Convergence In Measure

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Suppose $(X, \mathscr{M}, \mu)$ is a measure space, where X is a non-empty set, $\mathscr{M}$ is a σ-algebra of subsets of X and $\mu : \mathscr{M} \to [0, \infty]$ is a positive measure, i.e., μ is a function such that $\mu(\varnothing) = 0$ and μ is countably additive, that is, if $\{E_n\}_{n=1}^{\infty}$ is countable disjoint collection of subsets in $\mathscr{M}$, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$.

Suppose $\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ is a sequence of real valued functions. Then we have the notion of the sequence, $\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$, *converging pointwise* to a function $f : X \to \mathbb{R}$ and also the notion of the sequence, $\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$, *converging uniformly* to $f : X \to \mathbb{R}$. We said the sequence $\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ *converges pointwise almost everywhere* if there exists a measurable set E in X such that $\mu(E) = 0$ and the sequence $\{f_n(x)\}$ converges for all x in $X - E$. We said the sequence $\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ *converges uniformly almost everywhere* if there exists a measurable set E in X such that $\mu(E) = 0$ and the sequence $\{f_n : X - E \to \mathbb{R}\}_{n=1}^{\infty}$ converges uniformly on $X - E$.

For the case of a sequence of extended real valued functions on X, $\{f_n : X \to \overline{\mathbb{R}}\}_{n=1}^{\infty}$, we have the notion of the sequence, $\{f_n : X \to \overline{\mathbb{R}}\}_{n=1}^{\infty}$, *converging pointwise* to an extended real-valued function, $f : X \to \overline{\mathbb{R}}$. However, examining the definition of uniform convergence of a sequence of functions, we do not have the notion of a sequence of extended real valued functions, $\{f_n : X \to \overline{\mathbb{R}}\}_{n=1}^{\infty}$, converging uniformly on a subset E of X, unless each f_n is finite valued on E. $\{f_n : X \to \overline{\mathbb{R}}\}_{n=1}^{\infty}$ *converges pointwise almost everywhere* if there exists a measurable set E in X such that $\mu(E) = 0$ and the sequence $\{f_n(x)\}$ converges for all x in $X - E$. $\{f_n : X \to \overline{\mathbb{R}}\}_{n=1}^{\infty}$ *converges uniformly almost everywhere*, if there exists a measurable set E in X such that $\mu(E) = 0$, each $f_n : X - E \to \overline{\mathbb{R}}$ is finite valued and the sequence $\left\{f_n : X - E \to \overline{\mathbb{R}}\right\}_{n=1}^{\infty}$ converges uniformly on *X–E*. Note that in this case the limiting function, *f*, is necessarily finite valued on *X–E*.

Note that for convergence in the extended real numbers, we say the sequence, $\{a_n\}$ converges to an extended real number if $\limsup_{n\to\infty}\{a_n\} = \liminf_{n\to\infty}\{a_n\}$. If the limit, that is, $\lim_{n\to\infty} a_n = \limsup_{n\to\infty}\{a_n\} = \liminf_{n\to\infty}\{a_n\}$ is finite, then this coincide with the usual definition of the finite limit of a sequence. The pointwise convergence for a sequence of extended real valued functions $\left\{f_n : X \to \overline{\mathbb{R}}\right\}_{n=1}^{\infty}$ is based on this meaning of convergence. (For details about lim sup and lim inf see my article, *All About Lim Sup and Lim Inf*.) Note that almost everywhere uniform convergence of a sequence of functions, $\left\{f_n : X \to \overline{\mathbb{R}}\right\}_{n=1}^{\infty}$, implies almost everywhere pointwise convergence of the sequence but not necessarily the converse.

Suppose $\left\{f_n : X \to \overline{\mathbb{R}}\right\}_{n=1}^{\infty}$ is a sequence of μ measurable functions and *p* is a non-negative integer. If each

$$f_n \in L^p(X,\mu,\overline{\mathbb{R}}) = \left\{g : X \to \overline{\mathbb{R}}; g \text{ measurable and } \int_X |g|^p \, d\mu < \infty\right\},$$

and $f \in L^p(X,\mu,\overline{\mathbb{R}})$, then we have the notion of convergence in the *p*-th mean, if

$\left(\int_X |f_n - f|^p \, d\mu\right)^{\frac{1}{p}} \to 0$, in which case, we say f_n converges to *f* in the *p*-th mean with respect to μ. Note that f_n, f are necessarily finite valued almost everywhere with respect to the measure μ. The almost everywhere equivalent classes of measurable functions in $L^p(X,\mu,\overline{\mathbb{R}})$ form a normed vector space with the *p*-th norm, $\|g\|_{p,\mu} = \left(\int_X |g|^p \, d\mu\right)^{\frac{1}{p}}$. With the metric induced by the *p*-th norm, the equivalence classes of measurable functions in $L^p(X,\mu,\overline{\mathbb{R}})$ is a complete metric space, a Banach space. We shall denote these equivalence classes by the same symbol, $L^p(X,\mu,\overline{\mathbb{R}})$. Thus, the sequence $\left\{f_n : X \to \overline{\mathbb{R}}\right\}_{n=1}^{\infty}$ in $L^p(X,\mu,\overline{\mathbb{R}})$ converges in the *p*-th mean if, and only if, $\left\{f_n : X \to \overline{\mathbb{R}}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\left(L^p(X,\mu,\overline{\mathbb{R}}), \|\ \|_{p,\mu}\right)$. Note that if $\left\{f_n : X \to \overline{\mathbb{R}}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\left(L^p(X,\mu,\overline{\mathbb{R}}), \|\ \|_{p,\mu}\right)$, then there is a function $f \in L^p(X,\mu,\overline{\mathbb{R}})$ such that f_n

converges to f in the p-th mean with respect to μ. Note that a sequence $\{f_n : X \to \overline{\mathbb{R}}\}_{n=1}^{\infty}$ is convergent in the p-th mean does not necessarily imply that $\{f_n : X \to \overline{\mathbb{R}}\}_{n=1}^{\infty}$ is convergent almost everywhere. Likewise, $\{f_n : X \to \overline{\mathbb{R}}\}_{n=1}^{\infty}$ is convergent almost everywhere does not necessarily imply that it is convergent in the p-th mean. However, it is true that if a sequence, $\{f_n : X \to \overline{\mathbb{R}}\}_{n=1}^{\infty}$, is convergent in the p-th mean, then it has a subsequence, $\{f_{n_k} : X \to \overline{\mathbb{R}}\}_{k=1}^{\infty}$, which converges pointwise almost everywhere. We can deduce this as follows. Since each f_n is measurable and finite valued almost everywhere, we may assume that each f_n is real valued and measurable. Hence, $\{f\}_n$ is a Cauchy sequence in

$$L^p(X, \mu, \mathbb{R}) = \left\{ g : X \to \mathbb{R}; g \text{ measurable and } \int_X |g|^p \, d\mu < \infty \right\} .$$

The existence of a subsequence $\{f_{n_k} : X \to \mathbb{R}\}_{k=1}^{\infty}$, which is almost everywhere pointwise convergent to a function in $L^p(X, \mu, \mathbb{R})$, is shown in the proof of Theorem 11, in my article, *Convex Function, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*.

Definition 1.

