Denjoy Saks Young Theorem for Arbitrary Function

By

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This article is the third of a series of articles towards the proof of the Denjoy Saks Young Theorem. The first two articles are "*Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem*" and "*Functions of Bounded Variation and de La Vallée Poussin's Theorem*". The definition of Dini derivates is given in "*Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem*" and we shall use the notation given there. The total variation function of a function of bounded variation is defined in "*Functions of Bounded Variation and de La Vallée Poussin's Theorem*" and we shall adopt the notation therein.

Suppose *A* is an arbitrary subset of $\mathbb{R}$ and $f: A \to \mathbb{R}$ is a finite valued function defined on *A*. We shall begin with the case when the right upper Dini derivate of *f* is less than ∞.

Let $B = \left\{ x \in A : {}_A D^+ f(x) < \infty \right\}$. For simplicity, we may assume that every point of *A* is a two-sided limit point of *A*, for non-limit points or only one-sided limit points constitute at most a denumerable set.

We assume that $m^*(B) > 0$.

The function *f* need not be bounded. We shall limit the domain to where *f* is bounded and pass to the whole space by extending the domain. To begin this approach, take a strictly increasing positive sequence of real numbers, (M_n), such that $M_n \nearrow \infty$. For each positive integer *n*, let $E_n = \left\{ x \in A : |f(x)| < M_n \right\}$.

Then plainly, $E_n \subseteq E_{n+1}$ and $A = \bigcup_{n=1}^{\infty} E_n$. Thus, $m^*(E_n) \to m^*(A)$, by the continuity from below property of Lebesgue outer measure. For each integer $k \geq 1$, let

$$E_{n,k} = \left\{ x \in B : \frac{f(x+h) - f(x)}{h} < M_n, x+h \in A \cap \left(x, x + \frac{1}{k} \right) \right\}$$

$$= \left\{ x \in B : \frac{f(x+h) - f(x)}{h} < M_n, 0 < h < \frac{1}{k}, x+h \in A \right\}$$

$$=\left\{x\in B:\frac{f(t)-f(x)}{t-x}<M_n, t\in A\cap\left(x,x+\frac{1}{k}\right)\right\}.$$

Since $M_n < M_{n+1}$, $E_{n,k} \subseteq E_{n+1,k}$ and $E_{n,k} \subseteq E_{n,k+1}$. Thus,

$$E_{n,k} \subseteq E_{n,k+1} \subseteq E_{n+1,k+1} \subseteq \cdots \qquad \text{------------------ } (*)$$

We claim that $B=\bigcup_{n,k}^{\infty} E_{n,k}$.

Suppose $x\in B$. Then ${}_AD^+ f(x) < M_n$ for some integer $n > 0$. That is to say,

$$\limsup_{t\to x^+, t\in A}\frac{f(t)-f(x)}{t-x}=\limsup_{h\to 0^+}\left\{\frac{f(x+h)-f(x)}{h}: x+h\in A\right\}<M_n .$$

This implies that there exists $\delta_x > 0$ such that

$$\sup\left\{\frac{f(x+h)-f(x)}{h}: 0<h<\delta_x, x+h\in A\right\}<M_n .$$

Therefore, $\frac{f(x+h)-f(x)}{h}<M_n$, for all $x+h\in A$ and $0<h<\delta_x$. Let k be a positive integer such that $\frac{1}{k}<\delta_x$. Then we have

$$\frac{f(x+h)-f(x)}{h}<M_n, \text{ for all } x+h\in A \text{ and } 0<h<\frac{1}{k}.$$

Consequently, $x\in E_{n,k}$. Hence, we conclude that $B\subseteq\bigcup_{n,k}^{\infty}E_{n,k}$.

Conversely, suppose $x\in E_{n,k}$ for some positive n and k. Then

$$\frac{f(x+h)-f(x)}{h}<M_n, \text{ for all } x+h\in A \text{ and } 0<h<\frac{1}{k}.$$

Hence,

$$\sup\left\{\frac{f(x+h)-f(x)}{h}<M_n : 0<h<\frac{1}{k}, x+h\in A\right\}\le M_n$$

and so,

$$\limsup_{k\to\infty}\left\{\frac{f(x+h)-f(x)}{h} < M_n : 0 < h < \frac{1}{k}, x+h \in A\right\} \le M_n .$$

Therefore, ${}_AD^+ f(x) \le M_n < \infty$ and so $x \in B$. Thus, we can conclude that $B = \bigcup_{n,k}^{\infty} E_{n,k}$.

Observe that $\bigcup_{n,k}^{\infty} E_{n,k} = \bigcup_{n,n}^{\infty} E_{n,n}$. This is because if $k < n$, then $E_{n,k} \subseteq E_{n,n}$ and if $k > n$, then $E_{n,k} \subseteq E_{k,k}$. Thus, $\bigcup_{n,k}^{\infty} E_{n,k} \subseteq \bigcup_{n,n}^{\infty} E_{n,n}$ and so equality follows.

Thus, by the continuity from below property of Lebesgue outer measure,

$$m^*(B) = \lim_{n\to\infty} m^*(E_{n,n}) \ .$$

Note that $B = B \cap A = B\left(\cap \bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty}\left(B \cap E_n\right)$.

We observe that $B = \bigcup_{n=1}^{\infty}\left(E_{n,n} \cap E_n\right)$. We deduce this as follows.

Take any x in B. Then $x \in E_{k,k}$ for some integer k and $x \in E_L$. If, $L \le k$, then $x \in E_k$, so that $x \in E_{k,k} \cap E_k$ and if $L > k$, then $x \in E_{L,L}$ so that $x \in E_{L,L} \cap E_L$. It follows that $B \subseteq \bigcup_{n=1}^{\infty}\left(E_{n,n} \cap E_n\right)$. Consequently, $B = \bigcup_{n=1}^{\infty}\left(E_{n,n} \cap E_n\right)$.

Note that $E_{n,n} \cap E_n \subseteq E_{n+1,n+1} \cap E_{n+1} \subseteq \cdots$ and so we have also that

$$m^*(B) = \lim_{n\to\infty} m^*\left(E_{n,n} \cap E_n\right).$$

Suppose that $m^*(B) < \infty$ or that the set B is bounded. Hence, for an ε > 0 such that $m^*(B) - \varepsilon > 0$ there exists a positive integer N such that $n \ge N \Rightarrow 0 < m^*(B) - \varepsilon < m^*(E_{n.n} \cap E_n) \le m^*(B)$. Thus, taking $n = N$, we have that

$$0 < m^*(B) - \varepsilon < m^*(E_{N,N} \cap E_N) \le m^*(B) .$$

Suppose $m^*(B) = \infty$, then for any $K > 0$, there exists integer N such that $n \ge N \Rightarrow m^*(E_{n.n} \cap E_n) \ge K$.

Let $A_1 = E_{N,N} \cap E_N$. Then $A_1 \subseteq B$. Moreover, for $x \in A_1$, we have for any $0 < \delta < \dfrac{1}{N}$,

$$\left| f(x) \right| < M_N \,,\ \frac{f(x+h) - f(x)}{h} < M_N \text{ for } 0 < h < \delta \text{ and } x + h \in A. \text{ ----------- (1)}$$

We have the following technical lemma.

Lemma 1. The function *f* is locally of bounded variation on A_1. That is to say, for any $c < d$, with *c*, *d* in A_1 and $d - c < \dfrac{1}{N}$. The restriction of *f* on $A_1 \cap [c,d]$ is of bounded variation on $A_1 \cap [c,d]$. More generally, for any positive integer, *n*, *f* is locally of bounded variation on $E_{n,n} \cap E_n$.

