Complex Measure, Dual Space of L^p Space, Radon-Nikodym Theorem and Riesz Representation Theorems

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Our aim is to show how to identify the dual or conjugate space of $L^p(X,\mu)$ and $C_0(X)$, the space of continuous complex functions on a locally compact topological space X, which vanish at infinity. There are some very useful and basic results used. I should mention the Lebesgue decomposition of a bounded positive measure with respect to another bounded positive measure. This is analogous to the Lebesgue decomposition of an increasing function into sum of an absolutely continuous increasing function, an increasing singular function and a saltus type function. The Radon-Nikodym Theorem provides the seed for the identification of a bounded complex linear functional on a L^p space, through the Radon Nikodym derivative. The Radon-Nikodym derivative is also the key to integration over complex measure, through the polar decomposition of complex measure. With this we can then represent a bounded complex linear functional on $C_0(X)$ by a Lebesgue integral over a complex measure. The aim is to find this measure and show that it is unique. This is the Riesz Representation Theorem for complex measure. The proof of the Radon-Nikodym Theorem uses a crucial result in Hilbert space theory, more specifically that the bounded linear functional of a Hilbert space is determined by inner product with a unique element of the Hilbert space. This is played out here by the Hilbert space, $L^2(X,\lambda)$, where λ is a positive measure. We have added the integral representation of continuous real linear functional on the space of bounded continuous real-valued functions on a normal Hausdorff topological space. We end the article with a brief discussion on Riesz type representation theorems for the topological dual of the space of bounded continuous function on a completely regular Hausdorff space.

Recall that if X is a set and $\mathscr{M}$ a σ-algebra on X, then a *positive measure* on $\mathscr{M}$ is a countably additive function $\mu: \mathscr{M} \to \overline{\mathbb{R}^+}$ mapping the σ-algebra $\mathscr{M}$ into the extended positive real numbers, a *real measure* on $\mathscr{M}$ is a countably additive function $\mu: \mathscr{M} \to \mathbb{R}$ mapping the σ-algebra $\mathscr{M}$ into the real numbers and a *complex measure* on $\mathscr{M}$ is a countably additive function $\mu: \mathscr{M} \to \mathbb{C}$ mapping the σ-algebra $\mathscr{M}$ into the complex numbers. Hence, a real measure is a complex measure but a positive measure is not necessarily a real measure nor a complex measure.

Suppose μ: $\mathscr{M} \to \mathbb{C}$ is a complex measure. Then for any $E \in \mathscr{M}$, $\mu(E) \in \mathbb{C}$ and so $|\mu(E)| < \infty$. Our first consideration is to find a smallest positive measure λ such that $|\mu(E)| \leq \lambda(E)$ for all $E \in \mathscr{M}$. Suppose $\{E_i\}_{i=1}^{\infty}$ is a countable disjoint collection of sets in $\mathscr{M}$. Then $E = \bigcup_{i=1}^{\infty} E_i \in \mathscr{M}$. Hence, by countable additivity,

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i).$$

This means $\sum_{i=1}^{\infty} \mu(E_i)$ is convergent and as the summation is independent of the order of E_i, the summation $\sum_{i=1}^{\infty} \mu(E_i)$ must be absolutely convergent.

Theorem 1. Let μ: $\mathscr{M} \to \mathbb{C}$ be a complex measure on the measure space $(X, \mathscr{M})$, where X is a set and $\mathscr{M}$ is a σ-algebra on X. Define

$$|\mu|(E) = \sup_{\text{all partitions } \{E_i\} \text{ of } E} \sum |\mu(E_i)|.$$

Then $|\mu|$ is a measure on $\mathscr{M}$, called the *total variation measure* of μ.

Note that for any E in $\mathscr{M}$, $|\mu|(E) \geq |\mu(E)|$.

Proof. Plainly, $|\mu|(\varnothing) = 0$. We shall show that $|\mu|$ is countably additive. Take $E \in \mathscr{M}$.

Suppose $\{F_i\}$ is a partition of E by disjoint sets in $\mathscr{M}$. We shall show that

$$|\mu|(E) = \sum_i |\mu|(F_i).$$

We show that $\sum_i |\mu|(F_i) \leq |\mu|(E)$ as follows.

For each integer i, choose $t_i < |\mu|(F_i)$. Then by definition of $|\mu|(F_i)$, there exists a partition $\{G_{i,j}\}_j$ of F_i such that

$$t_i < \sum_j |\mu(G_{i,j})| \quad (\leq |\mu|(F_i)) \quad \text{-------------------------} \quad (1)$$

Then $\{G_{i,j}\}_{i,j}$ is a partition of E. Now, $\sum_{i,j}\mu(G_{i,j})$ is an absolutely convergent double series. Therefore,

$$|\mu|(E) \geq \sum_{i,j}\left|\mu(G_{i,j})\right| \quad \text{by definition of } |\mu|(E),$$

$$= \sum_i\sum_j\left|\mu(G_{i,j})\right|$$

$$> \sum_i t_i \quad \text{by (1) for all } t_i < |\mu|(F_i).$$

It follows that $|\mu|(E) \geq \sum_i |\mu|(F_i)$.

Next, we show that $|\mu|(E) \leq \sum_i |\mu|(F_i)$.

Let $\{H_j\}$ be any other partition of E. Then for each j, $\{F_i \cap H_j\}_i$ is a partition of H_j and $\{F_i \cap H_j\}_j$ is a partition of F_i. It follows that

$$\sum_j\left|\mu(H_j)\right| = \sum_j\left|\sum_i \mu(F_i \cap H_j)\right| \leq \sum_j\sum_i\left|\mu(F_i \cap H_j)\right| = \sum_i\sum_j\left|\mu(F_i \cap H_j)\right|$$

$$\leq \sum_i |\mu|(F_i).$$

This holds for any partition $\{H_j\}$ of E. Therefore, $|\mu|(E) \leq \sum_i |\mu|(F_i)$. It follows that $|\mu|(E) = \sum_i |\mu|(F_i)$ and so $|\mu|$ is countably additive on $\mathscr{M}$ and is therefore a positive measure on $\mathscr{M}$.

Our next result is an assertion that the total variation measure of a complex measure is a finite positive measure.

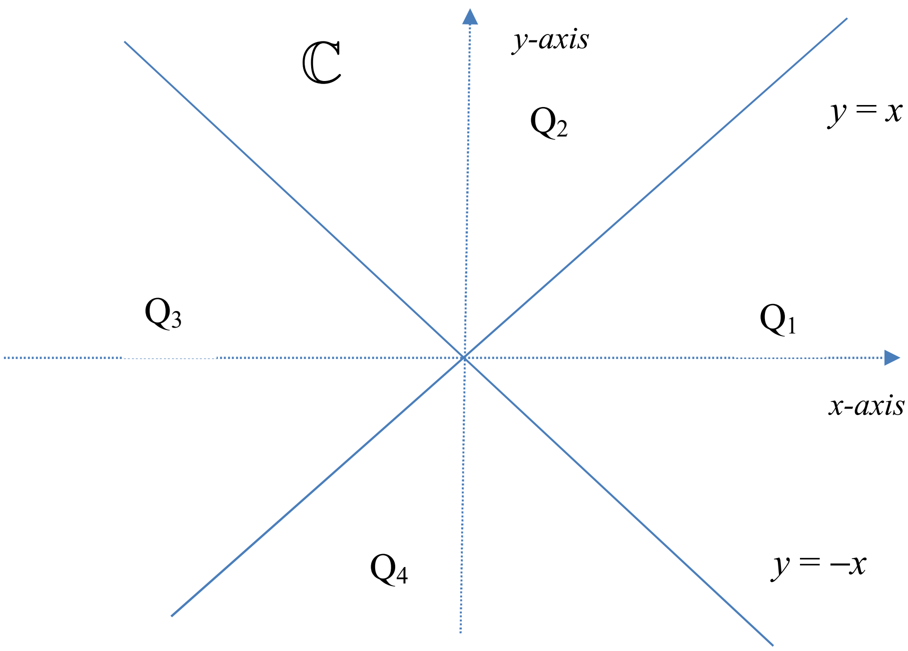
Proposition 2. Let $\mu: \mathscr{M} \to \mathbb{C}$ be a complex measure on the measure space (X, $\mathscr{M}$), where X is a set and $\mathscr{M}$ is a σ-algebra on X. Then the total variation measure of μ, $|\mu|$, is a finite positive measure.

We already knew that $|\mu|$ is a positive measure. We only need to show that it is finite.

We shall need the following technical lemma.

Lemma 3. Let $z_1, z_2, \cdots, z_n \in \mathbb{C}$. Then there exists a subset $S \subseteq \{z_1, z_2, \cdots, z_n\}$ such that $\sum_{i=1}^{n} |z_i| \le 6 \left| \sum_{i \in S} z_i \right|$.

Proof. Let $w = \sum_{i=1}^{n} |z_i|$. The two lines $y = \pm x$ divide the complex plane into 4 quadrants, Q_1, Q_2, Q_3 and Q_4 as shown in the diagram below.



Examine the points in each quadrant.

In one of the four quadrants, we must have that the sum of the modulus of the points is greater than or equal to $\frac{1}{4}w$. Suppose it occurs in Q_1. Then

$\sum_{z_i \in Q_1} |z_i| \ge \frac{1}{4} w$.

Observe that if $z \in Q_1$, then $\operatorname{Re} z = |z| \cos(\theta) \ge |z| \frac{1}{\sqrt{2}}$ since $|\theta| \le \frac{\pi}{4}$. Then

$$\left| \sum_{i \in S = Q_1} z_i \right| \ge \operatorname{Re} \sum_{i \in Q_1} z_i = \sum_{i \in Q_1} \operatorname{Re} z_i \ge \sum_{i \in Q_1} |z_i| \frac{1}{\sqrt{2}} \ge \frac{1}{\sqrt{2}} \frac{1}{4} \sum_{i=1}^{n} |z_i| \ge \frac{1}{6} \sum_{i=1}^{n} |z_i|.$$

Suppose $\sum_{z_i \in Q_3} |z_i| \geq \frac{1}{4} w$. Then, similarly, for $z \in Q_3$, we get $|\operatorname{Re} z| = -\operatorname{Re} z = |z| \cos(\theta)$, where $|\theta| \leq \frac{\pi}{4}$. Thus $-\operatorname{Re} z \geq |z| \frac{1}{\sqrt{2}}$. It follows that

$$\left| \sum_{i \in S = Q_3} z_i \right| \geq -\operatorname{Re} \sum_{i \in Q_3} z_i = \sum_{i \in Q_3} (-\operatorname{Re} z_i) \geq \sum_{i \in Q_3} |z_i| \frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{2}} \frac{1}{4} \sum_{i=1}^{n} |z_i| \geq \frac{1}{6} \sum_{i=1}^{n} |z_i|.$$

Suppose $\sum_{z_i \in Q_2} |z_i| \geq \frac{1}{4} w$. If $z \in Q_2$, then $\operatorname{Im} z = |z| \cos(\theta) \geq 0$, where $|\theta| \leq \frac{\pi}{4}$. Hence, $|\operatorname{Im} z| = |z| \cos(\theta) \geq |z| \frac{1}{\sqrt{2}}$, since $|\theta| \leq \frac{\pi}{4}$. Then

$$\left| \sum_{i \in S = Q_2} z_i \right| \geq \operatorname{Im} \sum_{i \in Q_2} z_i = \sum_{i \in Q_2} \operatorname{Im} z_i \geq \sum_{i \in Q_2} |z_i| \frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{2}} \frac{1}{4} \sum_{i=1}^{n} |z_i| \geq \frac{1}{6} \sum_{i=1}^{n} |z_i|.$$

Suppose $\sum_{z_i \in Q_4} |z_i| \geq \frac{1}{4} w$. If $z \in Q_4$, then $|\operatorname{Im} z| = -\operatorname{Im} z = |z| \cos(\theta) \geq 0$, where $|\theta| \leq \frac{\pi}{4}$. Hence, $|\operatorname{Im} z| = |z| \cos(\theta) \geq |z| \frac{1}{\sqrt{2}}$, since $|\theta| \leq \frac{\pi}{4}$. Then

$$\left| \sum_{i \in S = Q_4} z_i \right| \geq -\operatorname{Im} \sum_{i \in Q_4} z_i = \sum_{i \in Q_4} (-\operatorname{Im} z_i) \geq \sum_{i \in Q_4} |z_i| \frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{2}} \frac{1}{4} \sum_{i=1}^{n} |z_i| \geq \frac{1}{6} \sum_{i=1}^{n} |z_i|.$$

So, we can take S to be one of Q_1, Q_2, Q_3 and Q_4 intersection with $\{z_1, z_2, \cdots, z_n\}$, whose sum is greater than or equal to $\frac{1}{4} w$. This completes the proof of Lemma 3.

Proof of Proposition 2.

This is a proof by contradiction.

Suppose there exists $B_0 \in \mathscr{M}$ such that $|\mu|(B_0) = \infty$.

We shall show that we can decompose $B_0 = A_1 \cup B_1$, a disjoint union with $A_1, B_1 \in \mathscr{M}$, $|\mu(A_1)| > 1$ and $|\mu|(B_1) = \infty$.

Repeat this procedure to B_1 and inductively to B_n to get a sequence (A_i), where $|\mu(A_i)|>1$, $\{A_i\}$ are pairwise disjoint and a sequence (B_i) with $|\mu|(B_i)=\infty$ for all integer $i \geq 1$.

Let $C=\bigcup_{i=1}^{\infty} A_i$. Then $C\in \mathscr{M}$ and the collection $\{A_i\}$ consists of pairwise disjoint sets and so by the countable additivity of μ,

$$\mu(C)=\sum_{i=1}^{\infty}\mu(A_i).$$

But the series $\sum_{i=1}^{\infty}\mu(A_i)$ cannot converge absolutely as $|\mu(A_i)|>1$. But we know that the series must converge absolutely, since it is independent of the order of the A_i. This contradiction shows that there does not exist a member $B_0 \in \mathscr{M}$ with $|\mu|(B_0)=\infty$ and so $|\mu|$ is a finite positive measure.

Suppose $|\mu|(B_0)=\infty$. Then for any real number $t > 0$, there exists a partition $\{E_i\}$ of B_0 such that

$$\sum_{i=1}^{\infty}|\mu(E_i)|>t .$$

This implies that there exists an integer N such that $\sum_{i=1}^{N}|\mu(E_i)|>t$. For if $\sum_{i=1}^{n}|\mu(E_i)|\leq t$ for all integer $n \geq 1$, then $\sum_{i=1}^{\infty}|\mu(E_i)|\leq t$.

Note that $|\mu(B_0)|<\infty$. So, we can take $t=6\left(1+|\mu(B_0)|\right)$. Hence, there exists an integer N such that

$$\sum_{i=1}^{N}|\mu(E_i)|>t=6\left(1+|\mu(B_0)|\right).$$

Therefore, by Lemma 3, there exists a subset $S\subseteq\{1,2,\cdots,N\}$ such that

$$6\left|\sum_{i\in S}\mu(E_i)\right|\geq\sum_{i=1}^{N}|\mu(E_i)|>6\left(1+|\mu(B_0)|\right).$$

Now, let $A_1 = \bigcup_{i\in S} E_i$. Then $A_1 \subseteq B_0$ and $6|\mu(A_1)| = 6\left|\sum_{i\in S}\mu(E_i)\right|$, since $\{E_i\}$ are disjoint. It follows that $6|\mu(A_1)| > 6(1+|\mu(B_0)|)$. Hence, $|\mu(A_1)| > 1+|\mu(B_0)| \geq 1$.

Let $B_1 = B_0 - A_1$. Then $|\mu(B_1)| = |\mu(B_0) - \mu(A_1)| \geq |\mu(A_1)| - |\mu(B_0)| > 1$.

Finally, since $|\mu|$ is a measure, $|\mu|(A_1) + |\mu|(B_1) = |\mu|(B_0) = \infty$. So, one of $|\mu|(A_1)$ or $|\mu|(B_1)$ must equal ∞. Arrange for this to be B_1 and the other to be A_1. Rename, if necessary.

Thus, if $|\mu|(X) = \infty$, then call $X = B_0$ and apply the above process to B_1 and inductively to B_n for $n > 1$, to obtain a collection of disjoint sets, $\{A_n\}_{n=1}^{\infty}$ and a collection of sets $\{B_n\}_{n=0}^{\infty}$ with $B_{n-1} = A_n \cup B_n$, $A_n \cap B_n = \varnothing$, $|\mu(A_n)| \geq 1$, for $n \geq 1$, $|\mu|(B_n) = \infty$ for $n \geq 0$. Let $C = \bigcup_{i=1}^{\infty} A_i$. Then $C \in \mathscr{M}$ and by the countable additivity of μ,

$$\mu(C) = \sum_{i=1}^{\infty} \mu(A_i).$$

But the series $\sum_{i=1}^{\infty}\mu(A_i)$ cannot converge absolutely as $|\mu(A_i)| > 1$ contradicting that for a complex measure μ, $\sum_{i=1}^{\infty}\mu(A_i)$ must converge absolutely. Thus $|\mu|(X) < \infty$. It follows that for all $E \in \mathscr{M}$, $|\mu|(E) \leq |\mu|(X) < \infty$ and so $|\mu|$ is a finite measure.

Corollary 4. Let μ: $\mathscr{M} \to \mathbb{C}$ be a complex measure on the measure space (X, $\mathscr{M}$), where X is a set and $\mathscr{M}$ is a σ-algebra on X. Then $\{\mu(E) : E \in \mathscr{M}\}$ is a bounded subset of the complex plane. Thus, every complex measure μ is of bounded variation.

Proof. This is because $|\mu(E)| \leq |\mu|(E) \leq |\mu|(X) < \infty$.

Now, let $M(\mathscr{M})$ be the collection of complex measures on the σ-algebra $\mathscr{M}$. If λ, $\mu \in M(\mathscr{M})$, define $(\lambda+\mu)(E) = \lambda(E) + \mu(E)$ for all $E \in \mathscr{M}$ and

$(\alpha\lambda)(E) = \alpha\lambda(E)$ for all $E \in \mathscr{M}$ and any $\alpha \in \mathbb{C}$. Then this makes $M(\mathscr{M})$ into a complex linear space with norm given by $\|\lambda\| = |\lambda|(X)$ for $\lambda \in M(\mathscr{M})$. We verify that $\|.\|$ is a norm on $M(\mathscr{M})$. For any $E \in \mathscr{M}$, take a partition $\{F_i\}$ of E by disjoint sets in $\mathscr{M}$. Then $\sum_i |(\lambda+\mu)(F_i)| \le \sum_i |\lambda(F_i)| + \sum_i |\mu(F_i)| \le |\lambda|(E) + |\mu|(E)$.
Thus, by the definition of $|(\lambda+\mu)|(E)$, $|(\lambda+\mu)|(E) \le |\lambda|(E) + |\mu|(E)$. It follows that $\|(\lambda+\mu)\| = |(\lambda+\mu)|(X) \le |\lambda|(X) + |\mu|(X) = \|\lambda\| + \|\mu\|$. Plainly for any complex number c and any $\lambda \in M(\mathscr{M})$, $\|c\lambda\| = |c\lambda|(X) = |c||\lambda|(X) = |c|\|\lambda\|$. Also $\|\lambda\| = 0$ implies $|\lambda|(X) = 0$ which in turn implies that $|\lambda|(E) = 0$ for all $E \in \mathscr{M}$, It follows that $|\lambda(E)| = 0$ for all $E \in \mathscr{M}$ and so $\lambda = 0$. Thus, $M(\mathscr{M})$ is a normed linear space.

When is $M(\mathscr{M})$ a Banach space? If X is a Hausdorff topological space and $\mathscr{M}$ is a σ-algebra containing the Borel sets of X, we can associate to each complex measure, μ, in $M(\mathscr{M})$, a linear functional Λ on $C_c(X)$ with the sup norm, defined by $\Lambda(f) = \int_X f\, d\mu$ for $f \in C_c(X)$, when we can make sense of integration over a complex measure. (See Definition 11). By Proposition 2, the total variation measure of μ is a bounded positive measure and so Λ is a bounded linear functional. However, we do not know if this association is one to one and neither do we know if the association is onto. When X is a locally compact Hausdorff topological space, we shall investigate this question in due course in Theorem 20 (Riesz Representation Theorem).

Now we look at the situation of two measures, which are basically independent of one another, meaning each one is non-zero on a set which is disjoint from the set on which the other is nonzero. We describe such a situation as follows. Let μ be a Lebesgue measure on $\mathbb{R}$. Let $L^1(\mathbb{R},\mu)$ be the equivalence classes of absolutely integrable complex functions on $\mathbb{R}$ with the $L^1(\mu)$ norm, $\|f\| = \int_{\mathbb{R}} |f| d\mu < \infty$. Now for a fixed $f \in L^1(\mathbb{R},\mu)$, define $\lambda(E) = \int_E f\, d\mu$ for Lebesgue measurable set E. Then λ is a measure. Suppose now we have two functions $f_1, f_2 \in L^1(\mathbb{R},\mu)$ such that $f_1 \cdot f_2 = 0$. Let $A_i = \{x : f_i(x) \ne 0\}$. Then $A_1 \cap A_2 = \varnothing$. Let $\lambda_i(E) = \int_E f_i\, d\mu$ for $i = 1, 2$. Then we have,

$$\lambda_i(E) = \int_E f_i\, d\mu = \int_{E\cap A_i} f_i\, d\mu = \lambda_i(E \cap A_i),\ i = 1, 2,$$

$\lambda_1(E\cap A_2)=\int_{E\cap A_2} f_1\,d\mu=0$ and $\lambda_2(E\cap A_1)=\int_{E\cap A_1} f_2\,d\mu=0$.

We abstract the properties of the two measures above in the following definition.

Definition 5. Let $(X, \mathscr{M})$ be a measure space, where $\mathscr{M}$ is a σ-algebra on X. Let μ be a positive measure on $\mathscr{M}$ and λ be a complex or positive measure on $\mathscr{M}$.

(a) We say λ is *absolutely continuous* with respect to μ and write $\lambda \ll \mu$ if $E\in \mathscr{M}$ and $\mu(E)=0$ implies that $\lambda(E)=0$.

(b) We say λ is *concentrated on A* for some $A \in \mathscr{M}$, if for all $E \in \mathscr{M}$, $\lambda(E)=\lambda(E\cap A)$.

(c) Suppose λ_1 is concentrated on A_1 and λ_2 is concentrated on A_2 with $A_1\cap A_2=\varnothing$. Then we say λ_1 and λ_2 are *mutually singular* and write $\lambda_1 \perp \lambda_2$. If λ is any complex measure concentrated on some set of μ-measure aero, then we write $\lambda \perp \mu$.

Note that if $\lambda \ll \mu$, then $E \in \mathscr{M}$ and $\mu(E)=0 \Rightarrow |\lambda|(E)=0$. This is because for $E \in \mathscr{M}$ and any partition, $\{F_i\}$, of E by disjoint sets in $\mathscr{M}$,

$$\mu(E)=0 \Rightarrow \mu(F_i)=0 \Rightarrow \sum_i |\lambda(F_i)|=0 \Rightarrow |\mu|(E)=0.$$

Proposition 6. Let $(X, \mathscr{M})$ be a measure space, where $\mathscr{M}$ is a σ-algebra on X. Let μ be a positive measure on $\mathscr{M}$ and λ be a complex or a real measure on $\mathscr{M}$.

$\lambda \ll \mu$, if and only if, given any $\varepsilon > 0$, there exists $\delta > 0$, such that for all $E \in \mathscr{M}$, $\mu(E)<\delta \Rightarrow |\lambda(E)|<\varepsilon$.

Proof. Suppose given any $\varepsilon > 0$, there exists $\delta > 0$, such that for all $E \in \mathscr{M}$, $\mu(E)<\delta \Rightarrow |\lambda(E)|<\varepsilon$. Thus, there exists $\delta_n>0$ such that $\mu(E)<\delta_n \Rightarrow |\lambda(E)|<\frac{1}{n}$. If $\mu(E)=0$, then $\mu(E)<\delta_n$ for all integer $n \geq 1$. Hence, $|\lambda(E)|<\frac{1}{n}$ for all integer $n \geq 1$. It follows that $|\lambda(E)|=0$ and so $\lambda(E)=0$. This means $\lambda \ll \mu$.

Conversely, suppose $\lambda \ll \mu$. We shall prove that given any $\varepsilon > 0$, there exists $\delta > 0$, such that for all $E \in \mathscr{M}$, $\mu(E) < \delta \Rightarrow |\lambda(E)| < \varepsilon$. We show this by way of contradiction. Suppose there exists an $\varepsilon > 0$ such that for any $\delta > 0$, there exists $E_\delta \in \mathscr{M}$, with $\mu(E_\delta) < \delta$ but $|\lambda(E_\delta)| \geq \varepsilon$. So, taking $\delta = \frac{1}{2^r}$, r an integer ≥ 1,there exists $E_r \in \mathscr{M}$, with $\mu(E_r) < \frac{1}{2^r}$ but $|\lambda(E_r)| \geq \varepsilon$.

Let $F_r = \bigcup_{s=r}^{\infty} E_s$ and $E = \bigcap_{r=1}^{\infty} F_r$. Then $F_r \in \mathscr{M}$, for each integer $r \geq 1$, $E \in \mathscr{M}$ and

$\mu(F_r) \leq \sum_{s=r}^{\infty} \mu(E_s) < \sum_{s=r}^{\infty} \frac{1}{2^s} = \frac{1}{2^{r-1}}$. Hence, $\mu(E) \leq \mu(F_r) < \frac{1}{2^{r-1}}$ for integer $r \geq 1$. It follows that $\mu(E) = 0$. If λ is a complex measure, then λ is of bounded variation so that $|\lambda(X)| < \infty$. Since $|\lambda|$ is a finite positive measure by Proposition 2, by the continuity from above property of a measure,

$$|\lambda|(E) = |\lambda|\left(\bigcap_{r=1}^{\infty} F_r\right) = \lim_{r\to\infty} |\lambda|(F_r).$$

Since $|\lambda|(F_r) \geq |\lambda|(E_r) \geq |\lambda(E_r)| \geq \varepsilon$ for integer $r \geq 1$, we must have $|\lambda|(E) \geq \varepsilon > 0$. But $\mu(E) = 0$ and $\lambda \ll \mu$ implies that $|\lambda|(E) = 0$. So, we have arrived at a contradiction and this means that given any $\varepsilon > 0$, there exists $\delta > 0$, such that for all $E \in \mathscr{M}$, $\mu(E) < \delta \Rightarrow |\lambda(E)| < \varepsilon$. If λ is a real measure, then λ is a complex measure and we obtain the same contradiction as above for the converse.

We have the following immediate consequence of Definition 5.

Lemma 7. Let $(X, \mathscr{M})$ be a measure space, where $\mathscr{M}$ is a σ-algebra on X. Let μ be a positive measure on $\mathscr{M}$. Suppose λ, λ_1 and λ_2 are complex measures on $\mathscr{M}$.

(a) λ is concentrated on $A \Leftrightarrow$ for all $E \in \mathscr{M}$, $E \cap A = \varnothing$ implies $\lambda(E) = 0$.

(b) If λ is concentrated on A, then so is its total variation $|\lambda|$.

(c) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.

(d) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $(\lambda_1 + \lambda_2) \perp \mu$.

(e) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $(\lambda_1 + \lambda_2) \ll \mu$.

(f) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.

(g) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.

(h) If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.

Proof.

(a) If λ is concentrated on A, $E \cap A = \varnothing \Rightarrow \lambda(E) = \lambda(E \cap A) = \lambda(\varnothing) = 0$. Conversely, suppose for all $E \in \mathscr{M}$, $E \cap A = \varnothing$ implies $\lambda(E) = 0$. Then for all $E \in \mathscr{M}$, $\lambda(E) = \lambda\big((E \cap A) \cup (E - A)\big) = \lambda(E \cap A) + \lambda(E - A) = \lambda(E \cap A) + 0 = \lambda(E \cap A)$. Hence, λ is concentrated on A.

(b) Suppose λ is concentrated on A. Then for all $E \in \mathscr{M}$, $\lambda(E) = \lambda(E \cap A)$.

Take a partition, $\{F_i\}$, of E by disjoint sets in $\mathscr{M}$. Then

$\sum_i |\lambda(F_i)| = \sum_i |\lambda(F_i \cap A)| \le |\lambda|(E \cap A)$ since $\{F_i \cap A\}$ is a partition of $E \cap A$.

Since this is true for any partition, $\{F_i\}$, of E by disjoint sets in $\mathscr{M}$, $|\lambda|(E) \le |\lambda|(E \cap A)$. Since $E \cap A \subseteq E$ and $|\lambda|$ is a finite positive measure, $|\lambda|(E \cap A) \le |\lambda|(E)$. Therefore, $|\lambda|(E \cap A) = |\lambda|(E)$ for any $E \in \mathscr{M}$. Hence, $|\lambda|$ is concentrated on A.

(c) If $\lambda_1 \perp \lambda_2$, then λ_1 is concentrated on A_1 and λ_2 is concentrated on A_2 for some A_1 and A_2 in $\mathscr{M}$ with $A_1 \cap A_2 = \varnothing$. By part (b), $|\lambda_1|$ is concentrated on A_1 and $|\lambda_2|$ is concentrated on A_2. Hence, $|\lambda_1| \perp |\lambda_2|$.

(d) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then λ_1 is concentrated on A_1 and λ_2 is concentrated on A_2 for some A_1 and A_2 in $\mathscr{M}$ such that $\mu(A_1) = \mu(A_2) = 0$. For all $E \in \mathscr{M}$, $\lambda_1(E) = \lambda_1(E \cap A_1)$ and $\lambda_2(E) = \lambda_2(E \cap A_2)$. Now,

$E\cap(A_1\cup A_2)=(E\cap A_1)\cup(E\cap(A_2-A_1))$ and $E\cap A_1$ and $E\cap(A_2-A_1)$ are disjoint and belong to $\mathscr{M}$. Therefore,

$\lambda_1(E\cap(A_1\cup A_2))=\lambda_1(E\cap A_1)+\lambda_1(E\cap(A_2-A_1))=\lambda_1(E\cap A_1)+0$,

since by part (a), $\lambda_1(E\cap(A_2-A_1))=0$ as $(E\cap(A_2-A_1))\cap A_1=\varnothing$,

$=\lambda_1(E)$.

Similarly, we have $\lambda_2(E\cap(A_1\cup A_2))=\lambda_2(E)$.

Hence, for all $E\in\mathscr{M}$, $(\lambda_1+\lambda_2)(E\cap(A_1\cup A_2))=(\lambda_1+\lambda_2)(E)$. Moreover, $\mu(A_1\cup A_2)\leq\mu(A_1)+\mu(A_2)=0\Rightarrow\mu(A_1\cup A_2)=0$. Thus, $(\lambda_1+\lambda_2)\perp\mu$.

(e) Suppose $\lambda_1\ll\mu$ and $\lambda_2\ll\mu$. Then for any $E\in\mathscr{M}$, $\mu(E)=0\Rightarrow\lambda_1(E)=\lambda_2(E)=0$. Therefore, $(\lambda_1+\lambda_2)(E)=\lambda_1(E)+\lambda_2(E)=0$. This means $(\lambda_1+\lambda_2)\ll\mu$.

(f) We have already proved this immediately after Definition 5.

(g) Suppose $\lambda_1\ll\mu$ and $\lambda_2\perp\mu$. Suppose λ_2 is concentrated on A_2 for some A_2 in $\mathscr{M}$ with $\mu(A_2)=0$. It follows that for all E in $\mathscr{M}$ and $E\subseteq A_2$ $\mu(E)=0$. As $\lambda_1\ll\mu$, this implies that for all E in $\mathscr{M}$ and $E\subseteq A_2$, $\lambda_1(E)=0$. Thus λ_1 is concentrated on some set in the complement of A_2 because for any E in $\mathscr{M}$

$$\lambda_1(E)=\lambda_1((E\cap A_2^c)\cup(E\cap A_2))=\lambda_1(E\cap A_2^c)+\lambda_1(E\cap A_2)=\lambda_1(E\cap A_2^c).$$

Since $A_2\cap A_2^c=\varnothing$, $\lambda_1\perp\lambda_2$.

(h) Suppose $\lambda\ll\mu$ and $\lambda\perp\mu$. By part (g) $\lambda\perp\lambda$. This can only happen if $\lambda=0$.

We can verify this directly. $\lambda\perp\mu$ implies that λ is concentrated on A for some A in $\mathscr{M}$ with $\mu(A)=0$. For any E in $\mathscr{M}$,

$\lambda(E\cap A)=\lambda(E)=\lambda(E\cap A)+\lambda(E\cap A^c)\Rightarrow\lambda(E\cap A^c)=0$. But since $\mu(A)=0$, $\mu(E\cap A)=0$ and as $\lambda\ll\mu$, $\lambda(E\cap A)=0$. Therefore, $\lambda(E)=0$.

Theorem 8. Let $(X, \mathscr{M})$ be a measure space, where $\mathscr{M}$ is a σ-algebra on X. Suppose λ and μ are two positive bounded measures on $\mathscr{M}$. Then

(a) (The Lebesgue Decomposition Theorem)

There is a unique pair of measures, λ_a (the *absolutely continuous part* of λ with respect to μ) and λ_s (the *singular part* of λ with respect to μ) such that

$$\lambda = \lambda_a + \lambda_s ,$$

where $\lambda_a \ll \mu$ and $\lambda_s \perp \mu$. Moreover, $\lambda_a \perp \lambda_s$ and both measures are positive.

(b) (Radon-Nikodym Theorem)

There is a function h in $L^1(X,\mu)$ such that

$$\lambda_a(E) = \int_E h d\mu \text{ for all } E \text{ in } \mathscr{M}$$

and h is almost everywhere unique with respect to μ.

Here $L^1(X,\mu) = \left\{ f : X \to \mathbb{C}; f \text{ is measurable and } \int_X |f| d\mu < \infty \right\}$.