Now we consider the notion of convergence in measure. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose E is a $\mathscr{M}$- measurable subset of X. Suppose $f_n : E \to \overline{\mathbb{R}}$, n =1 ,2, … , and $f : E \to \overline{\mathbb{R}}$ are $\mathscr{M}$- measurable functions. We say the sequence $\{f_n : E \to \overline{\mathbb{R}}\}_{n=1}^{\infty}$ *converges in measure* (μ), with respect to μ, on E, to $f : E \to \overline{\mathbb{R}}$, if given any ε > 0,

$$\lim_{n\to\infty} \mu\left\{ x \in E : \left| f_n(x) - f(x) \right| \ge \varepsilon \right\} = 0 .$$

Observe that this definition makes sense only for functions $f_n : E \to \overline{\mathbb{R}}$ and $f : E \to \overline{\mathbb{R}}$, which are finite almost everywhere on E.

Note that this definition is equivalent to:

Given any δ > 0, there exists integer N such that

$$n \geq N \Rightarrow \mu\{x \in E : |f(x) - f_n(x)| \geq \varepsilon\} < \delta \ .$$

Remark. When f_n and f are random variables, convergence in measure is also known as convergence in probability.

We note that if f_n converges in measure to *f*, then *f* is unique almost everywhere with respect to μ in *X*. That is to say, if f_n converges in measure to *f* and f_n converges in measure to *g*, then *f* = *g* almost everywhere in *X* with respect to μ. We show this below.

Firstly, observe that given any $\varepsilon > 0$,

$$\{x : |f(x) - g(x)| > \varepsilon\} \subseteq \left\{x : |f_n(x) - f(x)| > \frac{\varepsilon}{2}\right\} \cup \left\{x : |f_n(x) - g(x)| > \frac{\varepsilon}{2}\right\} \ .$$

Hence,

$$\mu\left(\{x : |f(x) - g(x)| > \varepsilon\}\right) \leq \mu\left(\left\{x : |f_n(x) - f(x)| > \frac{\varepsilon}{2}\right\}\right) + \mu\left(\left\{x : |f_n(x) - g(x)| > \frac{\varepsilon}{2}\right\}\right) \ .$$

By definition of convergence in measure, there exists an integer *N* such that

$$n \geq N \Rightarrow \mu\left\{x \in E : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} < \frac{\delta}{2}$$

And there exists an integer *M* such that

$$n \geq M \Rightarrow \mu\left\{x \in E : |f_n(x) - g(x)| \geq \frac{\varepsilon}{2}\right\} < \frac{\delta}{2} \ .$$

It follows that

$$n \geq \max(N, M) \Rightarrow \mu\left(\left\{x : |f_n(x) - f(x)| > \frac{\varepsilon}{2}\right\}\right) + \mu\left(\left\{x : |f_n(x) - g(x)| > \frac{\varepsilon}{2}\right\}\right) < \delta \, .$$

Hence, for any $\delta > 0$, $\mu\left(\{x : |f(x) - g(x)| > \varepsilon\}\right) < \delta$. Since δ is arbitrary, we conclude that $\mu\left(\{x : |f(x) - g(x)| > \varepsilon\}\right) = 0$ for any $\varepsilon > 0$.

Now $\{x : |f(x) - g(x)| > 0\} = \bigcup_{n=1}^{\infty}\left\{x : |f(x) - g(x)| > \frac{1}{n}\right\}$ and so by the continuity from below property of measure (Proposition 18, *Introduction To Measure Theory*), $\mu\{x : |f(x) - g(x)| > 0\} = 0$. It follows that $f = g$ almost everywhere in X with respect to μ.

In general, convergence in measure does not imply convergence almost everywhere nor is it implied by convergence almost everywhere. However, convergence in measure does imply the existence of a subsequence converging almost everywhere.

Theorem 2. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose E is a $\mathscr{M}$-measurable subset of X. Suppose $f_n : E \to \overline{\mathbb{R}}$, $n = 1, 2, \ldots$ and $f : E \to \overline{\mathbb{R}}$ are $\mathscr{M}$ - measurable functions, which are defined and finite almost everywhere on E, with respect to the measure μ. Suppose $\{f_n\}$ converges in measure to f. Then there is a subsequence $\{f_{n_i}\}$ converging to f almost everywhere with respect to μ.

Proof.

Since $f_n \to f$ in measure, given $j = 1,2, \ldots$, there exists an integer n_j such that for all $n \geq n_j$,

$$\mu\left(\left\{x \in E : |f_n(x) - f(x)| \geq \frac{1}{2^j}\right\}\right) < \frac{1}{2^j} \quad . \quad \text{--------------------- (1)}$$

We may assume that the sequence $\{n_j\}$ is monotonically increasing. (Having chosen n_j, we can always choose $n_{j+1} > n_j$.)

For each integer $j \geq 1$, let $E_j = \left\{x \in E : \left|f_{n_j}(x) - f(x)\right| \geq \frac{1}{2^j}\right\}$ and for each integer $m \geq 1$, let $H_m = \bigcup_{j=m}^{\infty} E_j$. Note that $\mu\left(E_j\right) < \frac{1}{2^j}$. It follows that

$$\mu(H_m) = \mu\left(\bigcup_{j=m}^{\infty} E_j\right) \leq \sum_{j=m}^{\infty} \frac{1}{2^j} = \frac{1}{2^{m-1}} \quad .$$

Now, for all $x \in E - E_j$, $\left|f_{n_j}(x) - f(x)\right| < \frac{1}{2^j}$. Thus, if $j \ge m$, then

$$\left|f_{n_j}(x) - f(x)\right| < \frac{1}{2^j} \text{ for } x \text{ in } E - H_m .$$

As $\frac{1}{2^j} \to 0$, this means that $f_{n_j}(x) \to f(x)$ for x in $E - H_m$. Therefore,

$$x \in E - \bigcap_{m=1}^{\infty} H_m \Rightarrow f_{n_j}(x) \to f(x) .$$

Since, $\mu\left(\bigcap_{m=1}^{\infty} H_m\right) \le \mu\left(H_j\right) < \frac{1}{2^{j-1}}$ for $j \ge 1$, $\mu\left(\bigcap_{m=1}^{\infty} H_m\right) = 0$. It follows that $f_{n_j}(x)$ converges to $f(x)$ for x in E except perhaps for x in $\bigcap_{m=1}^{\infty} H_m$, which is a set of μ measure 0. That is, $\left\{f_{n_i}\right\}$ converges to f almost everywhere on E with respect to μ.

We introduce another notion of convergence involving measure.

Definition 3. Suppose (X, $\mathscr{M}$, μ) is a measure space. Suppose E is a $\mathscr{M}$-measurable subset of X. Suppose $f_n : E \to \overline{\mathbb{R}}$, n =1 ,2, … and $f : E \to \overline{\mathbb{R}}$ are $\mathscr{M}$ - measurable functions, which are defined and finite almost everywhere on E, with respect to the measure μ.

We say f_n *converges almost uniformly* to f on E if given $\varepsilon > 0$, there is a measurable subset $A \subseteq E$ with $\mu(A) < \varepsilon$ such that $f_n \to f$ uniformly on $E - A$.

$\{f_n\}$ is a *Cauchy sequence almost uniformly*, if given $\varepsilon > 0$, there is a measurable subset $A \subseteq E$ with $\mu(A) < \varepsilon$ such that $\{f_n\}$ is uniformly Cauchy on $E - A$.

