Chapter 9 Uniform Convergence, Integration and Power Series

Recall that Theorem 7 of Chapter 8 says that if a sequence of continuous functions (g_n) converges uniformly on $[a, b]$ to a function g, then the integral of the limiting function g is the limit of the integral of g_n over $[a, b]$ as n tends to ∞. We shall investigate here when we relax the requirement of continuity on (g_n) to one of integrability, whether the conclusion still holds. Indeed it is the case when we have uniform convergence. In fact we can even replace the condition of uniform convergence by a notion of domination by a suitable integrable function, i.e., that there exists an integrable function h such that $|g_n| \leq h(x)$ for all integer $n \geq 1$ and for all x in $[a, b]$ and if each g_n is (Riemann) integrable and $g_n \to g$ pointwise, then if g is (Riemann) integrable, the integral of g is the limit of the integral of g_n over $[a, b]$ as n tends to ∞. Note that the pointwise limit of a sequence of Riemann integrable functions need not be Riemann integrable and so if we relax the condition of uniform convergence we would have to assume integrability of the limiting function g. This result is known as the Arzelà's Dominated Convergence Theorem (also known as the Riemann Dominated Convergence Theorem because it applies to Riemann integrals) and is a special case of the Lebesgue Dominated Convergence Theorem. An elementary proof of the Arzelà's Dominated Convergence Theorem without using the idea of Lebesgue measure is difficult. We shall not go into the proof of this result or discuss Lebesgue theory. We concern ourselves with the consequence of the uniform convergence and the use of the Arzelà's Dominated Convergence Theorem when uniform convergence is lacking.

9.1 Uniform Convergence and Integration

Theorem 1. Suppose ($f_k : [a, b] \to \mathbf{R}$, $k = 1, 2, \ldots$) is a sequence of Riemann integrable functions. Suppose (f_n) converges uniformly to a function $f : [a, b] \to \mathbf{R}$. Then f is Riemann integrable, $\lim_{n\to\infty} \int_a^b |f_n - f| = 0$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n . \qquad \text{-----------------------------} \quad (A)$$

Proof. If we assume the Riemann integrability of the limiting function f, then the proof is similar to that of Theorem 7 of Chapter 8. Note that f has a good chance to be Riemann integrable since the uniform limit of a sequence of bounded function is bounded. We deduce this as follows.
Sinec $f_n \to f$ uniformly on $[a, b]$., there exists a positive integer N such that for all $n \geq N$ and for all x in $[a, b]$,

$$|f_n(x) - f(x)| < 1 .$$

Hence, for all $n \geq N$ and for all x in $[a, b]$,

$$|\,|f_n(x)| - |f(x)|\,| \leq |f_n(x) - f(x)| < 1.$$

Therefore,

$$|f(x)| < |f_N(x)| + 1 \leq K + 1 \text{ for all } x \text{ in } [a, b]$$

for some $K > 0$, since f_N is bounded because it is Riemann integrable.
Thus f is bounded on $[a, b]$.
Now we shall prove (A) assuming the integrability of f.
Given any $\varepsilon > 0$, since $f_n \to f$ uniformly on $[a, b]$, there exists a positive integer M such that for all $n \geq M$ and for all x in $[a, b]$,

$$|f_n(x)-f(x)|<\frac{\varepsilon}{2(b-a)}.$$

Therefore, since f_n and f are both Riemann integrable on $[a, b]$, $f_n - f$ is Riemann integrable on $[a, b]$ by Theorem 30 Chapter 5 and consequently by Theorem 53, $|f_n - f|$ is Riemann integrable on $[a, b]$. Hence for all $n \geq M$,

$$\int_a^b |f_n(x)-f(x)|dx \leq \int_a^b \frac{\varepsilon}{2(b-a)} = \frac{\varepsilon}{2} < \varepsilon. \qquad \text{--------------- (1)}$$

Thus, $\lim_{n\to\infty} \int_a^b |f_n(x)-f(x)|dx = 0.$

Using inequality (1) for all $n \geq M$,

$$\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\right| \leq \int_a^b |f_n - f|$$

by Theorem 53 Chapter 8

$$\leq \int_a^b \frac{\varepsilon}{2(b-a)} = \frac{\varepsilon}{2} < \varepsilon.$$

Hence, by definition, $\int_a^b f_n \to \int_a^b f$.

(We can also use the Comparison Test for sequences to conclude that , $\int_a^b f_n \to \int_a^b f$ since $\left|\int_a^b f_n - \int_a^b f\right| \leq \int_a^b |f_n - f|$ and $\lim_{n\to\infty} \int_a^b |f_n(x)-f(x)|dx = 0$.)

Now we show that f is Riemann integrable. Recall that f is Riemann integrable if and only if f is bounded and the upper Darboux integral is equal to the lower Darboux integral. All we need show now is that the upper Darboux integral is equal to the lower Darboux integral of f.