Not all measure spaces are bounded measure spaces, for example, the Lebesgue measure on $\mathbb{R}$ is not bounded. But $\mathbb{R}$ is a countable union of sets of finite Lebesgue measure. We say a measure space $(X, \mathscr{M}, \mu)$, where μ is a positive measure, is *σ-finite or μ* is a *σ-finite* positive measure if every set E in $\mathscr{M}$ is at most a countable union of sets in E with finite μ-measure.

Remarks.

1. After proving this theorem, we shall immediately extend to the case where μ is a positive and σ-finite measure (for example, when μ is the Lebesgue measure on $\mathbb{R}^k$) and λ is a complex measure.

2. It is helpful to think of μ as a Lebesgue measure on [0,1].

3. Obviously, if λ is defined by $\lambda(E) = \int_E f\, d\mu$ for E in $\mathscr{M}$ and a fixed $f \in L^1(X,\mu)$, then $\lambda \ll \mu$. The point of the part (b) of the theorem (Radon Nikodym Theorem) is that the converse is also true.

4. The Radon Nikodym Theorem is often abbreviated to

$$d\lambda_a = h\,d\mu \quad \text{or} \quad h = \frac{d\lambda_a}{d\mu}$$

and h is called the *Radon Nikodym derivative* of λ_a with respect to μ.

5. The uniqueness part of the theorem may be proven easily as follows.

Suppose we have $\lambda = \lambda_a + \lambda_s = \lambda_a' + \lambda_s'$, where $\lambda_a, \lambda_a' \ll \mu$ and $\lambda_a, \lambda_a' \perp \lambda_s$.

Then $\lambda_a - \lambda_a' = \lambda_s' - \lambda_s$. Let $v = \lambda_a - \lambda_a' = \lambda_s' - \lambda_s$. Then by property (e) Lemma 7, $v = \lambda_a - \lambda_a' \ll \mu$ and $v = \lambda_s' - \lambda_s \perp \mu$ by property (d) Lemma 7. It follows then by property (h) Lemma 7 that $v = 0$. Consequently, $\lambda_a = \lambda_a'$ and $\lambda_s = \lambda_s'$.

In part (b) of the theorem (Radon Nikodym Theorem), that h is almost everywhere unique with respect to μ, is deduced as follows. Suppose h' is another function in $L^1(X,\mu)$ such that $\lambda_a(E) = \int_E h' d\mu$ for all E in $\mathscr{M}$. Then

$$0 = \lambda_a(E) - \lambda_a(E) = \int_E h\,d\mu - \int_E h' d\mu = \int_E (h - h')\,d\mu \text{ for all } E \text{ in } \mathscr{M}.$$

Therefore, $h - h' = 0$ almost everywhere with respect to μ and so $h = h'$ almost everywhere with respect to μ.

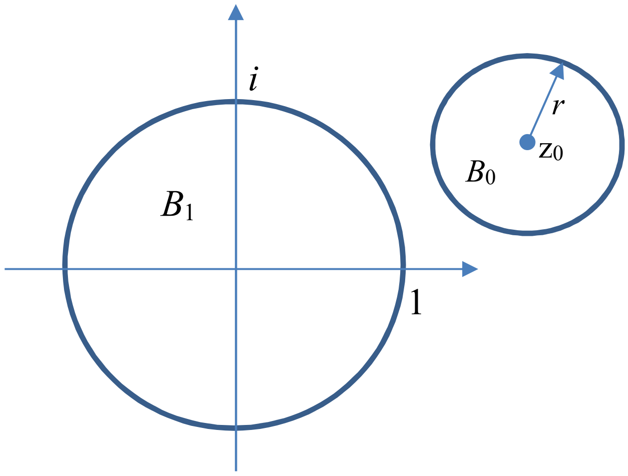
We shall need the following technical lemma for the proof of Theorem 8.

Lemma 9. If μ is a bounded positive measure on the measure space (X, $\mathscr{M}$) and $f \in L^1(X,\mu)$ is such that

$$\frac{1}{\mu(E)}\left|\int_E f\,d\mu\right| \le 1 ,$$

for all E in $\mathscr{M}$ with $\mu(E) > 0$, then $0 \le |f| \le 1$ almost everywhere with respect to μ. That is to say, if all the averages of f over all E in $\mathscr{M}$ belong to the unit disk, then almost all values of f belong to the unit disk.

Proof. Let $B_1 = \{z : |z| \le 1\}$ be the unit disk. Take z_0 outside the unit disk B_1 and a real number r such that $0 < r < |z_0| - 1$. Let $B_0 = \{z : |z - z_0| < r\}$. Then $B_1 \cap B_0 = \varnothing$.



Let $E_0 = f^{-1}(B_0)$. Then $E_0 \in \mathscr{M}$, since $f \in L^1(X,\mu)$. We shall show that $\mu(E_0) = 0$. Assuming this is true for B_0, then it is true for any open disk in the complement of B_1. As B_1^c is open and so is a countable union of such disks, it follows that $\mu\left(f^{-1}\left(B_1^c\right)\right) = 0$ as $f^{-1}\left(B_1^c\right)$ is a countable union of sets of μ measure zero. Hence, $0 \le |f| \le 1$ almost everywhere with respect to μ.

Now we show that $\mu(E_0) = 0$. Suppose on the contrary that $\mu(E_0) > 0$.

Then $\left|\frac{1}{\mu(E_0)}\int_{E_0} f\,d\mu - z_0\right| = \frac{1}{\mu(E_0)}\left|\int_{E_0}(f - z_0)\,d\mu\right| \le \frac{1}{\mu(E_0)}\int_{E_0}|f - z_0|\,d\mu$

$$< \frac{1}{\mu(E_0)}\int_{E_0} r\,d\mu = r.$$

But $\left|\frac{1}{\mu(E_0)}\int_{E_0} f\,d\mu - z_0\right| \ge |z_0| - \frac{1}{\mu(E_0)}\left|\int_{E_0} f\,d\mu\right| \ge |z_0| - 1$ because $0 \le \frac{1}{\mu(E_0)}\left|\int_{E_0} f\,d\mu\right| \le 1$.

This means $r > |z_0| - 1$. This contradicts that $r < |z_0| - 1$. Hence, $\mu(E_0) = 0$.

Remark.

In the proof of Lemma 9, we can use any closed disk in place of the closed unit disk B_1.

Lemma 9*. If μ is a bounded positive measure on the measure space $(X, \mathscr{M})$ and $f \in L^1(X,\mu)$ is such that

$$\left|\frac{1}{\mu(E)}\int_E f\,d\mu - \frac{1}{2}\right| \le \frac{1}{2},$$

for all E in $\mathscr{M}$ with $\mu(E) > 0$, then $0 \le \left|f - \frac{1}{2}\right| \le \frac{1}{2}$ almost everywhere with respect to μ.

For the proof, we may replace B_1 by $B_1 = \left\{z : \left|z - \frac{1}{2}\right| \le \frac{1}{2}\right\}$. Take z_0 outside the disk B_1, that is $\left|z_0 - \frac{1}{2}\right| > \frac{1}{2}$. Take $0 < r < \left|z_0 - \frac{1}{2}\right| - \frac{1}{2}$. Let $B_0 = \{z : |z - z_0| < r\}$. Let $E_0 = f^{-1}(B_0)$. Then as above we can show that $\mu(E_0) = 0$. It follows that $\mu\left(f^{-1}\left(B_1^{\,c}\right)\right) = 0$ and so $0 \le \left|f - \frac{1}{2}\right| \le \frac{1}{2}$.

Suppose on the contrary that $\mu(E_0) > 0$.

Then $$\left|\frac{1}{\mu(E_0)}\int_{E_0} f\,d\mu - z_0\right| = \frac{1}{\mu(E_0)}\left|\int_{E_0}(f - z_0)\,d\mu\right| \le \frac{1}{\mu(E_0)}\int_{E_0}|f - z_0|\,d\mu$$

$$< \frac{1}{\mu(E_0)}\int_{E_0} r\,d\mu = r.$$

But

$$\left|\frac{1}{\mu(E_0)}\int_{E_0} f\,d\mu - z_0\right| = \left|\frac{1}{\mu(E_0)}\int_{E_0} f\,d\mu - \frac{1}{2} + \frac{1}{2} - z_0\right| \ge \left|z_0 - \frac{1}{2}\right| - \left|\frac{1}{\mu(E_0)}\int_{E_0} f\,d\mu - \frac{1}{2}\right|$$

$$\ge \left|z_0 - \frac{1}{2}\right| - \frac{1}{2}.$$

Hence, $r > \left|z_0 - \frac{1}{2}\right| - \frac{1}{2}$. This contradicts that $0 < r < \left|z_0 - \frac{1}{2}\right| - \frac{1}{2}$. Hence, $\mu(E_0) = 0$.

For the proof of Theorem 8, we shall use a very useful property of a Hilbert space.

Let H be a Hilbert space with inner product $\langle x, y\rangle$ satisfying the following properties:

(1) $\langle x, x\rangle \geq 0; \langle x, x\rangle = 0 \Leftrightarrow x = 0$;

(2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$, the complex conjugate of $\langle y, x\rangle$;

(3) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;

(4) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$.

If the norm of a Banach space V arises from an inner product, then it is called a *Hilbert space*.

More precisely, an *inner product* on a (real or complex) linear space V is a scalar valued function on $V \times V$, whose value on (x, y) in $V \times V$ is denoted by $\langle x, y\rangle$ and the function satisfies the following properties:

(1) $\langle x, x\rangle \geq 0; \langle x, x\rangle = 0 \Leftrightarrow x = 0$;

(2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$, the complex conjugate of $\langle y, x\rangle$;

(3) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;

(4) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$.

The norm on H is given by $\|x\| = \sqrt{\langle x, x\rangle}$ for $x \in H$. We have the Schwarz Inequality for inner product: $|\langle x, y\rangle| \leq \|x\|\|y\|$ for all x, $y \in H$. With respect to the metric associated with the norm, H is a *Banach* space, i.e., a complete metric space.

Define for a fixed y in H, the linear functional, $\Lambda_y : H \to \mathbb{C}$, given by $\Lambda_y(x) = \langle x, y\rangle$ for all x in H. Then Λ_y is a bounded (complex) linear functional.

As $|\Lambda_y(x)| = |\langle x, y\rangle| \leq \|x\|\|y\|$, $\|\Lambda_y\| = \sup\left\{\frac{|\Lambda_y(x)|}{\|x\|} : x \neq 0\right\} \leq \|y\|$ and on account of $|\Lambda_y(y)| = |\langle y, y\rangle| = \|y\|\|y\|$, $\|\Lambda_y\| = \|y\|$.

The basic result in Hilbert space theory is that the converse is also true. Suppose $\Lambda : H \to \mathbb{C}$ is a bounded linear functional, then $\Lambda = \Lambda_y$ for some *y* in *H*.

It is in this way that we set up the conjugate linear isometry.

Suppose H^* is the collection of all bounded complex linear functional on *H*.

The association of the bounded complex linear functional Λ with *y* as $\Lambda = \Lambda_y$:

$$\Gamma : H^* \to H$$

given by $\Gamma(\Lambda) = y$, where $\Lambda = \Lambda_y$, is a linear isometry preserving norm. Note that $\Gamma(c\Lambda) = \bar{c}\, y$ for any complex scalar *c*. Note also that $\|\Gamma(\Lambda)\| = \|y\| = \|\Lambda\|$.

The proof of this result is independent of measure theory.

We briefly give the proof here. If $\Lambda = 0$, then take $y = 0$ and plainly, $\Lambda = \Lambda_y$.

Suppose $\Lambda \neq 0$. Let $N = \{x \in H : \Lambda(x) = 0\} \neq H$. It is easily seen that *N* is a closed subspace of *H*. As *H* is complete, *N* being a closed subspace of *H*, is complete.

The orthogonal complement of *N* must contain a nonzero *g*. We may choose *g* such that $\Lambda(g) = 1$. Then $\Lambda(x) = 0$ implies that $x \in N$ and so $\langle x, g \rangle = 0$.

For each $x \in H$, $\Lambda\big(x - \Lambda(x)g\big) = \Lambda(x) - \Lambda(x)\Lambda(g) = \Lambda(x) - \Lambda(x) = 0$ and so $x - \Lambda(x)g \in N$. Hence $\langle x - \Lambda(x)g, g \rangle = 0$. It follows that

$$\langle x, g \rangle - \Lambda(x)\langle g, g \rangle = 0.$$

This means $\Lambda(x)\|g\|^2 = \langle x, g \rangle$ for all *x* in *H* and as $\|g\|^2 \neq 0$,

$$\Lambda(x) = \left\langle x, \frac{g}{\|g\|^2} \right\rangle \text{ for all } x \text{ in } H.$$

Thus, if $y = \dfrac{g}{\|g\|^2}$, then $\Lambda(x) = \langle x, y \rangle$ for all *x* in *H*. Note that *y* is unique.

For if $y' \in H$ is such that $\Lambda(x) = \langle x, y' \rangle$ for all *x* in *H*, then $\langle x, y - y' \rangle = 0$ for all *x* in *H*. Hence, $\|y - y'\|^2 = \langle y - y', y - y' \rangle = 0$. Therefore, $y' = y$.

Proof of Theorem 8.

Let $\varphi = \lambda + \mu$. Since λ and μ are bounded positive measures, φ is a bounded positive measure on $\mathscr{M}$. Then for any $\mathscr{M}$ measurable function $f : X \to \mathbb{C}$,

$$\int_X f\,d\varphi = \int_X f\,d\lambda + \int_X f\,d\mu . \text{ -------------------------- } (1)$$

Evidently, (1) is true if $f = \chi_E$ for $E \in \mathscr{M}$. It then follows that (1) holds for measurable simple functions since any simple function is a complex linear combination of measurable characteristic functions. If f is real valued, non-negative and measurable, then there exists an increasing sequence of non-negative measurable simple functions $\{s_n\}_{n=1}^{\infty}$ such that $s_n \to f$ pointwise on X. Then we have $\int_X s_n\,d\varphi = \int_X s_n\,d\lambda + \int_X s_n\,d\mu$. Then applying the Lebesgue Monotone Convergence Theorem, we have

$$\int_X f\,d\varphi = \lim_{n\to\infty}\int_X s_n\,d\varphi = \lim_{n\to\infty}\int_X s_n\,d\lambda + \lim_{n\to\infty}\int_X s_n\,d\mu = \int_X f\,d\lambda + \int_X f\,d\mu .$$

Now if $f \in L^1(X,\varphi)$, i.e., $\int_X |f|\,d\varphi < \infty$, then $\mathrm{Re} f$ and $\mathrm{Im} f$ are measurable and $(\mathrm{Re} f)^+, (\mathrm{Re} f)^-, (\mathrm{Im} f)^+$ and $(\mathrm{Im} f)^-$ are all measurable and φ integrable, since

$(\mathrm{Re} f)^+, (\mathrm{Re} f)^-, (\mathrm{Im} f)^+, (\mathrm{Im} f)^- \le |f|$. Since (1) holds for non-negative real valued measurable functions, (1) holds for $f \in L^1(X,\varphi)$. Moreover, $f \in L^1(X,\lambda)$ and $f \in L^1(X,\mu)$. Conversely, suppose $f \in L^1(X,\lambda)$ and $f \in L^1(X,\mu)$. Then

$$\int_X \mathrm{Re} f\,d\varphi = \int_X \mathrm{Re} f\,d\lambda + \int_X \mathrm{Re} f\,d\mu < \infty$$

and $\int_X \mathrm{Im} f\,d\varphi = \int_X \mathrm{Im} f\,d\lambda + \int_X \mathrm{Im} f\,d\mu < \infty$. It follows that $f \in L^1(X,\varphi)$ and

$$\int_X f\,d\varphi = \int_X f\,d\lambda + \int_X f\,d\mu .$$

Now we take the Hilbert space,

$$H = L^2(X,\varphi) = \left\{ f : X \to \mathbb{C};\, f \text{ is measurable and } \int_X |f|^2\,d\varphi < \infty \right\},$$

with inner product $\langle f, g\rangle = \int_X f \cdot \overline{g}\,d\varphi$ and norm $\|f\|_{2,\varphi} = \left(\int_X |f|^2\,d\varphi\right)^{\frac{1}{2}}$. For the proof that H is a Hilbert space, see Theorem 11 of *Convex Function, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*.

We define a complex linear functional $\Lambda : H \to \mathbb{C}$ by

$$\Lambda(f) = \int_X f \, d\lambda \ .$$

Note that this is well defined. By Hölders Inequality (Theorem 10, *Convex Function, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*),

$$\int_X |f| \, d\varphi = \|f \cdot 1\|_{1,\varphi} \le \|1\|_{2,\varphi} \|f\|_{2,\varphi} = \left(\int_X 1 d\varphi\right)^{\frac{1}{2}} \left(\int_X |f|^2 \, d\varphi\right)^{\frac{1}{2}} = (\varphi(X))^{\frac{1}{2}} \left(\int_X |f|^2 \, d\varphi\right)^{\frac{1}{2}} < \infty ,$$

as $\varphi(X) = \lambda(X) + \mu(X) < \infty$, since λ and μ are bounded measures and $\left(\int_X |f|^2 \, d\varphi\right)^{\frac{1}{2}} < \infty$ for $f \in H$. Therefore, $\left|\int_X f \, d\lambda\right| \le \int_X |f| \, d\lambda \le \int_X |f| \, d\varphi < \infty$ since μ is positive. Hence, $\int_X f \, d\lambda$ exists.

Moreover,

$$|\Lambda(f)| = \left|\int_X f \, d\lambda\right| \le \int_X |f| \, d\lambda \le \int_X |f| \, d\varphi ,$$

since μ is positive and $\int_X |f| \, d\varphi = \int_X |f| \, d\lambda + \int_X |f| \, d\mu$,

$$\le (\varphi(X))^{\frac{1}{2}} \left(\int_X |f|^2 \, d\varphi\right)^{\frac{1}{2}} = (\varphi(X))^{\frac{1}{2}} \|f\|_{2,\varphi} .$$

Hence, $\|\Lambda\| \le (\varphi(X))^{\frac{1}{2}}$ and Λ is a bounded complex linear functional.

Since *H* is a Hilbert space, there exists $g \in H$, *g* is unique almost everywhere with respect to φ such that $\Lambda(f) = \langle f, g \rangle_\varphi$ for all *f* in *H*. That is,

$$\int_X f \, d\lambda = \int_X f \, \overline{g} \, d\varphi . \quad \text{---------------------------} \quad (*)$$

We shall next show that g is real and unique almost everywhere with respect to φ and that $0 \le g \le 1$.

Now, for $E \in \mathscr{M}$, $0 \le \int_X \chi_E d\lambda = \lambda(E)$. Substitute $f = \chi_E$ in (*), we get

$$0 \le \int_X \chi_E d\lambda = \int_X \chi_E \overline{g} \, d\varphi = \lambda(E) \le \varphi(E) .$$

Hence, if we take $E \in \mathscr{M}$ such that $\varphi(E) > 0$, then

$$0 \le \frac{1}{\varphi(E)} \int_E \overline{g} \, d\varphi = \frac{1}{\varphi(E)} \int_X \chi_E \overline{g} \, d\varphi \le 1 .$$

Therefore, $\operatorname{Im}\frac{1}{\varphi(E)}\int_E \bar{g}\,d\varphi = 0$. Hence, $\int_E \operatorname{Im}\bar{g}\,d\varphi = 0$ for all $E\in\mathscr{M}$. Therefore, $\operatorname{Im} g = \operatorname{Im}\bar{g} = 0$ almost everywhere with respect to φ. Thus, g is real almost everywhere with respect to φ and for all $E\in\mathscr{M}$ such that $\varphi(E) > 0$,

$$0 \le \frac{1}{\varphi(E)}\int_E g d\varphi \le 1 .$$

This means $\left|\frac{1}{\varphi(E)}\int_E g d\varphi - \frac{1}{2}\right| \le \frac{1}{2}$. Therefore, by Lemma 9*, $\left|g - \frac{1}{2}\right| \le \frac{1}{2}$. Since g is real almost everywhere with respect to φ, $0 \le g \le 1$ almost everywhere with respect to φ.

We now redefine g to take the value 0, where g is not real and where $0 \le g(x) \le 1$ does not hold. This function is obviously equal to g almost everywhere with respect to φ. We shall now assume that g is real, $0 \le g(x) \le 1$ and $g \in H = L^2(X,\varphi)$. If $f \in H = L^2(X,\varphi)$, then as $\varphi = \lambda + \mu$, λ and μ are bounded positive measures, $f, 1-g \in H = L^2(X,\lambda)$ and by the Hölders Inequality, $(1-g) f \in L^1(X,\lambda)$ and

$$\int_X (1-g) f\, d\lambda = \int_X f\, d\lambda - \int_X g\, f\, d\lambda = \Lambda(f) - \int_X g\, f\, d\lambda$$

$$= \int_X f\, g\, d\varphi - \int_X g\, f\, d\lambda$$

$= \int_X f\, g\, d\mu$, as $\int_X f\, g\, d\varphi = \int_X f\, g\, d\lambda + \int_X f\, g\, d\mu$.

Hence, we have for all $f \in H = L^2(X,\varphi)$,

$$\int_X (1-g) f\, d\lambda = \int_X f\, g\, d\mu . \quad \text{-----------------------------} \quad (**)$$

Note that $f, g \in H = L^2(X,\varphi) \Rightarrow f, g \in L^2(X,\mu) \Rightarrow f\, g \in L^1(X,\mu)$.

Let $A = \{x \in X : 0 \le g(x) < 1\}$ and $S = \{x \in X : g(x) = 1\}$. Then A and S are measurable, $X = A \cup S$ and $A \cap S = \varnothing$.

Let $\lambda_a(E) = \lambda(A \cap E)$ and $\lambda_s(E) = \lambda(S \cap E)$ for all $E \in \mathscr{M}$. Then plainly, λ_a is concentrated on A and λ_s is concentrated on S. Thus $\lambda_a \perp \lambda_s$.

Put $f = \chi_S$ in (**),we get

$$\int_X (1-g)\chi_S d\lambda = \int_X \chi_S g\, d\mu .$$

Since $X = A \cup S$, $1-g=0$ on S and $\chi_S = 0$ on A, we have then

$$0 = \int_X (1-g)\chi_S d\lambda = \int_X \chi_S g\, d\mu = \int_S g\, d\mu = \int_S d\mu = \mu(S) .$$

It follows that $\lambda_s \perp \mu$.

Note that g is bounded and so g^n is bounded for any integer $n \geq 1$. Since $\varphi(X) < \infty$, $g^n \in H = L^2(X,\varphi)$. Now, putting $f = (1+g+g^2+\cdots+g^n)\chi_E$ in (**), we get,

$$\int_E (1-g)(1+g+g^2+\cdots+g^n) d\lambda = \int_E (1+g+g^2+\cdots+g^n) g\, d\mu ,$$

i.e.,

$$\int_E (1-g^{n+1}) d\lambda = \int_E (g+g^2+\cdots+g^{n+1})\, d\mu . \quad \text{------------------------} \quad (***)$$

If $x \in S$, then $g(x)=1$ so that $1-g^{n+1}(x)=0$. If $x \in A$, then $0 \leq g(x) < 1$ and so $g^n(x) \searrow 0$ on A. Therefore, by the Lebesgue Monotone Convergence Theorem,

$$\int_E (1-g^{n+1}) d\lambda = \int_E \chi_A (1-g^{n+1}) d\lambda \to \int_E \chi_A d\lambda = \lambda(E \cap A) = \lambda_a(E) \text{, as } n \to \infty,$$

for any $E \in \mathscr{M}$.

The integrand on the right hand side of (***), $g+g^2+\cdots+g^{n+1}$, increases monotonically to some function h pointwise and h is non-negative. So, as h is a pointwise limit of an increasing sequence of non-negative measurable functions, h is measurable and by the Lebesgue Monotone Convergence Theorem,

$$\int_E (g+g^2+\cdots+g^{n+1})\, d\mu \nearrow \int_E h\, d\mu \text{ as } n \to \infty.$$

Therefore, $\lambda_a(E) = \int_E h\, d\mu$ for any $E \in \mathscr{M}$.

If $E = X$, then $\lambda_a(X) = \int_X h\, d\mu = \int_X |h|\, d\mu$ and as $\lambda_a(X) = \lambda\left(A \cap X\right) = \lambda(A) < \infty$, $\int_X |h|\, d\mu < \infty$. Hence, $h \in L^1(X,\mu)$. Plainly, if $\mu(E) = 0$, then

$$\lambda_a(E) = \int_E h\, d\mu = 0 .$$

Hence, $\lambda_a \ll \mu$.

This completes the proof of Theorem 8.

We now discuss the various extensions of Theorem 8.

Various Extensions of the Radon Nikodym Theorem

(1) To where λ is positive and bounded but μ is positive and σ-finite. For example, μ may be the Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^k$ as $\mathbb{R}$ or $\mathbb{R}^k$ is σ-compact for $\mathbb{R}$ or $\mathbb{R}^k$ is a countable union of sets of finite Lebesgue measure.

We write $X = \bigcup_{n=1}^{\infty} X_n$, where $\mu(X_n) < \infty$. We may suppose that the countable family $\{X_n\}$ are pairwise disjoint. If not, we may replace X_n by $X_n - \bigcup_{i=1}^{n-1} X_i$. Then apply the theorem to each X_n . We get, $\lambda_{a,n}, \lambda_{s,n}$ and h_n each defined on X_n .

Extend the definition to X trivially. Define value to be zero on $X - X_n$. Then splice together so that $h(x) = h_n(x)$ if $x \in X_n$. Since $0 \le h_n \le 1$, $0 \le h \le 1$. As $\lambda(X) < \infty$, $h \in L^1(X, \mu)$.

(2) To when μ is positive and bounded but λ is real.

Write $\lambda = \lambda^+ - \lambda^-$, $\lambda^+ = \frac{1}{2}(|\lambda| + \lambda)$ and $\lambda^- = \frac{1}{2}(|\lambda| - \lambda)$. Then λ^+ and λ^- are positive and bounded measures. Applying the theorem to the positive and negative parts of λ, we get $\lambda_a^+, \lambda_s^+, h^+$ and $\lambda_a^-, \lambda_s^-, h^-$. Then let $\lambda_a = \lambda_a^+ - \lambda_a^-$ and $\lambda_s = \lambda_s^+ - \lambda_s^-$. Then $\lambda = \lambda_a + \lambda_s$. As $\lambda_a^+ \ll \mu$ and $\lambda_a^- \ll \mu$, by property (e), $\lambda_a = \lambda_a^+ - \lambda_a^- \ll \mu$. Also, as $\lambda_s^+ \perp \mu$ and $\lambda_s^- \perp \mu$, $\lambda_s = \lambda_s^+ - \lambda_s^- \perp \mu$ by property (d). Thus, by property (g), $\lambda_a \perp \lambda_s$.

Now, for $E \in \mathscr{M}$, $\lambda_a^+(E) = \int_E h^+ d\mu$, $\lambda_a^-(E) = \int_E h^- d\mu$ so that $\lambda_a(E) = \lambda_a^+(E) - \lambda_a^-(E) = \int_E (h^+ - h^-) d\mu$. Let $h = h^+ - h^-$.

(3) To when μ is positive and bounded and λ is complex.

Let λ_R be the real part of λ and λ_I be the imaginary part of λ. Now apply part (2) and spliced together similarly.

(4) To when λ is real but μ is positive and σ-finite.

Write $\lambda = \lambda^+ - \lambda^-$, $\lambda^+ = \frac{1}{2}(|\lambda| + \lambda)$ and $\lambda^- = \frac{1}{2}(|\lambda| - \lambda)$. Then λ^+ and λ^- are positive and bounded measures. Apply extension (1) to λ^+ and λ^- to get $\lambda_a^+, \lambda_s^+, h^+$ and $\lambda_a^-, \lambda_s^-, h^-$. Then splice together as in (2) as follows. Let $\lambda_a = \lambda_a^+ - \lambda_a^-$ and $\lambda_s = \lambda_s^+ - \lambda_s^-$. Then $\lambda = \lambda_a + \lambda_s$. As $\lambda_a^+ \ll \mu$ and $\lambda_a^- \ll \mu$, by property (e), $\lambda_a = \lambda_a^+ - \lambda_a^- \ll \mu$. Also, as $\lambda_s^+ \perp \mu$ and $\lambda_s^- \perp \mu$, $\lambda_s = \lambda_s^+ - \lambda_s^- \perp \mu$ by property (d). Thus, by property (g), $\lambda_a \perp \lambda_s$. Now, for $E \in \mathscr{M}$, $\lambda_a^+(E) = \int_E h^+ d\mu$, $\lambda_a^-(E) = \int_E h^- d\mu$ so that $\lambda_a(E) = \lambda_a^+(E) - \lambda_a^-(E) = \int_E (h^+ - h^-) d\mu$. Let $h = h^+ - h^-$.

(5) To when λ is complex but μ is positive and σ-finite. Write $\lambda = \operatorname{Re}\lambda + i \operatorname{Im}\lambda$. Then $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$ are real measures. Apply extension (4) to $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$ separately and then splice together.

(6) To when both λ and μ are positive and σ-finite.

Write $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}$ are pairwise disjoint, $\lambda(X_n) < \infty$, $\mu(X_n) < \infty$ for each integer $n \geq 1$. Applying (1), we get $\lambda_{a,n}, \lambda_{s,n}$ and h_n each defined on X_n. Then splice together as in (1). We obtain $h(x) = h_n(x)$ for $x \in X_n$. But we cannot conclude that $h \in L^1(X, \mu)$, we can only assert that $h|_{X_n} \in L^1(X_n, \mu)$, that is, h is locally in $L^1(X, \mu)$.

(7) To when μ is complex and λ is a bounded positive measure.

We need to define $\int_X f\, d\mu$ for complex measure. Once defined, the extension is immediate.

If μ is a complex measure on the measure space $(X, \mathscr{M})$, then its total variation measure, $|\mu|$, by Proposition 2, is a bounded or finite positive measure. Note that for E in $\mathscr{M}$, $\mu(E) = 0 \Leftrightarrow |\mu|(E) = 0$. Following Definition 5, we say λ is *absolutely continuous* with respect to μ if $\mu(E) = 0 \Rightarrow \lambda(E) = 0$. This is equivalent to λ is absolutely continuous with respect to $|\mu|$. Thus, for a complex measure, $\lambda \ll \mu \Leftrightarrow \lambda \ll |\mu|$. Similarly, we say $\lambda \perp \mu$, i.e., if λ is concentrated on A with $\mu(A) = 0$. This is equivalent to $\lambda \perp |\mu|$.

Thus, if λ is a bounded positive measure and μ is a complex measure, then by Theorem 8, $\lambda = \lambda_a + \lambda_s$, where $\lambda_a \ll |\mu|$, $\lambda_s \perp |\mu|$ and there exists $h_1 \in L^1(X, |\mu|)$

such that $\lambda_a(E) = \int_E h_1 d|\mu|$. Thus, $\lambda_a \ll \mu$ and $\lambda_s \perp \mu$. By the polar decomposition of μ (see Theorem 10 below), there exists $h \in L^1(X, |\mu|)$ such that $|h| = 1$ and $d\mu = hd|\mu|$. Therefore,

$$\lambda_a(E) = \int_E h_1 d|\mu| = \int_E h_1 \left(\bar{h}h\right) d|\mu| = \int_E \left(h_1 \bar{h}\right) hd|\mu| = \int_E g d\mu ,$$

where $g = h_1 \bar{h}$. The function $h_1 \bar{h} \in L^1(X, |\mu|)$ since h is bounded and so $g \in L^1(X, \mu)$. (See Definition 11 below.)

Some applications of the Radon-Nikodym Theorem

Observe that for a complex number *z*, we can write $z = |z| e^{i\theta}$, where $\left|e^{i\theta}\right| = 1$. For a complex function, f, we can similarly write $f = |f| h$, where $|h| = 1$. For an $n \times n$ complex matrix *A*, we can write $A = UR$, where *U* is unitary and *R* is positive semi-definite Hermitian. For a complex measure μ, we can write, as we shall show later,

$$d\mu = h\, d|\mu| ,$$

where $|h| = 1$.

All these representations are known as *polar decomposition* in analogy with the polar representation of complex numbers.

The Polar Decomposition of a Complex Measure

Theorem 10 (Polar Decomposition). If μ is a complex measure on the measure space $(X, \mathscr{M})$, then there exists a measurable complex function $h : X \to \mathbb{C}$ such that $h \in L^1(X, |\mu|)$, $|h| = 1$ and $d\mu = h\, d|\mu|$. More precisely, for any $E \in \mathscr{M}$,

$$\mu(E) = \int_E h\, d|\mu| .$$

Proof. Plainly, $\mu \ll |\mu|$. By Proposition 2, $|\mu|$ is a bounded positive measure. Therefore, by the Radon-Nikodym Theorem (Theorem 8, part (b)), there exists

$h \in L^1(X,|\mu|)$ such that $\mu(E) = \int_E h\, d|\mu|$. (We are using the extension (3) of Radon-Nikodym Theorem, discussed above.)

We may prove it directly here.

Note that $\operatorname{Re}\mu \ll |\mu|$ and $\operatorname{Im}\mu \ll |\mu|$. Therefore, by Theorem 8, there exist $h_{\mathrm{Re}} \in L^1(X,|\mu|)$ and $h_{\mathrm{Im}} \in L^1(X,|\mu|)$ such that for all $E \in \mathscr{M}$,

$$\operatorname{Re}\mu(E) = \int_E h_{\mathrm{Re}} d|\mu| \text{ and } \operatorname{Im}\mu(E) = \int_E h_{\mathrm{Im}} d|\mu|.$$

Therefore,

$$\mu(E) = \operatorname{Re}\mu(E) + i\operatorname{Im}\mu(E) = \int_E (h_{\mathrm{Re}} + i\, h_{IM})\, d|\mu| = \int_E hd|\mu|,$$

where $h = h_{\mathrm{Re}} + i\, h_{IM}$. Note that h is measurable and $h \in L^1(X,|\mu|)$.