$\{f_n\}$ is a *Cauchy sequence in measure*, if given $\varepsilon > 0$, $\delta > 0$, there is an integer N such that $n, m \ge N \Rightarrow \mu\left(\left\{x : \left|f_n(x) - f_m(x)\right| > \varepsilon\right\}\right) < \delta$.

We have immediately,

Proposition 4. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose E is a $\mathscr{M}$-measurable subset of X. Suppose $f_n : E \to \overline{\mathbb{R}}$, n =1 ,2, … and $f : E \to \overline{\mathbb{R}}$ are $\mathscr{M}$ - measurable functions, which are defined and finite almost everywhere on E, with respect to the measure μ. Then

f_n converges to f in measure implies that $\{f_n\}$ is a Cauchy sequence in measure.

Proof.

By definition of convergence in measure, given any $\varepsilon > 0$, $\delta > 0$, there exists an integer N such that

$$n \geq N \Rightarrow \mu\left\{x \in E : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} < \frac{\delta}{2} .$$

Next, observe that given any $\varepsilon > 0$,

$$\{x : |f_n(x) - f_m(x)| > \varepsilon\} \subseteq \left\{x : |f_n(x) - f(x)| > \frac{\varepsilon}{2}\right\} \cup \left\{x : |f_m(x) - f(x)| > \frac{\varepsilon}{2}\right\}.$$

Hence, for n, $m \geq N$,

$$\mu\left(\{x : |f(x_n) - f_m(x)| > \varepsilon\}\right) \leq \mu\left(\left\{x : |f_n(x) - f(x)| > \frac{\varepsilon}{2}\right\}\right) + \mu\left(\left\{x : |f_m(x) - f(x)| > \frac{\varepsilon}{2}\right\}\right)$$

$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta .$$

Thus, $\{f_n\}$ is a Cauchy sequence in measure.

Proposition 5. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose E is a $\mathscr{M}$-measurable subset of X. Suppose $f_n : E \to \overline{\mathbb{R}}$, n =1 ,2, … are $\mathscr{M}$- measurable functions, which are defined and finite almost everywhere on E, with respect to the measure μ.

If $\{f_n\}$ is a Cauchy sequence in measure, then $\{f_n\}$ has a subsequence, $\{f_{n_i}\}$, which is a Cauchy sequence almost uniformly.

Proof.

Since $\{f_n\}$ is a Cauchy sequence in measure, for each integer $k \geq 1$, there exists an integer n_k such that

$$n \geq n_k \Rightarrow \mu\left(\left\{x \in E : |f_n(x) - f(x)| \geq \frac{2^{-k}}{2}\right\}\right) < \frac{2^{-k}}{2}. \text{ -------------- } (1)$$

We may assume that $n_{k+1} > n_k$. Then since

$$\{x : |f_n(x) - f_m(x)| \geq 2^{-k}\} \subseteq \left\{x : |f_n(x) - f(x)| \geq \frac{2^{-k}}{2}\right\} \cup \left\{x : |f_m(x) - f(x)| > \frac{2^{-k}}{2}\right\},$$

it follows that

$$n, m \geq n_k \Rightarrow \mu\left(\{x : |f_n(x) - f_m(x)| \geq 2^{-k}\}\right) \leq \mu\left(\left\{x : |f_n(x) - f(x)| \geq \frac{2^{-k}}{2}\right\}\right) + \mu\left(\left\{x : |f_m(x) - f(x)| > \frac{2^{-k}}{2}\right\}\right)$$
$$\leq 2^{-k}.$$

For each integer $k \geq 1$, let $E_k = \left\{x : \left|f_{n_k}(x) - f_{n_{k+1}}(x)\right| \geq 2^{-k}\right\}$. Then $\mu(E_k) \leq 2^{-k}$.

For each integer $m \geq 1$, let $H_m = \bigcup_{i=m}^{\infty} E_i$. Then

$$\mu(H_m) = \mu\left(\bigcup_{i=m}^{\infty} E_i\right) \leq \sum_{i=m}^{\infty} \mu(E_i) \leq \sum_{i=m}^{\infty} \frac{1}{2^i} = \frac{1}{2^{m-1}}.$$

Thus, given any $\varepsilon > 0$, we can choose an integer M such that $\frac{1}{2^{M-1}} < \varepsilon$. Then

$\mu(H_M) < \varepsilon$. Note that as $E - H_M = E - \bigcup_{i=M}^{\infty} E_i = \bigcap_{i=M}^{\infty} (E - E_i)$, $x \in E - H_M$ $x \in E - H_m$ implies that $x \in E - E_i$ for integer $i \geq M$. Hence, for all $x \in E - H_M$,

$$\left|f_{n_i}(x) - f_{n_{i+1}}(x)\right| < \frac{1}{2^i}, \text{ for } i \geq M.$$

It follows that for $i \geq j \geq M$ and for all $x \in E - H_M$,

$$\left|f_{n_j}(x) - f_{n_i}(x)\right| \leq \sum_{m=j}^{m=i-1} \left|f_{n_m}(x) - f_{n_{m+1}}(x)\right| < \sum_{m=j}^{m=i-1} \frac{1}{2^m} < \frac{1}{2^{j-1}}.$$

Now, given any $\delta > 0$, choose integer $N \geq M$ such that $\frac{1}{2^{N-1}} < \delta$. Then

$$i \geq j \geq N \Rightarrow \left|f_{n_j}(x) - f_{n_i}(x)\right| \leq \frac{1}{2^{j-1}} \leq \frac{1}{2^{N-1}} < \delta \text{ for all } x \in E - H_M.$$

Thus, $\{f_{n_i}\}$ is uniformly Cauchy on $E - H_M$ and $\mu(H_M) < \varepsilon$. That is, $\{f_{n_i}\}$ is a Cauchy sequence almost uniformly.

Proposition 6. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose *E* is a $\mathscr{M}$-measurable subset of *X*. Suppose $f_n : E \to \overline{\mathbb{R}}$, n =1 ,2, … and $f : E \to \overline{\mathbb{R}}$ are $\mathscr{M}$ - measurable functions, which are defined and finite almost everywhere on *E*, with respect to the measure μ.

If $\{f_n\}$ converges to *f* almost uniformly, then $\{f_n\}$ converges to *f* in measure.

Proof.

If $\{f_n\}$ converges to *f* almost uniformly, then by definition, given $\varepsilon > 0$, $\delta > 0$, there exists a measurable set *A* in *E* such that $\mu(A) < \delta$ and $f_n \to f$ uniformly on *E*–*A*. It follows that there is an integer *N* such that

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon \text{ for all } x \text{ in } E\text{–}A.$$

Hence, for $n \geq N$, the set $\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\} \subseteq A$ and so

$$\mu\left(\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}\right) \leq \mu(A) < \delta \quad \text{for } n \geq N.$$

This means $f_n \to f$ in measure.

It is to be expected that almost uniformly convergence implies convergence almost everywhere.

Proposition 7. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose *E* is a $\mathscr{M}$-measurable subset of *X*. Suppose $f_n : E \to \overline{\mathbb{R}}$, n =1 ,2, … and $f : E \to \overline{\mathbb{R}}$ are $\mathscr{M}$ - measurable functions, which are defined and finite almost everywhere on *E*, with respect to the measure μ.