Take any $\varepsilon > 0$, since $f_n \to f$ uniformly on $[a, b]$, there exists a positive integer N such that for all $k \geq N$ and for all x in $[a, b]$,

$$|f_k(x)-f(x)| < \varepsilon.$$

That is, for all $k \geq N$ and for all x in $[a, b]$,

$$f_k(x) - \varepsilon < f(x) < f_k(x) + \varepsilon.$$

(We have already observed that f is bounded. This also follows from the above inequality because we can take $n = N$ and we already knew that f_N is bounded because f_N is Riemann integrable.)

Thus for any partition P : $a = x_0 < x_1 < \ldots < x_n = b$ for $[a, b]$ and for each $k > N$,

$$\sup\{f(x) : x \in [x_{i-1}, x_i]\} \leq \sup\{f_k(x) : x \in [x_{i-1}, x_i]\} + \varepsilon .$$

Let $M_i(f, P) = \sup\{f(x): x \in [x_{i-1}, x_i]\}$ for $1 \leq i \leq n$. Then it follows from above that

$$M_i(f, P) \leq M_i(f_k, P) + \varepsilon$$

for $1 \leq i \leq n$ and for $k \geq N$.

Therefore, for $k \geq N$,

$$U(f,P) = \sum_{i=1}^{n} M_i(f,P)(x_i - x_{i-1})$$

$$\leq \sum_{i=1}^{n} M_i(f_k,P)(x_i - x_{i-1}) + \sum_{i=1}^{n} \varepsilon(x_i - x_{i-1}) = U(f_k,P) + (b-a)\varepsilon.$$

Since we know f is bounded both the upper and lower Darboux integrals of f exist (see Definition 14 Chapter 5). Hence for $k \geq N$ and for any partition P for $[a, b]$

$$U\int_a^b f \leq U(f,P) \leq U(f_k,P) + (b-a)\varepsilon,$$

where $U\int_a^b f$ is the upper Darboux integral of f.

Thus for all $k \geq N$,

$$U\int_a^b f \leq U\int_a^b f_k + (b-a)\varepsilon \qquad \text{-------------------------- (1)}$$

Similarly, for any partition P : $a = x_0 < x_1 < \ldots < x_n = b$ for $[a, b]$ and for each $k > N$,

$$\inf\{f(x) : x \in [x_{i-1}, x_i]\} \ge \inf\{f_k(x) : x \in [x_{i-1}, x_i]\} - \varepsilon$$
for $1 \le i \le n$.

If we denote $\inf\{f(x) : x \in [x_{i-1}, x_i]\}$ by $m_i(f, P)$ and $\inf\{f_k(x) : x \in [x_{i-1}, x_i]\}$ by $m_i(f_k, P)$ for $k \ge N$, then we have for $1 \le i \le n$.
$$m_i(f, P) \ge m_i(f_k, P) - \varepsilon .$$
Therefore, for $k \ge N$,
$$L(f,P) = \sum_{i=1}^{n} m_i(f,P)(x_i - x_{i-1})$$
$$\ge \sum_{i=1}^{n} m_i(f_k,P)(x_i - x_{i-1}) - \sum_{i=1}^{n} \varepsilon(x_i - x_{i-1}) = L(f_k,P) - (b-a)\varepsilon.$$
Consequently, for any partition P for $[a, b]$,
$$L\int_a^b f \ge L(f,P) \ge L(f_k,P) - (b-a)\varepsilon.$$
Hence,for $k \ge N$,
$$L\int_a^b f \ge L\int_a^b f_k - (b-a)\varepsilon \quad \text{---------------------------} \ (2)$$
Hence, $\quad 0 \le U\int_a^b f - L\int_a^b f \le U\int_a^b f_N - L\int_a^b f_N + 2(b-a)\varepsilon$ by (1) and (2)

$\le 2(b-a)\varepsilon$

since f_N is Riemann integrable on $[a, b]$ so that $U\int_a^b f_N = L\int_a^b f_N$.
Since ε is arbitrarily small, $0 \le U\int_a^b f - L\int_a^b f \le 0$. Thus $U\int_a^b f = L\int_a^b f$. Hence f is Riemann integrable on $[a, b]$.

Alternative proof of Theorem 1
(We may also use the equivalent condition (3) in Theorem 21 Chapter 5 to show that f is Riemann integrable.)
As in the above proceeding, by uniform convergence, there exists a positive integer N such that for all $k \ge N$ and for all x in $[a, b]$,
$$|f_k(x) - f(x)| < \frac{\varepsilon}{4(b-a)}.$$
And using the above inequality, for any partition P of $[a, b]$ and for any $k \ge N$,
$$U(f,P) = \sum_{i=1}^{n} M_i(f,P)(x_i - x_{i-1})$$
$$\le \sum_{i=1}^{n} M_i(f_k,P)(x_i - x_{i-1}) + \sum_{i=1}^{n} \frac{\varepsilon}{4(b-a)}(x_i - x_{i-1}) = U(f_k,P) + \frac{\varepsilon}{4}$$
and
$$L(f,P) = \sum_{i=1}^{n} m_i(f,P)(x_i - x_{i-1})$$
$$\ge \sum_{i=1}^{n} m_i(f_k,P)(x_i - x_{i-1}) - \sum_{i=1}^{n} \frac{\varepsilon}{4(b-a)}(x_i - x_{i-1}) = L(f_k,P) - \frac{\varepsilon}{4}.$$
Hence, for any partition P of $[a, b]$ and for any $k \ge N$,
$$U(f,P) - L(f,P) \le U(f_k,P) - L(f_k,P) + \frac{\varepsilon}{2},$$
and so
$$U(f,P) - L(f,P) \le U(f_N,P) - L(f_N,P) + \frac{\varepsilon}{2}.$$
By Theorem 21 (3) since f_N is Riemann integrable, there exists a partition Q of $[a, b]$ such that
$$U(f_N,Q) - L(f_N,Q) < \frac{\varepsilon}{2}.$$
Therefore, $U(f,Q) - L(f,Q) \le U(f_N,Q) - L(f_N,Q) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$
Hence by Theorem 21 (3), f is Riemann integrable.