Now we show that $|h| = 1$ almost everywhere with respect to $|\mu|$.

Firstly, we show that $|h| \geq 1$ almost everywhere with respect to $|\mu|$.

Let $A = \{x \in X : |h(x)| < r\}$, for $0 < r < 1$.

Let $\{A_i\}$ be a partition of A by disjoint sets in $\mathscr{M}$. Then

$$\sum_i |\mu(A_i)| = \sum_i \left|\int_{A_i} hd|\mu|\right| \leq \sum_i \int_{A_i} |h|\, d|\mu| < r\sum_i \int_{A_i} d|\mu| = r\sum_i |\mu|(A_i) = r|\mu|(A).$$

Therefore, $|\mu|(A) \leq r|\mu|(A)$ for $0 < r < 1$. Hence, $|\mu|(A) = 0$ for $0 < r < 1$. This implies that $|h| \geq 1$ almost everywhere with respect to $|\mu|$.

Next, we show that $|h| \leq 1$ almost everywhere with respect to $|\mu|$.

Suppose $E \in \mathscr{M}$ and $|\mu|(E) > 0$. As $\int_E hd|\mu| = \mu(E)$, $\left|\int_E hd|\mu|\right| = |\mu(E)| \leq |\mu|(E)$. Therefore,

$$\frac{1}{|\mu|(E)}\left|\int_E hd|\mu|\right| = \frac{|\mu(E)|}{|\mu|(E)} \leq 1.$$

It follows then by Lemma 9, that $0 \leq |h| \leq 1$ almost everywhere with respect to $|\mu|$. Hence, $|h| = 1$ almost everywhere with respect to $|\mu|$. Thus, $\mu\{x \in X : |h| \neq 1\} = 0$.

Redefine h so that on this set $\{x \in X : |h| \neq 1\}$, $h(x) = 1$. Then we have $|h(x)| = 1$ for all x in X.

We shall now proceed to define integration over a complex measure.

For a measurable simple function, $s = \sum_{i=1}^{n} a_i \chi_{E_i}$, where E_i is measurable and a_i is a complex number for $1 \leq i \leq n$,

$$\int_X s\, d\mu = \sum_{i=1}^{n} a_i \mu(E_i) = \sum_{i=1}^{n} a_i \int_{E_i} h\, d|\mu| = \sum_{i=1}^{n} a_i \int_X \chi_{E_i} d\mu$$

$$= \sum_{i=1}^{n} a_i \int_X \chi_{E_i} h\, d|\mu| = \int_X \left(\sum_{i=1}^{n} a_i \chi_{E_i} \right) h\, d|\mu|$$

$$= \int_X s\, h\, d|\mu| .$$

If f is real valued, non-negative and measurable, then there exists an increasing sequence of measurable non-negative simple functions $\{s_n\}$ such that $s_n \nearrow f$.

We can write $h = \operatorname{Re} h + i \operatorname{Im} h$, $\operatorname{Re} h = (\operatorname{Re} h)^+ - (\operatorname{Re} h)^-$ and $\operatorname{Im} h = (\operatorname{Im} h)^+ - (\operatorname{Im} h)^-$. It then follows from the Lebesgue Monotone Convergence Theorem that

$\int_X s_n (\operatorname{Re} h)^+ d|\mu| \nearrow \int_X f (\operatorname{Re} h)^+ d|\mu|$, $\int_X s_n (\operatorname{Re} h)^- d|\mu| \nearrow \int_X f (\operatorname{Re} h)^- d|\mu|$,
$\int_X s_n (\operatorname{Im} h)^+ d|\mu| \nearrow \int_X f (\operatorname{Im} h)^+ d|\mu|$, and $\int_X s_n (\operatorname{Im} h)^- d|\mu| \nearrow \int_X f (\operatorname{Im} h)^- d|\mu|$.

So for a non-negative function f, we say $\int_X f d\mu$ exists if $\int_X f (\operatorname{Re} h) d|\mu|$ and $\int_X f (\operatorname{Im} h) d|\mu|$ exist, i.e., $\int_X f\, d\mu = \int_X f\, h d|\mu|$. For a real value measurable function, we can write $f = f^+ - f^-$ and define $\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$ and finally for measurable complex function f, $\int_X f d\mu = \int_X \operatorname{Re} f d\mu + i \int_X \operatorname{Im} f d\mu$.

Definition 11. In summary, we may define for f a complex measurable function, μ a complex measure,

$$\int_X f d\mu = \int_X f\, h d|\mu| .$$

So, $f \in L^1(X,\mu)$, if and only if, $\int_X |\mathrm{Re}\, f||h| d|\mu| < \infty$ and $\int_X |\mathrm{Im}\, f||h| d|\mu| < \infty$, if and only if, $\int_X |f| d|\mu| < \infty$.

If λ and μ are complex measures, the relation,

$$\int_X f\, d(\lambda+\mu) = \int_X f\, d\lambda + \int_X f\, d\mu \text{, ------------------------ (*)}$$

holds whenever f is a bounded measurable function or when $f \in L^1(X,|\lambda|)$ and $f \in L^1(X,|\mu|)$.

Plainly, (*) holds for f a measurable characteristic function. This is because

$$\int_X \chi_E\, d(\lambda+\mu) = (\lambda+\mu)(E) = \lambda(E)+\mu(E) = \int_X \chi_E\, d\lambda + \int_X \chi_E\, d\mu$$

Hence, (*) is true for measurable simple functions. Then (*) holds for any bounded measurable function f. Evidently, if $f \in L^1(X,|\lambda|)$ and $f \in L^1(X,|\mu|)$, then $f \in L^1(X,|\lambda+\mu|)$ and so (*) holds.

We may define complex measure by using any fixed complex function f in $L^1(X,\mu)$, where μ is a positive measure on $\mathscr{M}$.

Proposition 12. Suppose μ is a positive measure on the measure space $(X, \mathscr{M})$ and $f \in L^1(X,\mu) = \left\{ f : X \to \mathbb{C}; \int_X |f| d\mu < \infty \right\}$. If $d\lambda = f\, d\mu$, then $d|\lambda| = |f| d\mu$. That is, if $\lambda(E) = \int_E f\, d\mu$, then $|\lambda|(E) = \int_E |f| d\mu$.

Proof. It is easy to show that λ is a complex measure on $\mathscr{M}$. By Theorem 10 (Polar Decomposition of Complex Measure), there exists a measurable function, $h: X \to \mathbb{C}$, such that $h \in L^1(X,|\lambda|)$, $|h|=1$ and $d\lambda = h\, d|\lambda|$. More precisely, for any $E \in \mathscr{M}$,

$$\lambda(E) = \int_E h\, d|\lambda| \ .$$

By hypothesis, $\lambda(E) = \int_E f\, d\mu$. Now for a characteristic function χ_E, where E is measurable, $\int_X \chi_E d\lambda = \lambda(E) = \int_X \chi_E f\, d\mu$. Therefore, for a measurable simple function s, $\int_X s d\lambda = \int_X s f d\mu$. It follows that for a bounded measurable function g, $\int_X g d\lambda = \int_X g f d\mu$. Since h is a bounded measurable function, $\int_X \bar{h} d\lambda = \int_X \bar{h} f d\mu$. Hence, for any measurable $E \in \mathscr{M}$, $\int_E \bar{h} d\lambda = \int_E \bar{h} f d\mu$. Similarly, by using $\lambda(E) = \int_E h d|\lambda|$, we get

$$\int_E \bar{h} d\lambda = \int_E \bar{h} h d|\lambda| = \int_E d|\lambda| = |\lambda|(E) \geq 0 .$$

It follows that $\int_E \bar{h} f d\mu = |\lambda|(E) \geq 0$ for all $E \in \mathscr{M}$. Therefore, $\bar{h} f \geq 0$ almost everywhere with respect to μ. Therefore, $\bar{h} f = |\bar{h} f| = |f|$ almost everywhere with respect to μ. Hence, $|\lambda|(E) = \int_E |f| d\mu$ for any $E \in \mathscr{M}$.

This completes the proof.

Now we use Theorem 10 for a real measure on (X, $\mathscr{M}$).

Theorem 13. Hahn-Jordan Decomposition Theorem.

Let μ be a real measure on (X, $\mathscr{M}$).

(Jordan) Write $\mu^+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu^- = \frac{1}{2}(|\mu| - \mu)$, where $|\mu|$ is the total variation measure of μ. Then $\mu = \mu^+ - \mu^-$, μ^+ and μ^- are bounded positive measure.

(Hahn) $\mu^+ \perp \mu^-$.

That is, $X = A^+ \cup A^-$ with $A^+ \cap A^- = \varnothing$ and for any $E \in \mathscr{M}$,

$$\mu^+(E) = \mu(E \cap A^+) \text{ and } \mu^-(E) = \mu(E \cap A^-).$$

Proof. By Theorem 10 (Polar Decomposition), there exists a measurable function, $h : X \to \mathbb{C}$, such that $h \in L^1(X, |\mu|)$, $|h| = 1$ and $d\mu = h d|\mu|$. More precisely, for any $E \in \mathscr{M}$,

$$\mu(E) = \int_E h d|\mu| .$$

Since μ is real, we may assume that h is real. Hence, $h = \pm 1$. Let $A^+ = \{x \in X : h(x) = 1\}$ and $A^- = \{x \in X : h(x) = -1\}$. Then A^+ and A^- are measurable. Plainly, $A^+ \cap A^- = \varnothing$. Since $\pm\mu \le |\mu|$ and $|\mu|$ is a bounded positive measure, by Proposition 2, μ^+ and μ^- are bounded positive measures on $\mathscr{M}$ and $\mu = \mu^+ - \mu^-$.

Now, $\frac{1}{2}(1+h) = \begin{cases} h(x), x \in A^+ \\ 0, x \in A^- \end{cases}$ and $\frac{1}{2}(1-h) = \begin{cases} 0, x \in A^+ \\ -h(x), x \in A^- \end{cases}$.

If $E \in \mathscr{M}$,

$$\mu^+(E) = \int_E \frac{1}{2} d\left(|\mu| + \mu\right) = \int_E \frac{1}{2}(1+h)d|\mu| = \int_{E\cap A^+} hd|\mu| = \int_{E\cap A^+} d\mu = \mu(E \cap A^+).$$

Similarly,

$$\mu^-(E) = \int_E \frac{1}{2} d\left(|\mu| - \mu\right) = \int_E \frac{1}{2}(1-h)d|\mu| = \int_{E\cap A^-} -hd|\mu| = \int_{E\cap A^-} d\mu = \mu(E \cap A^-).$$

Therefore, μ^+ is concentrated on A^+ and μ^- is concentrated on A^- and so $\mu^+ \perp \mu^-$.

Next, we show that the Jordan Decomposition is optimal in the following sense.

Corollary 14. Let μ be a real measure on $(X, \mathscr{M})$. Suppose $\mu = \lambda_1 - \lambda_2$, where λ_1 and λ_2 are positive measures. Then $\lambda_1 \ge \mu^+$ and $\lambda_2 \ge \mu^-$.

Proof.

Recall that $\mu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$.

For any $E \in \mathscr{M}$, $|\mu(E)| = |\lambda_1(E) - \lambda_2(E)| \le |\lambda_1(E)| + |\lambda_2(E)| = \lambda_1(E) + \lambda_2(E)$.

Therefore, for any partition, $\{E_i\}$, of E by disjoint sets in $\mathscr{M}$,

$$\sum_i |\mu(E_i)| \le \sum_i \lambda_1(E_i) + \sum_i \lambda_2(E_i) = \lambda_1(E) + \lambda_2(E).$$

Hence, by definition of $|\mu|$, $|\mu|(E) \le \lambda_1(E) + \lambda_2(E)$ for all $E \in \mathscr{M}$. Now, $\mu = \mu^+ - \mu^- = \lambda_1 - \lambda_2$ so that $\lambda_1 - \mu^+ = \lambda_2 - \mu^-$.

Observe that $\mu^+(E) = \frac{1}{2}\left(|\mu|(E) + \mu(E)\right) = \frac{1}{2}\left(|\mu|(E) + \lambda_1(E) - \lambda_2(E)\right)$ so that

$$2\mu^+(E) = |\mu|(E) + \lambda_1(E) - \lambda_2(E) \le 2\lambda_1(E).$$

Thus, $\lambda_1(E) \ge \mu^+(E)$ for all $E \in \mathscr{M}$. This means $\lambda_1 \ge \mu^+$. Now, $\lambda_2 - \mu^- = \lambda_1 - \mu^+ \ge 0$ and so $\lambda_2 \ge \mu^-$.

The Dual Space or Conjugate Space of $L^p(X, \mu)$

Suppose $(X, \mathscr{M})$ is a measure space and $\mu: \mathscr{M} \to \overline{\mathbb{R}^+}$ is a positive measure.

Suppose $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$, where $1 < p < \infty$ and $1 < q < \infty$ are conjugate indices such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the Hölders inequality (see Theorem 10, *Convex Function, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*),

$$\|f\,g\|_{1,\mu} \le \|f\|_{p,\mu} \|g\|_{q,\mu} < \infty,$$

where $\|h\|_{n,\mu}$ is the $L^n(X, \mu)$ norm given by $\|h\|_{n,\mu} = \left(\int_X |h|^n \, d\mu\right)^{\frac{1}{n}}$ for $n \ge 1$.
Therefore, $f\,g \in L^1(X, \mu)$.

Define $\Phi_g : L^p(X, \mu) \to \mathbb{C}$ by $\Phi_g(f) = \int_X f\,g d\mu$. Then for any $f \in L^p(X, \mu)$

$$|\Phi_g(f)| = \left|\int_X f\,g d\mu\right| \le \int_X |f\,g| \, d\mu \le \|g\|_{q,\mu} \|f\|_{p,\mu}.$$

Hence, Φ_g is a bounded complex linear functional on $L^p(X, \mu)$ and the norm of this linear functional, $\|\Phi_g\| \le \|g\|_{q,\mu}$. Recall that for a linear functional $\Phi : V \to \mathbb{C}$ on a norm space V with norm $\|\cdot\|$, $\|\Phi\| = \sup\left\{\frac{|\Phi(x)|}{\|x\|} : x \in V, x \ne 0\right\}$.

We investigate if the converse is true. Is any bounded complex linear functional $\Phi : L^p(X, \mu) \to \mathbb{C}$ expressible as $\Phi = \Phi_g$ for some $g \in L^q(X, \mu)$?

One case is clear. Take $p = q = 2$ and we know $L^2(X, \mu)$ is a Hilbert pace with inner product $\langle f, g \rangle = \int_X f \cdot \overline{g} \, d\varphi$. By a non-measure theoretic argument, if $\Phi : H \to \mathbb{C}$ is a bounded linear functional on a Hilbert space *H*, then there exists $y \in H$ such that $\Phi(x) = \langle x, y \rangle$ for all *x* in *H*. As $H = L^2(X, \mu)$ is a Hilbert space, for a linear functional $\Phi : L^2(X,\mu) \to \mathbb{C}$, there exists a function $g \in L^2(X,\mu)$, unique almost everywhere with respect to μ, such that $\Phi = \Phi_g$.

One case is false. This is the case when $p = \infty$. For $L^\infty(X,\mu)$, the answer is false because $L^1(X,\mu)$ does not furnish all bounded linear functionals on $L^\infty(X,\mu)$. (See Example 25.16, *Principle of Real Analysis, Aliprantis and Burkinshaw.*)

If $1 < p < \infty$, the answer is always 'yes'. However, we will prove this together with the case p =1 with the additional hypothesis that the measure μ be σ-finite. Subsequently, we shall prove the case for $1 < p < \infty$, without the σ-finiteness condition on the measure μ. One case is "usually" yes, except in very big spaces (i.e., where open sets are not σ-finite), for p =1.

We note that for $g \in L^1(X,\mu)$, we can define a bounded complex linear functional $\Phi_g : L^\infty(X,\mu) \to \mathbb{C}$ by $\Phi_g(f) = \int_X f\, g d\mu$ for $f \in L^\infty(X,\mu)$.

This is because since $f \in L^\infty(X,\mu)$, there exists a set B of μ-measure zero such that $|f(x)| \le \|f\|_{\infty,\mu} < \infty$ for all $x \in B^c$ so that $|f\, g| \le \|f\|_{\infty,\mu} |g|$ almost everywhere with respect to μ and so $f\, g \in L^1(X,\mu)$. Thus,

$$|\Phi_g(f)| = \left|\int_X f\, g d\mu\right| \le \int_X |f\, g|\, d\mu \le \int_X |g| \|f\|_{\infty,\mu}\, d\mu$$

$$\le \|f\|_{\infty,\mu} \int_X |g|\, d\mu = \|g\|_{1,\mu} \|f\|_{\infty,\mu}.$$

Therefore, by the definition of $\|\Phi_g\|$, $\|\Phi_g\| \le \|g\|_{1,\mu} < \infty$. Hence, Φ_g is a bounded complex linear functional. Next, we shall show that $\|\Phi_g\| = \|g\|_{1,\mu}$.

Since g is measurable, there exists a measurable function h such that $|h| = 1$ and $g = |g| h$ so that $\bar{h} g = |g|$. Indeed, we can define h as follows. Let $B_0 = \{x \in X : g(x) = 0\}$. Then B_0 and $V = B_0^c$ are measurable since g is measurable. Let $\phi : \mathbb{C} - \{0\} \to \mathbb{C}$ be defined by $\phi(z) = \frac{z}{|z|}$. Then ϕ is continuous on $\mathbb{C} - \{0\}$. Let $g_1 = g + \chi_{B_0}$. Then g_1 is measurable and $g_1 \ne 0$. Define $h = \phi \circ g_1$, i.e.,

$h(x) = \phi \circ g_1(x) = \phi(g(x) + \chi_E(x))$. The function h is measurable because ϕ is continuous on $\mathbb{C} - \{0\}$. In particular, $|h| = 1$. Moreover,

$$|g|(x)h(x) = |g|(x)\frac{g(x)+\chi_{B_0}(x)}{|g(x)+\chi_{B_0}(x)|} = \begin{cases} |g|(x)\dfrac{g(x)}{|g(x)|} = g(x), x \in V \\ |g|(x)\cdot\dfrac{1}{1} = 0, x \in B_0 \end{cases} = g(x).$$

Let $f(x) = \overline{h}(x)$. Plainly, f is measurable and $|f| = 1$ and so it is essentially bounded and $\|f\|_{\infty,\mu} = 1$. Therefore,

$$\Phi_g(f) = \int_X f\, g d\mu = \int_X \overline{h}\, g d\mu = \int_X |g|\, d\mu = \|g\|_{1,\mu}.$$

Hence, $|\Phi_g(f)| = \|g\|_{1,\mu} = \|g\|_{1,\mu}\|f\|_{\infty,\mu}$. Therefore, $\|\Phi_g\| = \|g\|_{1,\mu}$ as $\|\Phi_g\| = \sup\left\{|\Phi_g(f)| : \|f\|_{\infty,\mu} = 1, f \in L^\infty(X,\mu)\right\}$.

Likewise, if $g \in L^\infty(X,\mu)$, then the complex linear functional $\Phi_g : L^1(X,\mu) \to \mathbb{C}$ defined by $\Phi_g(f) = \int_X f\, g d\mu$ for $f \in L^1(X,\mu)$ is a bounded linear functional as

$$|\Phi_g(f)| = \left|\int_X f\, g d\mu\right| \le \int_X |f\, g|\, d\mu \le \int_X |f|\|g\|_{\infty,\mu}\, d\mu \le \|g\|_{\infty,\mu}\|f\|_{1,\mu},$$

so that $\|\Phi_g\| \le \|g\|_{\infty,\mu} < \infty$.

For any $E \in \mathscr{M}$, $\Phi_g(\chi_E) = \int_X \chi_E g d\mu = \int_E g d\mu$ and

$$\left|\int_E g d\mu\right| = \|\Phi_g(\chi_E)\| \le \|\Phi_g\|\|\chi_E\|_{1,\mu} = \|\Phi_g\|\mu(E).$$

Hence, for $\mu(E) \ne 0$, $\frac{1}{\mu(E)}\left|\int_E g d\mu\right| \le \|\Phi_g\|$. Then by Lemma 9, with closed disk of radius $\|\Phi_g\|$, we conclude that $|g| \le \|\Phi_g\|$ almost everywhere with respect to μ. Consequently, $\|g\|_{\infty,\mu} \le \|\Phi_g\|$. Therefore, $\|\Phi_g\| = \|g\|_{\infty,\mu}$.

Now, we assume that $1 < p < \infty$. Take $g \in L^q(X,\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$. We have already shown that $\|\Phi_g\| \le \|g\|_{q,\mu}$, We now show by a similar argument as above that $\|\Phi_g\| = \|g\|_{q,\mu}$. Take $f = |g|^{q-1}\overline{h}$. Then f is measurable. Moreover, $|f|^p = |g|^{p(q-1)} = |g|^q$ and so $\int_X |f|^p \, d\mu = \int_X |g|^q \, d\mu < \infty$. It follows that $f \in L^p(X,\mu)$.

Now,

$$\Phi_g(f) = \Phi_g(|g|^{q-1}\overline{h}) = \int_X \left(|g|^{q-1}\overline{h}\right) g d\mu = \int_X |g|^q \, d\mu$$

$$= \left(\int_X |g|^q \, d\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q \, d\mu\right)^{\frac{1}{q}} = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q \, d\mu\right)^{\frac{1}{q}} = \|g\|_{q,\mu} \|f\|_{p,\mu}.$$

Hence, $|\Phi_g(f)| = \|g\|_{q,\mu} \|f\|_{p,\mu}$. Thus, if $\|g\|_{q,\mu} \neq 0$, by definition of $\|\Phi_g\|$ and that $\|\Phi_g\| \leq \|g\|_{q,\mu}$, $\|\Phi_g\| = \|g\|_{q,\mu}$. If $\|g\|_{q,\mu} = 0$, then $g = 0$ almost everywhere with respect to μ. Hence, $\Phi_g = 0$ and so $\|\Phi_g\| = \|g\|_{q,\mu} = 0$. In summary, we have the following result.

Theorem 15. Suppose $(X, \mathscr{M})$ is a measure space and $\mu\colon \mathscr{M} \to \overline{\mathbb{R}^+}$ is a positive measure. Suppose $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ are conjugate indices such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $g \in L^q(X,\mu)$, the complex linear functional, $\Phi_g : L^p(X,\mu) \to \mathbb{C}$, defined by $\Phi_g(f) = \int_X f\, g d\mu$, is a bounded complex linear functional such that $\|\Phi_g\| = \|g\|_{q,\mu}$.

For a measure space with a σ-finite measure we have the following representation of bounded complex linear functional.

Theorem 16. Suppose $(X, \mathscr{M})$ is a measure space and $\mu\colon \mathscr{M} \to \overline{\mathbb{R}^+}$ is a σ-finite positive measure. Let $1 \leq p < \infty$ and $1 < q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $\Phi : L^p(X,\mu) \to \mathbb{C}$ is a bounded complex linear functional. Then there exists a unique g in $L^q(X,\mu)$ such that

$$\Phi(f) = \int_X f\, g d\mu = \Phi_g(f),$$

for all $f \in L^p(X,\mu)$. Moreover, $\|\Phi_g\| = \|g\|_{q,\mu}$. More precisely, the dual space of $L^p(X,\mu)$, $\left(L^p(X,\mu)\right)^*$, is isometric isomorphic with $L^q(X,\mu)$, under a Banach space isomorphism preserving norm.

Proof.

The proof is difficult and we shall do it in two steps. The uniqueness part is easy and we shall dispose of this presently.

Suppose g and g' in $L^q(X,\mu)$ are such that they both satisfy the conclusion of the theorem.

Take any $f=\chi_E$ for any E in $\mathscr{M}$ with $\mu(E)<\infty$. Then

$$0=\Phi(f)-\Phi(f)=\int_X \chi_E g\,d\mu-\int_X \chi_E g'\,d\mu=\int_X \chi_E\left(g-g'\right)d\mu=\int_E\left(g-g'\right)d\mu .$$

It follows that for all E in $\mathscr{M}$, $\int_E\left(g-g'\right)d\mu=0$. Hence, $g=g'$ almost everywhere with respect to μ.

By Theorem 15, $\|\Phi\|=\|g\|_{q,\mu}$.

We shall use the functional Φ to define a measure on $\mathscr{M}$. We shall use the Radon Nikodym Theorem.

First of all, if $\|\Phi\|=0$, then we can just take g to be zero almost everywhere with respect to μ. So we assume that $\|\Phi\|>0$.

Step 1. We consider the special case when μ is a finite positive measure, i.e., $\mu(X)<\infty$.

For any E in $\mathscr{M}$, χ_E plainly belongs to $L^p(X,\mu)$, since $\int_X \chi_E d\mu=\mu(E)\le\mu(X)<\infty$.
We define a measure λ on $\mathscr{M}$ by $\lambda(E)=\Phi(\chi_E)$ for E in $\mathscr{M}$.

We check that this defines a measure on $\mathscr{M}$. Trivially, $\lambda(\varnothing)=0$. Plainly λ is finitely additive, for E_1 and $E_2\in\mathscr{M}$ with $E_1\cap E_2=\varnothing$,

$$\lambda(E_1\cup E_2)=\Phi\left(\chi_{E_1\cup E_2}\right)=\Phi\left(\chi_{E_1}+\chi_{E_2}\right)=\Phi\left(\chi_{E_1}\right)+\Phi\left(\chi_{E_2}\right)=\lambda(E_1)+\lambda(E_2).$$

Thus, by induction, we obtain that, if $\{E_i\}_{i=1}^n$ is any finite collection of disjoint measurable sets in $\mathscr{M}$, then $\lambda\left(\bigcup_{i=1}^n E_i\right)=\sum_{i=1}^n\lambda(E_i)$. Now we show that λ is countably additive. Suppose $E=\bigcup_{i=1}^{\infty}E_i$ is a disjoint union of countably infinite members of $\mathscr{M}$. Let $A_n=\bigcup_{i=1}^{n}E_i$. Then

$$\left\|\chi_E-\chi_{A_n}\right\|_{p,\mu}=\left|\int_X\chi_{E-A_n}d\mu\right|^{\frac{1}{p}}=\left|\mu(E-A_n)\right|^{\frac{1}{p}}\to 0 \text{ as } n\to\infty,$$

by the “continuity from below” property of the measure μ. So, since Φ is bounded and so is continuous, $\Phi(\chi_{A_n})\to\Phi(\chi_E)$ as $n\to\infty$, because

$$\left|\Phi\left(\chi_{A_n}\right)-\Phi\left(\chi_E\right)\right|=\left|\Phi\left(\chi_E-\chi_{A_n}\right)\right|\le\|\Phi\|\left\|\chi_E-\chi_{A_n}\right\|_p\to 0 \text{ as } n\to\infty.$$

Hence, $\lambda(A_n)=\Phi\left(\chi_{A_n}\right)=\lambda\left(\bigcup_{i=1}^{n}E_i\right)=\sum_{i=1}^{n}\lambda(E_i)\to\Phi(E)=\lambda\left(\bigcup_{i=1}^{\infty}E_i\right)$. Therefore, $\lambda(E)=\sum_{i=1}^{\infty}\lambda(E_i)$. This proves that λ is a complex measure.

If $E\in\mathscr{M}$ and $\mu(E)=0$, then $\chi_E=0$ almost everywhere with respect to μ. Hence, $\lambda(E)=\Phi(\chi_E)=\Phi(0)=0$. Thus, $\lambda\ll\mu$, i,e., λ is absolutely continuous with respect to μ. Therefore, by the Radon Nikodym Theorem (Theorem 8 Extension (3) for positive finite μ and complex λ), $\lambda=\lambda_a+\lambda_s$, $\lambda_a\ll\mu$, $\lambda_s\perp\mu$ and there exists a measurable function $g\in L^1(X,\mu)$ such that $\lambda_a(E)=\int_E g\,d\mu$. Hence, $\lambda_s=\lambda-\lambda_a\ll\mu$ by Lemma 7 (e) and so as $\lambda_s\perp\mu$, $\lambda_s=0$, by Lemma 7 (h).

Therefore, $\lambda=\lambda_a$ and

$$\lambda(E)=\int_E g\,d\mu\,.$$

It follows that, for any $E\in\mathscr{M}$,

$$\Phi(\chi_E)=\lambda(E)=\int_E gd\mu=\int_X\chi_E gd\mu\,.$$

We shall extend this equality to arbitrary $f\in L^p(X,\mu)$,

$$\Phi(f)=\int_X f\,gd\mu\,. \quad \text{--------------------------------} \quad (*)$$

We have just shown that (*) is true for measurable characteristic functions. Therefore, (*) is true for measurable simple functions. We then claim that (*) holds for every f in $L^\infty(X,\mu)$. Note that (*) holds for non-negative measurable

bounded function, because if f is bounded and non-negative, then there exists an increasing sequence of measurable simple non-negative functions $\{s_n\}$ such that $s_n \nearrow f$. Because f is bounded, $s_n \nearrow f$ uniformly. (See Theorem 17). As $\Phi(s_n) = \int_X s_n \, g d\mu$, by the Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty} \Phi(s_n) = \lim_{n\to\infty} \int_X s_n g d\mu = \int_X f \, g d\mu$. Note that as $s_n \nearrow f$ uniformly, $\|s_n - f\|_{p,\mu} \to 0$ and so $\lim_{n\to\infty} \Phi(s_n) = \Phi(f)$. Thus $\Phi(f) = \int_X f \, g d\mu$ and so (*) is true for a bounded measurable non-negative function. If f is a bounded measurable real valued function, then we can write $f = f^+ - f^-$, where f^+ and f^- are bounded non-negative measurable functions and so by linearity, (*) holds for bounded measurable real valued function. Finally, if f is a bounded measurable complex function, then write $f = \operatorname{Re} f + i \operatorname{Im} f$, where $\operatorname{Re} f$ and $\operatorname{Im} f$ are bounded real valued measurable functions. Therefore, by linearity (*) holds for any bounded measurable functions. If $f \in L^\infty(X,\mu)$, then there exists measurable subset B of X such that $|f(x)| \le \|f\|_{\infty,\mu} < \infty$ for all $x \in B^c$ and $\mu(B) = 0$. Let $f_1(x) = \begin{cases} f(x), & x \in B^c \\ 0, & x \in B \end{cases}$.
Then $f_1 = f$ almost everywhere with respect to μ and f_1 is bounded and measurable. Therefore, $\Phi(f) = \Phi(f_1) = \int_X f_1 g d\mu = \int_X f \, g d\mu$.

Now we shall show that $g \in L^q(X,\mu)$.

We consider the **case $p = 1$.**

For any E in $\mathscr{M}$, $\left|\int_E g \, d\mu\right| = |\Phi(\chi_E)| \le \|\Phi\| \|\chi_E\|_{1,\mu} = \|\Phi\| \mu(E)$. Therefore, for $\mu(E) > 0$,

$$\frac{1}{\mu(E)}\left|\int_E g \, d\mu\right| \le \|\Phi\|.$$

Therefore, by Lemma 9, with closed disk of radius $\|\Phi\|$, $|g| \le \|\Phi\|$ almost everywhere with respect to μ. Hence, $g \in L^\infty(X,\mu)$ and $\|g\|_{\infty,\mu} \le \|\Phi\|$.

Now for the case **$1 < p < \infty$.**

As shown in the proof of Theorem 15, there exists a measurable function h such that $\bar{h} g = |g|$ and $|\bar{h}| = 1$. For each integer $n \ge 1$, let $E_n = \{x : |g(x)| \le n\}$ and $f_n = \bar{h} |g|^{q-1} \chi_{E_n}$. Plainly, f_n is measurable and bounded and so $f_n \in L^\infty(X,\mu)$. It follows that $f_n \in L^p(X,\mu)$.

Now, $|f_n|^p = \begin{cases} |g|^{(q-1)p} = |g|^q \text{ on } E_n, \\ 0 \text{ on } E_n{}^c \end{cases}$. Putting f_n in (*), we get

$$\Phi(f_n) = \int_X f_n\, g d\mu = \int_{E_n} |g|^{q-1}\,\overline{h}\, g d\mu = \int_{E_n} |g|^q\, d\mu .$$

We also have, $|\Phi(f_n)| \le \|\Phi\|\|f_n\|_{p,\mu} = \|\Phi\|\left(\int_X |g|^{(q-1)p}\,\chi_{E_n} d\mu\right)^{\frac{1}{p}} = \|\Phi\|\left(\int_{E_n} |g|^q\, d\mu\right)^{\frac{1}{p}}$.

Hence, $\left(\int_{E_n} |g|^q\, d\mu\right)^{1-\frac{1}{p}} = \left(\int_{E_n} |g|^q\, d\mu\right)^{\frac{1}{q}} \le \|\Phi\|$. Letting *n* tends to ∞, we get

$$\left(\int_X |g|^q\, d\mu\right)^{\frac{1}{q}} \le \|\Phi\| .$$

This is because $|g|^q\,\chi_{E_n} \nearrow |g|^q$ monotonically and so by the Lebesgue Monotone Convergence Theorem, $\lim_{n\to\infty}\int_{E_n} |g|^q\, d\mu = \lim_{n\to\infty}\int_X |g|^q\,\chi_{E_n} d\mu = \int_X |g|^q\, d\mu$.

It follows that $g \in L^q(X,\mu)$. Now we shall show that $\Phi = \Phi_g$. We recall that the collection, $S = \left\{ s : X \to \mathbb{C}; s \text{ is a simple measurable function with } \mu\left(\{x : s(x) \ne 0\}\right) < \infty \right\}$ is dense in $L^p(X,\mu)$ in the $L^p(X,\mu)$ metric. (See Proposition 16, *Convex Function, L^p Soaces, Space of Continuous Functions, Lusin's Theorem*.) Since $\mu(X) < \infty$, every simple measurable function is in *S*. We have already shown that (*) holds for all simple measurable functions and that means Φ and Φ_g agrees on *S*. As Φ and Φ_g are both continuous on $L^p(X,\mu)$, $\Phi = \Phi_g$. By Theorem 15, $\|\Phi\| = \|\Phi_g\| = \|g\|_{q,\mu}$.