If $\{f_n\}$ converges to *f* almost uniformly, then $\{f_n\}$ converges to *f* almost everywhere in *E*.

Proof.

If $\{f_n\}$ converges to f almost uniformly, then by definition, given any integer $m \geq 1$, there exists a measurable set A_m in E such that $\mu(A_m) < \frac{1}{m}$ and $f_n \to f$ uniformly on $E - A_m$. Let $H = \bigcup_{m=1}^{\infty} (E - A_m)$. Then for any $x \in H$, $x \in E - A_m$ for some integer m so that $f_n(x) \to f(x)$. Therefore, $f_n \to f$ pointwise on H. Now $E - H = \bigcap_{m=1}^{\infty} A_m$. Therefore, $\mu(E-H) = \mu\left(\bigcap_{m=1}^{\infty} A_m\right) \leq \mu(A_n) < \frac{1}{n}$ for each integer $n \geq 1$. Since $\frac{1}{n} \to 0$, $\mu(E-H) = 0$. It follows that $f_n \to f$ pointwise almost everywhere in E.

Proposition 8. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose E is a $\mathscr{M}$-measurable subset of X. Suppose $f_n : E \to \overline{\mathbb{R}}$, n =1 ,2, … are $\mathscr{M}$- measurable functions, which are defined and finite almost everywhere on E, with respect to the measure μ .

If $\{f_n\}$ is a Cauchy sequence almost uniformly, then $\{f_n\}$ is a Cauchy sequence in measure.

Proof.

If $\{f_n\}$ is a Cauchy sequence almost uniformly, then by definition, given $\varepsilon > 0$, $\delta > 0$, there exists a measurable set A in E such that $\mu(A) < \delta$ and $\{f_n\}$ is a Cauchy sequence uniformly on E–A. It follows that there is an integer N such that

$$n, m \geq N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon \quad \text{for all } x \text{ in } E–A.$$

Hence, for n, $m \geq N$, the set $\{x \in E : |f_n(x) - f_m(x)| \geq \varepsilon\} \subseteq A$ and so

$$\mu(\{x \in E : |f_n(x) - f_m(x)| \geq \varepsilon\}) \leq \mu(A) < \delta \quad \text{for } n \geq N.$$

This means $\{f_n\}$ is a Cauchy sequence in measure.

Proposition 9. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose E is a $\mathscr{M}$-measurable subset of X. Suppose $f_n : E \to \overline{\mathbb{R}}$, n =1 ,2, … are $\mathscr{M}$- measurable functions, which are defined and finite almost everywhere on E, with respect to the measure μ.

If $\{f_n\}$ is a Cauchy sequence in measure, then there exists a measurable function, g, finite almost everywhere on E, such that f_n converges to g in measure.

Proof.

If $\{f_n\}$ is a Cauchy sequence in measure, then by Proposition 5, $\{f_n\}$ has a subsequence, $\{f_{n_i}\}$, which is a Cauchy sequence almost uniformly.

As in the proof of Proposition 5,

$\{f_{n_i}\}$ is uniformly Cauchy on $E - H_k$ and $\mu(H_k) < \frac{1}{2^{k-1}}$. Thus, $\{f_{n_i}\}$ converges uniformly to a function g on $E - H_k$ and so it converges pointwise to g on $E - H_k$. Let $H = \bigcap_{k=1}^{\infty} H_k$. Then H is measurable and $E - H = E - \bigcap_{k=1}^{\infty} H_k = \bigcup_{k=1}^{\infty} (E - H_k)$. Therefore, g is defined and finite on $E - H$. Moreover, $\mu(H) = 0$. Thus, g is defined and measurable on $E - H$. Define $g(x) = 0$ for x in H. Then g is measurable on E. It follows that $\{f_{n_i}\}$ converges almost uniformly to g on E.

Now, take $\varepsilon > 0$. Observe that

$$\{x : |f_n(x) - g(x)| \geq \varepsilon\} \subseteq \left\{x : |f_n(x) - f_{n_k}(x)| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x : |f_{n_k}(x) - g(x)| \geq \frac{\varepsilon}{2}\right\} \text{ . ----- (1)}$$

Since $\{f_n\}$ is a Cauchy sequence in measure, given $\varepsilon > 0$, $\delta > 0$, there is an integer N such that

$$n, m \geq N \Rightarrow \mu\left(\left\{x : |f_n(x) - f_m(x)| \geq \frac{\varepsilon}{2}\right\}\right) < \frac{\delta}{2} \text{ . -------- (2)}$$

Since $\{f_{n_i}\}$ converges almost uniformly to g on E, there exists an integer M and a measurable subset set E_δ in E such that $\mu(E_\delta) < \frac{\delta}{2}$ and

$$\left|f_{n_k}(x)-g(x)\right|<\frac{\varepsilon}{2} \text{ for all } k\geq M \text{ and for all } x \text{ in } E-E_\delta \ .$$

From (2), if $n, n_k \geq N$, then $\mu\left(\left\{x:\left|f_n(x)-f_{n_k}(x)\right|\geq\frac{\varepsilon}{2}\right\}\right)<\frac{\delta}{2}$.

Thus if $k\geq \max(N,M)$, then by (1),

$$\mu\left(\left\{x:\left|f_k(x)-g(x)\right|\geq\varepsilon\right\}\right)\leq\mu\left(\left\{x:\left|f_k(x)-f_{n_k}(x)\right|\geq\frac{\varepsilon}{2}\right\}\right)+\mu\left(\left\{x:\left|f_{n_k}(x)-g(x)\right|\geq\frac{\varepsilon}{2}\right\}\right)$$

$$\leq\mu\left(\left\{x:\left|f_k(x)-f_{n_k}(x)\right|\geq\frac{\varepsilon}{2}\right\}\right)+\mu\left(E_\delta\right)<\frac{\delta}{2}+\frac{\delta}{2}=\delta \ .$$

It follows that f_n converges to g in measure.

Remark. The term *Cauchy sequence in measure* does live up to its name. By Proposition 4, if f_n converges to f in measure, then $\{f_n\}$ is a Cauchy sequence in measure. Proposition 9 says that if $\{f_n\}$ is a Cauchy sequence in measure, then there is a measurable function g such that f_n converges to g in measure. Hence $f = g$ almost everywhere on E. Thus, with the hypothesis of Proposition 9, $\{f_n\}$ is convergent in measure if, and only if, $\{f_n\}$ is a Cauchy sequence in measure.

Now we state a relation between convergence in the pth mean and convergence in measure.

Theorem 10. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Let $L^p(X,\mu,\overline{\mathbb{R}})$ be the collection of all $\mathscr{M}$ measurable extended real valued functions $g: X\to\overline{\mathbb{R}}$, which are finite almost everywhere on X and $\int_X |g|^p d\mu<\infty$.

If $\{f_n\}$ is a Cauchy sequence in the $L^p(X,\mu,\overline{\mathbb{R}})$ norm, then $\{f_n\}$ is a Cauchy sequence in measure. Suppose f is a measurable extended real valued function, which is finite almost everywhere. If $f_n\to f$ in the p-th mean, i.e., in the $L^p(X,\mu,\overline{\mathbb{R}})$ norm, then $f_n\to f$ in measure.