The following is a specialization to series of functions.

Corollary 2. Suppose for each integer $n \geq 1$, $f_n : [a, b] \to \mathbf{R}$ is a Riemann integrable function. Suppose the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$ to a function $f : [a, b] \to \mathbf{R}$. Then f is Riemann integrable, $\lim_{n\to\infty} \int_a^b \left| \sum_{k=1}^{n} f_k - f \right| = 0$ and

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n .$$

Proof. For each integer $n \geq 1$, let $s_n = \sum_{k=1}^{n} f_k$. By assumption $s_n \to f$ uniformly on $[a, b]$. Therefore, since each s_n is Riemann integrable as it is a finite sum of Riemann integrable functions, by Theorem 1, $\lim_{n\to\infty} \int_a^b \left| \sum_{k=1}^{n} f_k - f \right| = \lim_{n\to\infty} \int_a^b |s_n - f| = 0$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b s_n = \lim_{n\to\infty} \int_a^b \sum_{k=1}^{n} f_k = \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b f_k = \sum_{k=1}^{\infty} \int_a^b f_k .$$

This completes the proof.

To apply Theorem 8 of Chapter 8, we have to check the uniform convergence of the sequence (f_n'). To apply Theorem 1, we need to check the uniform convergence of the sequence (f_n). Besides the Weierstrass M Test, which is a Test for absolute convergence, we have the more delicate Abel's and Dirichlet's Test for uniform convergence. These are useful when the series of functions is not absolutely convergent.

9.2 Abel's Test for Uniform Convergence

Theorem 3 (Abel's Test).
Let $(f_n : E \to \mathbf{R})$, where E is an interval, be a decreasing sequence of functions. That is to say, $f_{n+1}(x) \leq f_n(x)$ for all x in E and for all integer $n \geq 1$. Further suppose (f_n) is uniformly bounded, i.e., $|f_n(x)| \leq K$ for some real number $K > 0$ for all x in E and for all integer $n \geq 1$. If $(g_n : E \to \mathbf{R})$ is a sequence of functions such that the series $\sum_{n=1}^{\infty} g_n$ converges uniformly on E, then the series of functions $\sum_{n=1}^{\infty} f_n \cdot g_n$ also converges uniformly on E.

Proof. The proof is reminiscence of Abel's Theorem (Theorem 18) in Chapter 8. We shall show that $\sum_{n=1}^{\infty} f_n \cdot g_n$ is uniformly Cauchy on E. We shall make use of Abel's summation formula.

For each integer $n \geq 1$, let $s_n = \sum_{k=1}^{n} b_k$. Then

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= \sum_{k=2}^{n} a_k (s_k - s_{k-1}) + a_1 s_1 = \sum_{k=2}^{n} a_k s_k - \sum_{k=2}^{n} a_k s_{k-1} + a_1 s_1 \\ &= \sum_{k=2}^{n} a_k s_k - \sum_{k=1}^{n-1} a_{k+1} s_k + a_1 s_1 = \sum_{k=2}^{n-1} (a_k - a_{k+1}) s_k + a_n s_n - a_2 s_1 + a_1 s_1 \\ &= \sum_{k=1}^{n-1} (a_k - a_{k+1}) s_k + a_n s_n\end{aligned}$$

$$= \sum_{k=1}^{n}(a_k - a_{k+1})s_k + a_{n+1}s_n. \qquad \text{-------------------------------} \quad (1)$$

Observe that $\sum_{k=1}^{n}(a_{k+1} - a_k) = a_{n+1} - a_1$. Then we have $a_{n+1} = \sum_{k=1}^{n}(a_{k+1} - a_k) + a_1$.

It then follows from (1) that

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n}(a_k - a_{k+1})s_k + a_{n+1}s_n = -\sum_{k=1}^{n}(a_{k+1} - a_k)s_k + \sum_{k=1}^{n}(a_{k+1} - a_k)s_n + a_1 s_n$$
$$= \sum_{k=1}^{n}(a_{k+1} - a_k)(s_n - s_k) + a_1 s_n. \quad \text{--} \ (2)$$

Formula (2) is known as *Abel's summation formula*.