Step 2.

Now we move on to the case when the measure space $(X, \mathscr{M}, \mu)$ is σ-finite. We may assume that $X = \bigcup_{i=1}^{\infty} X_i$, a disjoint union of measurable sets $\{X_i\}$ with $\mu(X_i) < \infty$ for each $i \ge 1$.

Let $Y_n = \bigcup_{i=1}^{n} X_i$. Then $\mu(Y_n) = \mu\left(\bigcup_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mu(X_i) < \infty$.

Note that for any $E \in \mathscr{M}$, we can define $\Psi : L^p(X,\mu) \to \mathbb{C}$ by

$$\Psi(f) = \Phi(\chi_E f) \quad \text{for} \quad f \in L^p(X,\mu).$$

It is easy to see that Ψ is a complex linear functional.

Ψ satisfies $|\Psi(f)| = |\Phi(\chi_E f)| \le \|\Phi\| \|\chi_E f\|_{p,\mu} \le \|\Phi\| \|f\|_{p,\mu}$ for all $f \in L^p(X,\mu)$. Therefore, $\|\Psi\| \le \|\Phi\|$.

Take $E = X_n$. Let $\Psi_n : L^p(X,\mu) \to \mathbb{C}$ be defined as above by $\Psi_n(f) = \Phi(\chi_{X_n} f)$ and we have $\|\Psi_n\| \le \|\Phi\|$. Note that $\chi_{X_n} f \in L^p(X_n, \mu|_{X_n})$, where $(X_n , \mathscr{M}|_{X_n})$ is the sub-measure space of $(X, \mathscr{M}, \mu)$. Consider

$$\tilde{\Psi}_n : L^p(X_n, \mu|_{X_n}) \to \mathbb{C}$$

defined by $\tilde{\Psi}_n(h) = \Phi(\chi_{X_n} \tilde{h})$, for any $h \in L^p(X_n, \mu|_{X_n})$ and $\tilde{h} : X \to \mathbb{C}$ is given by $\tilde{h}(x) = \begin{cases} h(x), x \in X_n, \\ 0, x \notin X_n \end{cases}$. Plainly, $\tilde{h} : X \to \mathbb{C}$ is μ-measurable. Obviously $\tilde{\Psi}_n$ is a linear functional. Moreover, as noted above,

$$\left|\tilde{\Psi}_n(h)\right| = \left|\Phi(\chi_{X_n} \tilde{h})\right| \le \|\Phi\| \left\|\chi_{X_n} \tilde{h}\right\|_{p,\mu} = \|\Phi\| \|h\|_{p,\mu|_{X_n}}.$$

It follows that $\left\|\tilde{\Psi}_n\right\| \le \|\Phi\|$ and $\tilde{\Psi}_n$ is a bounded linear functional on $L^p(X_n, \mu|_{X_n})$.

As $\mu(X_n) < \infty$, by what we have just proved for finite measure, there exists $g_n \in L^q(X_n, \mu|_{X_n})$ such that for any $h \in L^p(X_n, \mu|_{X_n})$,

$$\tilde{\Psi}_n(h) = \Phi(\chi_{X_n} \tilde{h}) = \int_{X_n} h g_n d\mu|_{X_n}$$

and $\left\|\tilde{\Psi}_n\right\| = \|g_n\|_{q,\mu|_{X_n}} = \left\|\tilde{g}_n\right\|_{q,\mu}$, where $\tilde{g}_n : X \to \mathbb{C}$ is a measurable extension of g_n to X defined by $\tilde{g}_n(x) = \begin{cases} g_n(x), x \in X_n, \\ 0, x \notin X_n \end{cases}$.

Thus,

$$\Psi_n(f) = \Phi\left(\chi_{X_n} f\right) = \tilde{\Psi}_n\left(\chi_{X_n} f\right) \text{, where } \chi_{X_n} f \text{ is considered as a function on } X_n,$$

$$= \int_{X_n} \left(\chi_{X_n} f\right) g_n d\mu|_{X_n} = \int_{X_n} \left(\chi_{X_n} f\right) \tilde{g}_n d\mu = \int_X \left(\chi_{X_n} f\right) \tilde{g}_n d\mu.$$

Therefore, for all $f \in L^p(X,\mu)$,

$$\left|\Psi_n(f)\right| = \left|\int_X \left(\chi_{X_n} f\right)\tilde{g}_n d\mu\right| \le \left\|\chi_{X_n} f\right\|_{p,\mu} \left\|\tilde{g}_n\right\|_{q,\mu} \le \|f\|_{p,\mu} \left\|\tilde{g}_n\right\|_{q,\mu}$$

Hence, $\|\Psi_n\| \le \left\|\tilde{g}_n\right\|_{q,\mu}$.

Now, consider $\Lambda_n : L^p(X,\mu) \to \mathbb{C}$ defined by $\Lambda_n(f) = \Phi\left(\chi_{Y_n} f\right)$.

Then $\Lambda_n(f) = \Phi\left(\chi_{Y_n} f\right) = \Phi\left(\sum_{i=1}^{n} \chi_{X_i} f\right) = \sum_{i=1}^{n} \Phi\left(\chi_{X_i} f\right) = \sum_{i=1}^{n} \Psi_i(f) = \sum_{i=1}^{n} \int_X \left(\chi_{X_i} f\right)\tilde{g}_i d\mu$.

$$= \sum_{i=1}^{n} \int_X f\,\tilde{g}_i d\mu = \int_X (\tilde{g}_1 + \tilde{g}_2 + \ldots + \tilde{g}_n) f\, d\mu .$$

Let $\overline{\Lambda_n} : L^p(Y_n, \mu|_{Y_n}) \to \mathbb{C}$ be defined by $\overline{\Lambda_n}(f) = \Lambda_n(\tilde{f})$ for $f \in L^p(Y_n, \mu|_{Y_n})$ and $\tilde{f}$ is the obvious extension of f to X by defining $\tilde{f}(x) = f(x)$, when $x \in Y_n$ and $\tilde{f}(x) = 0$ when $x \notin Y_n$. $\overline{\Lambda_n}$ is obviously a complex linear functional and for all $f \in L^p(Y_n, \mu|_{Y_n})$,

$$\overline{\Lambda_n}(f) = \int_X (\tilde{g}_1 + \tilde{g}_2 + \ldots + \tilde{g}_n)\tilde{f}\, d\mu = \int_{Y_n} (\tilde{g}_1 + \tilde{g}_2 + \ldots + \tilde{g}_n)\chi_{Y_n} f\, d\mu|_{Y_n} .$$

Thus, $\left|\overline{\Lambda_n}(f)\right| = \left|\Lambda_n(\tilde{f})\right| = \left|\int_X (\tilde{g}_1 + \tilde{g}_2 + \ldots + \tilde{g}_n)\tilde{f}\, d\mu\right|$

$$\le \left\|\tilde{f}\right\|_{p,\mu} \left\|\tilde{g}_1 + \tilde{g}_2 + \ldots + \tilde{g}_n\right\|_{q,\mu} = \|f\|_{p,\mu|_{Y_n}} \left\|\tilde{g}_1 + \tilde{g}_2 + \ldots + \tilde{g}_n\right\|_{q,\mu|_{Y_n}} .$$

Therefore, $\overline{\Lambda_n}$ is a bounded linear functional and $\left\|\overline{\Lambda_n}\right\| = \left\|\tilde{g}_1 + \tilde{g}_2 + \ldots + \tilde{g}_n\right\|_{q,\mu}$.

Now $\left|\overline{\Lambda_n}(f)\right| = \left|\Lambda_n(\tilde{f})\right| = \left|\Phi\left(\chi_{Y_n}\tilde{f}\right)\right| \le \|\Phi\| \left\|\chi_{Y_n}\tilde{f}\right\|_{p,\mu} = \|\Phi\| \|f\|_{p,\mu|_{Y_n}}$ so that $\left\|\overline{\Lambda_n}\right\| \le \|\Phi\|$. It follows that $\left\|\tilde{g}_1 + \tilde{g}_2 + \ldots + \tilde{g}_n\right\|_{q,\mu} \le \|\Phi\|$. In particular, $\left\|\tilde{g}_i\right\|_{q,\mu} \le \|\Phi\|$ for each integer $i \ge 1$.

Let $g = \tilde{g}_1 + \tilde{g}_2 + \ldots + \tilde{g}_n + \cdots = \sum_{i=1}^{\infty} \tilde{g}_i$. Note that this is well defined. For any $x \in X$, $x \in X_n$ for some integer n so that $\tilde{g}_i(x) = 0$ for $i \ne n$ and $\sum_{i=1}^{k} \tilde{g}_i(x) = \tilde{g}_n(x)$ for $k \ge n$.

As $\sum_{i=1}^{n} \tilde{g}_i$ is measurable, g is measurable.

Observe that by definition of $\tilde{g}_i$, $|g|(x)=\left|\sum_{i=1}^{\infty}\tilde{g}_i\right|(x)=\liminf_{n\to\infty}\left|\sum_{i=1}^{n}\tilde{g}_i\right|(x)$. Therefore, by Fatou's Lemma,

$$\int_X |g|^q d\mu \le \liminf_{n\to\infty}\int_X\left|\sum_{i=1}^{n}\tilde{g}_i\right|^q d\mu = \liminf_{n\to\infty}\left(\left\|\tilde{g}_1+\tilde{g}_2+\ldots+\tilde{g}_n\right\|_{q,\mu}\right)^q \le \|\Phi\|^q .$$

Hence, $\|g\|_{q,\mu}=\left(\int_X |g|^q d\mu\right)^{\frac{1}{q}} \le \|\Phi\|$. It follows that $g\in L^q(X,\mu)$.

Now $\Lambda_n(f)=\Phi\left(\chi_{Y_n}f\right)\to\Phi\left(f\right)$ as $n\to\infty$.

For all $f\in L^p(X,\mu)$,

$$\int_X g\,f\,d\mu=\int_{\bigcup_{i=1}^{\infty}X_i} g\,f\,d\mu=\sum_{i=1}^{\infty}\int_{X_i} g\,fd\mu=\sum_{i=1}^{\infty}\int_{X_i}\tilde{g}_i\,fd\mu$$

$$=\lim_{n\to\infty}\int_{Y_i}\left(\tilde{g}_1+\tilde{g}_2+\cdots+\tilde{g}_n\right)fd\mu=\lim_{n\to\infty}\Phi\left(\chi_{Y_n}f\right)=\Phi\left(f\right).$$

Hence, by Theorem 15, $\|\Phi\|=\|g\|_{q,\mu}$.

Now, we consider the case $\boldsymbol{p}=1$.

The preceding argument applies to the case $p=1$, yielding, $\Psi_n : L^1(X,\mu)\to\mathbb{C}$, $\tilde{\Psi}_n : L^1(X_n,\mu|_{X_n})\to\mathbb{C}$, $g_n\in L^\infty(X_n,\mu|_{X_n})$ such that $\tilde{\Psi}_n(h)=\Phi(\chi_{X_n}\tilde{h})=\int_{X_n} h\,g_n d\mu|_{X_n}$ for $h\in L^1(X_n,\mu|_{X_n})$, $\|\tilde{\Psi}_n\|=\|g_n\|_{\infty,\mu|_{X_n}}=\left\|\tilde{g}_n\right\|_{\infty,\mu}$. We also have $\Lambda_n : L^1(X,\mu)\to\mathbb{C}$, $\overline{\Lambda_n} : L^1(Y_n,\mu|_{Y_n})\to\mathbb{C}$, with $\overline{\Lambda_n}(f)=\Lambda_n(\tilde{f})$ for $f\in L^1(Y_n,\mu|_{Y_n})$ and $\Lambda_n(f)=\Phi\left(\chi_{Y_n}f\right)=\sum_{i=1}^{n}\Psi_i\left(f\right)=\sum_{i=1}^{n}\int_X\left(\chi_{X_i}f\right)\tilde{g}_i d\mu=\int_X(\tilde{g}_1+\tilde{g}_2+\ldots+\tilde{g}_n)f\,d\mu$. We have also deduced that

$$\left|\overline{\Lambda_n}(f)\right|=\left|\int_X(\tilde{g}_1+\tilde{g}_2+\ldots+\tilde{g}_n)\tilde{f}\,d\mu\right|\le\left\|\tilde{g}_1+\tilde{g}_2+\ldots+\tilde{g}_n\right\|_{\infty,\mu}\left\|\tilde{f}\right\|_{1,\mu}=\left\|\tilde{g}_1+\tilde{g}_2+\ldots+\tilde{g}_n\right\|_{\infty,\mu}\left\|\tilde{f}\right\|_{1,\mu|_{Y_n}}$$

so that $\overline{\Lambda_n}$ is a bounded linear functional and so $\left\|\overline{\Lambda_n}\right\|=\left\|\tilde{g}_1+\tilde{g}_2+\ldots+\tilde{g}_n\right\|_{\infty,\mu}$.
Since $\left\|\overline{\Lambda_n}\right\|\le\|\Phi\|$, we deduce as before that $\left\|\tilde{g}_1+\tilde{g}_2+\ldots+\tilde{g}_n\right\|_{\infty,\mu}\le\|\Phi\|$. Hence, there exists a measurable set B_n such that $\mu(B_n)=0$ and

$$\left|\tilde{g}_1(x)+\tilde{g}_2(x)+\ldots+\tilde{g}_n(x)\right|\le\|\Phi\| \text{ for all } x\in B_n^{\ c} .$$

Let $B = \bigcup_{n=1}^{\infty} B_n$. Then $\mu(B) = 0$ and $B^c = \left(\bigcup_{n=1}^{\infty} B_n\right)^c = \bigcap_{n=1}^{\infty} B_n{}^c \subseteq B_n{}^c$. Therefore, for all $x \in B^c$,

$$\left|\tilde{g}_1(x) + \tilde{g}_2(x) + \ldots + \tilde{g}_n(x)\right| \leq \|\Phi\| \text{ for all integer } n \geq 1.$$

Hence,

$$\lim_{n\to\infty}\left|\tilde{g}_1(x) + \tilde{g}_2(x) + \ldots + \tilde{g}_n(x)\right| = |g(x)| \leq \|\Phi\| \text{ for all } x \in B^c \text{ so that } \|g\|_{\infty,\mu} \leq \|\Phi\|.$$

This implies that $g \in L^\infty(X, \mu)$, $\Phi(f) = \int_X f\, g d\mu$ and $\|\Phi\| = \|g\|_{\infty,\mu}$.

Theorem 17. Suppose $(X, \mathscr{M}, \mu)$ is a measure space and $f : X \to \mathbb{C}$ is a measurable function. If f is integrable with respect to μ, then the set $\{x : f(x) \neq 0\}$ is σ-finite.

It follows easily that for any $f \in L^p(X, \mu)$, $1 \leq p < \infty$, the set $\{x : f(x) \neq 0\}$ is σ-finite.

Proof.

The function f is integrable means that f is measurable and $\int_X |f| d\mu < \infty$.

Partition $(0, \infty)$ by $\left\{\left[\frac{1}{1+n}, \frac{1}{n}\right)\right\}_{n=1}^{\infty} \cup [1, \infty)$. Let $E_n = |f|^{-1}\left[\frac{1}{n+1}, \frac{1}{n}\right)$ for integer $n \geq 1$ and $E_0 = |f|^{-1}[1, \infty)$. Plainly, E_n is measurable for $0 \leq n < \infty$. Since $\int_X |f| d\mu < \infty$, $\int_{E_n} |f| d\mu < \infty$. Note that $\int_{E_n} |f| d\mu \geq \frac{1}{n+1}\mu(E_n)$ for integer $n \geq 0$ and so $\mu(E_n) < \infty$ for integer $n \geq 0$. As $\{x : f(x) \neq 0\} = \{x : |f|(x) \neq 0\} = \bigcup_{n=0}^{\infty} E_n$, $\{x : f(x) \neq 0\}$ is σ-finite.

If $f \in L^p(X, \mu)$, then $\int_X |f|^p d\mu < \infty$ and as $\{x : f(x) \neq 0\} = \left\{x : |f|^p(x) \neq 0\right\} = \bigcup_{n=0}^{\infty} E_n$, it follows that $\{x : f(x) \neq 0\}$ is σ-finite. Here we note that $\int_{E_n} |f|^p d\mu \geq \left(\frac{1}{n+1}\right)^p \mu(E_n)$ for each integer $n \geq 0$ and the same argument applies to give the same conclusion.

We now show that in Theorem 16, for $1 < p < \infty$, we may drop the condition that the measure μ be σ-finite.

Theorem 18.

Suppose $(X, \mathscr{M})$ is a measure space and $\mu: \mathscr{M} \to \overline{\mathbb{R}^+}$ is a positive measure. Let $1 < p < \infty$ and $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $\Phi : L^p(X, \mu) \to \mathbb{C}$ is a bounded complex linear functional. Then there exists a unique g in $L^q(X, \mu)$ such that

$$\Phi(f) = \int_X f\, g d\mu = \Phi_g(f),$$

for all $f \in L^p(X, \mu)$. Moreover, $\|\Phi_g\| = \|g\|_{q,\mu}$. More precisely, the dual space of $L^p(X, \mu)$, $\left(L^p(X, \mu)\right)^*$, is isometric isomorphic with $L^q(X, \mu)$, under a Banach space isomorphism preserving norm.

Proof.

Let S be the collection of σ-finite measurable subsets of $\mathscr{M}$. That is,

$$S = \{ E \in \mathscr{M} : E \text{ is } \sigma\text{-finite}\}.$$

Now for each E in S, by Theorem 16, there exists a unique $\widetilde{g_E}$ vanishing outside of E such that for any $f \in L^p(X, \mu)$ and f vanishing outside of E, such that

$$\Phi(f) = \int_E (f \chi_E) \widetilde{g_E} d\mu = \int_X f\, \widetilde{g_E} d\mu .$$

This is because

$$\Phi_E : L^p(E, \mu|_E) \to \mathbb{C} \ ,$$

defined by $\Phi_E(f) = \Phi\left(\widetilde{f}\right)$, for $f \in L^p(E, \mu|_E)$, where $\widetilde{f}(x) = \begin{cases} f(x), x \in E \\ 0, x \in E^c \end{cases}$, is a bounded complex linear functional. Since $f : E \to \mathbb{C}$ is $\mathscr{M}|_E$ measurable, $\widetilde{f}$ is $\mathscr{M}$ measurable. Actually, Theorem 16 gives a unique $g_E \in L^q(E, \mu|_E)$ such that $\Phi_E(f) = \int_E f\, g_E d\left(\mu|_E\right)$ for $f \in L^p(E, \mu|_E)$.

Note that $g_E : E \to \mathbb{C}$ is $\mathscr{M}|_E$ measurable and so the extension $\widetilde{g_E}(x) = \begin{cases} g_E(x), x \in E \\ 0, x \in E^c \end{cases}$ is $\mathscr{M}$ measurable. Hence, $\Phi_E(f) = \int_X \widetilde{f}\, \widetilde{g_E} d\mu$. Moreover, for $f \in L^p(E, \mu|_E)$,

$\int_E |f|^p d(\mu|_E) = \int_E |\widetilde{f}|^p \chi_E d\mu = \int_X |\widetilde{f}|^p d\mu < \infty$ so that $\widetilde{f} \in L^p(X,\mu)$. We note that if $B \subseteq E$, then $\widetilde{g_B} = \widetilde{g_E}$ almost everywhere with respect to μ on B by uniqueness.

For each E in S, define $\lambda(E) = \int_X |\widetilde{g_E}|^q d\mu < \infty$. Now, for $f \in L^p(E,\mu|_E)$,

$$|\Phi_E(f)| = |\Phi(\widetilde{f})| \le \|\Phi\| \|\widetilde{f}\|_{p,\mu} = \|\Phi\| \|f\|_{p,\mu|_E}.$$

Hence $\|\Phi_E\| \le \|\Phi\|$. By Theorem 16, $\|g_E\|_{q,\mu|_E} = \|\widetilde{g_E}\|_{q,\mu} = \|\Phi_E\| \le \|\Phi\|$.

It follows that $\{\lambda(E) : E \in S\}$ is bounded above by $\|\Phi\|^q$. Let $\alpha = \sup\{\lambda(E) : E \in S\}$. Then there exists a sequence $\{E_i\}$ of σ-finite measurable sets in S such that $\lambda(E_n) \to \alpha$. Let $H = \bigcup_{n=1}^{\infty} E_n$. Plainly, H is σ-finite and so $H \in S$ and as $\lambda(E_n) \le \lambda(H)$ for $E_n \subseteq H$, $\lambda(H) = \alpha$. Let $g = \widetilde{g_H} = \begin{cases} g_H(x), x \in H \\ 0, x \in H^c \end{cases}$. Therefore, $g \in L^q(X,\mu)$.

Note that if E is any set of σ-finite measure and contains H, then by uniqueness, $\widetilde{g_E} = \widetilde{g_H}$ almost everywhere on H with respect to μ. On the other hand,

$$\int_E |\widetilde{g_E}|^q d\mu = \int_E |g_E|^q d(\mu|_E) = \int_X |\widetilde{g_E}|^q d\mu = \lambda(E) \le \alpha = \int_X |\widetilde{g_H}|^q d\mu = \int_X |g|^q d\mu$$

and

$$\int_E |\widetilde{g_E}|^q d\mu = \int_H |\widetilde{g_E}|^q d\mu + \int_{E-H} |\widetilde{g_E}|^q d\mu$$

$$= \int_H |\widetilde{g_H}|^q d\mu + \int_{E-H} |\widetilde{g_E}|^q d\mu = \int_X |g|^q d\mu + \int_{E-H} |\widetilde{g_E}|^q d\mu$$

and so $\int_{E-H} |\widetilde{g_E}|^q d\mu = 0$. This implies that $\widetilde{g_E} = 0$ almost everywhere on $E - H$. Thus, $g = \widetilde{g_E}$ almost everywhere on E with respect to μ.

Now we take any function f in $L^p(X,\mu)$. Let $G = \{x : |f|(x) \ne 0\}$. Then G is measurable and σ-finite by Theorem 17. Therefore, $E = G \cup H$ is σ-finite. Let $\overline{f} = f|_E$. Then

$$\Phi(f)=\Phi_E(\overline{f})=\int_E \overline{f} g_E d\left(\mu|_E\right)=\int_E f\widetilde{g_E} d\mu=\int_E fgd\mu=\int_X fgd\mu .$$

Thus, we have shown that for any $f\in L^p(X,\mu)$, $\Phi(f)=\int_X fgd\mu$. As in Theorem 16, we can deduce that $\|\Phi\|=\|g\|_{q,\mu}$.

This completes the proof.

However, for p =1, we may not relax the σ-finiteness condition for the measure μ in Theorem 16. For there is an example of a measure space (X, $\mathscr{M}$, μ) with μ not σ-finite and a bounded linear functional, Φ, on $L^1(X,\mu)$ such that there does not exist $g\in L^\infty(X,\mu)$ satisfying $\Phi(f)=\int_X fgd\mu$.

The next theorem is a result, which we have used, about approximation of measurable non-negative function by simple measurable functions.

Theorem 19. Suppose $f:X\to\overline{\mathbb{R}^+}$ is a non-negative measurable function, where (X, $\mathscr{M}$) is a measure space. Then there exists an increasing sequence of measurable simple functions (s_n) converging pointwise to f. If f is bounded, then (s_n) converges uniformly to f .

Proof.

We construct the sequence (s_n) as follows. For each integer $n \geq 1$, divide the interval [0, n] into $n\times 2^n$ sub-intervals of length $\frac{1}{2^n}$.

Let $E_{n,i}=f^{-1}\left(\left[\frac{i-1}{2^n},\frac{i}{2^n}\right)\right)$, $i=1,2,\cdots,n2^n$, $F_n=f^{-1}\left([n,\infty)\right)$ and

$$s_n=\sum_{i=1}^{n2^n}\frac{i-1}{2^n}\chi_{E_{n,i}}+n\chi_{F_n} .$$

Since f is measurable, the sets $E_{n,i}$ and F_n are measurable.

Note that $E_{n,i} = E_{n+1,j} \cup E_{n+1,j+1}$, where $\frac{j-1}{2^{n+1}} = \frac{i-1}{2^n}$ or $j = 2i-1$. On the set $E_{n,i}$, $s_{n+1}(x)$ takes on the value $\frac{j-1}{2^{n+1}} = \frac{i-1}{2^n}$ when x is in $E_{n+1,j}$ and the value $\frac{j}{2^{n+1}} > \frac{i-1}{2^n}$ when x is in $E_{n+1,j+1}$. Observe also that

$$F_n = f^{-1}([n,\infty)) = f^{-1}([n+1,\infty)) \cup f^{-1}([n,n+1)) = F_{n+1} \cup f^{-1}([n,n+1))$$

and $f^{-1}([n,n+1)) = \bigcup\{E_{n+1,i} : i = n2^{n+1}+1 \text{ to } (n+1)2^{n+1}\}$.

Thus, on the set F_{n+1}, $s_{n+1}(x)$ takes on the value $n+1$ when x is in $E_{n+1,j}$ and on the set $f^{-1}([n,n+1))$, $s_{n+1}(x)$ takes on values $\geq n$, when $s_n(x)$ is defined and is equal to n. Therefore, $s_{n+1} \geq s_n$.

Since $f(x) < \infty$, take an integer N such that $N > f(x)$, then for all $n \geq N$, $s_{n+1}(x) \leq N$ and so the sequence is pointwise convergence. Moreover, for each integer $n > f(x), f(x)$ lies in $\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$ for some i such that $1 \leq i \leq n2^n$ and so $s_n(x) \leq f(x)$. Furthermore, $s_n(x) \geq f(x) - \frac{1}{2^n}$. Hence $\lim_{n\to\infty} s_n(x) = f(x)$.

Now, suppose f is bounded such that $0 \leq f < K$ and $K \geq 1$.

First of all, note that $F_n = \varnothing$ for all integer $n \geq K$. For any integer $n > K$, $E_{n,i} = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) = \varnothing$ if $2^n K + 1 \leq i \leq n2^n$.

This means for $0 \leq f < K$, we effectively partition the interval [0, K] into $2^n K$ sub-intervals each of length $\frac{1}{2^n}$.

Observe that since $f(x) < K$, for any integer $N \geq K$, $N > f(x)$ for all x, and so for all $n \geq N$, $s_{n+1}(x) \leq N$ for all x and so the sequence is uniformly bounded. Moreover, for each integer $n \geq N$, $f(x)$ lies in $\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$ for some i such that $1 \leq i \leq n2^n$ so that, $s_n(x) \geq f(x) - \frac{1}{2^n}$ for all x. Hence, for all $n \geq N$ and for all x, $f(x) \geq s_n(x) \geq f(x) - \frac{1}{2^n}$. This means that (s_n) converges uniformly to f .

The Riesz Representation Theorem - The Complex Version

In *Positive Borel Measure and Riesz Representation Theorem*, we represent a positive (complex) linear functional, $\Lambda : C_c(X) \to \mathbb{C}$, where *X* is a locally compact Hausdorff topological space and $C_c(X)$ is the space of continuous complex functions on *X* with compact support with the uniform norm, by

$$\Lambda(f) = \int_X f \, d\lambda ,$$

for some positive measure λ , which is *almost* regular and complete on a σ-algebra $\mathscr{M}$ containing all the Borel sets of *X*.

There was no question of Λ being continuous, i.e., bounded. Actually, in some cases, with additional condition on *X*, it is true that Λ is positive implies that Λ is bounded. Note that $C_c(X)$ is endowed with the uniform sup norm. If the representing measure λ satisfies $\lambda(X) < \infty$, then for any $f \in C_c(X)$,

$$|\Lambda(f)| = \left|\int_X f \, d\lambda\right| \le \int_X |f| \, d\lambda \le \|f\|_u \int_X d\lambda = \|f\|_u \, \lambda(X) .$$

Recall that $\|f\|_u = \sup\{|f(x)| : x \in X\} < \infty$, since f is continuous with compact support. It follows that $\|\Lambda\| \le \lambda(X) < \infty$ and so Λ is bounded. This means that if *X* is compact, by Theorem 1 (Riesz Representation Theorem) of *Positive Borel Measure and Riesz Representation Theorem*, the representing measure is finite and so the positive complex linear functional is bounded and so is continuous. If we specialize to positive real linear functional $\Lambda : C_{c,\mathbb{R}}(X) \to \mathbb{R}$, where $C_{c,\mathbb{R}}(X)$ is the space of continuous real valued function on *X* with compact support, then as a consequence of the representation theorem, Λ is a bounded real linear functional *if* the representing measure λ is finite. But we would need some additional condition, for example when *X* is compact, to obtain a finite representing measure. However, a real linear functional on the normed linear space $C_{\mathbb{R}}(X)$ with the sup norm is continuous if and only if it is bounded. When *X* is compact and Hausdorff, a real linear functional on the normed linear space $C_{\mathbb{R}}(X)$ can be represented by a regular finite real Borel measure expressible as the difference of two regular finite positive measures. (See Theorem 3, *Finite Borel Measure and Riesz Representation Theorem*.)

Now we want to consider any bounded complex linear functional $\Phi : C_c(X) \to \mathbb{C}$ and represent Φ as $\Phi(f) = \int_X f \, d\mu$ for some complex measure on a σ-algebra $\mathscr{M}$ containing all the Borel sets of *X*. Since $\Phi : C_c(X) \to \mathbb{C}$ is bounded, we can extend Φ to the completion of $\left(C_c(X), \|\cdot\|_u\right)$, i.e., $\left(C_0(X), \|\cdot\|_u\right)$ the space of continuous complex functions on *X* which vanishes at infinity. (See Proposition 25, *Convex Functions, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*.) Hence, we might as well consider the representation of bounded complex linear functional $\Phi : C_0(X) \to \mathbb{C}$ on $\left(C_0(X), \|\cdot\|_u\right)$.

Theorem 20. Riesz Representation Theorem - The Complex Version

Let *X* be a locally compact Hausdorff topological space and $\Phi : C_0(X) \to \mathbb{C}$ a bounded complex linear functional on $C_0(X)$ with the uniform sup norm. Then there exists a unique regular complex Borel measure μ such that

$$\Phi(f) = \int_X f \, d\mu .$$

Moreover, $\|\Phi\| = |\mu|(X)$. That is to say, the dual space or conjugate space of $C_0(X)$, $(C_0(X))^* \cong M(X)$, where $M(X)$ is the collection of all regular Borel complex measures with norm given by $\|\mu\| = |\mu|(X)$ and "$\cong$" here means Banach space isomorphism preserving norm.

Recall that a complex measure μ is *regular* if $|\mu|$ is regular as a positive measure. μ is *finite* if $|\mu|$ is finite as a positive measure.

If $X = [0,1]$, then $(C_0[0,1])^* = (C[0,1])^*$ is the space of all regular complex Borel measures on [0, 1].

Before we prove the theorem, we present a technical result concerning the regularity of the sum of regular complex measures.

Proposition 21. Suppose X is a topological space and $(X, \mathscr{M})$ is a measure space, where $\mathscr{M}$ is a σ-algebra containing all the Borel sets of X.

Suppose μ_1 and μ_2 are two regular complex Borel measures. Then $\mu_1 + \mu_2$ is also a regular complex Borel measure.

Proof.

Plainly, $\mu_1 + \mu_2$ is a complex Borel measure. The measures μ_1 and μ_2 are regular means that $|\mu_1|$ and $|\mu_2|$ are regular.

We show that $|\mu_1 + \mu_2|$ is inner regular.

$|\mu_1|$ is inner regular implies that for any $E \in \mathscr{M}$, given $\varepsilon > 0$, there exists compact $K_1 \subseteq E$ such that $|\mu_1|(E) - \varepsilon < |\mu_1|(K_1)$. That is to say,

$$|\mu_1|(E - K_1) = |\mu_1|(E) - |\mu_1|(K_1) < \varepsilon . \text{ ----------------------- (1)}$$

Similarly, as $|\mu_2|$ is inner regular, for any $E \in \mathscr{M}$, given $\varepsilon > 0$, there exists compact $K_2 \subseteq E$ such that

$$|\mu_2|(E - K_2) = |\mu_2|(E) - |\mu_2|(K_2) < \varepsilon . \text{ ---------------------- (2)}$$

Let $K = K_1 \cup K_2$. Then K is compact and $K \subseteq E$.

$$|\mu_1 + \mu_2|(E - K) \leq |\mu_1|(E - K) + |\mu_2|(E - K) < 2\varepsilon .$$

Hence, $|\mu_1 + \mu_2|(E) - 2\varepsilon < |\mu_1 + \mu_2|(K) \leq |\mu_1 + \mu_2|(E)$. This implies that

$$|\mu_1 + \mu_2|(E) = \sup\left\{|\mu_1 + \mu_2|(K),\ K \text{ compact and } K \subseteq E\right\}.$$

Thus, for any $E \in \mathscr{M}$, $|\mu_1 + \mu_2|(E) = \sup\left\{|\mu_1 + \mu_2|(K),\ K \text{ compact and } K \subseteq E\right\}$. It follows that $|\mu_1 + \mu_2|$ is inner regular.

We now show that $|\mu_1 + \mu_2|$ is outer regular. $|\mu_1|$ and $|\mu_2|$ are both outer regular. This means for any $E \in \mathscr{M}$, given $\varepsilon > 0$, there exists an open set $V_1 \supseteq E$ such that $|\mu_1|(V_1) < |\mu_1|(E) + \varepsilon$. Therefore, $|\mu_1|(V_1 - E) < \varepsilon$. Similarly, there exists an open set $V_2 \supseteq E$ such that $|\mu_2|(V_2 - E) < \varepsilon$. Let $V = V_1 \cap V_2$. Then V is open and

$V \supseteq E$. Therefore, $|\mu_1 + \mu_2|(V - E) \le |\mu_1|(V - E) + |\mu_2|(V - E) < 2\varepsilon$. Hence, $|\mu_1 + \mu_2|(V) = |\mu_1 + \mu_2|(V - E) + |\mu_1 + \mu_2|(E) < |\mu_1 + \mu_2|(E) + 2\varepsilon$.