Proof. Recall that the $L^p(X,\mu,\overline{\mathbb{R}})$ norm is given by, $\|g\|_p = \left(\int_X |g|^p d\mu\right)^{\frac{1}{p}}$ for g in $L^p(X,\mu,\overline{\mathbb{R}})$. Suppose $\{f_n\}$ is a Cauchy sequence in the $L^p(X,\mu,\overline{\mathbb{R}})$ norm. Then given any $\eta > 0$, there exists an integer N such that

$$n,m \geq N \Rightarrow \|f_n - f_m\|_p = \left(\int_X |f_n - f_m|^p\right)^{\frac{1}{p}} < \eta \quad \text{------------------------ (1)}$$

For any $\varepsilon > 0$, $\delta > 0$, let $E_{n,m} = \{x : |f_n(x) - f_m(x)| \geq \varepsilon\}$. Then $E_{n,m}$ is measurable and

$$\int_{E_{n,m}} |f_n - f_m|^p d\mu \geq \int_{E_{n,m}} \varepsilon^p d\mu = \varepsilon^p \mu(E_{n,m}) \quad \text{--------------------- (2)}$$

Choose $\eta > 0$ such that $\eta^p < \delta\varepsilon^p$. It follows then from (1) and (2) that for n, $m \geq N$, $\varepsilon^p \mu(E_{n,m}) \leq \int_{E_{n,m}} |f_n - f_m|^p d\mu \leq \int_X |f_n - f_m|^p d\mu < \eta^p$ implying that

$$\mu(E_{n,m}) < \int_{E_{n,m}} |f_n - f_m|^p d\mu \leq \int_X |f_n - f_m|^p d\mu < \frac{\eta^p}{\varepsilon^p} < \delta .$$

Hence, for any $\varepsilon > 0$, $\delta > 0$, there exists an integer N such that

$$n,m \geq N \Rightarrow \mu(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}) < \delta .$$

This means that $\{f_n\}$ is a Cauchy sequence in measure.

Suppose $f_n \to f$ in the $L^p(X,\mu,\overline{\mathbb{R}})$ norm. Then given any $\eta > 0$, there exists an integer N such that

$$n \geq N \Rightarrow \|f_n - f\|_p = \left(\int_X |f_n - f|^p\right)^{\frac{1}{p}} < \eta . \quad \text{------------------------ (3)}$$

For any $\varepsilon > 0$, $\delta > 0$, let $H_n = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$. Then H_n is measurable and

$$\int_{H_n} |f_n - f|^p d\mu \geq \int_{H_n} \varepsilon^p d\mu = \varepsilon^p \mu(H_n). \quad \text{--------------------- (4)}$$

Thus, taking any $\eta > 0$ such that $\eta^p < \delta\varepsilon^p$, we have that

$$n \geq N \Rightarrow \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon^p} \int_{H_n} |f_n - f|^p d\mu \leq \frac{1}{\varepsilon^k} \int_X |f_n - f|^p d\mu < \frac{\eta^p}{\varepsilon^p} < \delta .$$

This proves that $f_n \to f$ in measure.

Theorem 11 (Egoroff's Theorem). Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose *E* is a $\mathscr{M}$- measurable subset of *X* and $\mu(E) < \infty$. Suppose $f_n : E \to \overline{\mathbb{R}}$, *n* =1 ,2, … , and $f : E \to \overline{\mathbb{R}}$ are $\mathscr{M}$- measurable functions, which are defined and finite almost everywhere on *E*, with respect to the measure μ.

If $\{f_n\}$ converges to *f* almost everywhere on *E*, then $\{f_n\}$ converges to *f* almost uniformly in *E*.

Proof.

By omitting a subset of zero measure, we may assume that f_n and *f* are finite on *E* and that $f_n(x) \to f(x)$ for all *x* in *E*.

For integers, *m*, $n \geq 1$, let

$$E_n^m = \bigcap_{i=n}^{\infty} \left\{ x : \left| f_i(x) - f(x) \right| < \frac{1}{m} \right\} .$$

Then plainly, for each integer $m \geq 1$, $E_n^m \subseteq E_{n+1}^m \subseteq E_{n+2}^m \subseteq \cdots$ is an increasing sequence of measurable sets. Since $f_n(x) \to f(x)$ for all *x* in *E*, this sequence converges to *E*. That is to say, $\bigcup_{i=1}^{\infty} E_i^m = E$. Hence, by continuity from below property of positive measure, $\lim_{i\to\infty} \mu\left(E_i^m\right) = \mu(E)$ (see Proposition 18, *Introduction to Measure Theory*). Therefore, since $\mu(E) < \infty$, there exists an integer, N_m, depending on *m* such that

$$i \geq N_m \Rightarrow \mu(E) - \mu\left(E_i^m\right) < \frac{\varepsilon}{2^m} \Rightarrow \mu\left(E - E_i^m\right) < \frac{\varepsilon}{2^m} .$$

Let $F_\varepsilon = \bigcup_{m=1}^{\infty} \left(E - E_{N_m}^m\right)$. Then $\mu\left(F_\varepsilon\right) = \mu\left(\bigcup_{m=1}^{\infty} \left(E - E_{N_m}^m\right)\right) \leq \sum_{m=1}^{\infty} \mu\left(E - E_{N_m}^m\right) < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon$.

Observe that $E - F_\varepsilon = E - \bigcup_{m=1}^{\infty} \left(E - E_{N_m}^m\right) = \bigcap_{m=1}^{\infty} E_{N_m}^m$. Therefore, for all *x* in $E - F_\varepsilon$, $x \in E_{N_m}^m$ for all integer $m \geq 1$. Given any $\delta > 0$, choose integer *M* such that $\frac{1}{M} < \delta$. Hence, for all integer $i \geq N_M$, $\left| f_i(x) - f(x) \right| < \frac{1}{M} < \delta$ for all *x* in $E - F_\varepsilon$ as

$E - F_\varepsilon \subseteq E^M_{N_M}$. It follows that $f_n \to f$ uniformly on $E - F_\varepsilon$. Hence, $\{f_n\}$ converges to *f* almost uniformly in *E*.

Remark. If $(X, \mathscr{M}, \mu)$ is a finite measure space, i.e., $\mu(X) < \infty$, then by Theorem 11, $\{f_n\}$ converges to *f* almost everywhere on *E*, implies that $\{f_n\}$ converges to *f* almost uniformly in *E* and by Proposition 7, $\{f_n\}$ converges to *f* almost uniformly in *E* implies that $\{f_n\}$ converges to *f* almost everywhere on *E*.

Thus, when $\mu(E) < \infty$, under the hypothesis of Egoroff's Theorem, convergence almost uniformly in *E* is equivalent to convergence almost everywhere in *E*. Thus, for a probability measure, these two notions coincide.

Corollary 12. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose *E* is a $\mathscr{M}$-measurable subset of *X* and $\mu(E) < \infty$. Suppose $f_n : E \to \overline{\mathbb{R}}$, $n = 1, 2, \dots$, and $f : E \to \overline{\mathbb{R}}$ are $\mathscr{M}$- measurable functions, which are defined and finite almost everywhere on *E*, with respect to the measure μ.

If $\{f_n\}$ converges to *f* almost everywhere on *E*, then $\{f_n\}$ converges to *f* in measure in *E*.

Proof. This is a consequence of Theorem 11 and Proposition 6.