Proof.

Take any $x \in A_1 \cap [c,d]$ with $c < x \le d$ and $d - c = \delta < \dfrac{1}{N}$. For any partition

$Q : c = x_0 < x_1 < \cdots < x_n = x$ by points in $A_1 \cap [c,d]$,

The positive variation with respect to the partition *Q*, by (1), satisfies

$$P(Q) = \sum_{i=1}^{n} \max\left(f(x_i) - f(x_{i-1}), 0 \right) \le \sum_{i=1}^{n} \left(x_i - x_{i-1} \right) M_N = (x - x_0) M_N < \frac{1}{N} M_N .$$

The negative variation with respect to the partition *Q* is

$$N(Q) = \sum_{i=1}^{n} \min\left(f(x_i) - f(x_{i-1}), 0 \right)$$

$$\ge -2 \sum_{i=1}^{n} \left(x_i - x_{i-1} \right) M_N = -2(x - c) M_N .$$

Thus, the negative variation is bounded below. It follows that the negative variation function of the restriction of *f* to $A_1 \cap [c,d]$,

$$n(x) = \inf\left\{ N(Q) : Q \text{ any partition of } [c,x] \text{ by points in } A_1 \cap [c,d] \right\} \ge -2(x-c) M_N .$$

Therefore, with *c* as the anchor point, the positive variation function, *p*(*x*) of the restriction of *f* to $A_1 \cap [c,d]$ satisfies that $p(x) \le \delta M$ with $0 < \delta < \dfrac{1}{N}$ and $n(x) \ge -2(d-c) M_N$ (For the definition of positive variation, negative variation and total variation, please refer to the definition in the proof of Theorem 6 in *Functions of Bounded Variation and de La Vallée Poussin's Theorem*.) Hence, the total variation function,

$$v_f(x) = p(x) - n(x) \le \delta M_N + 2(d-c)M_N < 3\delta M_N < 3\frac{M_N}{N}.$$

Thus, the restriction of *f* to $A_1 \cap [c,d]$ is of bounded variation on $A_1 \cap [c,d]$.

The proof for $E_{n,n} \cap E_n$ is exactly the same.

Remark. In Lemma 1, we need not choose the closed interval to have end points in *A*. However, we assume that $A_1 \cap [c,d] \ne \varnothing$. We may just pick any point, *a*, in $A_1 \cap [c,d]$ as anchor point for the definition of total variation function of *f*.

Now for $x > a$ with $x \in A_1 \cap [c,d]$, $v_{f,a}(x) = p(x) - n(x) \le (x-a)M_N + 2(x-a)M_N$ and for $y < a$ with $y \in A_1 \cap [c,d]$, $|v_{f,a}(y)| = |v_{f,y}(a)| \le (a-y)M_N + 2(a-y)M_N$. Hence, the total absolute variation for *f* on $(x,y) \cap A_1$ is less than

$$(x-y)M_N + 2(x-y)M_N < 3\delta M_N < 3\frac{M_N}{N}.$$

It follows that *f* is of bounded variation on $A_1 \cap [c,d]$ with bound $3M_N$.

It is customary to proceed from finite Dini derivates to possibly infinite ones. So, our next result will be on set with finite Dini derivates. We shall make similar definitions of $E_{n,k}$ and E_n with *B* replaced by $\{x \in A : -\infty < {}_AD^+ f(x) < \infty\}$

and $E_{n,k} = \left\{x \in B : -M_n < \dfrac{f(x+h) - f(x)}{h} < M_n, x+h \in A \cap \left(x, x+\dfrac{1}{k}\right)\right\}$. Analogous properties that have been stated for $E_{n,k}$ and E_n hold true and that Lemma 1 holds when $B = \{x \in A : -\infty < {}_AD^+ f(x) < \infty\}$ with the corresponding set $E_{n,n} \cap E_n$. Note that the definition of E_n remains unchanged, $E_n = \{x \in A : |f(x)| < M_n\}$.

We verify these facts below:

Suppose $B = \{x \in A : -\infty < {}_AD^+ f(x) < \infty\}$.

Let (M_n) be a strictly increasing positive sequence of real numbers, (M_n) , such that $M_n \nearrow \infty$. As before let $E_n = \{x \in A : |f(x)| < M_n\}$. We shall use the same notation as before. Define

$$E_{n,k} = \left\{ x \in B : -M_n < \frac{f(x+h) - f(x)}{h} < M_n, x + h \in A \cap \left(x, x + \frac{1}{k} \right) \right\}$$

$$= \left\{ x \in B : -M_n < \frac{f(x+h) - f(x)}{h} < M_n, 0 < h < \frac{1}{k}, x + h \in A \right\}$$

$$= \left\{ x \in B : -M_n < \frac{f(t) - f(x)}{t - x} < M_n, t \in A \cap \left(x, x + \frac{1}{k} \right) \right\}.$$

Since $M_n < M_{n+1}$, $E_{n,k} \subseteq E_{n+1,k}$ and $E_{n,k} \subseteq E_{n,k+1}$. It follows that $E_{n,n} \subseteq E_{n+1,n+1} \subseteq \cdots$.

We can show as before that $B = \bigcup_{n=1}^{\infty} E_{n,n}$.

Suppose $x \in B$. Then $-M_n < {}_A D^+ f(x) < M_n$ for some integer $n > 0$. That is to say,

$$-M_n < \limsup_{t \to x^+, t \in A} \frac{f(t) - f(x)}{t - x} = \limsup_{h \to 0^+} \left\{ \frac{f(x+h) - f(x)}{h} : x + h \in A \right\} < M_n .$$

This implies that $\sup_{\delta > h > 0} \left\{ \frac{f(x+h) - f(x)}{h} : x + h \in A \right\} > -M_n$ and for any $\delta > 0$ there exists $\delta_x > 0$ such that

$$\sup \left\{ \frac{f(x+h) - f(x)}{h} : 0 < h < \delta_x, x + h \in A \right\} < M_n .$$

Therefore, $-M_n < \frac{f(x+h) - f(x)}{h} < M_n$, for all $x + h \in A$ and $0 < h < \delta_x$. Let k be a positive integer such that $\frac{1}{k} < \delta_x$. Then we have

$$-M_n < \frac{f(x+h) - f(x)}{h} < M_n, \text{ for all } x + h \in A \text{ and } 0 < h < \frac{1}{k}.$$

Consequently, $x \in E_{n,k}$. Hence, we conclude that $B \subseteq \bigcup_{n,k}^{\infty} E_{n,k}$.