For each integer $n \geq 1$, let $s_n(x) = \sum_{k=1}^{n} g_k(x)$ and denote the n-th partial sum for $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ by $t_n(x) = \sum_{k=1}^{n} f_k(x)g_k(x)$. Then using (2) with $b_n = g_n(x)$ and $a_n(x) = f_n(x)$, we obtain,

$$t_n(x) = \sum_{k=1}^{n} f_k(x)g_k(x) = \sum_{k=1}^{n}(f_{k+1}(x) - f_k(x))(s_n(x) - s_k(x)) + f_1(x)s_n(x) \ \text{----------} \ (3)$$

We want to show that ($t_n(x)$) is uniformly Cauchy on E. So we examine $t_n(x) - t_m(x)$ for $n > m$.

From (3) we get for integers $n > m$,

$$\begin{aligned} t_n(x) - t_m(x) &= \sum_{k=1}^{n}(f_{k+1}(x) - f_k(x))(s_n(x) - s_k(x)) + f_1(x)(s_n(x) - s_m(x)) \\ &\quad - \sum_{k=1}^{m}(f_{k+1}(x) - f_k(x))(s_m(x) - s_k(x)) \\ &= \sum_{k=m+1}^{n}(f_{k+1}(x) - f_k(x))(s_n(x) - s_k(x)) \\ &\quad + \sum_{k=1}^{m}(f_{k+1}(x) - f_k(x))(s_n(x) - s_m(x)) + f_1(x)(s_n(x) - s_m(x)) \\ &= \sum_{k=m+1}^{n}(f_{k+1}(x) - f_k(x))(s_n(x) - s_k(x)) \\ &\quad + (s_n(x) - s_m(x)) \sum_{k=1}^{m}(f_{k+1}(x) - f_k(x)) + f_1(x)(s_n(x) - s_m(x)) \\ &= \sum_{k=m+1}^{n}(f_{k+1}(x) - f_k(x))(s_n(x) - s_k(x)) \\ &\quad + (s_n(x) - s_m(x))(f_{m+1}(x) - f_1(x)) + f_1(x)(s_n(x) - s_m(x)) \\ &= \sum_{k=m+1}^{n}(f_{k+1}(x) - f_k(x))(s_n(x) - s_k(x)) + f_{m+1}(x)(s_n(x) - s_m(x)) . \end{aligned}$$

Hence for all integers n , m with $n > m$ and for all x in E,

$$|t_n(x) - t_m(x)| \leq \sum_{k=m+1}^{n} |f_{k+1}(x) - f_k(x)||s_n(x) - s_k(x)| + |f_{m+1}(x)||s_n(x) - s_m(x)|. \ \text{----} \ (4)$$

Now we bring in the Cauchy condition for the series $\sum_{n=1}^{\infty} g_n$. Since $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on E, it is uniformly Cauchy by Theorem 3 Chapter 8. Therefore, given any $\varepsilon > 0$, there exists a positive integer N such that for all integers n *and* m,

$$n, m \geq N \Rightarrow |s_n(x) - s_m(x)| < \frac{\varepsilon}{3K} .$$

It then follow from (4) that for all integers n , m with $n > m \geq N$ and for all x in E,

$$|t_n(x) - t_m(x)| < \sum_{k=m+1}^{n} |f_{k+1}(x) - f_k(x)| \frac{\varepsilon}{3K} + K\frac{\varepsilon}{3K}$$

since $|f_{m+1}(x)| \le K$.
But since ($f_n : E \to \mathbf{R}$) is a decreasing sequence of functions, $|f_{k+1}(x) - f_k(x)| = f_k(x) - f_{k+1}(x)$. Thus, for all integers n , m with $n > m \ge N$ and for all x in E,

$$|t_n(x) - t_m(x)| < \sum_{k=m+1}^{n} (f_k(x) - f_{k+1}(x))\frac{\varepsilon}{3K} + \frac{\varepsilon}{3} = (f_{m+1}(x) - f_{n+1}(x))\frac{\varepsilon}{3K} + \frac{\varepsilon}{3}$$
$$\le (K + K)\frac{\varepsilon}{3K} + \frac{\varepsilon}{3} = \varepsilon$$

because $|f_j(x)| \le K$ for all integer $j \ge 1$. This proves that ($t_n(x)$) is uniformly Cauchy on E. Hence ($t_n(x)$) converges uniformly on E.

The Weierstrass M Test is a test for absolute convergence as well as uniform convergence. Thus for a series of functions that does not necessarily converge absolutely for every x, we may need for instance, either Abel's Test or Dirichlet's Test, which we shall describe later.

Example 4. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-nx}$ converges uniformly on $[0, \infty)$ and so we can integrate this series function termwise on $[0, \infty)$.

Proof. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges uniformly (since it is independent of x) by Leibnitz's Alternating Series Test. Moreover for all $x \ge 0$, e^{nx} is increasing, i.e, $e^{(n+1)x} \ge e^{nx}$ for all integers $n \ge 1$. Hence $e^{-(n+1)x} \le e^{-nx}$ for all integers $n \ge 1$. Thus the sequence (e^{-nx}) is a decreasing sequence of function on $[0, \infty)$. Note that $|e^{-nx}| = \frac{1}{e^{nx}} \le 1$ for all integer $n \ge 1$ and all $x \ge 0$. Therefore, (e^{-nx}) is uniformly bounded by 1. Hence by Abel's Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-nx}$ converges uniformly on $[0, \infty)$. [Here we take $f_n(x) = e^{-nx}$ and $g_n(x) = \frac{(-1)^n}{n}$ for integer $n \ge 1$.]