Theorem 13. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose *E* is a measurable set and $\{f_n : E \to \mathbb{R}\}$ is a sequence of measurable functions defined on *E*. Suppose f_n converges almost everywhere to a measurable function $f : E \to \mathbb{R}$. Suppose there exists a Lebesgue integrable function $g : E \to [0, \infty)$ such that, $|f_n(x)| \le g(x)$ for all integer $n \ge 1$ and for almost all *x* in *E*. Then f_n and *f* are Lebesgue integrable, and f_n converges almost uniformly to *f*. Consequently, f_n converges in measure to *f*.

Proof.

By omitting a subset of zero measure, we may assume that f_n and f are finite on E, $f_n(x) \to f(x)$ almost everywhere in E, $|f_n(x)| \leq g(x)$ for almost all x in E. Therefore, $|f| \leq g$ almost everywhere in E. Hence, f_n and f are Lebesgue integrable,

For integers, $m, n \geq 1$, let

$$E_n^m = \bigcap_{i=n}^{\infty} \left\{ x : |f_i(x) - f(x)| < \frac{1}{m} \right\} .$$

Then plainly, for each integer $m \geq 1$, $E_n^m \subseteq E_{n+1}^m \subseteq E_{n+2}^m \subseteq \cdots$ is an increasing sequence of measurable sets. Now, the set

$$\left\{ x \in E : \lim_{i \to \infty} f_i(x) = f(x) \right\} \subseteq \bigcup_{n=1}^{\infty} E_n^m$$

for each integer $m \geq 1$. Since f_n converges almost everywhere to f and

$$E - \bigcup_{n=1}^{\infty} E_n^m = \bigcap_{n=1}^{\infty} \left(E - E_n^m \right) \subseteq E - \left\{ x \in E : \lim_{i \to \infty} f_i(x) = f(x) \right\} \ ,$$

$$\mu\left(\bigcap_{n=1}^{\infty} \left(E - E_n^m \right) \right) \subseteq \mu\left(E - \left\{ x \in E : \lim_{i \to \infty} f_i(x) = f(x) \right\} \right) = 0 \text{ implies that}$$

$$\mu\left(\bigcap_{n=1}^{\infty} \left(E - E_n^m \right) \right) = 0 .$$

Now, $|f_n - f| \leq 2g$ almost everywhere in E for all integer $n \geq 1$. Let A be a measurable subset of E such $\mu(A) = 0$ and that on $E - A$, $|f_n - f| \leq 2g$. Then

$$\left(E - A \right) - E_n^m \subseteq \bigcup_{i=n}^{\infty} \left\{ x \in E - A : |f_i(x) - f(x)| \geq \frac{1}{m} \right\} \subseteq \left\{ x \in E - A : g(x) \geq \frac{1}{2m} \right\} .$$

Since g is integrable, $\mu\left(\left\{ x \in E - A : g(x) \geq \frac{1}{2m} \right\} \right) \leq \mu\left(\left\{ x \in E : g(x) \geq \frac{1}{2m} \right\} \right) < \infty$. It follows that $\mu\left(E - E_n^m \right) < \infty$. Therefore, by the continuity from above property of positive measure, μ, (see Proposition 18, *Introduction to Measure Theory*), $\mu\left(E - E_n^m \right) \to \mu\left(\bigcap_{n=1}^{\infty} \left(E - E_n^m \right) \right) = 0$ as $n \to \infty$. Hence, there exists an integer, N_m, depending on m such that $n \geq N_m \Rightarrow \mu\left(E - E_n^m \right) < \frac{\varepsilon}{2^m}$.

Let $F_\varepsilon = \bigcup_{m=1}^{\infty}\left(E - E^m_{N_m}\right)$. Then $\mu(F_\varepsilon) = \mu\left(\bigcup_{m=1}^{\infty}\left(E - E^m_{N_m}\right)\right) \le \sum_{m=1}^{\infty}\mu\left(E - E^m_{N_m}\right) < \sum_{m=1}^{\infty}\frac{\varepsilon}{2^m} = \varepsilon$.

Observe that $E - F_\varepsilon = E - \bigcup_{m=1}^{\infty}\left(E - E^m_{N_m}\right) = \bigcap_{m=1}^{\infty} E^m_{N_m}$. Therefore, for all *x* in $E - F_\varepsilon$, $x \in E^m_{N_m}$ for all integer $m \ge 1$. Given any $\delta > 0$, choose integer *M* such that $\frac{1}{M} < \delta$. Hence, for all integer $i \ge N_M$, $|f_i(x) - f(x)| < \frac{1}{M} < \delta$ for all *x* in $E - F_\varepsilon$ as $E - F_\varepsilon \subseteq E^M_{N_M}$. It follows that $f_n \to f$ uniformly on $E - F_\varepsilon$. Hence, $\{f_n\}$ converges to *f* almost uniformly in *E*.

> **Remark.** Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose *E* is a measurable set and $\{f_n : E \to \mathbb{R}\}$ is a sequence of measurable functions defined on *E* and dominated by an integrable non-negative function in *E*. Suppose $f : E \to \mathbb{R}$ is measurable. Then by Theorem 13 and Proposition 7, $\{f_n\}$ converges to *f* almost uniformly in *E* if, and only if, $\{f_n\}$ converges to *f* almost everywhere in *E*.

Theorem 14. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose $\{f_n : X \to [0,\infty]\}$ is an increasing sequence of non-negative measurable functions defined and finite almost everywhere with respect to μ on *X*. Suppose $\{f_n\}$ converges in measure to a measurable non-negative function *f* defined and finite almost everywhere in *X* with respect to the measure μ. Then

$$\int_X f_n \, d\mu \to \int_X f \, d\mu .$$

Proof.

Since $\{f_n\}$ converges in measure to *f*, by Theorem 2, Then there is a subsequence $\{f_{n_i}\}$ converging to *f* almost everywhere with respect to μ. Therefore, by the Lebesgue Monotone Convergence Theorem,

$$\int_X f_{n_j} \, d\mu \to \int_X f \, d\mu .$$

Since $\{f_n\}$ is an increasing sequence of non-negative measurable functions, $\lim_{n\to\infty}\int_X f_n d\mu = \lim_{j\to\infty}\int_X f_{n_j}\, d\mu$ and so $\int_X f_n\, d\mu \to \int_X f\, d\mu$.

Theorem 15. (Bounded Convergence)

Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose E is a measurable set of finite measure and $\{f_n : E \to \mathbb{R}\}$ is a sequence of measurable functions defined on E. Suppose there exists $M \geq 0$ such that $|f_n(x)| \leq M$ for all integer $n \geq 1$ and for x in E, i.e., $\{f_n\}$ is uniformly bounded on E. If f_n converges in measure to a measurable function $f : E \to \mathbb{R}$, then $\int_E f_n\, d\mu \to \int_E f\, d\mu$.

Proof.

Since $|f_n| \leq M$ and $\mu(E) < \infty$, f_n is Lebesgue integrable for all integer $n \geq 1$. Since f_n converges in measure to a measurable function $f : E \to \mathbb{R}$, by Theorem 2, there is a subsequence $\{f_{n_i}\}$ converging to f pointwise almost everywhere with respect to μ. Therefore, by the Lebesgue Dominated Convergence Theorem (See Theorem 33, *Introduction to Measure Theory and remark after the theorem*), $\int_E f_{n_i}\, d\mu \to \int_E f\, d\mu < \infty$.