Conversely, suppose $x \in E_{n,k}$ for some positive n and k. Then

$$-M_n < \frac{f(x+h)-f(x)}{h} < M_n, \text{ for all } x+h \in A \text{ and } 0 < h < \frac{1}{k}.$$

Hence,

$$-M_n < \sup\left\{\frac{f(x+h)-f(x)}{h} < M_n : 0 < h < \frac{1}{k}, x+h \in A\right\} \le M_n$$

and so,

$$-M_n \le \limsup_{k\to\infty}\left\{\frac{f(x+h)-f(x)}{h} < M_n : 0 < h < \frac{1}{k}, x+h \in A\right\} \le M_n.$$

Therefore, $-\infty < {}_A D^+ f(x) < \infty$ and so $x \in B$. Thus, we can conclude that

$$B = \bigcup_{n,k}^{\infty} E_{n,k} = \bigcup_{n=1}^{\infty} E_{n,n}.$$

Similarly, we deduce that

$$B = \bigcup_{n=1}^{\infty}\left(E_{n,n} \cap E_n\right).$$

We make the tacit understanding that when $B = \left\{x \in A : -\infty < {}_A D^+ f(x) < \infty\right\}$, $E_{n,k} = \left\{x \in B : -M_n < \frac{f(x+h)-f(x)}{h} < M_n, x+h \in A \cap \left(x, x+\frac{1}{k}\right)\right\}$ and when $B = \left\{x \in A : {}_A D^+ f(x) < \infty\right\}$, $E_{n,k} = \left\{x \in B : \frac{f(x+h)-f(x)}{h} < M_n, x+h \in A \cap \left(x, x+\frac{1}{k}\right)\right\}$.

Moreover, inequality (1) still holds, i.e., for $x \in E_{n,n}$ and for any $0 < \delta < \frac{1}{n}$, we have

$$|f(x)| < M_n, \ \frac{f(x+h)-f(x)}{h} < M_n \text{ for } 0 < h < \delta \text{ and } x+h \in A. \text{ ----------- (1)*}$$

We have that *f* is locally of bounded variation on $E_{n,n} \cap E_n$. This is a consequence of Lemma 1.

Lemma 2. Suppose $f : A \to \mathbb{R}$ is a finite-valued function. Let $B = \left\{x \in A : -\infty < {}_A D^+ f(x) < \infty\right\}$, $\left\{E_{n,n}\right\}$ and $\left\{E_n\right\}$ be defined as above. For a fixed positive integer *N*, let $A_1 = E_{N,N} \cap E_N$. Then there exists a set, $\widetilde{N} \subseteq A_1$ such that $m^*(\widetilde{N}) = m^*\left(f(\widetilde{N})\right) = 0$ and for all $x \in A_1 - \widetilde{N}$, ${}_A D^+ f(x) = {}_{A_1} Df(x) = {}_A D_- f(x)$ and is finite. Moreover, *f* is a Lusin function on $A_1 = E_{N,N} \cap E_N$.

Proof.

By Lemma 1, for any *c* < *d*, with *c*, *d* in A_1 and $d - c < \frac{1}{N}$, *f* is of bounded variations on $A_1 \cap [c,d]$. Therefore, by Theorem 15 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, there is a subset $N_1 \subseteq A_1 \cap [c,d]$ of *A* such that

$$m\left(f(N_1)\right) = m(N_1) = 0,$$

and for each $x \in A_1 \cap [c,d] - N_1$, ${}_{A_1} Df(x)$ exist and is finite since ${}_A D^+ f(x)$ is finite. We can cover A_1 by countable number of such closed interval of length strictly less than $\frac{1}{N}$. In the proof of Lemma 1, we need not use precisely the closed interval [*c*, *d*] with *c*, *d* in A_1 because the total absolute variation is independent of the anchor point for the definition of the total variation. (See remark after Lemma 1.) For instance, if *a* is the anchor point and *x* < *a* < *y* with $x, y \in A_1 \cap [c,d]$, the total absolute variation function is given by $\left|v_{f,a}(x)\right| + v_{f,a}(y)$ and so it is bounded and so the function is of bounded variation on $A_1 \cap [c,d]$. Hence, we can conclude that there is a subset $N_2 \subseteq A_1$ such that $m\left(f(N_2)\right) = m(N_2) = 0$ and that ${}_{A_1} Df(x)$ exist and is finite for all *x* in $A_1 - N_2$. Moreover, *f* is a Lusin function on A_1. This is because by Theorem 18, *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, *f* is a Lusin function on $A_1 - N_2$ and as $m\left(f(N_2)\right) = 0$, *f* is a Lusin function on A_1.

Let $A_2 \subseteq A_1$ be the points of density of A_1 at which ${}_{A_1} Df(x)$ exist and is finite.

Then by Theorem 9 of *Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem*, A_2 contains all of $A_1 - N_2$ except for a null set. This means for *x* in A_2,

$$\lim_{m(I_x)\to 0, I_x \text{ an interval containing } x} \frac{m^*(I_x \cap A_1)}{m(I_x)} = 1. \quad \text{-------------} \quad (1)$$

Now $x \in A_2$ implies that ${}_{A_1}Df(x)$ is finite. Suppose $x \in A_2$ and ${}_{A}D^+ f(x) > {}_{A_1}Df(x) + \eta$ for some $\eta > 0$. Then, by definition of ${}_{A}D^+ f(x)$, there exists a sequence (ξ_n) such that $\xi_n > x$, $\xi_n \to x$ and

$$\frac{f(\xi_n) - f(x)}{\xi_n - x} > {}_{A_1}Df(x) + \eta. \quad \text{------------------} \quad (2)$$

We claim that for a fixed integer *n* in (2) and with ξ_n sufficiently close to *x*, for any ξ in $(x, \xi_n) \cap A_1$,

$$\frac{f(\xi) - f(x)}{\xi - x} < {}_{A_1}Df(x) + \eta', \quad \text{-------------------------} \quad (3)$$

for $0 < \eta' < \eta$.

Proof of this claim.

As $x \in A_2$ is a point of density of A_1,

$$\limsup_{m(I)\to 0, I \text{ an interval containing } x} \frac{m^*(I \cap A_1)}{m(I)} = \liminf_{m(I)\to 0, I \text{ an interval containing } x} \frac{m^*(I \cap A_1)}{m(I)} = 1.$$

Therefore, given any $\hat{\xi} > 0$, there exists $\hat{\delta} > 0$ such that

$$1 - \hat{\xi} < \frac{m^*(I_\delta \cap A_1)}{m(I_\delta)} \le 1,$$

for all closed interval, I_δ containing *x* with length, $m(I_\delta) = \delta < \hat{\delta}$. Take $\hat{\xi}$ to be sufficiently small so that we have,

$$m^*(A_1 \cap I_\delta) > m(I_\delta) - \hat{\xi} m(I_\delta) = \left(1 - \hat{\xi}\right) m(I_\delta) > 0. \quad \text{----------------------} \quad (4)$$

By definition of ${}_{A_1}Df(x)$, for any $0 < \eta' < \eta$, there exists $\zeta > 0$ such that

$$\frac{f(\xi) - f(x)}{\xi - x} < {}_{A_1}Df(x) + \eta', \quad \text{-------------------} \quad (5)$$

for all $\xi \in A_1 \cap (x, \zeta)$. Thus, if we choose $\xi_n < \zeta$ and $\xi_n - x < \hat{\delta}$, then we get

$$\frac{f(\xi)-f(x)}{\xi - x} < {}_{A_1}Df(x)+\eta' ,$$

for all $\xi \in A_1 \cap (x,\xi_n)$. This proves our claim (3)

We note that we can choose $\xi \in A_1 \cap (x,\zeta)$ sufficiently close to ξ_n and $\xi < \xi_n$ so that $\ell = \frac{\xi_n - \xi}{\xi_n - x}$ can be made as small as we wish. We deduce this below.