Remark. Theorem 3 (Abel's Test) also holds true when ($f_n : E \to \mathbf{R}$), is an increasing sequence of functions, which is uniformly bounded. We deduce this as follows. If ($f_n : E \to \mathbf{R}$) is increasing and uniformly bounded, then ($-f_n : E \to \mathbf{R}$) is decreasing and also uniformly bounded. Therefore, by Theorem 3, $\sum_{n=1}^{\infty} -f_n(x)g_n(x)$ is uniformly convergent on E, when $\sum_{n=1}^{\infty} g_n(x)$ is uniformly convergent on E. Therefore, $\sum_{n=1}^{\infty} f_n(x)g_n(x) = -\sum_{n=1}^{\infty}(-f_n(x))g_n(x)$ is uniformly convergent if $\sum_{n=1}^{\infty} g_n(x)$ is uniformly convergent on E.

9.3 Dirichlet's Test for Uniform Convergence

Theorem 5. (Dirichlet's Test) Suppose E is a nontrivial interval.
Let ($f_n : E \to \mathbf{R}$) be a sequence of functions. For each integer $n \ge 1$, let $s_n : E \to \mathbf{R}$ be defined by $s_n(x) = \sum_{k=1}^{n} f_k(x)$ for x in E, that is to say, $s_n(x)$ is the n-th partial sum of the series $\sum_{k=1}^{\infty} f_k(x)$. Suppose (s_n) is uniformly bounded, i.e., $|s_n(x)| \le K$ for some

real number $K > 0$, for all x in E and for all integer $n \ge 1$. Suppose ($g_n : E \to \mathbf{R}$) is a sequence of non-negative functions such that $g_n \to 0$ uniformly on E. Suppose ($g_n : E \to \mathbf{R}$) is a decreasing sequence of functions, i.e., $g_{n+1}(x) \le g_n(x)$ for all x in E and all integer $n \ge 1$. Then the series of functions $\sum_{n=1}^{\infty} f_n(x) g_n(x)$ converges uniformly on E.

Proof.

For each integer $n \ge 1$, let $t_n(x) = \sum_{k=1}^{n} f_k(x) g_k(x)$. Then by the Abel's summation formula, (formula (1) in the proof of Theorem 3), with $b_k = f_k(x)$ and $a_k = g_k(x)$,

$$t_n(x) = \sum_{k=1}^{n} f_k(x) g_k(x) = \sum_{k=1}^{n} (g_k(x) - g_{k+1}(x)) s_k(x) + g_{n+1}(x) s_n(x).$$

Thus for integers $n > m$ and all x in E,

$$t_n(x) - t_m(x) = \sum_{k=m+1}^{n} (g_k(x) - g_{k+1}(x)) s_k(x) + g_{n+1}(x) s_n(x) - g_{m+1}(x) s_m(x). \text{ -------- } (1)$$

Hence from (1),

$$|t_n(x) - t_m(x)| \le \sum_{k=m+1}^{n} (g_k(x) - g_{k+1}(x)) |s_k(x)| + g_{n+1}(x) |s_n(x)| + g_{m+1}(x) |s_m(x)|$$

by the triangle inequality and that (g_n) is non-negative and decreasing

$$\le \sum_{k=m+1}^{n} (g_k(x) - g_{k+1}(x)) K + (g_{n+1}(x) + g_{m+1}(x)) K$$

$$\le (g_{m+1}(x) - g_{n+1}(x)) K + (g_{n+1}(x) + g_{m+1}(x)) K = 2K g_{m+1}(x). \text{ ------- } (2)$$

Since $g_n \to 0$ uniformly on E, given any $\varepsilon > 0$, there exists a positive integer N such that

$$n \ge N \Rightarrow |g_n(x)| = g_n(x) < \frac{\varepsilon}{2K} \text{ for all } x \text{ in } E.$$

Thus for integers $n > m \ge N$ and for all x in E,

$$|t_n(x) - t_m(x)| \le 2K g_{m+1}(x) < 2K \frac{\varepsilon}{2K} = \varepsilon.$$

Thus, $(t_n(x))$ is uniformly Cauchy on E and so $(t_n(x))$ converges uniformly on E.

Example 6. The series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ is uniformly convergent on the interval $[\delta, 2\pi - \delta]$, where $0 < \delta < \pi$.

We apply Dirichlet's Test with $f_n(x) = \sin(nx)$ and $g_n(x) = \frac{1}{n}$ for each integer $n \ge 1$. Then plainly $(g_n(x))$ is a non-negative, decreasing sequence and $g_n \to 0$ uniformly on any subset of $\mathbf{R}$.