Now there exists a set subset K of measure zero in E such that $f_{n_j}(x) \to f(x)$ for every x in $E - K$. As $|f_{n_j}(x)| \leq M$ for all x in E, $|f(x)| \leq M$ for all x in $E - K$. Therefore, $\int_E |f| d\mu \leq \int_E M d\mu = M\mu(E) < \infty$. Hence, $f_n - f$ is Lebesgue integrable for each integer $n \geq 1$. Note that

$$\left|\int_E f_n d\mu - \int_E f\, d\mu\right| \leq \int_E (f_n - f) d\mu \leq \int_E |f_n - f| d\mu .$$

For $\varepsilon > 0$, let $K_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}$ for each integer $n \geq 1$. Since f_n converges in measure to f, there exists an integer N such that

$$n \geq N \Rightarrow \mu(K_n) < \varepsilon .$$

It follows that for $n \geq N$,

$$\int_E |f_n - f| d\mu = \int_{E-K_n} |f_n - f| d\mu + \int_{K_n} |f_n - f| d\mu \leq \varepsilon\mu(E - K_n) + 2M\mu(K_n)$$

$$< \varepsilon\big(\mu\big(E - K_n\big) + 2M\big) \le \varepsilon(\mu(E) + 2M).$$

Since this is true for any $\varepsilon > 0$, $\int_E |f_n - f| d\mu \to 0$ as $n \to \infty$. Hence $\left|\int_E f_n d\mu - \int_E f d\mu\right| \to 0$ and so $\int_E f_n \, d\mu \to \int_E f d\mu$.

The following is a special form of Fatou's Lemma for convergence in measure.

Theorem 16. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose E is a measurable subset of X and $\{f_n : E \to [0, \infty]\}$ is a sequence of non-negative measurable functions defined and finite almost everywhere with respect to μ on X. Suppose $\{f_n\}$ converges in measure to a measurable non-negative function f defined and finite almost everywhere in E with respect to the measure μ. Then

$$\int_E f d\mu \le \liminf_{n\to\infty} \int_E f_n \, d\mu.$$

Proof:

Suppose that $\int_E f d\mu < \infty$ and $\int_E f d\mu > \liminf_{n\to\infty} \int_E f_n \, d\mu$. Then there exists a $\delta > 0$ and a sequence $\{n_i\}$ such that $\int_E f_{n_i} \, d\mu < \int_E f d\mu - \delta$ for all integer $i \ge 1$. (See Theorem 6, *All About Lim Sup and Lim Inf*.) Since $\{f_n\}$ converges in measure to f, $\{f_{n_i}\}$ also converges in measure to f. By Theorem 2, $\{f_{n_i}\}$ has a subsequence, $\{f_{n_{i_j}}\}$, that converges pointwise almost everywhere to f. Note that $\{f_{n_{i_j}}\}$ is also a subsequence of $\{f_n\}$. Therefore, by Fatou's Lemma,

$$\int_E f d\mu \le \liminf_{j\to\infty} \int_E f_{n_{i_j}} \, d\mu \le \int_E f d\mu - \delta,$$

giving a contradiction. Therefore, $\int_E f d\mu \le \liminf_{n\to\infty} \int_E f_n \, d\mu$,

Suppose now $\int_E f d\mu = \infty$ and that $\liminf_{n\to\infty} \int_E f_n \, d\mu < \infty$. Then by definition of lim inf, there exists a number $J > 0$ and a sub sequence, $\{f_{n_i}\}$, such that $\int_E f_{n_i} \, d\mu < J$. As above, we can find a subsequence $\{f_{n_{i_j}}\}$ of $\{f_{n_i}\}$ such that $\{f_{n_{i_j}}\}$ converges almost everywhere to f. It follows by Fatou's Lemma that $\liminf_{j\to\infty} \int_E f_{n_{i_j}} \, d\mu = \infty$ and so $\left\{\int_E f_n d\mu\right\}$ is unbounded above contradicting $\liminf_{j\to\infty} \int_E f_{n_{i_j}} \, d\mu < \infty$.

Theorem 17 (Dominated Convergence Theorem).

Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose E is a measurable set and $\{f_n : E \to \mathbb{R}\}$ is a sequence of measurable functions defined on E. Suppose f_n converges in measure to a measurable function $f : E \to \mathbb{R}$. Suppose there exists a Lebesgue integrable function $g : E \to [0, \infty)$ such that $|f(x)| \leq g(x)$, $|f_n(x)| \leq g(x)$ for all integer $n \geq 1$ and for almost all x in E. Then f_n and f are Lebesgue integrable, $\int_E f_n \, d\mu \to \int_E f \, d\mu$ and f_n converges in the mean to f, i.e., $\lim_{n\to\infty} \int_E |f_n - f| \, d\mu = 0$.

Proof.

Since f_n converges in measure to a measurable function $f : E \to \mathbb{R}$, by Theorem 2, there is a subsequence $\{f_{n_i}\}$ converging to f pointwise almost everywhere with respect to μ. Moreover, for each integer $i \geq 1$, $|f_{n_i}(x)| \leq g(x)$. Therefore, $|f(x)| \leq g(x)$ for almost all x in E. Hence, f is Lebesgue integrable on E. Note that for each integer $n \geq 1$, $f_n(x) + g(x) \geq 0$ almost everywhere in E. Furthermore, as f_n converges in measure to f, $f_n + g$ converges in measure to $f + g$. Therefore, by Theorem 16,

$$\int_E f \, d\mu + \int_E g \, d\mu \leq \liminf_{n\to\infty} \int_E (f_n + g) \, d\mu .$$

It follows that

$$\int_E f \, d\mu \leq \liminf_{n\to\infty} \int_E f_n \, d\mu . \quad \text{--------------------} \quad (1)$$

We also have that for each integer $n \geq 1$, $g(x) - f_n(x) \geq 0$ almost everywhere in E *and* $g - f_n$ converges in measure to $g - f$. Therefore, by Theorem 16,

$$\int_E g \, d\mu - \int_E f \, d\mu \leq \liminf_{n\to\infty} \int_E (g - f_n) \, d\mu .$$

It follows that

$$\int_E f \, d\mu \geq \limsup_{n\to\infty} \int_E f_n \, d\mu . \quad \text{--------------------} \quad (2)$$

Therefore, $\int_E f d\mu \le \liminf_{n\to\infty} \int_E f_n d\mu \le \limsup_{n\to\infty} \int_E f_n d\mu \le \int_E f d\mu$ which implies that

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu .$$

By definition of convergence in measure, f_n converges in measure to f implies that $|f_n - f|$ converges in measure to 0. Moreover, for each integer $n \ge 1$, $|f_n - f| \le 2g$ almost everywhere in *E*. Therefore, applying the previous conclusion to the sequence, $\{|f_n - f|\}$, we have

$$\lim_{n\to\infty} \int_E |f_n - f| d\mu = \int_E 0 d\mu = 0 .$$

Remark. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose *E* is a measurable set and $\{f_n : E \to \mathbb{R}\}$ is a sequence of measurable functions defined on *E* and dominated by an integrable non-negative function in *E*. Suppose $f : E \to \mathbb{R}$ is measurable. Then by Theorem 10 and Theorem 17, $\{f_n\}$ converges to *f* in the mean in *E* if, and only if, $\{f_n\}$ converges in measure to f in *E*.

