$, then there is a sequence ( $h_n$ ) such that $h_n \neq 0$ , $h_n \to 0$ and

$$DV = \lim_{n\to\infty} \frac{v_f(x+h_n) - v_f(x)}{h_n} .$$

Therefore, the sequence $\left( \frac{v_f(x+h_n) - v_f(x)}{h_n} \right)$ is bounded. Since we have for each $n$, $\left| \frac{f(x+h_n) - f(x)}{h_n} \right| \le \left| \frac{v_f(x+h_n) - v_f(x)}{h_n} \right|$, the sequence $\left( \frac{f(x+h_n) - f(x)}{h_n} \right)$ is also bounded. Hence by the Bolzano Weierstrass Theorem, $\left( \frac{f(x+h_n) - f(x)}{h_n} \right)$ has a convergent subsequence, $\left( \frac{f(x+h_{n_k}) - f(x)}{h_{n_k}} \right)$ and

$$Df_1 = \lim_{k\to\infty} \frac{f(x+h_{n_k}) - f(x)}{h_{n_k}}$$

is a derived number of $f$ at $x$. Moreover the subsequence $\left( \frac{v_f(x+h_{n_k}) - v_f(x)}{h_{n_k}} \right)$ converges to the same value $DV$ and so we have

$$|Df_1| \le DV$$

But $DV \le |Df_1|$ and so $DV = |Df_1|$. It follows that any derived number of $v_f$ at $x$ is equal to $\inf\{|Df| : Df \text{ is a derived number of } f \text{ at } x.\}$. Consequently there can be only one derived number of $v_f$ at $x$ and so $v_f$ is differentiable at $x$. It follows that for any derived number $Df$ of $f$ at $x$,

$$|Df| \le v_f'(x)$$

and $v_f'(x) \le |Df|$ because $v_f'(x)$ is the infimum of all absolute values of the derived numbers of $f$ at $x$. Thus $|Df| = v_f'(x)$ for any derived number $Df$ of $f$ at $x$. Therefore, any derived number of $f$ has one unique absolute value. Since $f$ has no derived number of opposite sign at $x$, it can have only one unique derived number at $x$. That is to say, $f$ is differentiable at $x$.

Suppose now that $v_f$ has an infinite derived number at $x$, then since $x$ is in $[a, b] - N$, any derived number $Df$ of $f$ at $x$ must have $|Df| = \infty$ . Consequently there is only one derived number of $v_f$ at $x$, namely $+\infty$. Since $f$ does not have derived number of opposite signs at $x$, it can have only one derived number at $x$ either $+\infty$ or $-\infty$.

We have thus proved that $f$ is differentiable (finite or infinite) at every point of $[a, b] - N$.

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