In general, take $\hat{\zeta}$ closed to ξ_n, with $x < \hat{\zeta} < \xi_n < \zeta$ such that $1-\varepsilon < \frac{\hat{\zeta}-x}{\xi_n - x} < 1-\theta$, with $\theta < \varepsilon$ and $\hat{\xi} < \theta$. If $(\hat{\zeta},\xi_n) \cap A_1 = \varnothing$ or $m^*\left((\hat{\zeta},\xi_n)\cap A_1\right) = 0$, then

$$1-\hat{\xi} < \frac{m^*([x,\xi_n]\cap A_1)}{m([x,\xi_n])} = \frac{m^*([x,\hat{\zeta}]\cap A_1)}{\xi_n - x} \le \frac{\hat{\zeta}-x}{\xi_n - x} < 1-\theta$$

and so $\hat{\xi} > \theta$ and we have a contradiction. This means that $(\hat{\zeta},\xi_n)\cap A_1 \ne \varnothing$. Thus, there exists a point $\xi \in (\hat{\zeta},\xi_n)\cap A_1$. This means there exists a point $\xi \in (\hat{\zeta},\xi_n)\cap A_1$ such that $\frac{f(\xi)-f(x)}{\xi - x} < {}_{A_1}Df(x)+\eta'$.

Now we show that for any $\xi \in (\hat{\zeta},\xi_n)\cap A_1$, $\frac{\xi_n - x}{\xi_n - \xi} > \frac{1}{\varepsilon}$.

Observe that $\frac{\xi_n - x}{\xi_n - \xi} > \frac{\xi_n - x}{\xi_n - \hat{\zeta}}$. We have

$$\frac{\xi_n - \hat{\zeta}}{\xi_n - x} = \frac{\xi_n - x - \left(\hat{\zeta}-x\right)}{\xi_n - x} = 1 - \frac{\hat{\zeta}-x}{\xi_n - x} < 1-(1-\varepsilon) = \varepsilon .$$

Hence,

$$\frac{\xi_n - x}{\xi_n - \xi} > \frac{\xi_n - x}{\xi_n - \hat{\zeta}} > \frac{1}{\varepsilon} . \quad \text{---------------------} \quad (6)$$

Indeed, we have that $\ell = \frac{\xi_n - \xi}{\xi_n - x} < \varepsilon$.

From inequality (2) we get,

$$f(\xi_n) - f(x) > \left(\xi_n - x\right) {}_{A_1}Df(x) + \eta\left(\xi_n - x\right). \quad \text{------------------} \quad (7)$$

From inequality (3), we get, for any $\xi \in (x, \xi_n) \cap A_1$,

$$f(\xi) - f(x) < (\xi - x)\, {}_{A_1}Df(x) + \eta'(\xi - x). \text{ ------------------ (8)}$$

Hence, from (7) and (8) we obtain,

$$f(\xi_n) - f(\xi) > (\xi_n - \xi)\, {}_{A_1}Df(x) + \eta(\xi_n - x) - \eta'(\xi - x)$$

$$= (\xi_n - \xi)\, {}_{A_1}Df(x) + (\xi_n - x)\left[\eta - \eta'\left(\frac{\xi - x}{\xi_n - x}\right)\right].$$

Therefore, for $\xi \in (x, \xi_n) \cap A_1$,

$$\frac{f(\xi_n) - f(\xi)}{\xi_n - \xi} > {}_{A_1}Df(x) + \left(\frac{\xi_n - x}{\xi_n - \xi}\right)\left[\eta - \eta'\left(\frac{\xi - x}{\xi_n - x}\right)\right]. \text{ --------------- (9)}$$

Note that $\xi_n \in A$ but $\xi \in A_1$. It follows that for all ξ_n with $|\xi_n - \xi| < \dfrac{1}{N}$,

$$\frac{f(\xi_n) - f(\xi)}{\xi_n - \xi} < M_N. \text{ ---------------------------- (10)}$$

From (9) we deduce that for our choice of $\xi \in (x, \xi_n) \cap A_1$,

$$\frac{f(\xi_n) - f(\xi)}{\xi_n - \xi} > {}_{A_1}Df(x) + \frac{1}{\ell}(\eta - \eta').$$

Since $\eta - \eta' > 0$, we can choose $\hat{\xi} > 0$ as small as we like and we can choose θ and ε to be as small as we like so that

$$\frac{f(\xi_n) - f(\xi)}{\xi_n - \xi} > {}_{A_1}Df(x) + \frac{1}{\ell}(\eta - \eta') > M_N.$$

This contradicts inequality (10) and so at points of A_2, ${}_{A}D^{+}f(x) \le {}_{A_1}Df(x)$. Since ${}_{A}D^{+}f(x) \ge {}_{A_1}Df(x)$, we conclude that ${}_{A}D^{+}f(x) = {}_{A_1}Df(x)$.

Now we consider ${}_{A}D_{-}f(x)$ at point $x \in A_2$.

Suppose $x \in A_2$ and ${}_{A}D_{-}f(x) < {}_{A_1}Df(x) - \eta$ for some $\eta > 0$. Then, by definition of ${}_{A}D_{-}f(x)$, there exists a sequence (ξ_n) such that $\xi_n < x$, $\xi_n \to x$ and

$$\frac{f(\xi_n)-f(x)}{\xi_n - x} < {}_{A_1}Df(x) - \eta \text{ . -------------------- (11)}$$

By definition of ${}_{A_1}D_{-}f(x) = {}_{A_1}Df(x)$, for any $0 < \eta' < \eta$, there exists $\zeta > 0$ such that

$$\frac{f(\xi)-f(x)}{\xi - x} > {}_{A_1}Df(x) - \eta' \text{ , -------------------- (12)}$$

for all $\xi \in A_1 \cap (\zeta, x)$.

We claim that for a fixed integer *n* in (2) and with ξ_n sufficiently close to *x*, there exists a ξ in $(\varsigma, \xi_n) \cap A_1$ such that

$$\frac{f(\xi)-f(x)}{\xi - x} > {}_{A_1}Df(x) - \eta' \text{ , -------------------------- (13)}$$

for $0 < \eta' < \eta$.

Proof of this claim.

As $x \in A_2$ is a point of density of A_1,

$$\limsup_{m(I)\to 0, I \text{ an interval containing } x} \frac{m^*(I \cap A_1)}{m(I)} = \liminf_{m(I)\to 0, I \text{ an interval containing } x} \frac{m^*(I \cap A_1)}{m(I)} = 1 .$$

Therefore, given any $\hat{\xi} > 0$, there exists $\hat{\delta} > 0$ such that

$$1 - \hat{\xi} < \frac{m^*(I_\delta \cap A_1)}{m(I_\delta)} \le 1 ,$$

for all closed interval, I_δ containing *x* with length, $m(I_\delta) = \delta < \hat{\delta}$. Take $0 < \hat{\xi} < \frac{1}{2}$ to be sufficiently small so that we have,

$$m^*(A_1 \cap I_\delta) > m(I_\delta) - \hat{\xi} m(I_\delta) = \left(1 - \hat{\xi}\right) m(I_\delta) > 0 \text{ . ----------------------- (14)}$$

Thus, if we take $\zeta < x$ and $x - \zeta < \hat{\delta}$, and choose $\zeta < \xi_n < x$, then

$$\frac{f(\xi)-f(x)}{\xi - x} > {}_{A_1}Df(x) - \eta' ,$$

for all $\xi \in A_1 \cap (\zeta, \xi_n)$.

We note that we can choose $\xi \in A_1 \cap (\zeta, x)$ sufficiently close to ξ_n so that $\zeta < \xi < \xi_n < x$ and $\ell = \dfrac{\xi_n - \xi}{x - \xi_n}$ can be made as small as we wish. We deduce this below.