It follows that $|\mu_1 + \mu_2|(E) = \inf\{|\mu_1 + \mu_2|(V),\ V \text{ open and } V \supseteq E\}$. As this holds for any $E \in \mathscr{M}$, $|\mu_1 + \mu_2|$ is outer regular.

Therefore, $|\mu_1 + \mu_2|$ is regular and so $\mu_1 + \mu_2$ is regular.

Proof of Theorem 20.

We prove the uniqueness part of the theorem.

Suppose μ_1 and μ_2 are two regular complex Borel measures satisfying the conclusion of the theorem. Then

$$0 = \Phi(f) - \Phi(f) = \int_X f\, d\mu_1 - \int_X f\, d\mu_2 = \int_X f\, d\nu \text{, where } \nu = \mu_2 - \mu_2 .$$

By Proposition 21, $\nu = \mu_2 - \mu_2$ is also a regular complex Borel measure.

By Theorem 10, there exists a measurable complex function $h : X \to \mathbb{C}$ such that $h \in L^1(X, |\nu|)$, $|h| = 1$ and $d\nu = h\, d|\nu|$. That is, for any $E \in \mathscr{M}$,

$$\nu(E) = \int_E h\, d|\nu|$$

and for any $f \in C_0(X)$, $\int_X f\, d\nu = \int_X f\, h\, d|\nu|$.

We shall show that $|\nu|(X) = 0$. Once we have shown this, then since for all $E \in \mathscr{M}$, $|\nu(E)| \le |\nu|(E) \le |\nu|(X) = 0$, $|\nu(E)| = 0$. It follows that for all $E \in \mathscr{M}$, $\nu(E) = 0$ and so $\mu_1 = \mu_2$.

Now,

$$|\nu|(X) = \int_X d|\nu| = \int_X h\bar{h}\, d|\nu| = \int_X h\bar{h}\, d|\nu| - \int_X f\, d\nu$$

$$= \int_X h\bar{h}\, d|\nu| - \int_X f\, h d|\nu| = \int_X h\left(\bar{h} - f\right) d|\nu|$$

$$\le \int_X |\bar{h} - f|\, d|\nu|, \text{ ----------------------------- } (*)$$

for any $f \in C_0(X)$.

Since $C_c(X)$ is dense in $L^1(X,|\nu|)$, (see Theorem23, *Convex Functions, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*), when $C_c(X)$ is endowed with the $L^1(X,|\nu|)$ norm and since $h \in L^1(X,|\nu|)$, we can take a sequence of functions $\{f_n\}$ in $C_c(X)$ such that $f_n \to \overline{h}$ in $L^1(X,|\nu|)$ so that

$$\int_X \left|\overline{h} - f_n\right| d|\nu| \to 0 \text{ as } n \to \infty .$$

It follows then from (*) that $|\nu|(X) = 0$.

Note that given a bounded linear functional Φ on $C_0(X)$, if $\|\Phi\| \neq 0$, we may normalise it by taking $\frac{1}{\|\Phi\|}\Phi$ so that its norm is unity. If $\|\Phi\| = 0$, we can just take the trivial Borel measure. So now we assume that $\|\Phi\| > 0$ and normalise it by considering $\frac{1}{\|\Phi\|}\Phi$. We shall thus assume without lost of generality that $\|\Phi\| = 1$.

The key to the proof is to use the positive measure version of the Riesz Representation Theorem (Theorem 1, *Positive Borel Measure and Riesz Representation Theorem*).

Assume that we can construct a positive complex linear functional Λ on $C_c(X)$ such that

$$|\Phi(f)| \leq \Lambda(|f|) \leq \|f\|_u . \qquad \text{-------------------------} \quad (1)$$

Then we can apply the positive measure version of Riesz Representation Theorem (Theorem 1, *Positive Borel Measure and Riesz Representation Theorem*) to Λ to give a positive complete Borel measure, λ, which is outer regular and inner regular with respect to open set and sets of finite measure, such that

$$\Lambda(f) = \int_X f \, d\lambda , \text{ for all } f \in C_c(X) .$$

Note that $\lambda(X) = \sup\{\Lambda(f) : f \in C_c(X), f \prec X\} = \sup\{\Lambda(f) : f \in C_c(X), 0 \le f \le 1\}$. It follows from (1) that for $f \in C_c(X)$

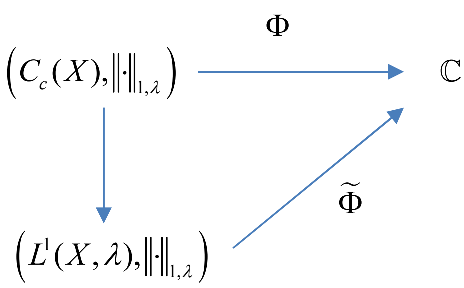
$$|\Lambda(f)| \le \int_X |f| \, d\lambda = \Lambda(|f|)$$

$$\le \|f\|_u \le 1, \text{ if } 0 \le f \le 1 .$$

It follows that $\lambda(X) \le 1 < \infty$ and so by Theorem 1 of *Positive Borel Measure and Riesz Representation Theorem*, λ is inner and outer regular for all measurable sets in $\mathscr{M}$, i.e., a finite regular Borel measure.

By (1), $|\Phi(f)| \le \Lambda(|f|) = \int_X |f| d\lambda = \|f\|_{1,\lambda}$ for all $f \in C_c(X)$.

Therefore, $\Phi : \left(C_c(X), \|\cdot\|_{1,\lambda}\right) \to \mathbb{C}$ is a bounded complex linear functional of norm $\|\Phi\|_1$ less than or equal to 1. Therefore, we can extend Φ by continuity to $L^1(X,\lambda)$, since $\left(C_c(X), \|\cdot\|_{1,\lambda}\right)$ is dense in $L^1(X,\lambda)$ (see Theorem 23, *Convex Functions, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*). Let the extension be denoted by $\widetilde{\Phi} : \left(L^1(X,\lambda), \|\cdot\|_{1,\lambda}\right) \to \mathbb{C}$. Then $\widetilde{\Phi}$ is a bounded complex linear functional on $L^1(X,\lambda)$.



Therefore, by Theorem 16, as λ is a finite positive measure, there exists a unique *g* in $L^\infty(X,\lambda)$ such that

$$\widetilde{\Phi}(f) = \int_X f\, g d\lambda , \quad \text{----------------------------} \quad (2)$$

for all $f \in L^1(X,\lambda)$. Moreover, $\|g\|_{\infty,\lambda} = \|\widetilde{\Phi}\|_1 = \|\Phi\|_1 \le 1$. It follows that $|g| \le 1$ almost everywhere with respect to λ.

(2) is of course valid for $\Phi : \left(C_c(X), \|\cdot\|_{1,\lambda}\right) \to \mathbb{C}$.

Note that $\Phi:\left(C_c(X),\|\cdot\|_u\right)\to\mathbb{C}$ is a bounded linear functional and

$$\Phi(f)=\int_X f\,g d\lambda \text{ for all } f\in C_c(X). \text{ ------------------- (3)}$$

Note that the left-hand side of (3) is the restriction to $C_c(X)$ of our original bounded (continuous) linear functional Φ on $C_0(X)$ with the uniform sup norm. The right-hand side of (3), $\int_X f\,g d\lambda$, is also a bounded linear functional on $C_0(X)$ with the uniform norm. Since $\left(C_c(X),\|\cdot\|_u\right)$ is dense in $\left(C_0(X),\|\cdot\|_u\right)$, (3) holds for all $f\in C_0(X)$.

Extend (3) to the completion of $C_c(X)$ in the uniform norm, $C_0(X)$. Therefore, we can write for all $f\in C_0(X)$,

$$\Phi(f)=\int_X f\,d\mu, \text{ where } d\mu=g\,d\lambda.$$

More precisely, for any $E\in\mathscr{M}$, $\mu(E)=\int_E g\,d\lambda$. It can be easily check that μ is a complex measure.

From (3), for all $f\in C_0(X)$, $|\Phi(f)|=\left|\int_X f\,g d\lambda\right|\le\|f\|_u\int_X|g|\,d\lambda\le\int_X|g|\,d\lambda$ if $\|f\|_u\le 1$. Thus,

$$\|\Phi\|=\sup\left\{\frac{|\Phi(f)|}{\|f\|_u}: f\in C_0(X), f\neq 0\right\}=\sup\left\{|\Phi(f)|: f\in C_0(X),\|f\|_u\le 1\right\}\le\int_X|g|d\lambda.$$

Since $\|\Phi\|=1$, we have $1=\|\Phi\|\le\int_X|g|d\lambda\le\int_X d\lambda=\lambda(X)$ since $|g|\le 1$ almost everywhere with respect to λ. We have previously shown that $\lambda(X)\le 1$ and so it follows that $\lambda(X)=1$ and $|g|=1$ almost everywhere with respect to λ. Since $d\mu=g\,d\lambda$ and $g\in L^1(X,\lambda)$, by Proposition 12, $d|\mu|=|g|\,d\lambda=d\lambda$. Therefore,

$$|\mu|(X)=\lambda(X)=1=\|\Phi\|.$$

As λ is a regular measure, i.e., it is inner and outer regular for all measurable sets in $\mathscr{M}$, it follows that $|\mu|$ is regular and so μ is a regular complex Borel measure.

It now remains to construct the positive complex linear functional Λ on $C_c(X)$ with the required property (1).

Let $C_c^+(X)$ denote the set of non-negative real-valued functions in $C_c(X)$ and define for $f \in C_c^+(X)$,

$$\Lambda(f) = \sup\{|\Phi(h)| : h \in C_c(X) \text{ and } |h| \le f\}.$$

Then plainly, $\Lambda(f) \ge 0$ for all $f \in C_c^+(X)$ and Λ satisfies

$$|\Phi(f)| \le \Lambda(|f|) \text{ for all } f \in C_c(X).$$

This is because for any $f \in C_c(X)$, $|f| \in C_c^+(X)$ and

$$\Lambda(|f|) = \sup\{|\Phi(h)| : h \in C_c(X) \text{ and } |h| \le |f|\}.$$

Now, for any $h \in C_c(X)$, $|\Phi(h)| \le \|\Phi\| \|h\|_u = \|h\|_u$, since $\|\Phi\| = 1$. If $|h| \le |f|$, $\|h\|_u \le \|f\|_u$. It follows that for all $f \in C_c(X)$,

$$\Lambda(|f|) = \sup\{|\Phi(h)| : h \in C_c(X) \text{ and } |h| \le |f|\} \le \|f\|_u.$$

Now, $\Lambda : C_c^+(X) \to \mathbb{R}$ is non-negative and by definition, if $f_1, f_2 \in C_c^+(X)$, $f_1 \le f_2 \Rightarrow \Lambda(f_1) \le \Lambda(f_2)$. Obviously, for any real number $c \ge 0$, $\Lambda(c f) = c\Lambda(f)$ for any $f \in C_c^+(X)$. We need to show that $\Lambda(f_1 + f_2) = \Lambda(f_1) + \Lambda(f_2)$ for $f_1, f_2 \in C_c^+(X)$.

We show that $\Lambda(f_1) + \Lambda(f_2) \le \Lambda(f_1 + f_2)$. By definition of $\Lambda(f_i)$, $i = 1, 2$, given $\varepsilon > 0$, there exists $h_i \in C_c(X)$ such that $|h_i| \le f_i$ and

$$\Lambda(f_i) - \varepsilon < |\Phi(h_i)| \quad \text{for } i = 1, 2.$$

Since $\Phi(h_i)$ is a complex number, we can write $|\Phi(h_i)| = \alpha_i \Phi(h_i)$ for some complex number α_i with $|\alpha_i| = 1$ for $i = 1, 2$. Then

$$\Lambda(f_1) + \Lambda(f_2) < |\Phi(h_1)| + |\Phi(h_2)| + 2\varepsilon = \alpha_1 \Phi(h_1) + \alpha_2 \Phi(h_2) + 2\varepsilon = \Phi(\alpha_1 h_1 + \alpha_2 h_2) + 2\varepsilon.$$

Hence, $\Lambda(f_1) + \Lambda(f_2) < |\Phi(\alpha_1 h_1 + \alpha_2 h_2)| + 2\varepsilon$.

Since $|\alpha_1 h_1 + \alpha_2 h_2| \le |h_1| + |h_2| \le f_1 + f_2$, $|\Phi(\alpha_1 h_1 + \alpha_2 h_2)| \le \Lambda(f_1 + f_2)$. It follows that

$$\Lambda(f_1) + \Lambda(f_2) < \Lambda(f_1 + f_2) + 2\varepsilon .$$

Since ε is arbitrary, $\Lambda(f_1) + \Lambda(f_2) \le \Lambda(f_1 + f_2)$.

Now, we show that $\Lambda(f_1 + f_2) \le \Lambda(f_1) + \Lambda(f_2)$.

Take $h \in C_c(X)$ with $|h| \le f_1 + f_2$. Let $V = \{x \in X : f_1(x) + f_2(x) > 0\}$. Then V is open in X. For i = 1, 2, let

$$h_i(x) = \begin{cases} \dfrac{f_i(x)h(x)}{f_1(x) + f_2(x)}, x \in V \\ 0, x \in V^c \end{cases} .$$

Then $|h_i(x)| \le f_i(x)$ for i = 1,2, since $\dfrac{|h(x)|}{f_1(x) + f_2(x)} \le 1$ for $x \in V$.

Note that $h_i(x) \ne 0 \Rightarrow h(x) \ne 0$ for x in V. Therefore, support $h_i \subseteq$ support h, which is compact. We claim that h_i is continuous for i =1 , 2. Plainly, h_i is continuous on V for i =1 , 2. For $a \in V^c$, $h_1(a) = 0$ but since $|h| \le f_1 + f_2$, $h(a) = 0$. As $|h_1| \le |h|$ and h is continuous at a with $h(a) = 0$, h_1 is continuous at a. Hence, h_1 is continuous on V^c and so is continuous on X. Similarly, we deduce that h_2 is continuous on X. Hence, $h_1, h_2 \in C_c(X)$. Observe that $h_1 + h_2 = h$.

Therefore,

$$|\Phi(h)| = |\Phi(h_1) + \Phi(h_2)| \le |\Phi(h_1)| + |\Phi(h_2)| \le \Lambda(f_1) + \Lambda(f_2) .$$

Since this holds for all $h \in C_c(X)$ with $|h| \le f_1 + f_2$, $\Lambda(f_1 + f_2) \le \Lambda(f_1) + \Lambda(f_2)$. Therefore, $\Lambda(f_1 + f_2) = \Lambda(f_1) + \Lambda(f_2)$. We can now extend Λ to $C_c(X)$ as follows. If $f \in C_c(X)$ is real valued, write f as $f = f^+ - f^-$, where $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$. Then $f^+, f^- \in C_c^+(X)$. Define

$$\Lambda(f) = \Lambda(f^+) - \Lambda(f^-).$$

If $f \in C_c(X)$ is complex valued, then write $f = \operatorname{Re} f + i \operatorname{Im} f$. Plainly, $\operatorname{Re} f$ and $\operatorname{Im} f$ are real valued and continuous with compact support. Define

$$\Lambda(f) = \Lambda(\operatorname{Re} f) + i\Lambda(\operatorname{Im} f).$$

We now check that we have thus defined a complex linear functional on $C_c(X)$.

Suppose $f \in C_c(X)$ and f is real valued. Then for $c > 0$,

$$c\Lambda(f) = c\Lambda(f^+) - c\Lambda(f^-) = \Lambda(cf^+) - \Lambda(cf^-) = \Lambda(cf).$$

For $c < 0$,

$$(cf)^+ = \frac{1}{2}(|cf| + cf) = c\frac{1}{2}(-|f| + f) = -c\frac{1}{2}(|f| - f) = -cf^- \quad \text{and}$$

$$(cf)^- = \frac{1}{2}(|cf| - cf) = c\frac{1}{2}(-|f| - f) = -c\frac{1}{2}(|f| + f) = -cf^+.$$

Therefore,

$$\Lambda(cf) = \Lambda((cf)^+) - \Lambda((cf)^-) = \Lambda(-c f^-) - \Lambda(-c f^+)$$

$$= -c\Lambda(f^-) + c\Lambda(f^+) = c\Lambda(f).$$

Thus for real valued $f \in C_c(X)$ and any real valued c, $\Lambda(cf) = c\Lambda(f)$.

Now suppose $f_1, f_2 \in C_c(X)$ and f_1 and f_2 are real-valued. We observe that

$$f_1 + f_2 = (f_1 + f_2)^+ - (f_1 + f_2)^- = f_1^+ + f_2^+ - (f_1^- + f_2^-).$$

Therefore,

$$(f_1 + f_2)^+ + (f_1^- + f_2^-) = (f_1 + f_2)^- + f_1^+ + f_2^+ \geq 0 \text{ and so}$$

$$\Lambda((f_1 + f_2)^+) + \Lambda(f_1^-) + \Lambda(f_2^-) = \Lambda((f_1 + f_2)^-) + \Lambda(f_1^+) + \Lambda(f_2^+).$$

It follows that

$$\Lambda\left((f_1+f_2)^+\right)-\Lambda\left((f_1+f_2)^-\right)=\Lambda\left(f_1^+\right)-\Lambda\left(f_1^-\right)+\Lambda\left(f_2^+\right)-\Lambda\left(f_2^-\right).$$

Hence, $\Lambda(f_1+f_2)=\Lambda(f_1)+\Lambda(f_2)$.

Suppose for any $f\in C_c(X)$ and f is complex valued, $f=\operatorname{Re}f+i\operatorname{Im}f$. For real scalar *c*,

$$\Lambda(cf)=\Lambda(c\operatorname{Re}f+ic\operatorname{Im}f)=\Lambda(c\operatorname{Re}f)+i\Lambda(c\operatorname{Im}f)=c\Lambda(\operatorname{Re}f)+ic\Lambda(\operatorname{Im}f)$$

$$=c\left(\Lambda(\operatorname{Re}f)+i\Lambda(\operatorname{Im}f)\right)=c\Lambda(f).$$

Suppose *c* is a complex scalar, $c=\operatorname{Re}c+i\operatorname{Im}c$.

Then $cf=\operatorname{Re}c\operatorname{Re}f-\operatorname{Im}c\operatorname{Im}f+i(\operatorname{Re}c\operatorname{Im}f+\operatorname{Im}c\operatorname{Re}f)$. It follows that

$$\Lambda(cf)=\operatorname{Re}c\Lambda(\operatorname{Re}f)-\operatorname{Im}c\Lambda(\operatorname{Im}f)+i\left(\operatorname{Re}c\Lambda(\operatorname{Im}f)+\operatorname{Im}c\Lambda(\operatorname{Re}f)\right)$$

$$=(\operatorname{Re}c+i\operatorname{Im}c)\left(\Lambda(\operatorname{Re}f)+i\Lambda(\operatorname{Im}f)\right)=c\Lambda(f).$$

Suppose $f_1,f_2\in C_c(X)$, $f_1=\operatorname{Re}f_1+i\operatorname{Im}f_1$ and $f_2=\operatorname{Re}f_2+i\operatorname{Im}f_2$.

$$\Lambda(f_1+f_2)=\Lambda(\operatorname{Re}f_1+\operatorname{Re}f_2)+i\Lambda(\operatorname{Im}f_1+\operatorname{Im}f_2)$$

$$=\Lambda(\operatorname{Re}f_1)+\Lambda(\operatorname{Re}f_2)+i\left(\Lambda(\operatorname{Im}f_1)+\Lambda(\operatorname{Im}f_2)\right)$$

$$=\Lambda(f_1)+\Lambda(f_2).$$

Hence Λ is a complex linear functional on $C_c(X)$.

Let $M(X)$ be the collection of all regular Borel complex measures with norm given by $\|\mu\|=|\mu|(X)$. Note that if μ_1 and μ_2 are regular complex measures, then $\mu_1+\mu_2$ is also a regular complex Borel measure, by Proposition 21. Obviously, for any complex number *c*, and $\mu\in M(X)$, $c\mu$ is a regular complex Borel measure. Thus, $M(X)$ is a complex linear space. Define a norm on $M(X)$, by $\|\mu\|=|\mu|(X)$ for $\mu\in M(X)$. We check that this is indeed a norm.

Plainly, for all $\mu\in M(X)$, $\|\mu\|=|\mu|(X)\geq 0$ and $\|c\mu\|=|c||\mu|(X)$. $\|\mu\|=|\mu|(X)=0\Leftrightarrow\mu=0$. Suppose $\mu_1,\mu_2\in M(X)$. For any $E\in\mathscr{M}$,

$$|\mu_1+\mu_2|(E) = \sup_{\text{all partitions } \{E_i\} \text{ of } E} \sum |(\mu_1+\mu_2)(E_i)| \le \sup_{\text{all partitions } \{E_i\} \text{ of } E} \sum \left(|\mu_1(E_i)| + |\mu_2(E_i)|\right)$$

$$\le \sup_{\text{all partitions } \{E_i\} \text{ of } E} \sum |\mu_1(E_i)| + \sup_{\text{all partitions } \{E_i\} \text{ of } E} \sum |\mu_2(E_i)| = |\mu_1|(E) + |\mu_2|(E).$$

Therefore, $\|\mu_1+\mu_2\| = |\mu_1+\mu_2|(X) \le |\mu_1|(X) + |\mu_2|(X) = \|\mu_1\| + \|\mu_2\|$. Therefore, $M(X)$ is a normed complex linear space. The Riesz Representation Theorem says that $M(X)$ is a Banach space. Let $(C_0(X))^*$ be the set of bounded complex linear functionals on $C_0(X)$ with the uniform sup norm. By the Riesz Representation Theorem, to each bounded complex linear functional Φ, there corresponds a unique complex regular measure μ such that $\Phi(f) = \int_X f\, d\mu$ for all $f \in C_0(X)$.

Let $\Gamma : (C_0(X))^* \to M(X)$ be this correspondence. That is, $\Gamma(\Phi) = \mu$. This correspondence is linear. Let $\Phi_1, \Phi_2 \in (C_0(X))^*$. Suppose $\Gamma(\Phi_i) = \mu_i$ for i =1, 2. Then for all $f \in C_0(X)$,

$$(\Phi_1+\Phi_2)(f) = \Phi_1(f) + \Phi_2(f) = \int_X f\, d\mu_1 + \int_X f\, d\mu_2 = \int_X f\, d(\mu_1+\mu_2).$$

Thus, $\Phi_1+\Phi_2$ is represented by $\mu_1+\mu_2$ and so by uniqueness, $\Gamma(\Phi_1+\Phi_2) = \mu_1+\mu_2 = \Gamma(\Phi_1)+\Gamma(\Phi_2)$. If c is a complex number, then for $\Phi \in (C_0(X))^*$ with $\Gamma(\Phi) = \mu$, for any $f \in C_0(X)$, $c\Phi(f) = c\int_X f\, d\mu = \int_X f\, d(c\mu)$. It follows by uniqueness that $c\Phi$ is represented by $c\mu = c\Gamma(\Phi)$. Hence $\Gamma(c\Phi) = c\Gamma(\Phi)$. Thus, Γ is a complex linear transformation. Note that Γ is norm preserving. We deduce this as follows. Suppose $\Gamma(\Phi) = \mu$, then $\|\Phi\| = |\mu|(X) = \|\mu\|$. This means $\|\Gamma(\Phi)\| = \|\Phi\|$. By uniqueness of the Riesz Representation, Γ is injective. Γ is also onto. Take any $\mu \in M(X)$. Then take $\Phi : C_0(X) \to \mathbb{C}$ given by $\Phi(f) = \int_X f\, d\mu$. Then Φ is a bounded complex linear functional and by the uniqueness of the Riesz Representation, $\Gamma(\Phi) = \mu$. Hence Γ is a norm preserving isometric isomorphism. Since any conjugate space is a Banach space, $(C_0(X))^*$ is a Banach space and so $M(X)$ is a Banach space.

We present a proof that $(C_0(X))^*$ is a Banach space modelled on the standard proof.

Let $\{\Phi_n\}$ be a Cauchy sequence in $(C_0(X))^*$. This means given any $\varepsilon > 0$, there exists an integer N, such that

$$n, m \geq N \Rightarrow \|\Phi_n - \Phi_m\| < \varepsilon \ .$$

For a fixed $f \in C_0(X)$, $\{\Phi_n(f)\}$ is a Cauchy sequence in $\mathbb{C}$. This is because

$|\Phi_n(f) - \Phi_m(f)| \leq \|\Phi_n - \Phi_m\| \|f\|_u < \varepsilon \|f\|_u$ for n, $m \geq N$. Therefore, $\lim_{n\to\infty} \Phi_n(f)$ exists for each $f \in C_0(X)$. Let $T(f) = \lim_{n\to\infty} \Phi_n(f)$. Then plainly, T is a complex linear functional. We claim that T is bounded. Since $\{\Phi_n\}$ is a Cauchy sequence, $\{\Phi_n\}$ is bounded. That is, there exists $K > 0$ such that $\|\Phi_n\| \leq K$ for all integer $n \geq 1$. Therefore, $|T(f)| = \lim_{n\to\infty} |\Phi_n(f)| \leq \lim_{n\to\infty} \|\Phi_n\| \|f\|_u \leq K \|f\|_u$. It follows that $\|T\| \leq K < \infty$ and so T is a bounded complex linear functional.

We now show that $\Phi_n \to T$ in norm.

For each $f \in C_0(X)$ with $\|f\|_u = 1$, for n, $m \geq N$.

$$|\Phi_n(f) - \Phi_m(f)| \leq \|\Phi_n - \Phi_m\| \|f\|_u = \|\Phi_n - \Phi_m\| < \varepsilon \ .$$

Letting $m \to \infty$, we get $|\Phi_n(f) - T(f)| \leq \varepsilon$ for $\|f\|_u = 1$. It follows that $\|\Phi_n - T\| \leq \varepsilon$. This is because for any linear functional, $H : V \to \mathbb{C}$, on a normed linear space $(V, \|\cdot\|)$, $\|H\| = \sup\left\{\frac{|H(x)|}{\|x\|} : x \neq 0\right\} = \sup\left\{\left|H\left(\frac{x}{\|x\|}\right)\right| : x \neq 0\right\} = \sup\{|H(x)| : \|x\| = 1\}$.

Therefore, $\Phi_n \to T$ in norm. Thus, any Cauchy sequence in $(C_0(X))^*$ converges in norm and so $(C_0(X))^*$ is a Banach space.

This completes the proof of Theorem 20.

We now state the real measure version of theorem 20.

Theorem 22. Riesz Representation Theorem - The Real Version

Let X be a locally compact Hausdorff topological space, $C_0(X, \mathbb{R})$ be the space of continuous real valued functions on X vanishing at infinity and $\Phi : C_0(X, \mathbb{R}) \to \mathbb{R}$

a continuous real linear functional on $C_0(X,\mathbb{R})$ with the uniform sup norm. Then there exists a unique regular real Borel measure μ such that

$$\Phi(f) = \int_X f\, d\mu .$$

Moreover, $\|\Phi\| = |\mu|(X)$. That is to say, the dual space or conjugate space of $C_0(X,\mathbb{R})$, $(C_0(X,\mathbb{R}))^* \cong M(X,\mathbb{R})$, where $M(X,\mathbb{R})$ is the collection of all regular Borel real measures with norm given by $\|\mu\| = |\mu|(X)$ and "$\cong$" here means isometric isomorphism preserving norm.

Proof. The proof is similar to that for Theorem 20. We use the corresponding results for real measure and real linear functional.

Uniqueness.

Suppose μ_1 and μ_2 are two regular real Borel measures satisfying the conclusion of the theorem. Then

$$0 = \Phi(f) - \Phi(f) = \int_X f\, d\mu_1 - \int_X f\, d\mu_2 = \int_X f\, d\nu \text{, where } \nu = \mu_2 - \mu_2 .$$

By Proposition 21, $\nu = \mu_2 - \mu_2$ is also a regular Borel measure.

By a real version of Theorem 10 (the proof of which is exactly the same via replacing open disks by open intervals), there exists a measurable real valued function $h : X \to \mathbb{R}$ such that $h \in L^1(X,|\nu|)$, $|h| = 1$ and $d\nu = h\, d|\nu|$. That is, for any $E \in \mathscr{M}$,

$$\nu(E) = \int_E h\, d|\nu|$$

and for any $f \in C_0(X,\mathbb{R})$, $\int_X f\, d\nu = \int_X f\, h\, d|\nu|$.

We shall show that $|\nu|(X) = 0$. Then since for all $E \in \mathscr{M}$, $|\nu(E)| \le |\nu|(E) \le |\nu|(X) = 0$, $|\nu(E)| = 0$. It follows that for all $E \in \mathscr{M}$, $\nu(E) = 0$ and so $\mu_1 = \mu_2$.

Now,

$$|\nu|(X) = \int_X d|\nu| = \int_X hhd|\nu| = \int_X hhd|\nu| - \int_X f\, d\nu$$

$$= \int_X hh\, d|\nu| - \int_X f\, hd|\nu| = \int_X h(h-f)d|\nu|$$

$$\leq \int_X |h-f|\, d|\nu| \, , \text{ ------------------------------ (*)}$$

for any $f \in C_0(X, \mathbb{R})$.

Since $C_c(X, \mathbb{R})$ is dense in $L^1(X, |\nu|)$, the space of all real valued $|\nu|$-integrable functions (see the real version of Theorem 23, *Convex Functions, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*), when $C_c(X, \mathbb{R})$ is endowed with the $L^1(X, |\nu|)$ norm and since $h \in L^1(X, |\nu|)$, we can take a sequence of functions $\{f_n\}$ in $C_c(X, \mathbb{R})$ such that $f_n \to h$ in $L^1(X, |\nu|)$ so that

$$\int_X |h - f_n|\, d|\nu| \to 0 \text{ as } n \to \infty .$$

It follows then from (*) that $|\nu|(X) = 0$.

Note that given a bounded linear functional Φ on $C_0(X, \mathbb{R})$, if $\|\Phi\| \neq 0$, we may normalise it by taking $\frac{1}{\|\Phi\|}\Phi$ so that its norm is unity. If $\|\Phi\| = 0$, we can just take the trivial Borel measure. So now we assume that $\|\Phi\| > 0$ and normalise it by considering $\frac{1}{\|\Phi\|}\Phi$. We shall thus assume without lost of generality that $\|\Phi\| = 1$.

As in the proof of Theorem 20, the key to the proof is to use the positive measure version of the Riesz Representation Theorem (Theorem 1, *Positive Borel Measure and Riesz Representation Theorem*).

Assume that we can construct a positive real linear functional Λ on $C_c(X, \mathbb{R})$ such that

$$|\Phi(f)| \leq \Lambda(|f|) \leq \|f\|_u . \qquad \text{-------------------------- (1)}$$

Then we can apply the positive measure version of Riesz Representation Theorem (Theorem 1, *Positive Borel Measure and Riesz Representation Theorem*) to Λ to give a positive Borel measure, λ, which is outer regular and inner regular with respect to open set and sets of finite measure, such that

$$\Lambda(f) = \int_X f\, d\lambda \text{, for all } f \in C_c(X, \mathbb{R}).$$

Note that $\lambda(X) = \sup\{\Lambda(f) : f \in C_c(X, \mathbb{R}), f \prec X\} = \sup\{\Lambda(f) : f \in C_c(X, \mathbb{R}), 0 \le f \le 1\}$. It follows from (1) that for $f \in C_c(X, \mathbb{R})$

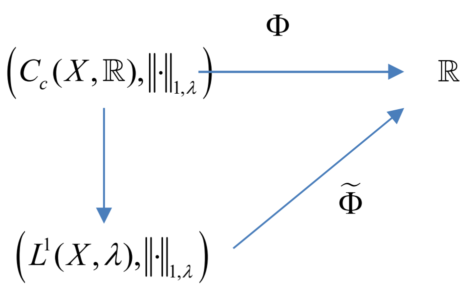
$$|\Lambda(f)| \le \int_X |f|\, d\lambda = \Lambda(|f|)$$

$$\le \|f\|_u \le 1 \text{, if } 0 \le f \le 1 .$$

Hence $\lambda(X) \le 1 < \infty$ and so by Theorem 1 of *Positive Borel Measure and Riesz Representation Theorem*, λ is inner and outer regular for all measurable sets in $\mathscr{M}$, i.e., a finite regular Borel measure.

By (1), $|\Phi(f)| \le \Lambda(|f|) = \int_X |f| d\lambda = \|f\|_{1,\lambda}$ for all $f \in C_c(X, \mathbb{R})$.

Therefore, $\Phi : \left(C_c(X, \mathbb{R}), \|\cdot\|_{1,\lambda}\right) \to \mathbb{R}$ is a bounded real linear functional of norm $\|\Phi\|_1$ less than or equal to 1. Therefore, we can extend Φ by continuity to $L^1(X, \lambda)$, since $\left(C_c(X, \mathbb{R}), \|\cdot\|_{1,\lambda}\right)$ is dense in $L^1(X, \lambda)$ (see Theorem 23, *Convex Functions, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*). Let the extension be denoted by $\tilde{\Phi} : \left(L^1(X, \lambda), \|\cdot\|_{1,\lambda}\right) \to \mathbb{R}$. Then $\tilde{\Phi}$ is a bounded real linear functional on $L^1(X, \lambda)$.



Therefore, by Theorem 16 (real version), as λ is a finite positive measure, there exists a unique g in $L^\infty(X, \lambda)$ such that

$$\tilde{\Phi}(f) = \int_X f\, g\, d\lambda \text{, ---------------------------- (2)}$$

for all $f \in L^1(X, \lambda)$. Moreover, $\|g\|_{\infty,\lambda} = \left\|\tilde{\Phi}\right\|_1 = \|\Phi\|_1 \le 1$. It follows that $|g| \le 1$ almost everywhere with respect to λ.

(2) is of course valid for $\Phi:\left(C_c(X,\mathbb{R}),\|\cdot\|_{1,\lambda}\right)\to\mathbb{R}$.