Note that for each n,

$$2\sin(\tfrac{1}{2}x) \sum_{k=1}^{n} \sin(kx) = \cos(\tfrac{1}{2}x) - \cos(nx + \tfrac{1}{2}x)$$

and so
$$\sum_{k=1}^{n} \sin(kx) = \frac{\cos(\frac{1}{2}x) - \cos(nx + \frac{1}{2}x)}{2\sin(\frac{1}{2}x)}$$

if x is not a multiple of 2π.

Thus for all x in $[\delta, 2\pi - \delta]$, where $0 < \delta < \pi$,

$$s_n(x) = \sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} \sin(kx) = \frac{\cos(\frac{1}{2}x) - \cos(nx + \frac{1}{2}x)}{2\sin(\frac{1}{2}x)}.$$

Hence, for all x in $[\delta, 2\pi - \delta]$,

$$|s_n(x)| \le \left| \frac{\cos(\frac{1}{2}x) - \cos(nx + \frac{1}{2}x)}{2\sin(\frac{1}{2}x)} \right| \le \frac{1}{|\sin(\frac{1}{2}x)|} .$$

Now $\delta < x < 2\pi - \delta \Rightarrow$

$\frac{\delta}{2} < \frac{x}{2} < \pi - \frac{\delta}{2} (< \pi) \Rightarrow \sin(\frac{x}{2}) > \sin(\frac{\delta}{2}) > 0 \Rightarrow \frac{1}{|\sin(\frac{1}{2}x)|} < \frac{1}{|\sin(\frac{1}{2}\delta)|}$.

Therefore, for all x in $[\delta, 2\pi - \delta]$, $|s_n(x)| \le \frac{1}{|\sin(\frac{1}{2}\delta)|}$.

Thus, by Dirichlet's Test, $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ is uniformly convergent on $[\delta, 2\pi - \delta]$, for $0 < \delta < \pi$.

Remark.

1. We can show similarly as in Example 6 that $\sum_{n=1}^{\infty} a_n \sin(nx)$ is uniformly convergent on $[\delta, 2\pi - \delta]$, where $0 < \delta < \pi$ for any decreasing sequence (a_n) having non-negative terms and which converges to 0 as n tends to infinity.
2. The conclusion of Theorem 5 also holds true if $(g_n(x))$ instead of being decreasing, is increasing on E, with $g_n(x) \le 0$ for all integer $n \ge 1$ and all x in E and $g_n \to 0$ uniformly on E. Just apply the test (Theorem 5) to $(-g_n)$ and (f_n) to conclude that $\sum_{n=1}^{\infty} f_n(x)(-g_n(x))$ converges uniformly on E and so it follows that $\sum_{n=1}^{\infty} f_n(x) g_n(x)$ converges uniformly on E.

Example 7. By Corollary 2, since we have shown that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ is uniformly convergent on $[\delta, 2\pi - \delta]$, where $0 < \delta < \pi$, we can integrate the series term by term on any subinterval in $[\delta, 2\pi - \delta]$. That is,

$$\sum_{n=1}^{\infty} \int_{\pi}^{x} \frac{\sin(nt)}{n} dt = \sum_{n=1}^{\infty} \left[-\frac{\cos(nt)}{n^2} \right]_{\pi}^{x} = \sum_{n=1}^{\infty} \left(\frac{\cos(n\pi)}{n^2} - \frac{\cos(nx)}{n^2} \right)$$

converges for any x in $[\delta, 2\pi - \delta]$.

Actually the series on the right hand side is uniformly convergent on **R** by the Weierstrass M Test.

On the other hand we cannot differentiate the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ term by term on the interval $[\delta, 2\pi - \delta]$. This is because $\sum_{n=1}^{\infty} \cos(nx)$ is divergent, as $\cos(nx) \not\to 0$ as $n \to \infty$ (Reference: Proposition 10 Chapter 6). We can easily deduce this as follows. If $a_n = \cos(nx) \to 0$, then $a_{2n} = \cos(2nx) \to 0$ too, because if a sequence converges to a value, then any subsequence converges to the same value.

But $a_{2n} = \cos(2nx) = 2\cos^2(nx) - 1 \to 2\times 0 - 1 = -1$ because $a_n^2 \to 0$. So we have a contradiction as we have just shown that $a_{2n} \to 0$. Hence for the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ we can integrate it term by term on $[\delta, 2\pi - \delta]$, with $0 < \delta < \pi$ but we cannot differentiate it term by term on any interval.

9.4 Integrating Power Series

Example 8. For integral that has no closed formula involving elementary functions where tables or algorithms are readily available, we may use power series expansion for the function to obtain a series expansion for the desired integral. As an example we shall do this for $\int_0^x e^{-t^2}dt$. We shall obtain a power series expansion for this integral.
Recall either in Example 9 or 12 of Chapter 8 that the exponential function e^x has a series expansion $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x in **R**. (We can take e^x as the solution to the differential equation $f'(x) = f(x)$ with initial condition $f(0) = 1$.)
Then $e^{-t^2} = \sum_{n=0}^{\infty}\frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{n!}$ for all t in **R**.
(By a simple Ratio Test we confirm that the radius of convergence of this series is $+\infty$.) Note that for any $x \neq 0$, by Theorem 11 Chapter 8 the series $\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{n!}$ is uniformly convergent on $[-|x|, |x|]$. Thus by Theorem 1, we have

$$\int_0^x e^{-t^2}dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^x t^{2n}dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{2n+1}}{2n+1}$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots$$

We can use this power series to calculate $\int_0^x e^{-t^2}dt$.