Take $\hat{\zeta}$ closed to ξ_n such that $\zeta < \hat{\zeta} < \xi_n < x$ and $1-\varepsilon < \dfrac{x-\xi_n}{x-\hat{\zeta}} < 1-\theta$, with $\theta < \varepsilon$ and $\hat{\xi} < \theta$. As shown above if $(\hat{\zeta}, \xi_n) \cap A_1 = \varnothing$, then

$$1-\hat{\xi} < \frac{m^*([\hat{\zeta}, x] \cap A_1)}{m([\hat{\zeta}, x)]} = \frac{m^*([\xi_n, x] \cap A_1)}{x-\hat{\zeta}} \le \frac{x-\xi_n}{x-\hat{\zeta}} < 1-\theta$$

and so $\hat{\xi} > \theta$ and we have a contradiction. Thus $(\hat{\zeta}, \xi_n) \cap A_1 \neq \varnothing$ and so take $\xi \in (\hat{\zeta}, \xi_n) \cap A_1$. Note that as $\hat{\xi}$ gets smaller, we can choose θ and ε to get smaller, for instance that they may be chosen to be less than $2\hat{\xi}$.

Note that $\dfrac{x-\xi_n}{\xi_n - \xi} > \dfrac{x-\xi_n}{\xi_n - \hat{\zeta}}$ and

$$\frac{\xi_n - \hat{\zeta}}{x-\xi_n} = \frac{x-\hat{\zeta}-(x-\xi_n)}{x-\xi_n} = \frac{x-\hat{\zeta}}{x-\xi_n} - 1 < \frac{1}{1-\varepsilon} - 1 = \frac{\varepsilon}{1-\varepsilon} < 2\varepsilon \quad \text{if} \quad 0 < \varepsilon < \frac{1}{2}.$$

Thus, $\ell = \dfrac{\xi_n - \xi}{x-\xi_n} < \dfrac{\xi_n - \hat{\zeta}}{x-\xi_n} < 2\varepsilon$ and so $\dfrac{x-\xi_n}{\xi_n - \xi} > \dfrac{1}{2\varepsilon}$.

Since $\xi_n - x$ and $\xi - x$ are negative, we have,

$$f(\xi_n) - f(x) > (\xi_n - x)\, {}_A Df(x) - \eta(\xi_n - x)$$

and $f(\xi) - f(x) < (\xi - x)\, {}_{A_1} Df(x) - \eta'(\xi - x)$. Subtracting these two inequalities we get $f(\xi_n) - f(\xi) > (\xi_n - \xi)\, {}_A Df(x) + (x - \xi_n)\left[\eta - \eta'\left(\dfrac{x-\xi}{x-\xi_n}\right)\right]$ and so

$$\frac{f(\xi_n) - f(\xi)}{\xi_n - \xi} > {}_A Df(x) + \frac{x-\xi_n}{\xi_n - \xi}\left[\eta - \eta'\left(\frac{x-\xi}{x-\xi_n}\right)\right].$$

Observe that $\left(\dfrac{x-\xi}{x-\xi_n}\right) > 1$ and so $\dfrac{x-\xi_n}{\xi_n-\xi}\left[\eta - \eta'\left(\dfrac{x-\xi}{x-\xi_n}\right)\right] \ge \dfrac{x-\xi_n}{\xi_n-\xi}(\eta - \eta') > \dfrac{1}{2\varepsilon}(\eta - \eta')$.

Note that $\eta - \eta' > 0$. We can choose $\hat{\xi}$ to be arbitrary small along with arbitrary small θ and ε with $\hat{\xi} < \theta < \varepsilon$. It follows that there exist $\xi \in A_1$ and ξ_n in A such that $\xi < \xi_n$ and $\dfrac{f(\xi_n) - f(\xi)}{\xi_n - \xi} > M_N$. Thus, we arrive at a contradiction. Hence, we must have ${}_A D_- f(x) \geq {}_{A_1} Df(x)$. As ${}_{A_1} Df(x) \geq D_- f(x)$, we have ${}_A D_- f(x) = {}_{A_1} Df(x)$. Suppose $A_2 = (A_1 - N_2) - N_3$, where $m(N_3) = 0$. If we now let $\widetilde{N} = N_2 \cup N_3$, then $m^*(\widetilde{N}) = m^*\left(f(\widetilde{N})\right) = 0$ and for all $x \in A_1 - \widetilde{N}$, ${}_A D^+ f(x) = {}_A D_- f(x)$ and is finite. This completes the proof of Lemma 2.

Corollary 3. Suppose $f : A \to \mathbb{R}$ is a finite valued function. Let $B = \left\{x \in A : {}_A D^+ f(x) < \infty\right\}$, $\left\{E_{n,n}\right\}$ and $\left\{E_n\right\}$ be defined as above. For a fixed positive integer *N*, let $A_1 = E_{N,N} \cap E_N$. Then there exists a set, $\widetilde{N} \subseteq A_1$ such that $m^*(\widetilde{N}) = m^*\left(f(\widetilde{N})\right) = 0$ and for all $x \in A_1 - \widetilde{N}$, ${}_A D^+ f(x) = {}_A D_- f(x)$ is finite or $-\infty$.

Moreover, the set $K_N = \left\{x \in A : {}_A D^+ f(x) = -\infty\right\}$ has measure zero and *f* is a Lusin function on $A_1 - \widetilde{N} - K_N$.

Proof.

The proof proceeds exactly as for the proof for Lemma 2. This time we have that except for a null set N_2, ${}_{A_1} Df(x)$ exists finitely or infinitely for $x \in A_1 - N_2$ and $m(N_2) = m(f(N_2)) = 0$. Since ${}_A D^+ f(x) < \infty$, ${}_{A_1} Df(x)$ can take on the value $-\infty$, when ${}_A D^+ f(x) = -\infty$.

In this case we have ${}_A D^+ f(x) = {}_A D_- f(x) = -\infty$. By Theorem 9, *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, the set $K_N = \left\{x \in A_1 : {}_{A_1} Df(x) = -\infty\right\} = \left\{x \in A_1 : {}_A D^+ f(x) = -\infty\right\}$ has measure zero. Let $A_2 \subseteq A_1 - N_2 - K_N$ be the points of density of A_1 at which ${}_{A_1} Df(x)$ exist and is finite. Suppose $A_2 = (A_1 - N_2 - K_N) - N_3$, where $N_3 \subseteq A_1 - N_2 - K_N$ and $m(N_3) = 0$. Let $\widetilde{N} = N_2 \cup N_3$.

Since *f* is a Lusin function on $A_1 - N_2 - K_N$, by Theorem 18, *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, $m\left(f(N_3)\right)=0$, as $m(N_3)=0$. Hence, $m^*(\widetilde{N})=m^*\left(f(\widetilde{N})\right)=0$ and ${}_AD^+f(x)={}_AD_-f(x)$ finitely or infinitely.

Theorem 4. Suppose $f:A\to\mathbb{R}$ is a finite-valued function. Suppose *B* is a subset of *A* such that at each point *x* of *B*, ${}_AD^+f(x)<\infty$. Then, except for a null set, $\widetilde{N}\subseteq B$, ${}_AD^+f(x)={}_AD_-f(x)$, its opposite derivate, and is either finite or -infinity. Moreover, $m^*(\widetilde{N})=m^*\left(f(\widetilde{N})\right)=0$. The set $K=\left\{x\in B: {}_AD^+f(x)=-\infty\right\}$ is a null set and *f* is a Lusin function on $B-K$.

Proof.

We shall use the notation used in Corollary 3.