Note that $\Phi:\left(C_c(X,\mathbb{R}),\|\cdot\|_u\right)\to\mathbb{R}$ is a bounded real linear functional and

$$\Phi(f)=\int_X f\,g d\lambda \text{ for all } f\in C_c(X,\mathbb{R}). \text{ ------------------- (3)}$$

Observe that the left-hand side of (3) is the restriction to $C_c(X,\mathbb{R})$ of our original bounded (continuous) linear functional Φ on $C_0(X,\mathbb{R})$ with the uniform sup norm. The right-hand side of (3), $\int_X f\,g d\lambda$, is also a bounded linear functional on $C_0(X,\mathbb{R})$ with the uniform norm. Since $\left(C_c(X,\mathbb{R}),\|\cdot\|_u\right)$ is dense in $\left(C_0(X,\mathbb{R}),\|\cdot\|_u\right)$, (3) holds for all $f\in C_0(X,\mathbb{R})$.

Extend (3) to the completion of $C_c(X,\mathbb{R})$ in the uniform norm, $C_0(X,\mathbb{R})$. Therefore, we can write for all $f\in C_0(X,\mathbb{R})$,

$$\Phi(f)=\int_X f\,d\mu, \text{ where } d\mu=g\,d\lambda.$$

More precisely, for any $E\in\mathcal{M}$, $\mu(E)=\int_E g\,d\lambda$. It can be easily check that μ is a real measure.

From (3), for all $f\in C_0(X,\mathbb{R})$, $|\Phi(f)|=\left|\int_X f\,g d\lambda\right|\le\|f\|_u\int_X|g|\,d\lambda\le\int_X|g|\,d\lambda$ if $\|f\|_u\le 1$. Thus,

$$\|\Phi\|=\sup\left\{\frac{|\Phi(f)|}{\|f\|_u}: f\in C_0(X,\mathbb{R}), f\ne 0\right\}=\sup\left\{|\Phi(f)|: f\in C_0(X,\mathbb{R}),\|f\|_u\le 1\right\}\le\int_X|g|d\lambda.$$

Since $\|\Phi\|=1$, we have $1=\|\Phi\|\le\int_X|g|d\lambda\le\int_X d\lambda=\lambda(X)$ since $|g|\le 1$ almost everywhere with respect to λ. We have previously shown that $\lambda(X)\le 1$ and so it follows that $\lambda(X)=1$ and $|g|=1$ almost everywhere with respect to λ. Since $d\mu=g\,d\lambda$ and $g\in L^1(X,\lambda)$, by Proposition 12, $d|\mu|=|g|\,d\lambda=d\lambda$. Therefore,

$$|\mu|(X)=\lambda(X)=1=\|\Phi\|.$$

As λ is a regular measure, i.e., it is inner and outer regular for all measurable sets in $\mathcal{M}$, it follows that $|\mu|$ is regular and so μ is a regular real Borel measure.

It now remains to construct the positive real linear functional Λ on $C_c(X,\mathbb{R})$ with the required property (1).

Let $C_c^+(X,\mathbb{R})$ denote the set of non-negative real-valued functions in $C_c(X,\mathbb{R})$ and define for $f \in C_c^+(X,\mathbb{R})$,

$$\Lambda(f) = \sup\left\{|\Phi(h)| : h \in C_c(X,\mathbb{R}) \text{ and } |h| \le f\right\}.$$

Then plainly, $\Lambda(f) \ge 0$ for all $f \in C_c^+(X,\mathbb{R})$ and Λ satisfies

$$|\Phi(f)| \le \Lambda(|f|) \text{ for all } f \in C_c(X,\mathbb{R}).$$

This is because $|f| \in C_c^+(X,\mathbb{R})$ and $\Lambda(|f|) = \sup\left\{|\Phi(h)| : h \in C_c(X,\mathbb{R}) \text{ and } |h| \le |f|\right\}$.

Now, for any $h \in C_c(X,\mathbb{R})$, $|\Phi(h)| \le \|\Phi\|\|h\|_u = \|h\|_u$, since $\|\Phi\| = 1$. If $|h| \le |f|$, $\|h\|_u \le \|f\|_u$. It follows that for all $f \in C_c(X,\mathbb{R})$,

$$\Lambda(|f|) = \sup\left\{|\Phi(h)| : h \in C_c(X,\mathbb{R}) \text{ and } |h| \le |f|\right\} \le \|f\|_u.$$

We have thus shown that (1) holds. It remains to show that Λ can be extended to $C_c(X,\mathbb{R})$.

Now, $\Lambda : C_c^+(X,\mathbb{R}) \to \mathbb{R}$ is non-negative and by definition, if $f_1, f_2 \in C_c^+(X,\mathbb{R})$, $f_1 \le f_2 \Rightarrow \Lambda(f_1) \le \Lambda(f_2)$. Obviously, for any real number $c \ge 0$, $\Lambda(cf) = c\Lambda(f)$ for any $f \in C_c^+(X,\mathbb{R})$. We need to show that $\Lambda(f_1 + f_2) = \Lambda(f_1) + \Lambda(f_2)$ for $f_1, f_2 \in C_c^+(X,\mathbb{R})$.

We show that $\Lambda(f_1) + \Lambda(f_2) \le \Lambda(f_1 + f_2)$. By definition of $\Lambda(f_i)$, $i = 1, 2$, given $\varepsilon > 0$, there exists $h_i \in C_c(X,\mathbb{R})$ such that $|h_i| \le f_i$ and

$$\Lambda(f_i) - \varepsilon < |\Phi(h_i)| \text{ for } i = 1, 2.$$

We can write $|\Phi(h_i)| = \alpha_i \Phi(h_i)$ for some number α_i with $\alpha_i = \pm 1$ for $i = 1, 2$. Then

$$\Lambda(f_1) + \Lambda(f_2) < |\Phi(h_1)| + |\Phi(h_2)| + 2\varepsilon = \alpha_1\Phi(h_1) + \alpha_2\Phi(h_2) + 2\varepsilon = \Phi(\alpha_1 h_1 + \alpha_2 h_2) + 2\varepsilon.$$

Hence, $\Lambda(f_1) + \Lambda(f_2) < |\Phi(\alpha_1 h_1 + \alpha_2 h_2)| + 2\varepsilon$.

Since $|\alpha_1 h_1 + \alpha_2 h_2| \le |h_1| + |h_2| \le f_1 + f_2$, $|\Phi(\alpha_1 h_1 + \alpha_2 h_2)| \le \Lambda(f_1 + f_2)$. It follows that

$$\Lambda(f_1) + \Lambda(f_2) < \Lambda(f_1 + f_2) + 2\varepsilon .$$

Since ε is arbitrary, $\Lambda(f_1) + \Lambda(f_2) \le \Lambda(f_1 + f_2)$.

Now, we show that $\Lambda(f_1 + f_2) \le \Lambda(f_1) + \Lambda(f_2)$.

Take $h \in C_c(X, \mathbb{R})$ with $|h| \le f_1 + f_2$. Let $V = \{x \in X : f_1(x) + f_2(x) > 0\}$. Then V is open in X. For i = 1, 2, let

$$h_i(x) = \begin{cases} \dfrac{f_i(x)h(x)}{f_1(x) + f_2(x)}, x \in V \\ 0, x \in V^c \end{cases} .$$

Then $|h_i(x)| \le f_i(x)$ for i = 1,2, since $\dfrac{|h(x)|}{f_1(x) + f_2(x)} \le 1$ for $x \in V$.

Note that $h_i(x) \ne 0 \Rightarrow h(x) \ne 0$ for x in V. Therefore, support $h_i \subseteq$ support h, which is compact. We claim that h_i is continuous for i =1 , 2. Plainly, h_i is continuous on V for i =1 , 2. For $a \in V^c$, $h_1(a) = 0$ but since $|h| \le f_1 + f_2$, $h(a) = 0$. As $|h_1| \le |h|$ and h is continuous at a with $h(a) = 0$, h_1 is continuous at a. Hence, h_1 is continuous on V^c and so is continuous on X. Similarly, we deduce that h_2 is continuous on X. Hence, $h_1, h_2 \in C_c(X, \mathbb{R})$. Observe that $h_1 + h_2 = h$.

Therefore,

$$|\Phi(h)| = |\Phi(h_1) + \Phi(h_2)| \le |\Phi(h_1)| + |\Phi(h_2)| \le \Lambda(f_1) + \Lambda(f_2).$$

Since this holds for all $h \in C_c(X, \mathbb{R})$ with $|h| \le f_1 + f_2$, $\Lambda(f_1 + f_2) \le \Lambda(f_1) + \Lambda(f_2)$. Therefore, $\Lambda(f_1 + f_2) = \Lambda(f_1) + \Lambda(f_2)$. We can now extend Λ to $C_c(X, \mathbb{R})$ as follows. For $f \in C_c(X, \mathbb{R})$, write f as $f = f^+ - f^-$, where $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$. Then $f^+, f^- \in C_c^+(X, \mathbb{R})$. Define

$$\Lambda(f) = \Lambda(f^+) - \Lambda(f^-).$$

It is easy to check that we have thus defined a real linear functional on $C_c(X, \mathbb{R})$.

Now for $c > 0$, $c\Lambda(f) = c\Lambda\left(f^+\right) - c\Lambda\left(f^-\right) = \Lambda\left(cf^+\right) - \Lambda\left(cf^-\right) = \Lambda\left(cf\right)$.

Suppose $c < 0$. Then for any $f \in C_c(X,\mathbb{R})$,

$$\left(cf\right)^+ = \frac{1}{2}\left(|cf| + cf\right) = c\frac{1}{2}\left(-|f| + f\right) = -c\frac{1}{2}\left(|f| - f\right) = -cf^- \text{ and}$$

$$\left(cf\right)^- = \frac{1}{2}\left(|cf| - cf\right) = c\frac{1}{2}\left(-|f| - f\right) = -c\frac{1}{2}\left(|f| + f\right) = -cf^+ .$$

Therefore,

$$\Lambda\left(cf\right) = \Lambda\left(\left(cf\right)^+\right) - \Lambda\left(\left(cf\right)^-\right) = \Lambda\left(-c\,f^-\right) - \Lambda\left(-c\,f^+\right)$$

$$= -c\Lambda\left(f^-\right) + c\Lambda\left(f^+\right) = c\Lambda(f) .$$

Now suppose $f_1, f_2 \in C_c(X,\mathbb{R})$. We note that

$$f_1 + f_2 = \left(f_1 + f_2\right)^+ - \left(f_1 + f_2\right)^- = f_1^+ + f_2^+ - \left(f_1^- + f_2^-\right).$$

Therefore,

$$\left(f_1 + f_2\right)^+ + \left(f_1^- + f_2^-\right) = \left(f_1 + f_2\right)^- + f_1^+ + f_2^+ \geq 0 \text{ and so}$$

$$\Lambda\left(\left(f_1 + f_2\right)^+\right) + \Lambda\left(f_1^-\right) + \Lambda\left(f_2^-\right) = \Lambda\left(\left(f_1 + f_2\right)^-\right) + \Lambda\left(f_1^+\right) + \Lambda\left(f_2^+\right).$$

It follows that

$$\Lambda\left(\left(f_1 + f_2\right)^+\right) - \Lambda\left(\left(f_1 + f_2\right)^-\right) = \Lambda\left(f_1^+\right) - \Lambda\left(f_1^-\right) + \Lambda\left(f_2^+\right) - \Lambda\left(f_2^-\right).$$

Hence, $\Lambda\left(f_1 + f_2\right) = \Lambda\left(f_1\right) + \Lambda\left(f_2\right)$.

It follows that Λ is a real linear functional on $C_c(X,\mathbb{R})$.

Let $M(X,\mathbb{R})$ be the collection of all regular real Borel measures with norm given by $\|\mu\| = |\mu|(X)$. Note that if μ_1 and μ_2 are regular Borel real measures, then $\mu_1 + \mu_2$ is also a regular real Borel measure, by Proposition 21. Obviously, for any real number c, and $\mu \in M(X,\mathbb{R})$, $c\mu$ is a regular real Borel measure. Thus,

$M(X,\mathbb{R})$ is a real linear space. Define a norm on $M(X,\mathbb{R})$, by $\|\mu\| = |\mu|(X)$ for $\mu \in M(X,\mathbb{R})$. As in the proof of Theorem 20, it is easy to check that this is indeed a norm. Thus, $M(X,\mathbb{R})$ is a normed real linear space. The Riesz Representation Theorem says that $M(X,\mathbb{R})$ is a Banach space as explained below. Let $(C_0(X,\mathbb{R}))^*$ be the set of all bounded real linear functionals on $C_0(X,\mathbb{R})$ with the uniform sup norm. By the Riesz Representation Theorem, to each bounded real linear functional Φ, there corresponds a unique real regular measure μ such that $\Phi(f) = \int_X f \, d\mu$ for all $f \in C_0(X,\mathbb{R})$.

Let $\Gamma : (C_0(X,\mathbb{R}))^* \to M(X,\mathbb{R})$ be this correspondence. That is, $\Gamma(\Phi) = \mu$. Γ is a norm preserving isometric isomorphism. Since any conjugate space is a Banach space, $(C_0(X,\mathbb{R}))^*$ is a Banach space and so $M(X,\mathbb{R})$ is a Banach space.

Remark.

Now a locally compact Hausdorff topological space is completely regular and a normal Hausdorff topological space is also completely regular. The proof of Theorem 20 uses indirectly Lusin's Theorem by using the result that $\left(C_c(X), \|\cdot\|_{1,\lambda}\right)$ is dense in $\left(L^1(X,\lambda), \|\cdot\|_{1,\lambda}\right)$, when *X* is locally compact and Hausdorff. There is a form of Lusin's Theorem for normal Hausdorff space *X* for *normal* measure μ on the Borel σ-algebra, which is outer regular, finite on closed sets and inner regular with respect to closed sets so that for a μ-measurable function $f : X \to \mathbb{C}$ and any $\varepsilon > 0$, there exists a bounded continuous function *g* such that $\mu\left(\{x \in X : g(x) \neq f(x)\}\right) < \varepsilon$. Note that the measure μ is special and is specified and not connected with any bounded positive linear functional on *BC*(*X*), the collection of all bounded continuous functions on *X*. For such a normal topological space, *X*, we use the following version of Urysohn's Lemma:

> Suppose *X* is a normal Hausdorff space, $U \subseteq X$ is open, *K* is closed with $K \subseteq U$. Then there exists $f \in BC(X)$ such that $K \prec f \prec U$.

However, if one starts with *BC*(*X*) and define the measure μ associated with a positive complex linear functional, $\Lambda : BC(X) \to \mathbb{C}$, as in the proof of Theorem 1 in *Positive Borel Measure and Riesz Represntation Theorem,* but using closed subsets instead of compact subsets and pursue the argument there, we find that we could not prove (1) (σ-subadditivity) there and hence we could not deduce countable additivity.

However, a normal Hausdorff space need not be locally compact and a locally compact Hausdorff space need not be normal with the exception of compact Hausdorff space, which is always normal.

For normal Hausdorff space X, and continuous real linear functional, it is possible to show that a continuous real linear functional, $\Lambda : BC(X,\mathbb{R}) \to \mathbb{R}$, where $BC(X,\mathbb{R})$ is the space of bounded continuous real valued functions endowed with the sup norm, is represented by a normal finitely-additive measure μ (i,e, μ: $\mathcal{M} \to \mathbb{R}$ is finitely additive, $|\mu|$ is outer regular with respect to open sets in $\mathcal{M}$ and also inner regular with respect to closed sets in $\mathcal{M}$ and $\mathcal{M}$ is the algebra generated by the open sets in X) with bounded variation, as

$$\Lambda(f) = \int_X f\,d\mu = \int_X f\,d\mu^+ - \int_X f\,d\mu^- \text{ for all } f \in BC(X,\mathbb{R}),$$

with $\|\Lambda\| = |\mu|(X)$.

Riesz Representation Theorem for normal Hausdorff space.

For normal Hausdorff space and positive complex linear functional on the vector space of bounded continuous complex function on X, without additional condition on X, we can only associate a finitely-additive measure on the algebra generated by the open sets of X. Thus, we shall discuss integration over a finitely-additive measure or charge.

Suppose A is an algebra of subsets of X. A set function $\mu : A \to [0,\infty]$ is said to be a *finitely-additive measure* or a *charge* if for any collection of finite number of pair-wise disjoint sets in A, $\{E_i\}_{1\le i<n}$, then $\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$. It is easy to see that for an additive measure μ, $\mu(\varnothing) = 0$ and μ is monotonic, that is, if B, $C \in A$ and $B \subseteq C$, $\mu(B) \le \mu(C)$. We shall use the terms, finitely additive measure and charge, interchangeably. A finitely-additive measure μ is said to be *finite* if $\mu(E) < \infty$ for all E in A. A *simple μ-measurable function*, s, is a linear combination of characteristic functions of sets in A of finite μ-measure, i.e., $s = \sum_{i=1}^{n} c_i \chi_{E_i}$, where $c_i \in \mathbb{R}$ or $\mathbb{C}$, $\mu\left(\chi_{E_i}\right) < \infty$ for $1 \le i \le n$. The *integral* of a simple function $s=\sum_{i=1}^{n} c_i \chi_{E_i}$ is defined as usual to be given by $\int_X s\,d\mu=\sum_{i=1}^{n} c_i \mu(E_i)$. It is easy to show that this integral is independent of the representation as simple function. Then by linearity, this definition of the integral is a linear functional on the collection of simple μ-measurable functions. In particular, this integral is a positive linear functional, that is, for a simple function φ, $\varphi \geq 0 \Rightarrow \int_X \varphi\,d\mu \geq 0$. Let $S(X)$ be the space of simple functions on X.

We now assume that μ is a finite charge, i.e., $\mu(X)<\infty$.

For a bounded real valued function f on X, we define the *lower Lebesgue integral* of f with respect to μ to be

$$\underline{\int_X} f\,d\mu=\sup\left\{\int_X \varphi\,d\mu : \varphi \leq f, \varphi \in S(X)\right\}$$

and the *upper Lebesgue integral* of f to be $\overline{\int_X} f\,d\mu=\inf\left\{\int_X \varphi\,d\mu : f \leq \varphi, \varphi \in S(X)\right\}$. Since f is bounded and $\mu(X)<\infty$,

$$-\infty<\underline{\int_X} f\,d\mu \leq \overline{\int_X} f\,d\mu<\infty .$$

We say a real valued function f is *μ-integrable* or simply *integrable* if the lower and upper Lebesgue integrals are the same and we denote the common value by $\int_X f\,d\mu$. A bounded complex valued function is μ-integrable if $\operatorname{Re} f$ and $\operatorname{Im} f$ are μ-integrable and

$$\int_X f\,d\mu=\int_X \operatorname{Re} f\,d\mu+i\int_X \operatorname{Im} f\,d\mu .$$

It is easy to prove the following regarding integration over a finitely-additive measure.

Theorem 23. The collection of all bounded μ-integrable functions on X with respect to a finite charge μ on an algebra of subsets of X is a vector space. Moreover, the integral is a continuous positive linear functional on the vector space of bounded μ-integrable functions with the sup norm.

In order to describe an integral not just for bounded function but for measurable function, we state the following version of the integrability of bounded measurable function over bounded finite charge.

Theorem 24. Let X be a non-empty set. Let A_X be an algebra of subsets of X. Let $A_{\mathbb{R}}$ be the algebra generated by the collection of half open intervals $\{[a,b) : a < b\}$. A bounded real valued function, f, is said to be $(A, A_{\mathbb{R}})$-measurable, if for any U in $A_{\mathbb{R}}$, $f^{-1}(U) \in A$.

Every bounded $(A, A_{\mathbb{R}})$-measurable real valued function f is integrable with respect to any finite charge μ on A.

Suppose f is a bounded complex function on X such that both Re f and Im f are (A_X, $A_{\mathbb{R}}$)-measurable, then f is integrable and $\int_X f\,d\mu = \int_X \operatorname{Re} f\,d\mu + i\int_X \operatorname{Im} f\,d\mu$.

Proof.

Let $f : X \to \mathbb{R}$ be a bounded (A_X, $A_{\mathbb{R}}$)-measurable function. Therefore, there exists $M > 0$ satisfying $-M < f(x) < M$ for each $x \in X$. Let $\varepsilon > 0$.

Given $\varepsilon > 0$, partition $[-M, M]$ as follows

$$-M = y_0 < y_1 < y_2 < \cdots < y_n = M \quad \text{with } y_i - y_{i-1} < \varepsilon \text{ for } 1 \le i \le n.$$

Let $E_i = \{x \in X : y_{i-1} \le f(x) < y_i\}$ for $1 \le i \le n$. That is, $E_i = f^{-1}([y_{i-1}, y_i))$. Since f is (A_X, $A_{\mathbb{R}}$)-measurable, each $E_i \in A_X$. Moreover, $\{E_i\}$ are pairwise disjoint and $\mu(E_i) \le \mu(X) < \infty$. The simple μ-measurable functions, $\varphi = \sum_{i=1}^{n} y_i \chi_{E_i}$ and $\psi = \sum_{i=1}^{n} y_{i-1} \chi_{E_i}$ satisfy $\psi \le f \le \varphi$. Then

$$\int_X (\varphi - \psi)\,d\mu = \sum_{i=1}^{n} (y_i - y_{i-1})\mu(E_i) < \varepsilon \sum_{i=1}^{n} \mu(E_i) = \varepsilon\mu(X).$$

It follows by the usual characterization of integrability that f is integrable.

Theorem 25. If X is a topological space and μ: $A_X \to [0,\infty)$ is a finite charge on the algebra A_X generated by the open sets of X, then $BC(X,\mathbb{R})$ the real vector space of bounded continuous real valued functions on X, is a partially ordered real vector space of all bounded μ-integrable real functions on X. The ordering is the usual ordering: $f,g \in BC(X,\mathbb{R})$, $f \le g \Leftrightarrow f(x) \le g(x)$ for all x in X.

Proof: If $f : X \to \mathbb{R}$ is continuous, then it is (A_X, $A_{\mathbb{R}}$)-measurable, since

$$f^{-1}([a,b)) = f^{-1}\left((-\infty,b)\right) \cap \left(f^{-1}\left((-\infty,a)\right)\right)^c \in A_X ,$$

The conclusion then follows from Theorem 23.

Proposition 26. Suppose Y is a normal Hausdorff topological space. Suppose U is open in Y and K is a closed subset such that $K \subseteq U$. Then there is an open neighbourhood of K, i.e., an open set V such that $K \subseteq V \subseteq \overline{V} \subseteq U$.

Proof.

Note that K and U^c are disjoint closed set. Therefore, by normality there exist open sets V and W such that $K \subseteq V$, $U^c \subseteq W$ and $V \cap W = \varnothing$. Hence $V \subseteq W^c \subseteq U$. Since W^c is closed, $\overline{V} \subseteq W^c \subseteq U$.

Lemma 27. Urysohn's Lemma for normal Hausdorff space

Suppose X is a normal Hausdorff space, $U \subseteq X$ is open, K is closed with $K \subseteq U$. Then there exists $f \in BC(X)$ such that $K \prec f \prec U$.

Proof.

We shall make use of the rational number in [0, 1] to construct the Urysohn function f. Take an enumeration $r : \mathbb{N} \to [0,1]$ of the rational numbers, i.e., a bijective function of $\mathbb{N}$ onto [0, 1] such that $r_1 = r(1) = 0$ and $r_2 = r(2) = 1$. We denote the image $r(k)$ by r_k.

Suppose K is closed, $K \subseteq U$ and U is open.

Let $U_{r_1} = U_0$ be the open neighbourhood of K as given by Proposition 26, such that

$$K \subseteq U_0 \subseteq \overline{U_0} \subseteq U \text{ ------------------ } (1).$$

Since $K \subseteq U_0$, let $U_{r_2} = U_1$ be the open neighbourhood of *K* as given by Proposition 26 such that

$$K \subseteq U_1 \subseteq \overline{U_1} \subseteq U_0 \text{ ------------------ (2).}$$

We shall inductively define the open set U_{r_k}.

Suppose $U_{r_1}, U_{r_2}, \cdots, U_{r_n}$ have been chosen so that if $r_i < r_j$, $j \le n$, then $U_{r_j} \subseteq \overline{U_{r_j}} \subseteq U_{r_i}$. Then arrange $r_1, r_2, \cdots, r_n$ in increasing order. Suppose in this sequence $r_i < r_{n+1} < r_j$. Then using $\overline{U_{r_j}} \subseteq U_{r_i}$, by Proposition 26 choose open $U_{r_{n+1}}$ such that

$$\overline{U_{r_j}} \subseteq U_{r_{n+1}} \subseteq \overline{U_{r_{n+1}}} \subseteq U_{r_i}. \text{ ---------------------------- (3)}$$

In this way we obtain a collection of open sets $\{U_r : r \text{ rational } \in [0,1]\}$ satisfying $\overline{U_s} \subseteq U_r$ whenever *s* > *r*, $K \subseteq U_1$ and $\overline{U_0} \subseteq U$.

Define a collection of functions $\{f_r : r \text{ rational } \in [0,1]\}$ by defining $f_r : X \to [0,1]$ by

$$f_r(x) = \begin{cases} r , & \text{if } x \in U_r \\ 0 , & \text{otherwise} \end{cases}$$

and a collection $\{g_s : s \text{ rational } \in [0,1]\}$ by defining $g_s : X \to [0,1]$ by

$$g_s(x) = \begin{cases} 1 , & \text{if } x \in \overline{U_s} \\ s , & \text{otherwise} \end{cases} .$$

Note that $f_r = r\chi_{U_r}$. Since U_r is open for each rational $r \in [0,1]$, f_r is lower semi-continuous for each rational $r \in [0,1]$. Observe that $\{x : g_s(x) < \alpha\} = X$ if α > 1, $\{x : g_s(x) < \alpha\} = \left(\overline{U_s}\right)^c$ if $s < \alpha \le 1$ and $\{x : g_s(x) < \alpha\} = \varnothing$ if $\alpha \le s$. Thus g_s is upper semi-continuous for each rational $s \in [0,1]$.

Therefore, by Proposition 21, *Convex Function, L^p Spaces, Space of Continuous Functions, Lusin's Theorem,* $f = \sup_r \{f_r : r \text{ rational } \in [0,1]\}$ is lower semi-continuous and $g = \inf_r \{g_r : r \text{ rational } \in [0,1]\}$ is upper semi-continuous.

We shall next show that $f = g$ and thus deduce that f is both lower and upper semi-continuous and so f is continuous.

Firstly, we show that $f \le g$.

Suppose on the contrary, there exists x in X such that $f(x) > g(x)$. Then by the definition of supremum, there exists r in $\mathbb{Q} \cap [0,1]$ such that $f_r(x) > g(x)$. Next by the definition of infimum, there exists s in $\mathbb{Q} \cap [0,1]$ such that $f_r(x) > g_s(x)$. This can only happen if $x \notin \overline{U_s}$, $x \in U_r$ and $r > s$. But $r > s$ implies that $U_r \subseteq \overline{U_r} \subseteq U_s \subseteq \overline{U_s}$ and so $x \in \overline{U_s}$ and we have a contradiction. This proves that $f \le g$.

Next, we show that $f \ge g$.

Suppose on the contrary, there exists x in X such that $f(x) < g(x)$. Then by the density of the rational numbers we can find rational numbers r and s such that $f(x) < s < r < g(x)$.

Since $f(x) < s$, $x \notin U_s$ and since $g(x) > r$, $x \in \overline{U_r}$. As $s < r$, $U_r \subseteq \overline{U_r} \subseteq U_s \subseteq \overline{U_s}$ and so $x \in U_s$ and we arrived at a contradiction and so we have $f \ge g$. Hence, $f = g$.

Plainly $0 \le f \le 1$. Now observe that $U_r \subseteq \overline{U_0}$ for all r in $\mathbb{Q} \cap [0,1]$. Therefore, $f(x) \ne 0 \Rightarrow x \in \overline{U_0}$ and it follows that support $f = \overline{\{x : f(x) \ne 0\}} \subseteq \overline{U_0} \subseteq U$ and so the support of f is in U. Hence $f \prec U$. As $K \subseteq U_r$ for all r in $\mathbb{Q} \cap [0,1]$, $f(x) = 1$ for all x in K. Therefore, $K \prec f$. It follows that $K \prec f \prec U$.

Our next technical lemma is a partition of unity for normal topological spaces.

Theorem 28. Partition of unity for normal Hausdorff space

Suppose X is a normal Hausdorff topological space. Then any closed subspace F of X is a normal space.

Suppose $\{U_1, U_2, \cdots, U_n\}$ is an open covering of the closed set F, i.e., U_i is open in X for $1 \le i \le n$ and $F \subseteq \bigcup_{i=1}^{n} U_i$. Then there exists $h_i \in BC(X, \mathbb{R})$, a continuous bounded real valued function on X, such that $h_i \prec U_i$ for $1 \le i \le n$ and $h_1 + h_2 + \cdots + h_n = 1$ on F. That is to say, $0 \le h_i \le 1$ and $h_i = 0$ on $(U_i)^c$ for $1 \le i \le n$. The collection of continuous functions, $\{h_1, h_2, \cdots, h_n\}$, is called a *partition of unity on F subordinate to the covering* $\{U_1, U_2, \cdots, U_n\}$ of F.

Proof.

Our first task is to shrink the covering $\{U_1, U_2, \cdots, U_n\}$ to another covering $\{V_1, V_2, \cdots, V_n\}$ of F such that $\overline{V_i} \subseteq U_i$ for $1 \le i \le n$.

We shall proceed to this by induction. Let $B = F - \bigcup_{i=2}^{n} U_i$. Then B is closed in X and

$$B = F \cap \left(\bigcup_{i=2}^{n} U_i \right)^c = F \cap \left(\bigcup_{i=2}^{n} U_i \right)^c = F \cap \left(\bigcup_{i=i}^{n} U_i \right) \cap \left(\bigcup_{i=2}^{n} U_i \right)^c \subseteq U_1 .$$

Therefore, by Proposition 25, there exists open set V_1 such that $B \subseteq V_1 \subseteq \overline{V_1} \subseteq U_1$.

Then $\{V_1, U_2, \cdots, U_n\}$ covers F. Next consider now $B = F - V_1 - \bigcup_{i=3}^{n} U_i$. Then B is closed in X and $B \subseteq U_2$. Again, by Proposition 26, there exists open set V_2 such that $B \subseteq V_2 \subseteq \overline{V_2} \subseteq U_2$. Then $\{V_1, V_2, U_3, \cdots, U_n\}$ covers F. In general, suppose $\{V_1, V_2, \cdots, V_k, U_{k+1}, \cdots, U_n\}$ covers F with $1 \le k < n$, with $V_i \subseteq \overline{V_i} \subseteq U_i$ for $1 \le i \le k$. Then let $B = F - \bigcup_{i=1}^{k} V_i - \bigcup_{i=k+2}^{n} U_i$ if $k \le n-2$. It follows that B is closed in X and $B \subseteq U_{k+1}$. Applying Proposition 26 again to give open set V_{k+1} such that $B \subseteq V_{k+1} \subseteq \overline{V_{k+1}} \subseteq U_{k+1}$. Proceeding in this way we get $\{V_1, V_2, \cdots, V_{n-1}, U_n\}$ covers F. Then $B = F - \bigcup_{i=1}^{n-1} V_i$ is closed in X and $B \subseteq U_n$. One more application of Proposition 26 gives an open set V_n with $B \subseteq V_n \subseteq \overline{V_n} \subseteq U_n$. Hence, $\{V_1, V_2, \cdots, V_n\}$ covers F.

Then by Lemma 27 (Urysohn's Lemma), there exists bounded continuous function g_i on X such that

$\overline{V_i} \prec g_i \prec U_i$ for $1 \le i \le n$.

Let $h_1 = g_1$, $h_2 = (1-g_1)g_2$, … , $h_n = (1-g_1)(1-g_2)\cdots(1-g_{n-1})g_n$. Since $g_i \prec U_i$, $h_i \prec U_i$ for $1 \le i \le n$. Now take any $x \in F$. Since $\{V_1, V_2, \cdots, V_n\}$ covers F, $x \in \overline{V_j}$ for some $1 \le j \le n$ and it follows that $g_j(x) = 1$. Now

$$h_1 + h_2 + \cdots + h_n = 1 - (1-g_1)(1-g_2)\cdots(1-g_n)\,. \quad \text{----------------------} \quad (*)$$

We can show this by induction. (*) is plainly true for n=1 and for n=2. If (*) is true for n−1, then

$$h_1 + h_2 + \cdots + h_{n-1} + h_n = 1 - (1-g_1)(1-g_2)\cdots(1-g_{n-1}) + g_n(1-g_1)(1-g_2)\cdots(1-g_{n-1})$$

$$= 1 - (1-g_n)(1-g_1)(1-g_2)\cdots(1-g_{n-1})\,.$$

For any $x \in F$, $(1-g_1(x))(1-g_2(x))\cdots(1-g_n(x)) = 0$ and so $h_1 + h_2 + \cdots + h_n = 1$ on F.

We are now ready to explore a Riesz type representation theorem for positive linear functional on the space $BC(X)$ with the uniform norm for normal Hausdorff space.

Theorem 29. (Riesz Representation Theorem For Normal Hausdorff Space)

Let X be a normal Hausdorff topological space. Let $BC(X) = \{f : X \to \mathbb{C};\, f \text{ is continuous and bounded}\}$. Let $\Lambda : BC(X) \to \mathbb{C}$ be a positive complex linear functional on $BC(X)$, i.e., whenever $f \in BC(X)$ and f is real valued with $f \ge 0$, then $\Lambda(f) \ge 0$. Then we have the following:

(a) There exists an algebra $\mathscr{M}$ on X, generated by the open sets of X and a unique finite finitely-additive measure μ, on $\mathscr{M}$, i.e., $\mu : \mathscr{M} \to \mathbb{R}$ is a finitely additive set function, such that

$$\Lambda(f) = \int_X f\, d\mu \quad \text{for all } f \in BC(X)\,.$$

(b) For all $E \in \mathscr{M}$, $\mu(E) < \infty$.