(Note that by Lemma 10 Chapter 8, this series has the same radius of convergence as the expansion for e^{-x^2}.)

Theorem 9. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and the series has radius of convergence r.
Then for any x in $(-r, r)$,

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} a_n \frac{1}{n+1}x^{n+1}$$

and $\sum_{n=0}^{\infty} a_n \frac{1}{n+1}x^{n+1}$ is absolutely convergent for all $|x| < r$ and diverges for $|x| > r$.

Proof. By Theorem 11 Chapter 8, for any real number K such that $0 < K < r$, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly and absolutely on $[-K, K]$. Since the n-th partial sum $s_n(x) = \sum_{k=0}^{n-1} a_k x^k$ is continuous, it is Riemann integrable on $[-K, K]$. Thus, for any x such that $|x| \leq K$, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, x]$ if $x > 0$ or $[x, 0]$ if $x < 0$ and so by Corollary 2,

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} a_n \frac{1}{n+1}x^{n+1}.$$

If $x = 0$, plainly the above equality is also true.
Hence, since for any x in $(-r, r)$, there exists a real number K such that $-r < -K < x < K < r$,

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} a_n \frac{1}{n+1} x^{n+1}$$

for all x in $(-r, r)$. Note that by Lemma 10 Chapter 8, $\sum_{n=0}^{\infty} a_n \frac{1}{n+1} x^{n+1}$ has the same radius of convergence r and so $\sum_{n=0}^{\infty} a_n \frac{1}{n+1} x^{n+1}$ converges for all $|x| < r$ and diverges for $|x| > r$.

Example 10. Consider the function $f(x) = \frac{1}{1+x^2}$.
We are going to obtain a power series expansion for the integral of f and use it to compute $\pi/4$.
We have the following formula:

$$(1 - a^n) = (1 - a)(1 + a^2 + \cdots + a^{n-1}) \quad \text{for integer } n \geq 1.$$

Thus for $a \neq 1$,

$$(1 + a + a^2 + \cdots + a^{n-1}) = \frac{(1 - a^n)}{1 - a} = \frac{1}{1 - a} - \frac{a^n}{1 - a}. \quad \text{------------- (1)}$$

Hence letting $a = -x^2$, from (1) we get, for integer $n \geq 1$,

$$(1 - x^2 + x^4 + \cdots + (-1)^{n-1} x^{2n-2}) = \frac{1}{1+x^2} - \frac{(-1)^n x^{2n}}{1+x^2}$$

or for integer $n \geq 0$,

$$(1 - x^2 + x^4 + \cdots + (-1)^n x^{2n}) = \frac{1}{1+x^2} - \frac{(-1)^{n+1} x^{2n+2}}{1+x^2}.$$

That is, for integer $n \geq 0$,

$$\frac{1}{1+x^2} = \sum_{k=0}^{n} (-1)^k x^{2k} + \frac{(-1)^{n+1} x^{2n+2}}{1+x^2}.$$

Now if $|x| < 1$, $x^{2n+2} \to 0$ as $n \to \infty$ and so, $\left| \frac{1}{1+x^2} - \sum_{k=0}^{n} (-1)^k x^{2k} \right| = \frac{x^{2n+2}}{1+x^2} \to 0$ as $n \to \infty$. Therefore, by the Comparison Test,

$$\sum_{k=0}^{n} (-1)^k x^{2k} \to \frac{1}{1+x^2} \quad \text{pointwise on } (-1, 1).$$

A simple ratio test confirms that the radius of convergence of the series $\sum_{k=0}^{\infty} (-1)^k x^{2k}$ is 1. Hence, by Theorem 9,

$$\int_0^x \frac{1}{1+t^2} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

for $|x| < 1$.
Therefore, for $|x| < 1$, by evaluating the integral we get,

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \quad \text{------------------ (2)}$$

Can we use this formula at $x = 1$ so that we can obtain an expansion for $\tan^{-1}(1) = \pi/4$?
Note that we have shown that

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} \quad \text{-------------------------- (3)}$$

for $|x| < 1$.
Plainly the series on the right hand side of (3) is divergent when $x = \pm 1$ as $|x^{2n}| \nrightarrow 0$ as $n \to \infty$ for $x = \pm 1$. But, the left hand side of (3) is valid for $x = \pm 1$.
On the other hand, the right hand side of (2) is convergent when $x = 1$ by the Leibnitz's Alternating Series Test (Theorem 20 Chapter 6). Can we just substitute $x = 1$ and conclude that (2) gives the value for $\tan^{-1}(1) = \frac{\pi}{4}$? Yes, we can indeed do so according to Abel's Theorem.