We note that $B=\bigcup_{n=1}^{\infty}\left(E_{n,n}\cap E_n\right)$, $E_{n,n}\cap E_n\subseteq E_{n+1,n+1}\cap E_{n+1}\subseteq\cdots$ and $m^*(B)=\lim_{n\to\infty}m^*\left(E_{n,n}\cap E_n\right)$.

For each positive integer, *n*, let $A_n=E_{n,n}\cap E_n$. Then by Corollary 3, there exists a set, $\widetilde{N}_n\subseteq A_n$ such that $m^*(\widetilde{N}_n)=m^*\left(f(\widetilde{N}_n)\right)=0$ and for all $x\in A_n-\widetilde{N}_n$, ${}_AD^+f(x)={}_AD_-f(x)$ and is finite or $-\infty$. Let $\widetilde{N}=\bigcup_{n=1}^{\infty}\widetilde{N}_n$. Then $m\left(\widetilde{N}\right)=0$. Since $m^*\left(f(\widetilde{N}_n)\right)=0$, for each positive integer, *n*, $m^*\left(f\left(\widetilde{N}\right)\right)=m^*\left(f\left(\bigcup_{n=1}^{\infty}\widetilde{N}_n\right)\right)=0$. Let $B''=\bigcup_n A_n-\bigcup_n\widetilde{N}_{\tilde{n}}=B-\widetilde{N}$. Take any $x\in B''$. Then $x\in A_k-\widetilde{N}_k$ and so ${}_AD^+f(x)={}_AD_-f(x)$ and is finite or $-\infty$. Observe that

$$m^*(B'')=m^*\left(B\right)=\lim_{n\to\infty}m^*(A_n).$$

Note that for each positive integer, *n*, $\widetilde{N}_n$ contains points, *x*, where *x* is not a point of density of A_n or ${}_{A_n}Df(x)$ does not exist finitely or infinitely.

Note that if ${}_AD^+f(x)=-\infty$, there exists an integer *N*, such that $x\in E_{n,n}$ for all $n>N$. This is because if ${}_AD^+f(x)=-\infty$, then

$\limsup_{h\to 0^+}\left\{\frac{f(x+h)-f(x)}{h}, x+h\in A\right\} = -\infty$ implies that

there exists $\hat{\delta} > 0$ such that $\sup_{0<\delta<\hat{\delta}}\left\{\frac{f(x+h)-f(x)}{h}, x+h\in A, 0<h<\delta\right\} < 0$. It follows that $\frac{f(x+h)-f(x)}{h} < 0$ for $x+h\in A, 0<h<\hat{\delta}$. Thus, $x\in E_{n,n}$ for all *n* such that $\frac{1}{n} < \hat{\delta}$. Take any integer *N* with $\frac{1}{N} < \hat{\delta}$ and we have $x \in E_{n,n}$ for all *n* > *N*.

Let $K = \left\{x\in B : {}_A D^+ f(x) = -\infty\right\}$. Then we claim that $K = \bigcup_n K_n$, where K_n is as given in Corollary 3. Note that $x\in K \Rightarrow x\in E_{n,n} \Rightarrow x \in K_n$ for some positive integer *n*. Therefore, $K \subseteq \bigcup_n K_n$. By definition, $K_n \subseteq K$ for each positive integer *n*. Hence, $\bigcup_n K_n \subseteq K$ and so $K = \bigcup_n K_n$. Therefore, $m^*(K) = m^*\left(\bigcup_n K_n\right) = 0$.

Now, *f* is a Lusin function on $B - K - \widetilde{N}$ by Theorem 5 below. Since $m\left(f(\widetilde{N})\right) = 0$, *f* is a Lusin function on $B - K$. This completes the proof of the theorem.

Theorem 5. Suppose $f : A \to \mathbb{R}$ is a finite valued function. Suppose *B* is a subset of *A* such that at each point *x* of *B*, ${}_A D^+ f(x)$ is finite, i.e., $-\infty < {}_A D^+ f(x) < \infty$. Then, except for a null set, $\widetilde{N} \subseteq B$, ${}_A D^+ f(x) = {}_A D_- f(x)$, its opposite derivate, and is finite. Moreover, $m^*(\widetilde{N}) = m^*\left(f(\widetilde{N})\right) = 0$ and *f* is a Lusin function on *B*.

Proof.

Now we can apply Lemma 1 and Lemma 2.

For each positive integer, *n*, let $A_n = E_{n,n} \cap E_n$. Then by Lemma 2, there exists a set, $\widetilde{N}_n \subseteq A_n$ such that $m^*(\widetilde{N}_n) = m^*\left(f(\widetilde{N}_n)\right) = 0$ and for all $x \in A_n - \widetilde{N}_n$, ${}_A D^+ f(x) = {}_A D_- f(x)$ and is finite. Let $\widetilde{N} = \bigcup_{n=1}^{\infty} \widetilde{N}_n$. Then $m\left(\widetilde{N}\right) = 0$. Since $m^*\left(f(\widetilde{N}_n)\right) = 0$, for each positive integer, *n*, $m^*\left(f\left(\widetilde{N}\right)\right) = m^*\left(f\left(\bigcup_{n=1}^{\infty}\widetilde{N}_n\right)\right) = 0$. Let

$B'' = \bigcup_n A_n - \bigcup_n \widetilde{N}_{\tilde{n}} = B - \widetilde{N}$. Take any $x \in B''$. Then $x \in A_k - \widetilde{N}_k$ and so ${}_A D^+ f(x) = {}_A D_- f(x)$ and is finite.

By Lemma 2, *f* is a Lusin function on $A_n = E_{n,n} \cap E_n$. Suppose $E \subseteq B$ is s null set in *B*. Then $m^*(E \cap A_n) = 0$. Therefore, $m^*\left(f(E \cap A_n)\right) = 0$. As $m^*\left(f(E)\right) = m^*\left(f(E \cap \bigcup_{n=1}^{\infty} A_n)\right) \le m^*\left(\bigcup_{n=1}^{\infty} f\left(E \cap A_n\right)\right)$, it follows that $m^*\left(f(E)\right) = 0$. Thus, *f* is a Lusin function on *B*. This completes the prof of Theorem 5.

Corollary 6. Suppose $f : A \to \mathbb{R}$ is a finite valued function. Suppose *B* is a subset of *A* such that at each point *x* of *B*, ${}_A D^+ f(x) < \infty$. Let $K = \left\{ x \in B : {}_A D^+ f(x) = -\infty \right\}$. Suppose $m(f(K)) = 0$. Then *f* is a Lusin function on *B*.

Proof.

By Theorem 4, *f* is a Lusin function on $B - \widetilde{N} - K$, where $\widetilde{N}$ is given in Theorem 4. Since $m^*\left(f(\widetilde{N} \cup K)\right) = m^*\left(f(\widetilde{N}) \cup f(K)\right) = 0$ as $m(f(K)) = m(f(\widetilde{N})) = 0$, *f* is a Lusin function on *B*.

We have corresponding results for similar condition on the Dini derivates.

We shall state these results without proof for the proof involves similar technique.

Theorem 7. Suppose $f : A \to \mathbb{R}$ is a finite valued function. Suppose *B* is a subset of *A* such that at each point *x* of *B*, ${}_A D_- f(x) > -\infty$. Then, except for a null set, $\widetilde{N} \subseteq B$, ${}_A D_- f(x) = {}_A D^+ f(x)$, its opposite derivate, and is either finite or + infinity. Moreover, $m^*(\widetilde{N}) = m^*\left(f(\widetilde{N})\right) = 0$. The set $K = \left\{ x \in B : {}_A D_-(x) = \infty \right\}$ is a null set and *f* is a Lusin function on $B - K$.