(c) For all $E \in \mathscr{M}$, $\mu(E) = \inf\{\mu(V) : V \supseteq E \text{ and } V \text{ is open}\}$. (Outer regularity)

(d) For all $E \in \mathscr{M}$,

$\mu(E) = \sup\{\mu(F) : F \subseteq E \text{ and } F \text{ is closed in } X\}$. (Inner regularity)

(e) $\mathscr{M}$ is μ-complete, i.e., for all $N \in \mathscr{M}$ such that $\mu(N) = 0$, for $E \subseteq N$, $E \in \mathscr{M}$.

Moreover, $\|\Lambda\| = \mu(X)$, when *BC*(*X*) is endowed with the sup norm.

Remark.

Assertion (d) is called *inner regular* only for finitely additive measure and is different from assertion (d) in the complex version of the Riesz representation theorem, where it means approximation from below by compact subsets and is sometimes refer to as *tight* measure. These two similarly named notions coincide when *X* is compact and Hausdorff.

A measure satisfying (c) and (d) in Theorem 29, is said to be *normal*.

Proof.

Firstly, we prove that the measure μ is unique. Then we show the existence of the measure μ. The remaining of the proof deals with the conclusions (b) (c) (d) and (e) of the theorem.

Uniqueness of μ.

Suppose μ_1 and μ_2 are two finite finitely-additive measures on $\mathscr{M}$ satisfying the conclusion of the theorem. Note that the value of the additive measure, μ, is entirely determined by the value of μ on closed subsets of *X* by part (d). Thus, it is sufficient to show that $\mu_1(F) = \mu_2(F)$ for any closed subset *F* of *X*.

Take any closed subset *F* of *X*. Then $F \in \mathscr{M}$. Note that, $\mu_1(F), \mu_2(F) < \infty$. Therefore, given any $\varepsilon > 0$, by part (c), there exists an open set *V* containing *F* such that

$$\mu_1(V) < \mu_1(F) + \varepsilon .$$

Now we use Urysohn's Lemma (Lemma 27). Since *X* is a normal Hausdorff topological space, and $F \subseteq V$, with *F* closed and *V* open, by Urysohn's Lemma, there exists a continuous function $f \in BC(X)$ such that $F \prec f \prec V$. This means that $\chi_F \le f \le \chi_V$. Note that

$$\mu_2(F) = \int_X \chi_F \, d\mu_2 \le \int_X f \, d\mu_2 = \Lambda(f) = \int_X f \, d\mu_1 \le \int_X \chi_V \, d\mu_1 = \mu_1(V) < \mu_1(F) + \varepsilon .$$

Since ε is arbitrary, it follows that $\mu_2(F) \le \mu_1(F)$.

Similarly, by reversing the role of μ_1 and μ_2, we can show that $\mu_1(F) \le \mu_2(F)$. Hence $\mu_1(F) = \mu_2(F)$ for any closed subset *F* of *X*. Thus, the uniqueness of the measure μ is established.

Now we shall define μ first on open set, then on any subset of *X*. Subsequently we shall define the algebra $\mathscr{M}$.

Let *V* be an open set of *X*. Define $\mu(V)$ by

$$\mu(V) = \sup\{\Lambda(f) : f \in BC(X) \text{ and } f \prec V\} .$$

For any subset $E \subseteq X$, define

$$\mu^*(E) = \inf\{\mu(V) : E \subseteq V \text{ and } V \text{ is open in } X\} .$$

Let $\mathscr{M}_F = \left\{E \subseteq X : \mu^*(E) < \infty \text{ and } \mu^*(E) = \sup\left\{\mu^*(F) : F \subseteq E \text{ and } F \text{ is closed in } X\right\}\right\}$
and

$\mathscr{M} = \{ E \subseteq X : E \cap F \in \mathscr{M}_F$ for all closed $F \subseteq X \}$.

Now suppose *U* and *V* are open subsets of *X* and $V \subseteq U$, then $\mu(V) \le \mu(U)$. This is because $\{f : f \in BC(X) \text{ and } f \prec V\} \subseteq \{f : f \in BC(X) \text{ and } f \prec U\}$ so that

$$\mu(V) = \sup\{\Lambda(f) : f \in BC(X) \text{ and } f \prec V\} \le \sup\{\Lambda(f) : f \in BC(X) \text{ and } f \prec U\} = \mu(U) .$$

Therefore, if *E* is open, $\mu(E) \le \mu(U)$ for all open *U* containing *E*. Hence,

$\mu^*(E) = \inf\{\mu(U) : E \subseteq U \text{ and } U \text{ is open in } X\} = \mu(E)$. Thus, our definition of $\mu^*(E)$ for any subset *E* of *X* is consistent with the open sets in *X*.

We shall prove in stages that μ^* is finitely additive on $\mathscr{M}$ and that $\mathscr{M}$ is an algebra generated by the open sets in *X*.

We note the following properties of the positive (real or complex) linear functional Λ and the function, μ^*, which is define on all subsets of *X*.

(1) Λ is *monotone*, i.e., for f and $g \in BC(X)$ and f and g are real valued, $f \leq g \Rightarrow \Lambda(f) \leq \Lambda(g)$. This is because by linearity, $\Lambda(g) = \Lambda(f) + \Lambda(g - f) \geq \Lambda(f)$ as $\Lambda(g - f) \geq 0$.

(2) μ^* is *monotone*, i.e., for any subsets A and B of X, $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$.

If $A \subseteq B$, then $\{V : B \subseteq V \text{ and } V \text{ is open in } X\} \subseteq \{V : A \subseteq V \text{ and } V \text{ is open in } X\}$. Therefore,

$$\mu^*(B) = \inf\{\mu(V) : B \subseteq V \text{ and } V \text{ is open in } X\}$$

$$\geq \inf\{\mu(V) : A \subseteq V \text{ and } V \text{ is open in } X\} = \mu^*(A).$$

(b)

Since $\mu(X) = \sup\{\Lambda(f) : f \in BC(X) \text{ and } f \prec X\}$ and as $1 \in BC(X)$, $\Lambda(f) \leq \Lambda(1)$ for $0 \leq f \leq 1$, we deduce that $\mu(X) \leq \Lambda(1) < \infty$. Hence, for all subset E of X, $\mu^*(E) \leq \mu^*(X) = \mu(X) < \infty$.

Trivially $\mu(\varnothing) = 0$.

We can prove part (e) easily.

Proof of part (e)

Suppose $\mu^*(E) = 0$. Plainly, by the monotonicity of μ^*, $E \in \mathscr{M}_F$ and that for any closed subset F of X, $\mu^*(E \cap F) = 0$ so that $E \cap F \in \mathscr{M}_F$. It follows that $E \in \mathscr{M}$. This means for any $N \subseteq E$, $E \in \mathscr{M}$. Thus, we may take $\mathscr{M}$ to be μ-complete.

Part (c) of the theorem plainly holds by the definition of μ^*.

Therefore, we only need to prove parts (a) and (d). That is, we need to prove that the restriction of μ^* to $\mathscr{M}$, also denoted by μ, is a positive charge or a finitely-additive positive measure on $\mathscr{M}$, $\mathscr{M}$ is an algebra, $\Lambda(f) = \int_X f\, d\mu$ for all $f \in BC(X)$ and μ satisfies part (d).

Note that μ^* is defined on all subsets of X. We need to show that μ^* is finitely additive on $\mathscr{M}$. We have the following consequence of the definition of μ^* on all subsets of X, which will contribute to part of the proof of the finite additivity of μ^* on $\mathscr{M}$.

(1) For any finite family $\{E_i\}_{1\le i<n}$ of subsets of *X*, $\mu^*\left(\bigcup_{i=1}^{n} E_i\right) \le \sum_{i=1}^{n} \mu^*(E_i)$.

To prove (1), we begin by considering open sets in *X*. If V_1 and V_2 are two open sets in *X*, then $\mu(V_1 \cup V_2) \le \mu(V_1) + \mu(V_2)$. We shall prove this as follows. Recall that $\mu(V_1 \cup V_2) = \sup\{\Lambda(g) : g \in BC(X) \text{ and } g \prec V_1 \cup V_2\}$. Suppose $g \in BC(X)$ and $g \prec V_1 \cup V_2$. Then support $g \subseteq V_1 \cup V_2$. Since support *g* is closed and plainly, $\{V_1, V_2\}$ is an open cover for support g, by Theorem 28, we can take a partition of unity $\{h_1, h_2\}$ on support *g* subordinate to the covering $\{V_1, V_2\}$, such that $h_i \in BC(X)$, $0 \le h_i \le 1$, $h_i \prec V_i$, $h_i\left((V_i)^c\right) = 0$, $i = 1, 2$ and $h_1 + h_2 = 1$ on support *g*. Note that support $h_i \subseteq V_i$, $i = 1, 2$. Hence, we get $h_i g \prec V_i$ for $i = 1, 2$ and $h_1 g + h_2 g = g$. Therefore,

$\Lambda(g) = \Lambda(h_1 g) + \Lambda(h_2 g) \le \mu(V_1) + \mu(V_2)$. This is true for any $g \in BC(X)$ with $g \prec V_1 \cup V_2$. Hence,

$$\mu(V_1 \cup V_2) = \sup\{\Lambda(g) : g \prec V_1 \cup V_2 \text{ and } g \in BC(X)\} \le \mu(V_1) + \mu(V_2).$$

It then follows by induction that for a finite family of open sets, $\{V_i\}_{1\le i\le n}$, $\mu\left(\bigcup_{i=1}^{n} V_i\right) \le \sum_{i=1}^{n} \mu(V_i)$. With this proven, we shall apply this to arbitrary family of subsets $\{E_i\}_{1\le i<n}$. We shall show that $\mu^*\left(\bigcup_{i=1}^{n} E_i\right) \le \sum_{i=1}^{n} \mu^*(E_i)$.

Note that $\mu^*(E_i) < \infty$ for all integer $1 \le i \le n$. By the definition of $\mu^*(E_i)$, given $\varepsilon > 0$, there exists open set V_i such that $E_i \subseteq V_i$

$$\mu(V_i) < \mu^*(E_i) + \frac{\varepsilon}{n} .$$

Let $V = \bigcup_{i=1}^{n} V_i$. Then *V* is an open subset of *X*. Take any $f \in BC(X)$ such that $f \prec V$. Since support *f* is closed and support *f* $\subseteq V$, $\{V_i\}_{1\le i<n}$ covers support *f*. Hence,

$$\Lambda(f) \le \mu\left(\bigcup_{i=1}^{n} V_i\right) \le \sum_{i=1}^{n} \mu(V_i) \le \sum_{i=1}^{n} \mu^*(E_i) + \sum_{i=1}^{n} \frac{\varepsilon}{n} \le \sum_{i=1}^{n} \mu^*(E_i) + \varepsilon .$$

It follows that $\mu(V) \leq \sum_{i=1}^{n} \mu^*(E_i) + \varepsilon$. Since $\bigcup_{i=1}^{n} E_i \subseteq V$,

$$\mu^*\left(\bigcup_{i=1}^{n} E_i\right) \leq \mu^*(V) = \mu(V) \leq \sum_{i=1}^{n} \mu^*(E_i) + \varepsilon .$$

As ε is arbitrary, $\mu^*\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mu^*(E_i)$.

(2) Every closed subset of X belongs to $\mathscr{M}_F$.

Take any closed subset F of X. Then $\mu^*(F) < \infty$. Plainly, by the monotonicity of μ^* on subsets of X, $\sup\{\mu^*(L) : L \subseteq F \text{ and } L \text{ is closed}\} = \mu^*(F)$ and so F is in $\mathscr{M}_F$.

(3) Every open subset V of X belongs to $\mathscr{M}_F$.

Take any open subset V of X with $\mu(V) < \infty$. By definition of μ on open subset, given $\varepsilon > 0$, there exists a bounded continuous function f such that $f \prec V$ and $\mu(V) - \varepsilon < \Lambda(f) \leq \mu(V)$. Let F = support f. Then $F \subseteq V$ and so $\mu^*(F) \leq \mu(V)$. Suppose now W is any open set containing F. Then $f \prec W$. By the definition of $\mu(W)$, $\Lambda(f) \leq \mu(W)$. Therefore, $\Lambda(f)$ is a lower bound for $\{\mu(V) : F \subseteq V \text{ and } V \text{ is open in } X\}$ and so

$$\Lambda(f) \leq \mu^*(F) = \inf\{\mu(V) : F \subseteq V \text{ and } V \text{ is open in } X\}.$$

It follows that $\mu(V) - \varepsilon < \Lambda(f) \leq \mu^*(F) \leq \mu^*(V) = \mu(V)$. This means $\mu(V) = \sup\{\mu^*(F) : F \subseteq V \text{ and } F \text{ is closed in } X\}$. Hence, $V \in \mathscr{M}_F$.

(4) μ^* is finitely additive on $\mathscr{M}_F$. That is, suppose $E_1, E_2, \ldots.., E_n$ are in $\mathscr{M}_F$ and are pairwise disjoint, then $\mu^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu^*(E_i)$. Moreover $\bigcup_{i=1}^{n} E_i \in \mathscr{M}_F$.

We shall prove this in stages, firstly, on closed subsets since closed subsets are contained in $\mathscr{M}_F$ by (2).

Suppose K_1, K_2 are disjoint closed subsets of X. We shall show that $\mu^*(K_1 \cup K_2) = \mu^*(K_1) + \mu^*(K_2)$. Since X is normal and Hausdorff, there exists open sets, $V_1 \supseteq K_1$ and $V_2 \supseteq K_2$ such that $V_1 \cap V_2 = \varnothing$. As $K_1 \cup K_2$ is closed, by (2)

$K_1 \cup K_2$ is in $\mathscr{M}_F$. As $\mu^*(K_1 \cup K_2) = \inf\{\mu(V) : K_1 \cup K_2 \subseteq V \text{ and } V \text{ is open in } X\}$, given $\varepsilon > 0$, there exists open set $W \supseteq K_1 \cup K_2$ such that

$$\mu^*(K_1 \cup K_2) \le \mu(W) < \mu^*(K_1 \cup K_2) + \varepsilon .$$

Note that $W \cap V_1$ and $W \cap V_2$ are open in X and are disjoint. As, $\mu(W \cap V_i) < \mu(W) < \infty$ for $i = 1, 2$, by the definition of μ on open set, there exists $f_i \in BC(X)$ such that $f_i \prec W \cap V_i$ and $\Lambda(f_i) > \mu(W \cap V_i) - \varepsilon$ for $i = 1, 2$.

Note that support $f_i \subseteq W \cap V_i$, for $i = 1, 2$, and so since $f_1, f_2 \ge 0$, $f_1 + f_2 \prec W$.

Now, $\mu^*(K_1) + \mu^*(K_1) \le \mu(W \cap V_1) + \mu(W \cap V_2)$ as $K_1 \subseteq W \cap V_1$ and $K_2 \subseteq W \cap V_2$,

$$\le \Lambda(f_1) + \varepsilon + \Lambda(f_2) + \varepsilon = \Lambda(f_1 + f_2) + 2\varepsilon$$

$$\le \mu(W) + 2\varepsilon \text{ , by definition of } \mu(W),$$

$$< \mu^*(K_1 \cup K_2) + 3\varepsilon .$$

Since ε is arbitrary, $\mu^*(K_1) + \mu^*(K_2) \le \mu^*(K_1 \cup K_2)$. We have already proved as in (1) that $\mu^*(K_1 \cup K_2) \le \mu^*(K_1) + \mu^*(K_2)$ and so $\mu^*(K_1 \cup K_2) = \mu^*(K_1) + \mu^*(K_2)$. By a simple mathematical induction, if $K_1, K_2, \ldots, K_n$ are closed subsets of X and are pairwise disjoint, then $\mu^*\left(\bigcup_{i=1}^{n} K_i\right) = \sum_{i=1}^{n} \mu^*(K_i)$

Now suppose $E_1, E_2, \ldots, E_n$ are in $\mathscr{M}_F$ and are pairwise disjoint. Let $E = \bigcup_{i=1}^{n} E_i$.

Then it follows by the inequality in part (1), $\mu^*(E) = \mu^*\left(\bigcup_{i=1}^{n} E_i\right) \le \sum_{i=1}^{n} \mu^*(E_i) < \infty$.

Since each $E_i \in \mathscr{M}_F$, $\mu^*(E_i) = \sup\{\mu^*(K) : K \subseteq E_i \text{ and } K \text{ is closed in } X\}$. Given $\varepsilon > 0$, there exists closed subset $K_i \subseteq E_i$ such that

$$\mu^*(E_i) \ge \mu^*(K_i) > \mu^*(E_i) - \frac{\varepsilon}{n} .$$

Let $H = \bigcup_{i=1}^{n} K_i$. Then $H \subseteq \bigcup_{i=1}^{n} E_i = E$. Therefore,

$$\mu^*(E) \ge \mu^*(H) = \mu^*\left(\bigcup_{i=1}^{n} K_i\right) = \sum_{i=1}^{n} \mu^*(K_i),$$

since $K_1, K_2, \ldots, K_n$ are pairwise disjoint closed sets,

$$> \sum_{i=1}^{n} \mu^*(E_i) - \varepsilon \sum_{i=1}^{n} \frac{1}{n} = \sum_{i=1}^{n} \mu^*(E_i) - \varepsilon .$$

It follows that $\mu^*(E) \geq \sum_{i=1}^{n} \mu^*(E_i) - \varepsilon$. Since ε is arbitrary, $\mu^*(E) \geq \sum_{i=1}^{n} \mu^*(E_i)$. Hence this together with part (1) gives $\mu^*(E) = \sum_{i=1}^{n} \mu^*(E_i)$. We now show that $E \in \mathscr{M}_F$.

Since H is closed, $H \subseteq E$ and $\mu^*(H) > \sum_{i=1}^{n} \mu^*(E_i) - \varepsilon = \mu^*(E) - \varepsilon$ It follows that $\mu^*(E) = \sup\{\mu^*(K) : K \subseteq E \text{ and } K \text{ is closed}\}$. Therefore, $E \in \mathscr{M}_F$.

(5) For all $E \in \mathscr{M}_F$, given $\varepsilon > 0$, there exists closed subset K of X and open subset V with $K \subseteq E \subseteq V$ such that $\mu^*(V - K) = \mu(V - K) < \varepsilon$.

For $E \in \mathscr{M}_F$, $\mu^*(E) = \sup\{\mu^*(K) : K \subseteq E \text{ and } K \text{ is closed}\}$. Hence given $\varepsilon > 0$, there exists closed subset $K \subseteq E$ such that

$$\mu^*(E) \geq \mu^*(K) > \mu^*(E) - \frac{\varepsilon}{2}.$$

Since $\mu^*(E) = \inf\{\mu(V) : E \subseteq V \text{ and } V \text{ is open in } X\}$, there exists open set V such that $E \subseteq V$ and

$$\mu(V) < \mu^*(E) + \frac{\varepsilon}{2}.$$

Hence, $\mu(V) - \frac{\varepsilon}{2} < \mu^*(E) < \mu^*(K) + \frac{\varepsilon}{2}$. By part (4), since K and $V - K \in \mathscr{M}_F$,

$\mu^*(V) = \mu^*(K) + \mu^*(V - K)$ and so $\mu^*(V - K) = \mu^*(V) - \mu^*(K) < \varepsilon$.

(6) If $A_1, A_2 \in \mathscr{M}_F$, then $A_1 - A_2$, $A_1 \cup A_2$ and $A_1 \cap A_2 \in \mathscr{M}_F$.

By (5), given ε > 0, there exist closed K_i, open V_i such that $K_i \subseteq A_i \subseteq V_i$ and $\mu(V_i - K_i) < \varepsilon$ for $i = 1, 2$.

Then $A_1 - A_2 \subseteq V_1 - K_2 \subseteq (V_1 - K_1) \cup (K_1 - V_2) \cup (V_2 - K_2)$. Therefore,

$$\mu^*\left(A_1 - A_2\right) \le \mu^*\left(V_1 - K_1\right) + {}^*\mu(K_1 - V_2) + \mu^*(V_2 - K_2)$$

$$< 2\varepsilon + \mu^*(K_1 - V_2) \le 2\varepsilon + \mu^*(K_1).$$

Note that $K_1 - V_2$ is closed, $K_1 - V_2 \subseteq A_1 - A_2$ and $\mu^*(K_1 - V_2) > \mu^*(A_1 - A_2) - 2\varepsilon$. This shows that given any $\varepsilon > 0$, there exists a closed set L such that $L \subseteq A_1 - A_2$ and $\mu^*(A_1 - A_2) \ge \mu^*(L) > \mu(A_1 - A_2) - \varepsilon$. Hence,

$$\mu^*(A_1 - A_2) = \sup\left\{\mu^*\left(L\right) : L \subseteq A_1 - A_2 \text{ and } L \text{ is closed}\right\}.$$

Therefore, $A_1 - A_2 \in \mathscr{M}_F$.

Now, $A_1 \cup A_2 = \left(A_1 - A_2\right) \cup A_2$ and as $A_1 - A_2, A_2 \in \mathscr{M}_F$ and $A_1 - A_2, A_2$ are disjoint and so by part (4), $A_1 \cup A_2 = \left(A_1 - A_2\right) \cup A_2 \in \mathscr{M}_F$.

Next, $A_1 \cap A_2 = A_1 - \left(A_1 - A_2\right) \in \mathscr{M}_F$, since $A_1 - A_2$ and $A_1 \in \mathscr{M}_F$.

(7) $\mathscr{M}$ is an algebra generated by open sets of X.

Recall that $A \in \mathscr{M}$ if $A \cap K \in \mathscr{M}_F$ for all closed K of X. Take $A \in \mathscr{M}$. We shall show that the complement $A^c \in \mathscr{M}$. Now $A^c \cap K = K - A \cap K \in \mathscr{M}_F$ by part (6) since K and $A \cap K \in \mathscr{M}_F$. Hence, $A^c \in \mathscr{M}$. If $\{A_i\}_{i=1}^n$ is a finite collection of sets in $\mathscr{M}$, then by part (6) for any closed K, $\left(\bigcup_{i=1}^n A_i\right) \cap K = \bigcup_{i=1}^n A_i \cap K \in \mathscr{M}_F$ and so $\bigcup_{i=1}^n A_i \in \mathscr{M}$.

Next we shall show that if $C \subseteq X$ is closed in X, then $C \in \mathscr{M}$. In particular, $X \in \mathscr{M}$.

If C is closed, then $C \in \mathscr{M}_F$ by part (2). Then $C \cap K$ is closed for any closed subset K of X and so $C \cap K \in \mathscr{M}_F$. Thus $C \in \mathscr{M}$. Hence, $X \in \mathscr{M}$ and $\mathscr{M}$ is an algebra containing all closed subsets of X, hence all open subsets of X.

(8) $\mathscr{M}_F = \mathscr{M}$.

Suppose $E \in \mathscr{M}_F$. Then by (6), since by (2) any closed $K \in \mathscr{M}_F$, $E \cap K \in \mathscr{M}_F$. Hence, $E \in \mathscr{M}$. That is, $\mathscr{M}_F \subseteq \mathscr{M}$.

Conversely, suppose $E \in \mathscr{M}$. Take any open V in X such that $E \subseteq V$. Since V is open, by (3), $V \in \mathscr{M}_F$. Hence, $\mu(V) = \sup\{\mu^*(K) : K \subseteq V \text{ and } K \text{ is closed in } X\}$. Therefore, given any $\varepsilon > 0$, there exists a closed set $K \subseteq V$ such that $\mu^*(K) > \mu(V) - \varepsilon$ so that $\mu(V - K) < \varepsilon$. Since by definition of $\mathscr{M}$, $E \cap K \in \mathscr{M}_F$, there exists closed $H \subseteq E \cap K$ such that $\mu^*(H) > \mu^*(E \cap K) - \varepsilon$. Since $E \subseteq (E \cap K) \cup (V - K)$,

$$\mu^*(E) \le \mu^*(E \cap K) + \mu^*(V - K) < \mu^*(H) + 2\varepsilon .$$

As H is closed and $H \subseteq E$, this shows that

$$\mu^*(E) = \sup\{\mu^*(H) : H \subseteq E \text{ and } H \text{ is closed in } X\}.$$

Therefore, $E \in \mathscr{M}_F$. Hence, $\mathscr{M} \subseteq \mathscr{M}_F$. Thus, $\mathscr{M} = \mathscr{M}_F$.

(9) μ^* is finitely additive on $\mathscr{M}$.

We have proved that μ^* is finitely additive on $\mathscr{M}_F$ and since $\mathscr{M} = \mathscr{M}_F$, μ^* is finitely additive on $\mathscr{M}$.

(11) For all $f \in BC(X)$, $\Lambda(f) = \int_X f\, d\mu$.

We note that it is sufficient to prove this for real f. For complex f we may write $f = \operatorname{Re} f + i \operatorname{Im} f$. Then the real part of f, $\operatorname{Re} f$, and the imaginary part of f, $\operatorname{Im} f$, are continuous bounded real valued functions. Then,

$$\Lambda(f) = \Lambda(\operatorname{Re} f + i \operatorname{Im} f) = \Lambda(\operatorname{Re} f) + i\Lambda(\operatorname{Im} f) = \int_X \operatorname{Re} f\, d\mu + i\int_X \operatorname{Im} f\, d\mu = \int_X f\, d\mu .$$

Let f be a bounded continuous real valued function in $BC(X)$. Let K = support f and so K is closed. Since f is bounded, $f(X)$ is contained in a bounded subset on the real line. Thus, we may assume that $f(X) \subseteq [a,b)$. Given $\varepsilon > 0$, partition $[a,b]$ as follows

$$y_0 = a < y_1 < y_2 < \cdots < y_n = b \text{ with } y_i - y_{i-1} < \varepsilon \text{ for } 1 \le i \le n .$$

Let $E_i = \{x \in X : y_{i-1} \le f(x) < y_i\}$ for $1 \le i \le n$. That is, $E_i = f^{-1}([y_{i-1}, y_i))$. Since f is continuous and so is ($\mathscr{M}$, $A_{\mathbb{R}}$)-measurable, it follows that each E_i is a Borel set in the algebra generated by the open and closed sets and is in $\mathscr{M}$. Moreover $\{E_i\}$ are pairwise disjoint and covers K. We assume that each $E_i \ne \varnothing$.

$\mu^*(E_i) \le \mu^*(X) < \infty$ and $\bigcup_{i=1}^{n} E_i = X$. Note that $E_i \in \mathscr{M}_F$ for $1 \le i \le n$. By the definition of $\mu^*(E_i)$, given $\varepsilon > 0$, there exists open set $W_i \supseteq E_i$ such that $\mu(W_i) < \mu^*(E_i) + \frac{\varepsilon}{n}$. Note that this holds even if $E_i = \varnothing$. Let $D_i = (y_{i-1} - \varepsilon, y_i)$. Then $U_i = f^{-1}(D_i)$ is open and if U_i is non-empty, $y_{i-1} - \varepsilon < f(x) < y_i$ for all x in U_i. Then $U_i \supseteq E_i$. Let $V_i = W_i \cap U_i$ and we have $\mu(V_i) \le \mu(W_i) < \mu^*(E_i) + \frac{\varepsilon}{n}$ and $y_{i-1} - \varepsilon < f(x) < y_i$ for all x in V_i when $V_i \ne \varnothing$. Note that $\bigcup_{i=1}^{n} V_i \supseteq \bigcup_{i=1}^{n} E_i = X$. Take a partition of unity $\{h_i\}_{1 \le i \le n}$ on X subordinate to the covering $\{V_i\}_{1 \le i \le n}$ such that, for $1 \le i \le n$, $0 \le h_i \le 1$, $h_i \prec V_i$ and $h_1 + \cdots + h_n = 1$ on X. Note that if $V_i = \varnothing$, then $h_i = 0$. Then we have

$\sum_{i=1}^{n} h_i f = f$ since $\sum_{i=1}^{n} h_i = 1$ on X, and for $1 \le i \le n$,

$h_i(x) f(x) \le h_i(x)(y_i)$ since $h_i \prec V_i$ and $f(x) < y_i$ for all x in V_i when $V_i \ne \varnothing$,

and

$y_{i-1} - \varepsilon < f(x) < y_i < y_{i-1} + \varepsilon$ for all x in E_i, when $E_i \ne \varnothing$

By linearity, $\Lambda(f) = \sum_{i=1}^{n} \Lambda(h_i f)$. As Λ is a positive linear functional and $h_i f \le h_i(y_i)$, $\Lambda(h_i f) \le \Lambda((y_i) h_i) = y_i \Lambda(h_i)$ for $1 \le i \le n$. Therefore,

$$\Lambda(f) = \sum_{i=1}^{n} \Lambda(h_i f) \le \sum_{i=1}^{n} y_i \Lambda(h_i).$$

Since $h_i \prec V_i$ for $1 \le i \le n$, by definition of $\mu(V_i)$, $\Lambda(h_i) \le \mu(V_i)$ for $1 \le i \le n$. For $1 \le i \le n$, $a \le y_i \le b$ so that $y_{i-1} + |a| \ge 0$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{n} (y_{i-1} + \varepsilon) \Lambda(h_i) &= \sum_{i=1}^{n} (y_{i-1} + \varepsilon + |a|) \Lambda(h_i) - \sum_{i=1}^{n} |a| \Lambda(h_i) \\
&= \sum_{i=1}^{n} (y_{i-1} + \varepsilon + |a|) \Lambda(h_i) - |a| \Lambda\left(\sum_{i=1}^{n} h_i \right) \\
&\le \sum_{i=1}^{n} (y_{i-1} + \varepsilon + |a|) \mu(V_i) - |a| \Lambda\left(\sum_{i=1}^{n} h_i \right)
\end{aligned}$$

$$\le \sum_{i=1}^{n}\left(y_{i-1}+\varepsilon+|a|\right)\mu^*(E_i)+\sum_{i=1}^{n}\left(y_{i-1}+\varepsilon+|a|\right)\frac{\varepsilon}{n}-|a|\Lambda\left(\sum_{i=1}^{n}h_i\right)$$

$$=\sum_{i=1}^{n}y_{i-1}\mu^*(E_i)+\left(\varepsilon+|a|\right)\sum_{i=1}^{n}\mu^*(E_i)+\sum_{i=1}^{n}\left(y_{i-1}+\varepsilon+|a|\right)\frac{\varepsilon}{n}-|a|\Lambda\left(\sum_{i=1}^{n}h_i\right)$$

$$\le \sum_{i=1}^{n}y_{i-1}\mu^*(E_i)+\left(\varepsilon+|a|\right)\mu^*(X)+\left(b+\varepsilon+|a|\right)\varepsilon-|a|\Lambda\left(\sum_{i=1}^{n}h_i\right)$$

$$\le \sum_{i=1}^{n}\int_{E_i} f\,d\mu+\varepsilon\left(\mu^*(X)+\left(b+\varepsilon+|a|\right)\right)+|a|\,\mu^*(X)-|a|\Lambda\left(\sum_{i=1}^{n}h_i\right)$$

$$=\int_X f\,d\mu+\varepsilon\left(\mu^*(X)+\left(b+\varepsilon+|a|\right)\right)+|a|\,\mu^*(X)-|a|\Lambda\left(\sum_{i=1}^{n}h_i\right).$$

We already knew that $\mu^*(X)\le\Lambda(1)=\Lambda\left(\sum_{i=1}^{n}h_i\right)$. Hence,

$$\Lambda(f)\le\int_X f\,d\mu+\varepsilon\left(\mu^*(X)+\left(b+\varepsilon+|a|\right)\right)+|a|\,\mu^*(X)-|a|\Lambda\left(\sum_{i=1}^{n}h_i\right)$$

$$\le\int_X f\,d\mu+\varepsilon\left(\mu^*(X)+\left(b+\varepsilon+|a|\right)\right).$$

Since ε is arbitrary, $\Lambda(f)\le\int_X f\,d\mu$.

As Λ is linear, $-\Lambda(f)=\Lambda(-f)\le\int_X(-f)\,d\mu=-\int_X f\,d\mu$ and so $\Lambda(f)\ge\int_X f\,d\mu$,
Thus, $\Lambda(f)=\int_X f\,d\mu$.

Now for any $f\in BC(X)$, $|\Lambda(f)|=\left|\int_X f\,d\mu\right|\le\int_X|f|\,d\mu\le\|f\|_u\,\mu(X)\le\mu(X)$ for $\|f\|_u\le 1$, where $\|f\|_u=\sup\{|f(x)|:x\in X\}$ is the uniform sup norm on $BC(X)$. Therefore,

$$\|\Lambda\|=\sup\left\{|\Lambda(f)|:f\in BC(X)\text{ and }\|f\|_u=1\right\}\le\mu(X)\le\Lambda(1).$$

Hence, $\|\Lambda\|=\mu(X)=\Lambda(1)$.

Note that the algebra generated by the open sets of X is a subalgebra of $\mathscr{M}$. We now denote this subalgebra of open sets also by the symbol $\mathscr{M}$ and called it the *Borel algebra* and sets in $\mathscr{M}$ the *Borel sets*. Denote μ to be the restriction of μ^* to the Borel algebra. Then μ is a finite finitely-additive measure or finite charge satisfying (c) and (d) and $\Lambda(f)=\int_X f\,d\mu$ for all f in $BC(X)$. Thus μ is outer regular and inner regular.

We say a positive charge or finitely-additive measure λ on an algebra containing all the open sets is *normal*, if the conclusion (c) and (d) holds for any set E in the algebra without any condition.

Hence, the finite positive charge or the finite finitely-additive measure μ on $\mathscr{M}$ given in the last theorem is a finite normal charge.

If need be we may choose the algebra $\mathscr{M}$ to be complete by part (e) .

This completes the proof of Theorem 29.

Now we consider bounded real linear functional on $BC(X,\mathbb{R})$ the space of bounded continuous real valued functions on X. For such a bounded linear functional, since $1\in BC(X,\mathbb{R})$, we can decompose the bounded real linear functional as the difference of two positive real linear functionals.