For a sequence of measurable functions, convergence almost uniformly in a measurable set *E*, does imply that after subtracting a set of measure zero, we can consider *E* as a countable union of measurable sets, where, in each of these sets, the sequence converges uniformly. We state this more precisely as follows:

Proposition 18. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Suppose *E* is a $\mathscr{M}$-measurable subset of *X*. Suppose $f_n : E \to \overline{\mathbb{R}}$, n =1 ,2, … and $f : E \to \overline{\mathbb{R}}$ are $\mathscr{M}$ - measurable functions, which are defined and finite almost everywhere on *E*, with respect to the measure μ .

If $\{f_n\}$ converges to *f* almost uniformly, then there exists a sequence of measurable sets, $\{E_i\}$ in *E*, such that $\mu\left(E - \bigcup_{i=1}^{\infty} E_i \right) = 0$ and $\{f_n\}$ converges uniformly to *f* on E_i for each integer $i \ge 1$.

Proof.

If $\{f_n\}$ converges to f almost uniformly, then by definition, given any integer $i \geq 1$, there exists a measurable set A_i in E such that $\mu(A_i) < \frac{1}{i}$ and $f_n \to f$ uniformly on $E - A_i$. For each integer $i \geq 1$, let $E_i = E - A_i$. Let $H = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty}(E - A_i)$. Then $f_n(x) \to f(x)$ uniformly on each E_i, $i \geq 1$. Now $E - H = \bigcap_{m=1}^{\infty} A_m$. Therefore,
$\mu(E - H) = \mu\left(\bigcap_{m=1}^{\infty} A_m\right) \leq \mu(A_n) < \frac{1}{n}$ for each integer $n \geq 1$. Since $\frac{1}{n} \to 0$,
$\mu(E - H) = 0$.

If E is of σ-finite measure, convergence almost everywhere in E does imply uniform convergence in some measurable subsets of E as in Proposition 18.

Proposition 19. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Let E be a $\mathscr{M}$-measurable subset of X. Suppose f, f_n, $n = 1, 2, \ldots$, are measurable and finite almost everywhere on E. Suppose E is of σ-finite measure and $f_n \to f$ almost everywhere on E. Then there exists a sequence of measurable sets $\{E_i\}$ such that $\mu\left(E - \bigcup_{i=1}^{\infty} E_i\right) = 0$ and $f_n \to f$ uniformly on each E_i.

Proof.

Since E is of σ-finite measure, we may assume that $E = \bigcup_{i=1}^{\infty} F_i$, where F_i is measurable and $\mu(F_i) < \infty$ for integer $i \geq 1$. Therefore, as $f_n \to f$ almost everywhere on E, $f_n \to f$ almost everywhere on each F_i. Hence, by Theorem 11 (Egoroff's Theorem), $\{f_n\}$ converges to f almost uniformly in each F_i. It follows then by Proposition 18, that for each integer $i \geq 1$, there exists measurable sets $\{E_j^i\}_{j=1}^{\infty}$ such that $f_n \to f$ uniformly on each E_j^i for $j = 1, 2, \ldots$ and $\mu\left(F^i - \bigcup_{j=1}^{\infty} E_j^i\right) = 0$. Note that $\mu\left(E - \bigcup_{i,j=1}^{\infty} E_j^i\right) = \mu\left(\bigcup_{i=1}^{\infty}\left(F_i - \bigcup_{j=1}^{\infty} E_j^i\right)\right) = 0$ and $f_n \to f$ uniformly on each E_j^i for $j = 1, 2, \ldots$, $i = 1, 2, \ldots$. The required sequence of measurable sets is $\{E_j^i\}$. Since it is countable, we may re-enumerate the sequence as $\{E_i\}$.

The Lebesgue measure on $\mathbb{R}$ is σ-finite. We have the following interesting generalization of the integral of a non-negative measurable function.

Proposition 20. Suppose $(X, \mathscr{M}, \mu)$ is a measure space. Let E be a $\mathscr{M}$-measurable subset of X. Suppose $f : E \to [0,\infty]$ be a non-negative measurable function. Let $B(f,E)$ denote the set of all bounded measurable functions $\xi : E \to \mathbb{R}$ satisfying $\xi \le f$ and $\mu\left(\xi^{-1}(\mathbb{R}-\{0\})\right) < \infty$. If E is of σ-finite measure, then $\int_E f\, d\mu = \sup\left\{\int_E \xi d\mu : \xi \in B(f,E)\right\}$.

Proof.

Recall that for a non-negative measurable function, $f : E \to [0,\infty]$, its Lebesgue integral is given by

$$\int_E f\, d\mu = \sup\left\{\int_E s\, d\mu : s \text{ is a measurable simple function and } 0 \le s \le f\right\}.$$

By Theorem 16 of *Introduction to Measure Theory*, there exists a monotone increasing sequence of (non-negative) measurable simple functions $s_n : E \to [0,\infty)$ converging pointwise to f. By Theorem 23 (Lebesgue Monotone Convergence Theorem) of *Introduction to Measure Theory*,

$$\int_E s_n\, d\mu \nearrow \int_E f\, d\mu \quad .$$

Since E is σ-finite, we may assume that $E = \bigcup_{i=1}^{\infty} K_i$, where $K_i \subseteq K_{i+1}$, K_i is measurable and $\mu(K_i) < \infty$ for each integer $i \ge 1$.

Then by Proposition 21 of *Introduction to Measure Theory*, $\varphi_{S_n}(A) = \int_A s_n d\mu$ for any measurable A, defines a positive measure on $\mathscr{M}$.

Therefore, by the continuity from below property of positive measure,

$$\int_E s_n d\mu = \varphi_{S_n}(E) = \varphi_{S_n}\left(\bigcup_{i=1}^{\infty} K_i\right) = \lim_{i\to\infty} \varphi_{S_n}(K_i)$$

$$= \lim_{i\to\infty} \int_E s_n \chi_{K_i} \, d\mu \, .$$

Note that $0 \le s_n \chi_{K_i} \le f$ and $s_n \chi_{K_i}$ is a bounded measurable function with

$$\left(s_n \chi_{K_i} \right)^{-1} (\mathbb{R} - \{0\}) \subseteq K_i \, .$$

Hence, $\mu\left(\left(s_n \chi_{K_i} \right)^{-1} (\mathbb{R} - \{0\}) \right) \le \mu\left(K_i\right) < \infty$. It follows that for each integer $i \ge 1$,

$$s_n \chi_{K_i} \in B(f, E) = \left\{ \xi : E \to \mathbb{R} \mid \xi \text{ is bounded and measurable}, \xi \le f \right\} .$$

Therefore, $\int_E s_n d\mu \le \sup\left\{ \int_E h d\mu : h \in B(f, E) \right\}$ for each integer $n \ge 1$. It follows that

$$\int_E f \, d\mu = \lim_{n\to\infty} \int_E s_n d\mu \le \sup\left\{ \int_E h d\mu : h \in B(f, E) \right\} .$$

But for each $h \in B(f, E)$, $h \le f$ so that $\int_E h d\mu \le \int_E f \, d\mu$. Hence,

$$\sup\left\{ \int_E h d\mu : h \in B(f, E) \right\} \le \int_E f \, d\mu \, .$$

Therefore, $\int_E f \, d\mu = \sup\left\{ \int_E h d\mu : h \in B(f, E) \right\}$.