Theorem 8. Suppose $f : A \to \mathbb{R}$ is a finite valued function. Suppose *B* is a subset of *A* such that at each point *x* of *B*, ${}_A D^- f(x) < \infty$. Then, except for a null set, $\widetilde{N} \subseteq B$, ${}_A D^- f(x) = {}_A D_+ f(x)$, its opposite derivate, and is either finite or

$-$infinity. Moreover, $m^*(\widetilde{N}) = m^*\left(f(\widetilde{N})\right) = 0$, the set $K = \left\{x \in B : {}_A D^- f(x) = -\infty\right\}$ is a null set and *f* is a Lusin function on $B - K$.

Theorem 9. Suppose $f : A \to \mathbb{R}$ is a finite valued function. Suppose *B* is a subset of *A* such that at each point *x* of *B*, ${}_A D_+ f(x) > -\infty$. Then, except for a null set, $\widetilde{N} \subseteq B$, ${}_A D_+ f(x) = {}_A D^- f(x)$, its opposite derivate, and is either finite or + infinity. Moreover, $m^*(\widetilde{N}) = m^*\left(f(\widetilde{N})\right) = 0$, the set $K = \{x \in B : {}_A D_+ f(x) = \infty\}$ is a null set and f is a Lusin function on $B - K$.

As a result of the above theorems we have,

Corollary 10. Suppose $f : A \to \mathbb{R}$ is a finite valued function. Then for almost all *x* in *A*, ${}_A D^+ f(x) > -\infty$, ${}_A D^- f(x) > -\infty$, ${}_A D_+ f(x) < \infty$ and ${}_A D_- f(x) < \infty$.

Theorem 11. Suppose $f : A \to \mathbb{R}$ is a finite valued function. Suppose *B* is a subset of *A* such that at each point *x* of *B*, *f* has either both finite Dini derivates on the same side or finite bilateral derivates ${}_A \overline{D} f(x)$ or ${}_A \underline{D} f(x)$. Then, *f* is differentiable almost everywhere on *B*, i.e., for almost all *x* in *B*, ${}_A Df(x)$ exists and is finite. Moreover, for the subset *E* of *B*, where ${}_A Df(x)$ does not exists, $m^*(E) = m^*\left(f(E)\right) = 0$. *f* is a Lusin function on *B*.

Proof.

Suppose ${}_A D^+ f(x)$ and ${}_A D_+ f(x)$ are finite for all *x* in *B*. Then by Theorem 5, since ${}_A D^+ f(x)$ is finite for all *x* in *B*, except for a null set, $\widetilde{N}_1 \subseteq B$, ${}_A D^+ f(x) = {}_A D_- f(x)$, its opposite derivate, and is finite. Moreover, $m^*(\widetilde{N}_1) = m^*\left(f(\widetilde{N}_1)\right) = 0$ and *f* is a Lusin function on *B*. As ${}_A D_+ f(x)$ is finite on *B*, by an analogue theorem to Theorem 5, except for a null set, $\widetilde{N}_2 \subseteq B$, ${}_A D_+ f(x) = {}_A D^- f(x)$, its opposite derivate, and is finite. We also have that $m^*(\widetilde{N}_2) = m^*\left(f(\widetilde{N}_2)\right) = 0$. Therefore, for

$x \in B - \left(\widetilde{N}_1 \cup \widetilde{N}_2\right)$, ${}_A D^+ f(x) = {}_A D_- f(x) \le {}_A D^- f(x) = {}_A D_+ f(x)$ and as ${}_A D_+ f(x) \le {}_A D^+ f(x)$, ${}_A D^+ f(x) = {}_A D_- f(x) = {}_A D^- f(x) = {}_A D_+ f(x)$. This means ${}_A Df(x)$ exists and is finite. Let $E = \widetilde{N}_1 \cup \widetilde{N}_2$. Then $m(E) = m(\widetilde{N}_1 \cup \widetilde{N}_2) = 0$. This means *f* is differentiable almost everywhere on *B*. We already knew that *f* is a Lusin function on *B*.

Suppose ${}_A D^- f(x)$ and ${}_A D_- f(x)$ are finite for all *x* in *B*. Then we can show similarly that for almost all *x* in *B*, ${}_A Df(x)$ exists and is finite and that the set where ${}_A Df(x)$ does not exists is a null set and that *f* is Lusin function on *B*.

Suppose $-\infty < {}_A \overline{D} f(x) < \infty$ for all *x* in *B*. Then ${}_A D^+ f(x) < \infty$ and ${}_A D^- f(x) < \infty$ for all *x* in *B*. Then, except for a null set, $\widetilde{N}_1 \subseteq B$, ${}_A D^+ f(x) = {}_A D_- f(x)$, its opposite derivate, and is either finite or –infinity. Moreover, $m^*(\widetilde{N}_1) = m^*\left(f(\widetilde{N}_1)\right) = 0$.

Similarly, except for a null set, $\widetilde{N}_2 \subseteq B$, ${}_A D^- f(x) = {}_A D_+ f(x)$, its opposite derivate, and is either finite or -infinity. Moreover, $m^*(\widetilde{N}_2) = m^*\left(f(\widetilde{N}_2)\right) = 0$.

Therefore, for $x \in B - \left(\widetilde{N}_1 \cup \widetilde{N}_2\right)$, ${}_A D^+ f(x) = {}_A D_- f(x)$ and ${}_A D^- f(x) = {}_A D_+ f(x)$.

Suppose ${}_A D^+ f(x) = -\infty$. Then ${}_A D^+ f(x) = {}_A D_- f(x) = {}_A D^- f(x) = {}_A D_+ f(x)$. But this contradicts that $-\infty < {}_A \overline{D} f(x) = \max\left({}_A D^+ f(x), {}_A D^- f(x)\right) < \infty$. Similarly, ${}_A D^- f(x) = -\infty$ leads to a contradiction and so for all $x \in B - \left(\widetilde{N}_1 \cup \widetilde{N}_2\right)$ ${}_A D^+ f(x) = {}_A D_- f(x) = {}_A D^- f(x) = {}_A D_+ f(x)$ and is finite. That is to say, *f* is differentiable on $B - \left(\widetilde{N}_1 \cup \widetilde{N}_2\right)$(with finite derivative). Moreover, denoting $E = \left(\widetilde{N}_1 \cup \widetilde{N}_2\right)$, $m^*(E) = m^*\left(f(E)\right) = 0$. By Theorem 5, *f* is a Lusin function on *B*–*E*. Therefore, *f* is a Lusin function on *B*.

Suppose $-\infty < {}_A \underline{D} f(x) < \infty$ for all *x* in *B*. Then ${}_A D_+ f(x) > -\infty$ and ${}_A D_- f(x) > -\infty$ for all *x* in *B*. We can show similarly that except for a null set *E* in *B*, *f* is differentiable with finite derivative. Moreover, $m^*(E) = m^*\left(f(E)\right) = 0$ and *f* is a Lusin function on *B*.

Theorem 12. Suppose $f: A \to \mathbb{R}$ is a finite valued function. Suppose *B* is a subset of *A* such that at each point *x* of *B*, $\left| {}_A D^+ f(x) \right| \le M$ for some non-negative number *M*. Then $m^*\big(f(B)\big) \le Mm^*(B)$.