Proposition 30. Suppose X is a Hausdorff topological space and $BC(X,\mathbb{R})=\{f:X\to\mathbb{R}; f \text{ is continuous and bounded}\}$. Suppose $\Phi:BC(X,\mathbb{R})\to\mathbb{R}$ is a bounded real linear functional. Then we can decompose Φ as $\Phi=\Phi^+-\Phi^-$ such that Φ^+ and Φ^- are positive real linear functionals and $\|\Phi\|=\|\Phi^+\|+\|\Phi^-\|=\Phi^+(1)+\Phi^-(1)$.

Proof.

Let $BC^+(X,\mathbb{R})$ denote the set of non-negative functions in $BC(X,\mathbb{R})$.

Define for f in $BC^+(X,\mathbb{R})$,

$$\Phi^+(f)=\sup\{\Phi(h):h\in BC(X,\mathbb{R}) \text{ and } 0\le h\le f\}$$

$$=\sup\{\Phi(h):h\in BC^+(X,\mathbb{R}) \text{ and } 0\le h\le f\}.$$

This is well defined since Φ is bounded so that the supremum above exists.

Since $\Phi(0)=0$, $\Phi^+(f)\geq 0$ for all $BC^+(X,\mathbb{R})$. Plainly, $\Phi^+(f)\geq \Phi(f)$ for all $f\in BC^+(X,\mathbb{R})$. Obviously, for $k>0$, $\Phi^+(kf)=k\Phi^+(f)$.

We need to show that $\Phi^+(f_1+f_2)=\Phi^+(f_1)+\Phi^+(f_2)$ for $f_1,f_2\in BC^+(X,\mathbb{R})$.

By definition of $\Phi^+(f_i)$, given ε > 0, there exists $h_i\in BC^+(X,\mathbb{R})$ such that $0\leq h_i\leq f_i$ and $\Phi^+(f_i)-\varepsilon<\Phi(h_i)$ for i = 1, 2. Then we have, as $0\leq h_1+h_2\leq f_1+f_2$,

$$\Phi^+(f_1)+\Phi^+(f_2)<\Phi(h_1)+\Phi(h_2)+2\varepsilon=\Phi(h_1+h_2)+2\varepsilon\leq\Phi^+(f_1+f_2)+2\varepsilon.$$

Since ε is arbitrary, $\Phi^+(f_1)+\Phi^+(f_2)\leq\Phi^+(f_1+f_2)$.

Take $h\in BC^+(X,\mathbb{R})$ with $0\leq h\leq f_1+f_2$. Let $V=\{x: f_1(x)+f_2(x)>0\}$. Then V is open in X.

Let $h_i(x)=\begin{cases}\dfrac{f_i(x)h(x)}{f_1(x)+f_2(x)}, & x\in V\\ 0, & x\in V^c\end{cases}$.

We claim that h_i is non-negative, continuous and bounded, $0\leq h_i\leq f_i$ for i = 1, 2.

Plainly, $h_i(x)\geq 0$ for all $x\in X$, $h_i(x)\leq f_i(x)$ for $x\in V$ and $h_i(x)=f_i(x)=0$ for $x\in V^c$ for i=1, 2.

Since $\dfrac{f_i(x)}{f_1(x)+f_2(x)}$ and $h(x)$ is continuous on the open set V, $\dfrac{f_i(x)h(x)}{f_1(x)+f_2(x)}$ is continuous on V so that h_i is continuous on V for i = 1, 2. Now we show that h_1 is continuous at any point $x_0\in V^c$. For such a $x_0\in V^c$, $h_1(x_0)=0$ and also $h(x_0)=0$. Since h is continuous at x_0, given any open interval $I=(-\delta,\delta)$, $\delta>0$, containing $h(x_0)=0$, there exists an open set U containing $x_0\in V^c$ such that $h(U)\subseteq I$. Now for $x\in U$, $|h_1(x)|\leq|h(x)|<\delta$ implies that $h_1(x)\in I$ and so $h_1(U)\subseteq I$. Hence, h_1 is continuous at x_0. Therefore, h_1 is continuous on X. Similarly, we can show that h_2 is continuous on X. Note that $h_i(x)\neq 0\Rightarrow h(x)\neq 0$ and $x\in V$. Therefore, support $h_i\subseteq$ support h and so $h_i\in BC^+(X,\mathbb{R})$ for i = 1,2. Then

$h_1(x)+h_2(x)=h(x)$ for all x in X and $\Phi(h)=\Phi(h_1)+\Phi(h_2)\leq\Phi^+(f_1)+\Phi^+(f_2)$.

This means that for all $h \in BC^+(X,\mathbb{R})$ with $0 \le h \le f_1 + f_2$, $\Phi(h) \le \Phi^+(f_1) + \Phi^+(f_2)$. Therefore, $\Phi^+(f_1 + f_2) \le \Phi^+(f_1) + \Phi^+(f_2)$. Thus, $\Phi^+(f_1 + f_2) = \Phi^+(f_1) + \Phi^+(f_2)$.

We now extend this definition of Φ^+ to all of $f \in BC(X,\mathbb{R})$. For $f \in BC(X,\mathbb{R})$, $|f(x)|$ is bounded above, say by a positive constant, *N*. Then $f + N \ge 0$. We define $\Phi^+(f) = \Phi^+(f+N) - \Phi^+(N)$. This is well defined. For suppose $f + M \ge 0$, then $\Phi^+(f+N+M) = \Phi^+(f+N) + \Phi^+(M) = \Phi^+(f+M) + \Phi^+(N)$ so that

$$\Phi^+(f+N) - \Phi^+(N) = \Phi^+(f+M) - \Phi^+(M).$$

It is clear that Φ^+ is linear on $BC(X,\mathbb{R})$. Suppose $f_1, f_2 \in BC(X,\mathbb{R})$ and $f_1 + N \ge 0, f_2 + M \ge 0$. Then $\Phi^+(f_1 + f_2) = \Phi^+(f_1 + f_2 + M + N) - \Phi^+(M+N)$

$$= \Phi^+(f_1 + N) + \Phi^+(f_2 + M) - \Phi^+(M) - \Phi^+(N)$$

$$= \Phi^+(f_1) + \Phi^+(f_2).$$

Plainly, $\Phi^+(0) = 0$ and for $c \ge 0$, $\Phi^+(c\,f) = c\,\Phi^+(f)$ for all $f \in C_{c,\mathbb{R}}(X)$.

In particular, for $f \in BC(X,\mathbb{R})$, $\Phi^+(f) + \Phi^+(-f) = \Phi^+(f + (-f)) = \Phi^+(0) = 0$ so that $\Phi^+(-f) = -\Phi^+(f)$. Thus, Φ^+ is a linear functional on $BC(X,\mathbb{R})$. Since $\Phi^+(f) \ge 0$ for $f \ge 0$, Φ^+ is a positive linear functional on $BC(X,\mathbb{R})$. Note that by definition of Φ^+ for $f \ge 0$, $\Phi(f) \le \Phi^+(f)$. Define $\Phi^-(f) = \Phi^+(f) - \Phi(f)$ for $f \in BC(X,\mathbb{R})$.

Then for $f \ge 0$, $\Phi^-(f) = \Phi^+(f) - \Phi(f) \ge 0$, it follows that Φ^- is also a positive linear functional on $BC(X,\mathbb{R})$ and $\Phi = \Phi^+ - \Phi^-$.

Note that

$$|\Phi(f)| = \left|\Phi^+(f) - \Phi^-(f)\right| \le \left|\Phi^+(f)\right| + \left|\Phi^-(f)\right|$$

$$\le \left\|\Phi^+\right\| \|f\|_u + \left\|\Phi^-\right\| \|f\|_u = \left(\left\|\Phi^+\right\| + \left\|\Phi^-\right\|\right) \|f\|_u.$$

Therefore, $\|\Phi\| \le \left\|\Phi^+\right\| + \left\|\Phi^-\right\|$. Note that for positive linear functionals, Φ^+ and Φ^-, $\left\|\Phi^+\right\| = \Phi^+(1)$ and $\left\|\Phi^-\right\| = \Phi^-(1)$. (If Λ is a positive linear functional, then for any *f* in $BC(X,\mathbb{R})$, $\Lambda(f), \Lambda(-f) \le \Lambda(|f|)$. Thus, if $|f| \le 1$, $|\Lambda(f)| \le \Lambda(|f|) \le \Lambda(1)$ as $1 \in BC(X,\mathbb{R})$ and since $\|\Lambda\| = \sup\left\{|\Lambda(f)| : \|f\|_u = 1 \text{ and } f \in BC(X,\mathbb{R})\right\}$, $\|\Lambda\| = \Lambda(1)$.)

Recall that $\Phi^+(1) = \sup\{\Phi(h) : h \in BC(X,\mathbb{R}) \text{ and } 0 \le h \le 1\}$.

Take any $h \in BC(X,\mathbb{R})$ with $0 \le h \le 1$. Then $-1 \le 2h-1 \le 1$. Therefore, by definition of $\|\Phi\|$, $|\Phi(2h-1)| \le \|\Phi\| \|2h-1\|_u \le \|\Phi\|$ so that $\Phi(2h-1) \le |\Phi(2h-1| \le \|\Phi\| \|2h-1\|_u \le \|\Phi\|$. This means, $2\Phi(h) - \Phi(1) = \Phi(2h-1) \le \|\Phi\|$ for all $h \in BC(X,\mathbb{R})$ such that $0 \le h \le 1$. Therefore, by definition of $\Phi^+(1)$, $2\Phi^+(1) - \Phi(1) \le \|\Phi\|$, that is to say,

$\Phi^+(1) + \Phi^-(1) = 2\Phi^+(1) - \Phi(1) \le \|\Phi\|$. Consequently, $\Phi^+(1) + \Phi^-(1) = \|\Phi\|$.

This completes the proof of Proposition 30.

Total variation measure for a finitely additive measure

For a real additive measure, μ, on an algebra of *A* of subsets of *X*, the variation measure of μ, is defined to be $|\mu|: A \to \overline{\mathbb{R}^+}$ given by

$$|\mu|(E) = \sup_{\text{All finite partitions } \{E_i\} \text{ of } E} \sum_i |\mu(E_i)| .$$

Note that for any *E* in *A*, $|\mu|(E) \ge |\mu(E)|$.

It is easy to see that if $U \subseteq V$, then $|\mu|(U) \le |\mu|(V)$.

Note that if $|\mu|$ is finite, that is, $|\mu|(X) < \infty$, $|\mu|$ is a finite finitely-additive measure.

Proposition 31. Suppose $\mu : A \to \mathbb{R}$ is a *signed finitely-additive measure* or *real finitely-additive measure* on the algebra *A* of subsets of a non-empty set *X*. If $|\mu|$ is bounded or finite, i.e., when $|\mu|(X) < \infty$, then $|\mu|$ is a finite finitely-additive positive measure.

The proof of this fact is similar to the proof when μ is a complex measure. It is easier as we shall deal only with finite partitions of *E* in the algebra. (See Theorem 1.)

Proof of Proposition 31.

Plainly, $|\mu|(\varnothing)=0$. We shall show that $|\mu|$ is finitely-additive. Take $E \in A$.

Suppose $\{F_i\}_{i=1}^{n}$ is a finite partition of *E* by disjoint sets in A. We shall show that

$$|\mu|(E)=\sum_{i=1}^{n}|\mu|(F_i).$$

We show that $\sum_{i=1}^{n}|\mu|(F_i)\le|\mu|(E)$ as follows.

For each integer i, choose $0<t_i<|\mu|(F_i)$ if $|\mu|(F_i)>0$ otherwise set $t_i=0$. Then by definition of $|\mu|(F_i)$, for $|\mu|(F_i)>0$, there exists a partition $\{G_{i,j}\}_{j=1}^{n_i}$ of F_i such that

$$t_i<\sum_{j=1}^{n_i}\left|\mu\left(G_{i,j}\right)\right| \quad (\ \le|\mu|(F_i)\ ). \quad \text{-------------------------} \quad (1)$$

If $|\mu|(F_i)=0$, then take the trivial partition $\{F_i\}=\{G_{i,1}\}$ for F_i.

Then $\{G_{i,j}\}_{i,j}$ is a finite partition of *E*. Now, $\sum_{i,j}\mu\left(G_{i,j}\right)$ is a finite sum.

Therefore,

$$|\mu|(E)\ge\sum_{i,j}\left|\mu\left(G_{i,j}\right)\right| \quad \text{by definition of } |\mu|(E),$$

$$=\sum_{i}\sum_{j}\left|\mu\left(G_{i,j}\right)\right|$$

$$\ge\sum_{i}t_i \ .$$

It follows that $|\mu|(E)\ge\sum_{i=1}^{n}|\mu|(F_i)$.

Next, we show that $|\mu|(E)\le\sum_{i}|\mu|(F_i)$.

Let $\{H_j\}_{j=1}^{m}$ be any other partition of *E*. Then for each *j*, $\{F_i\cap H_j\}_{i=1}^{n}$ is a partition of H_j and $\{F_i\cap H_j\}_{j=1}^{m}$ is a partition of F_i . It follows that

$$\sum_{j=1}^{m}\left|\mu\left(H_j\right)\right| = \sum_{j=1}^{m}\left|\sum_{i=1}^{n}\mu\left(F_i \cap H_j\right)\right| \le \sum_{j=1}^{m}\sum_{i=1}^{n}\left|\mu\left(F_i \cap H_j\right)\right| = \sum_{i=1}^{n}\sum_{j=1}^{m}\left|\mu\left(F_i \cap H_j\right)\right|$$

$$\le \sum_{i=1}^{n}|\mu|\left(F_i\right).$$

This holds for any finite partition $\left\{H_j\right\}_{j=1}^{m}$ of E. Therefore, $|\mu|(E) \le \sum_{i=1}^{n}|\mu|\left(F_i\right)$. It follows that $|\mu|(E) = \sum_{i=1}^{n}|\mu|\left(F_i\right)$ and so $|\mu|$ is finitely-additive on A and is therefore a finitely-additive positive measure on A.

Proposition 32.

Suppose $\lambda = \mu_1 - \mu_2$, where μ_1 and μ_2 are finite positive charge on A, which is an algebra of subset of a non-empty set X. Then λ has finite variation and so $|\lambda|$ is a finite finitely-additive positive measure. Moreover, $|\lambda|(E) \le \mu_1(E) + \mu_2(E)$ for all E in A.

Proof.

Suppose E is in A and $\{E_i\}_{i=1}^{n}$ is a partition of E by pairwise disjoints sets in A. Then

$$\sum_{i=1}^{n}\left|\lambda(E_i)\right| = \sum_{i=1}^{n}\left|\mu_1(E_i) - \mu_2(E_i)\right|$$

$$\le \sum_{i=1}^{n}\left(\mu_1(E_i) + \mu_2(E_i)\right) = \sum_{i=1}^{n}\mu_1(E_i) + \sum_{i=1}^{n}\mu_2(E_i) = \mu_1(E) + \mu_2(E) < \infty$$

Hence $|\lambda|(E) = \sup_{\text{All finite partitions } \{E_i\} \text{ of } E} \sum_{i}\left|\lambda(E_i)\right| \le \mu_1(E) + \mu_2(E) < \infty$. Thus, the variation measure of λ is bounded and so by Proposition 30, is a finite finitely-additive positive measure.

Suppose $\lambda = \mu_1 - \mu_2$, where μ_1 and μ_2 are finite finitely-additive positive measures. Then $\left|\lambda(E)\right| \le |\lambda|(E)$

For any finite partition $\{E_i\}_{i=1}^{n}$ of E by pairwise disjoints sets in A,

$$\lambda(E) = \mu_1(E) - \mu_2(E) = \sum_{i=1}^{n} \mu_1(E_i) - \sum_{i=1}^{n} \mu_2(E_i) = \sum_{i=1}^{n} \left(\mu_1(E_i) - \mu_2(E_i)\right)$$

$$\leq \sum_{i=1}^{n} \left|\mu_1(E_i) - \mu_2(E_i)\right|$$

$$\leq |\lambda|(E).$$

Similarly, $-\lambda(E) \leq |\lambda|(E)$. It follows that $|\lambda(E)| \leq |\lambda|(E)$.

Proposition 33. Suppose X is a topological space and $\mathscr{M}$ is the algebra generated by the open sets of X, i.e., Borel algebra.

Suppose μ_1 and μ_2 are two finite normal finitely-additive positive measures on $\mathscr{M}$, i.e., finite normal finitely-additive Borel measures. Then the total variation $|\mu_1 - \mu_2|$ is a finite normal finitely-additive Borel measure. We say a signed finite finitely-additive measure, μ, is *normal* if its variation is normal. Hence, $\mu_1 - \mu_2$ is normal.

Moreover, suppose μ_1 and μ_2 are two finite normal finitely-additive signed Borel measures on $\mathscr{M}$. Then $\mu_1 - \mu_2$ is also a finite normal finitely-additive signed measure.

Proof.

Plainly, if μ_1 and μ_2 are two finite normal finitely-additive positive measures on $\mathscr{M}$, then by Proposition 32, $\mu_1 - \mu_2$ is a finite signed finitely-additive Borel and $|\mu_1 - \mu_2|$ is a finite finitely-additive positive Borel measure. The measures μ_1 and μ_2 are normal means that $|\mu_1|$ and $|\mu_2|$ are normal.

We show that $|\mu_1 - \mu_2|$ is inner regular.

μ_1 is inner regular implies that for any $E \in \mathscr{M}$, given $\varepsilon > 0$, there exists closed set $K_1 \subseteq E$ such that $\mu_1(E) - \varepsilon < \mu_1(K_1)$. That is to say,

$$\mu_1(E - K_1) = \mu_1(E) - \mu_1(K_1) < \varepsilon . \quad \text{----------------------- (1)}$$

Similarly, as μ_2 is inner regular, for any $E \in \mathscr{M}$, given $\varepsilon > 0$, there exists closed set $K_2 \subseteq E$ such that

$$\mu_2(E - K_2) = \mu_2(E) - \mu_2(K_2) < \varepsilon . \text{ ---------------------- (2)}$$

Let $K = K_1 \cup K_2$ and K is closed and $K \subseteq E$. Then by Proposition 32,

$$|\mu_1 - \mu_2|(E - K) \le \mu_1(E - K) + \mu_2(E - K) < 2\varepsilon .$$

Hence, $|\mu_1 - \mu_2|(E) - 2\varepsilon < |\mu_1 - \mu_2|(K) \le |\mu_1 - \mu_2|(E)$. This implies that

$$|\mu_1 - \mu_2|(E) = \sup\{|\mu_1 - \mu_2|(K),\ K \text{ closed and } K \subseteq E\} .$$

Thus, for any $E \in \mathscr{M}$, $|\mu_1 - \mu_2|(E) = \sup\{|\mu_1 - \mu_2|(K),\ K \text{ closed and } K \subseteq E\}$. It follows that $|\mu_1 - \mu_2|$ is inner regular.

We now show that $|\mu_1 - \mu_2|$ is outer regular. μ_1 and μ_2 are both outer regular. This means for any $E \in \mathscr{M}$, given $\varepsilon > 0$, there exists an open set $V_1 \supseteq E$ such that $\mu_1(V_1) < \mu_1(E) + \varepsilon$. Therefore, $\mu_1(V_1 - E) < \varepsilon$. Similarly, there exists an open set $V_2 \supseteq E$ such that $\mu_2(V_2 - E) < \varepsilon$. Let $V = V_1 \cap V_2$. Then V is open and $V \supseteq E$. Therefore, by Proposition 32, $|\mu_1 - \mu_2|(V - E) \le \mu_1(V - E) + \mu_2(V - E) < 2\varepsilon$. Hence, $|\mu_1 - \mu_2|(V) = |\mu_1 - \mu_2|(V - E) + |\mu_1 - \mu_2|(E) < |\mu_1 - \mu_2|(E) + 2\varepsilon$.

It follows that $|\mu_1 - \mu_2|(E) = \inf\{|\mu_1 - \mu_2|(V),\ V \text{ open and } V \supseteq E\}$. As this holds for any $E \in \mathscr{M}$, $|\mu_1 - \mu_2|$ is outer regular.

Therefore, $|\mu_1 - \mu_2|$ is normal and so $\mu_1 - \mu_2$ is normal.

Suppose now μ_1 and μ_2 are two finite normal finitely-additive signed Borel measures on $\mathscr{M}$. That is to say, $|\mu_1|$ and $|\mu_2|$ are normal finitely-additive positive Borel measures.

Plainly, $\mu_1 - \mu_2$ is a finite signed finitely-additive Borel measure. The measures μ_1 and μ_2 are normal means that $|\mu_1|$ and $|\mu_2|$ are normal.

We show that $|\mu_1 - \mu_2|$ is inner regular.

$|\mu_1|$ is inner regular implies that for any $E \in \mathscr{M}$, given $\varepsilon > 0$, there exists closed $K_1 \subseteq E$ such that $|\mu_1|(E) - \varepsilon < |\mu_1|(K_1)$. That is to say,

$$|\mu_1|(E - K_1) = |\mu_1|(E) - |\mu_1|(K_1) < \varepsilon . \text{ ----------------------- } (1)$$

Similarly, as $|\mu_2|$ is inner regular, for any $E \in \mathscr{M}$, given $\varepsilon > 0$, there exists closed $K_2 \subseteq E$ such that

$$|\mu_2|(E - K_2) = |\mu_2|(E) - |\mu_2|(K_2) < \varepsilon . \text{ ---------------------- } (2)$$

Let $K = K_1 \cup K_2$. Then K is closed and $K \subseteq E$. Now for any finite partition $\{E_i\}_{i=1}^{n}$ of E by pairwise disjoints sets in $\mathscr{M}$,

$$\sum_{i=1}^{n} \left|(\mu_1 - \mu_2)(E_i)\right| \leq \sum_{i=1}^{n} |\mu_1(E_i) - \mu_2(E_i)| \leq \sum_{i=1}^{n} |\mu_1(E_i)| + \sum_{i=1}^{n} |\mu_2(E_i)| \leq |\mu_1|(E) + |\mu_2|(E) .$$

It follows that $|\mu_1 - \mu_2|(E) \leq |\mu_1|(E) + |\mu_2|(E) < |\mu_1|(X) + |\mu_2|(X) < \infty$.

Hence, by proposition 32, $|\mu_1 - \mu_2|$ is a finite finitely-additive positive Borel measure and from (1) and (2) we get,

$$|\mu_1 - \mu_2|(E - K) \leq |\mu_1|(E - K) + |\mu_2|(E - K) < 2\varepsilon .$$

Therefore, $|\mu_1 - \mu_2|(E) - 2\varepsilon < |\mu_1 - \mu_2|(K) \leq |\mu_1 - \mu_2|(E)$. This implies that

$$|\mu_1 - \mu_2|(E) = \sup\left\{|\mu_1 - \mu_2|(K),\ K \text{ closed and } K \subseteq E\right\} .$$

Thus, for any $E \in \mathscr{M}$, $|\mu_1 - \mu_2|(E) = \sup\left\{|\mu_1 - \mu_2|(K),\ K \text{ closed and } K \subseteq E\right\}$. It follows that $|\mu_1 - \mu_2|$ is inner regular.

We now show that $|\mu_1 - \mu_2|$ is outer regular. Now, $|\mu_1|$ and $|\mu_2|$ are both outer regular. This means for any $E \in \mathscr{M}$, given $\varepsilon > 0$, there exists an open set $V_1 \supseteq E$ such that $|\mu_1|(V_1) < |\mu_1|(E) + \varepsilon$. Therefore, $|\mu_1|(V_1 - E) < \varepsilon$. Similarly, there exists an open set $V_2 \supseteq E$ such that $|\mu_2|(V_2 - E) < \varepsilon$. Let $V = V_1 \cap V_2$. Then V is open and $V \supseteq E$. Therefore, $|\mu_1 - \mu_2|(V - E) \leq |\mu_1|(V - E) + |\mu_2|(V - E) < 2\varepsilon$. Hence, $|\mu_1 - \mu_2|(V) = |\mu_1 - \mu_2|(V - E) + |\mu_1 - \mu_2|(E) < |\mu_1 - \mu_2|(E) + 2\varepsilon$.

It follows that $|\mu_1 - \mu_2|(E) = \inf\{|\mu_1 - \mu_2|(V),\ V \text{ open and } V \supseteq E\}$. As this holds for any $E \in \mathscr{M}$, $|\mu_1 - \mu_2|$ is outer regular.

Therefore, $|\mu_1 - \mu_2|$ is normal and so $\mu_1 - \mu_2$ is normal.

Theorem 34. Suppose X is a normal Hausdorff topological space and $BC(X,\mathbb{R}) = \{f : X \to \mathbb{R}; f \text{ is continuous and bounded}\}$. Suppose $\Phi : BC(X,\mathbb{R}) \to \mathbb{R}$ is a bounded real linear functional. Then there exists an algebra $\mathscr{M}$ on X, containing all the open sets of X and a unique finite real finitely-additive measure (signed finitely-additive measure), λ, on $\mathscr{M}$, expressible as the difference of two finite normal finitely-additive positive measures, such that $\Phi(f) = \int_X f\, d\lambda$ and $\|\Phi\| = |\lambda|(X)$. Let M be the collection of all finite normal real finitely-additive Borel measures or finite normal real Borel charge, expressible as the difference of two finite normal finitely-additive positive Borel measures or finite normal positive Borel charge, with a norm on M given by $\|\mu\| = |\mu|(X)$ for μ in M. Then the association $\Gamma : BC(X,\mathbb{R})^* \to M$, where $BC(X,\mathbb{R})^*$ is the real dual space of $BC(X,\mathbb{R})$, given by $\Gamma(\Phi) = \lambda$, where $\Phi(f) = \int_X f\, d\lambda$, is a linear isometric isomorphism preserving norm.

Proof.

Suppose X is a normal Hausdorff topological space.

Suppose $\Phi : BC(X,\mathbb{R}) \to \mathbb{R}$ is a bounded real linear functional. Then by Proposition 8, we can decompose Φ as $\Phi = \Phi^+ - \Phi^-$ such that Φ^+ and Φ^- are positive real linear functional and $\|\Phi\| = \|\Phi^+\| + \|\Phi^-\| = \Phi^+(1) + \Phi^-(1)$. By the Riesz Representation Theorem (Theorem 29), there are unique *finite* normal finitely-additive positive Borel measures, μ_1 and μ_2, on $\mathscr{M}$, the algebra generated by the open sets of X, such that $\Phi^+(f) = \int_X f\, d\mu_1$ and $\Phi^-(f) = \int_X f\, d\mu_2$. Thus, $\Phi(f) = \int_X f\, d\mu_1 - \int_X f\, d\mu_2 = \int_X f\, d(\mu_1 - \mu_2)$. Let $\lambda = \mu_1 - \mu_2$. Then λ is a finitely-additive real measure. Moreover, by Proposition 33, λ is normal and $|\lambda|$ is a normal finitely-additive finite positive measure.

Then for all $f \in BC(X,\mathbb{R})$,

$$\Phi(f) = \int_X f\, d\lambda \text{ and}$$

$$|\Phi(f)| = \left|\int_X f\, d(\mu_1-\mu_2)\right| = \left|\int_X f^+\, d(\mu_1-\mu_2) - \int_X f^-\, d(\mu_1-\mu_2)\right|$$

$$\leq \left|\int_X f^+ d\lambda\right| + \left|\int_X f^- d\lambda\right| \leq \int_X f^+ d|\lambda| + \int_X f^- d|\lambda| = \int_X |f|\, d|\lambda|$$

$$\leq \|f\|_u \int_X d|\lambda| = \|f\|_u\, |\lambda|(X).$$

Hence, $\|\Phi\| \leq |\lambda|(X)$. But $|\lambda|(X) \leq \mu_1(X) + \mu_2(X) = \Phi^+(1) + \Phi^-(1) = \|\Phi\|$ and so $\|\Phi\| = |\lambda|(X) = \mu_1(X) + \mu_2(X)$.

We shall now show that λ is unique.

Suppose there exist finite normal finitely-additive real Borel measures, λ_1 and λ_2 such that $\Phi(f) = \int_X f\, d\lambda_1 = \int_X f\, d\lambda_2$. Let $\mu = \lambda_1 - \lambda_2$, then $\int_X f\, d\mu = 0$. By Proposition 33, μ is a finite normal finitely-additive real Borel measure. We can write $\mu = \mu^+ - \mu^-$, where $\mu^+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu^- = \frac{1}{2}(|\mu| - \mu)$ are finite finitely-additive positive Borel measures.

As $\int_X f\, d\mu = 0$ for all $f \in BC(X,\mathbb{R})$, $\int_X f\, d\mu^+ = \int_X f\, d\mu^-$ for all $f \in BC(X,\mathbb{R})$ and both define the same positive real linear functional. Therefore, by the uniqueness part of Theorem 29 (Riesz Representation Theorem), $\mu^+ = \mu^-$, consequently $\mu = 0$ and so $\lambda_1 = \lambda_2$.

Hence, we conclude that the real dual space of $BC(X,\mathbb{R})$, that is, the space of all bounded real linear functional on $BC(X,\mathbb{R})$ is isometrically isomorphic (i.e., via a norm preserving map) with the space of all normal finitely-additive real (signed) Borel measures (i.e., with bounded variation), expressible as the difference of two finite normal finitely-additive positive measures, on the algebra $\mathscr{M}$ on *X*, with norm given by $\|\mu\| = |\mu|(X)$.

Remark. The situation with bounded complex linear functional on $BC(X,\mathbb{C})$ with the sup norm is somewhat unclear.

Since both locally compact Hausdorff space and normal Hausdorff space are completely regular, it is natural to seek Riesz type representation theorem for positive or bounded linear functional on $BC(X)$. This is a many faceted problem. If we extend to the representation of continuous linear functional on $C^c(X)$, the algebra of complex continuous functions endowed with the "*c*" topology of compact convergence, we have a nice formulation of the representation of continuous linear functionals by complex Borel *measures with compact support*, attributed to Brooks and Dietrich, Jr. We describe this development as follows.

Theorem 35. Suppose X is a completely regular Hausdorff topological space. Let $M(X)$ be the vector space of all regular complex Borel measures on the Borel σ-algebra of X. Let $M_c(X)$ be the subspace of $M(X)$, consisting of all regular Borel measure, μ, which is concentrated on some compact set, i.e, the support of μ is compact. Then there is an isomorphism of vector space,

$$\Gamma : M_c(X) \to C^c(X)',$$

onto the topological dual of $C^c(X)$, the algebra of all continuous complex functions on X, under the topology $\mathscr{T}_c$ of compact convergence, where the topology $\mathscr{T}_c$ is defined by the m-semi-norms $\|f\|_K = \|f\,|_K\|$ for every $K \in \mathscr{K}$ = collection of all compact subspaces of X. $C^c(X)'$ is the space of continuous linear maps $C^c(X) \to \mathbb{C}$. Γ is given by $\Gamma(\mu) = \Lambda_\mu$, where $\Lambda_\mu(f) = \int_X f\, d\mu$, for all $f \in C^c(X)$. Moreover, $\|\Lambda_\mu\| = \|\mu\|(X)$ and Λ_μ is positive if and only if μ is positive.

For the details and proof see Chapter 6, Theorem 25.1 of *Topological Algebras with Involution* by Maria Fragoulopoulou.

In another direction if we take the space $BC(X)$ of bounded complex function on X, when X is a completely regular Hausdorff space, we can consider the topology, the strict topology $\mathscr{T}_{st}$ on $BC(X)$, in between the topology of compact convergence $\mathscr{T}_c$ and the uniform topology $\mathscr{T}_\infty$ given by the sup norm. For the definition of the strict topology see 2.10D *Locally Convex Spaces*, by Hans Jarchow.

Theorem 36 . Let X be a completely regular space, and let $\mathscr{T}_{st}$, be the strict topology on $BC(X)$, the space of bounded continuous complex functions. For every $\mathscr{T}_{st}$ -continuous linear form Λ on $BC(X)$, there is a unique regular complex measure $\mu \in M(X)$, the space of regular complex Borel measures on X, such that

$\Lambda(f)=\int_X f\,d\mu$ for all $f \in BC(X)$.

The map, $(BC(X), \mathscr{T}_{st})' \to M(X)$, obtained in this way is an isomorphism.

The three topologies we mentioned above satisfy the relation, $\mathscr{T}_c \subseteq \mathscr{T}_{st} \subseteq \mathscr{T}_\infty$.

For $BC(X)$ with the uniform topology, i.e., with the sup norm, we have the following.

Theorem 37. Suppose X is a completely regular Hausdorff topological space. Let $(BC(X), \mathscr{T}_\infty)'$ be the topological dual of $BC(X)$ with the uniform topology $\mathscr{T}_\infty$. Then for every Γ in $(BC(X), \mathscr{T}_\infty)'$, there exists $\tilde{\mu}$ a unique regular Borel measure of the Stone-Cech compactification βX of X such that

$$\Gamma(f)=\int_{\beta X} \tilde{f}\,d\tilde{\mu} ,$$

for every $f \in BC(X)$, where $\tilde{f}$ is the unique natural extension of f to βX.

(For the proof of Theorem 36 and Theorem 37, see Theorem 7.6.3 and Corollary 7.6.2 of *Locally Convex Spaces*, by Hans Jarchow. Theorem 36 is Theorem 2.6 of the article, *The σ-compact open topology and its relatives*, by Denny Gulick, Math Scand 30 (1972) 159-178 and also Theorem 2 of *A generalization of the strict topology,* Math Scand 30 (1972) 313-323 by J. Hoffman-JØRGENSEN.)

When X is a compact Hausdorff space and therefore, a completely regular Hausdorff space, Theorem 34, 35 and 36 coincide as $C(X) = BC(X)$, $C^c(X) = (BC(X), \mathscr{T}_\infty) = (BC(X), \mathscr{T}_{st})$, $\beta X = X$ and $\mathscr{T}_c = \mathscr{T}_{st} = \mathscr{T}_\infty$.