Proof.

We have $-M \le {}_A D^+ f(x) \le M$ for all *x* in *B*. By Theorem 5, ${}_A D^+ f(x) = {}_A D_- f(x)$ and is finite for all *x* in $B - \widetilde{N}$, where $\widetilde{N}$ is a subset of *B* such that $m^*(\widetilde{N}) = m^*\big(f(\widetilde{N})\big) = 0$. Therefore, for *x* in $B - \widetilde{N}$, as $\left| {}_A D^+ f(x) \right| \le M$, $-M \le {}_A D_- f(x) = {}_A D^+ f(x)$. It follows by Theorem 10 of *Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem*, that $m^*\big(f(B - \widetilde{N})\big) \le Mm^*(B - \widetilde{N})$. Hence,

$$m^*\big(f(B)\big) \le m^*\big(f(B - \widetilde{N})\big) + m^*\big(f(\widetilde{N})\big) = m^*\big(f(B - \widetilde{N})\big) \le Mm^*(B - \widetilde{N}) \le Mm^*(B).$$

Denjoy Saks Young Theorem

We are now ready to state the Denjoy Saks Young Theorem for arbitrary function on any subset of $\mathbb{R}$.

Theorem 13. Suppose $f: A \to \mathbb{R}$ is a finite valued function. Let

$$N = \left\{ x \in A : {}_A D^+ f(x) = -\infty \text{ or } {}_A D^- f(x) = -\infty \text{ or } {}_A D_+ f(x) = \infty \text{ or } {}_A D_- f(x) = \infty \right\},$$

$$S = \left\{ x \in A : {}_A Df(x) \text{ exists and is finite} \right\},$$

$$T = \left\{ x \in A : {}_A D^+ f(x) \text{ and } {}_A D_- f(x) \text{ are finite and equal}, \ {}_A D_+ f(x) = -\infty \text{ and } {}_A D^- f(x) = \infty \right\},$$

$$U = \left\{ x \in A : {}_A D^- f(x) \text{ and } {}_A D_+ f(x) \text{ are finite and equal}, \ {}_A D^+ f(x) = \infty \text{ and } {}_A D_- f(x) = -\infty \right\}$$

and $V = \left\{ x \in A : {}_A D^+ f(x) = {}_A D^- f(x) = \infty \text{ and } {}_A D_- f(x) = {}_A D_+ f(x) = -\infty \right\}$.

Then $A = N \cup S \cup T \cup U \cup V \cup E$, where *E* is a null set and $m\big(f(E)\big) = 0$.

Moreover, $m(N) = 0$ and *f* is a Lusin function on $S \cup T \cup U$.

Proof.

By Corollary 10, *N* is a null set. Consider now the set $A-N$. Thus, for all *x* in $A-N$, ${}_AD^+f(x)\neq-\infty$, ${}_AD^-f(x)\neq-\infty$, ${}_AD_+f(x)\neq\infty$ and ${}_AD_-f(x)\neq\infty$.

Consider the set $B_1=\{x\in A-N:{}_AD^+f(x)<\infty\}$. Then by Theorem 5, except for a null set, $\widetilde{N}_1\subseteq B_1$, ${}_AD^+f(x)={}_AD_-f(x)$, its opposite derivate, and is finite. Moreover, $m^*(\widetilde{N}_1)=m^*\big(f(\widetilde{N}_1)\big)=0$. Let $B_2=\{x\in A-N:{}_AD_+f(x)>-\infty\}$. By Theorem 7, except for a null set, $\widetilde{N}_2\subseteq B_2$, ${}_AD_+f(x)={}_AD^-f(x)$, its opposite derivate, and is finite. We also have $m^*(\widetilde{N}_2)=m^*\big(f(\widetilde{N}_2)\big)=0$. Let $B_3=\{x\in A-N:{}_AD_-f(x)>-\infty\}$. By Theorem 7, except for a null set, $\widetilde{N}_3\subseteq B_3$, ${}_AD_-f(x)={}_AD^+f(x)$, its opposite derivate, and is finite. We also have $m^*(\widetilde{N}_3)=m^*\big(f(\widetilde{N}_3)\big)=0$. Let $B_4=\{x\in A-N:{}_AD^-f(x)<\infty\}$. By Theorem 7, except for a null set, $\widetilde{N}_4\subseteq B_4$, ${}_AD^-f(x)={}_AD_+f(x)$, its opposite derivate, and is finite. Moreover, $m^*(\widetilde{N}_4)=m^*\big(f(\widetilde{N}_4)\big)=0$.

Let C_i be the complement of B_i in $A-N$, for *i* =1,2,3 and 4. Then

$C_1=\{x\in A-N:{}_AD^+f(x)=\infty\}$, $C_2=\{x\in A-N:{}_AD_+f(x)=-\infty\}$,
$C_3=\{x\in A-N:{}_AD_-f(x)=-\infty\}$ and $C_4=\{x\in A-N:{}_AD^-f(x)=\infty\}$.

The complement of $C_1\cup C_2\cup C_3\cup C_4$ is $B_1\cap B_2\cap B_3\cap B_4$. By Theorem 11, *f* is differentiable on $B_1\cap B_2\cap B_3\cap B_4$ except for a null subset E_1 in $B_2\cap B_3$, where $m\big(f(E_1)\big)=0$.

First of all,
$C_1\cap C_2\cap C_3\cap C_4=\{x\in A-N:{}_AD^+f(x)={}_AD^-f(x)=\infty \text{ and } {}_AD_+f(x)={}_AD_-f(x)=-\infty\}$.

Any intersection of three of the C_i's with the complement of the remaining C_i's gives a null set, *E*, such that $m\big(f(E)\big)=0$. This is because by Theorem 5 and its analogue, since the complement of the C_i is B_i, we will have a pair of equal finite opposite derivates and its intersection with the opposite Dini derivate being non-finite will result in a null set, whose image under *f* is also a null set. For instance, $B_1\cap C_3$, $B_2\cap C_4$, $B_3\cap C_1$ and $B_4\cap C_2$ are null sets whose images under *f* are also null sets. Similarly, any intersection of three of the B_i's with the complement of the remaining B_i's results in a null set, whose image under *f* is also a null set.

Note that $C_1 \cap C_4 \cap B_2 \cap B_3$ and $C_2 \cap C_3 \cap B_1 \cap B_4$ are null sets with null images under *f*.

$C_1 \cap C_3 \cap B_2 \cap B_4$

$= \left\{ x \in A - N : {}_A D^+ f(x) = \infty, {}_A D_- f(x) = -\infty, {}_A D^- f(x) = {}_A D_+ f(x) \text{ and is finite} \right\}$

and

$C_2 \cap C_4 \cap B_1 \cap B_3$

$= \left\{ x \in A - N : {}_A D^- f(x) = \infty, {}_A D_+ f(x) = -\infty, {}_A D^+ f(x) = {}_A D_- f(x) \text{ and is finite} \right\}.$

Let $S = B_1 \cap B_2 \cap B_3 \cap B_4$, $T = C_2 \cap C_4 \cap B_1 \cap B_3$, $U = C_1 \cap C_3 \cap B_2 \cap B_4$ and $V = C_1 \cap C_2 \cap C_3 \cap C_4$. Thus, $A = N \cup S \cup T \cup U \cup V \cup E$, where *E* is a null subset with $m\left(f(E) \right) = 0$.

By Theorem 5 and its analogue, *f* is a Lusin function on $S \cup T \cup U$. This completes the proof of the theorem.