By Corollary 19 Chapter 8 to Abel's Theorem,

$$\lim_{x\to 1^-}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

But $\lim_{x\to 1^-}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}=\lim_{x\to 1^-}\tan^{-1}(x)=\tan^{-1}(1)$ by the continuity of $\tan^{-1}(x)$ at $x = 1$.

Therefore, $$\frac{\pi}{4}=\tan^{-1}(1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

This is the famous Leibnitz's formula for $\frac{\pi}{4}$. Thus, even though the expansion for the integrand is not valid at x =1, the expansion for the integral is. This formula of Leibnitz (1674) though convergent, converges so slowly that to obtain an accuracy of 5 decimal places to compute π, would require more than 150,000 terms. The search for ever faster converging formula goes on. There is the Bailey-Crandall formula (2000) and other similar types of formula and a race for the computation of π to the largest known number of decimal places (the current record holder is 1.2411×10^{12} places by Kanada, Ushio and Kuroda (2002)).

9.5 Convergence Theorems for Riemann Integrals

We shall now consider relaxing the condition of uniform convergence in Theorem 1. The result we shall present next is now seen as a specialization of a theorem in Lebesgue integration. It is due to Cesare Arzelà (1847-1912) and is called the Arzelà Dominated Convergence Theorem. It is an easy consequence of the *Lebesgue Dominated Convergence Theorem* in Lebesgue theory (see Chapter 14, Theorem 31). Several proofs without using Lebesgue integration theory is available but by no means easy.

Theorem 11 (Arzelà, 1885, Arzelà's Dominated Convergence Theorem)
Let $(f_n : [a, b] \to \mathbf{R}, n = 1, 2, \ldots)$ be a sequence of Riemann integrable functions, converging pointwise on $[a, b]$ to a Riemann integrable function $f : [a, b] \to \mathbf{R}$. If (f_n) is uniformly bounded, i.e., $|f_n(x)| \le K$ for some real number $K > 0$ for all x in $[a, b]$ and for all integer $n \ge 1$, then

$$\lim_{n\to\infty}\int_a^b |f_n(x)-f(x)|dx=0.$$

In particular, $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.

The proof is omitted. (For a proof, see W.A.J. Luxemburg, Arzelà's Dominated Convergence Theorem for the Riemann integral, American Mathematical Monthly vol 78 (1971), 970-997. This article gives a good account of a proof of the theorem without using results from the theory of Lebesgue integration and historical account about elementary proof by F. Riesz, Bieberbach, Landau, Hausdorff, Eberlein, etc.)

Remark.

1. Pointwise convergence of a sequence of functions (f_n) does not guarantee that the limiting function f is Riemann integrable, even when (f_n) is uniformly bounded. Take for instance the sequence (f_n) of functions on $[a, b]$, where for each integer $n \ge 1$, $f_n : [a, b] \to \mathbf{R}$ is defined by $f_n(x) = 0$ for $x \ne a_k$, $k > n$ and $f_n(x) = 1$ for $x = a_k$, $k \le n$, where (a_n) is given by an enumeration injective map $a : \mathbf{N} \to [a, b]$ which maps $\mathbf{N}$ onto the set of rational numbers in $[a, b]$. Then it is easily seen that $f_n \to f$

pointwise on [*a*, *b*], where $f(x)=\begin{cases}1, x \text{ rational}\\ 0, x \text{ irrational}\end{cases}$. Note that f is not Riemann integrable on [*a*, *b*] (see Example 19 (1) Chapter 5). (We may also deduce this fact by noting that f is discontinuous at every irrational numbers in [*a*, *b*], which is of non-zero *measure* and invoking Lebesgue theorem (Theorem 33 Chapter 14) that any bounded function is Riemann integrable if and only if it is continuous except on a set of *measure* zero.)
Thus the requirement in Theorem 11 that the limiting function be Riemann integrable is necessary.

2. However, we may conclude by the Lebesgue Dominated Convergence Theorem (Theorem 31 Chapter 14) that if (f_n) is Riemann integrable and converges pointwise to a function f and if (f_n) is uniformly bounded, the limiting function f is Lebesgue integrable and

$$\int_a^b f_n(x)dx \to Lebesgue \int_a^b f(x)dx,$$

where $Lebesgue \int_a^b f(x)dx$ denotes the Lebesgue integral of f.
Hence for the sequence of function (f_n) given in Remark 1 above, the Riemann integrals $\int_a^b f_n(x)dx$ tends to the Lebesgue integral $Lebesgue \int_a^b f(x)dx = 0$, since f is zero almost everywhere on [*a*, *b*].

Corollary 12. Suppose for each integer $n \geq 1$, $f_n : [a, b] \to \mathbf{R}$ is a non-negative and Riemann integrable function. Suppose that there exists a non-negative integrable function f such that for integer $n \geq 1$, $0 \leq f_n \leq f_{n+1} \leq f$.
Then if $f_n \to f$ pointwise on [*a, b*],

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof. Note that since f is Riemann integrable f is bounded. Thus there is a real number K such that $|f(x)| \leq K$ for all x in [*a, b*]. Since $0 \leq f_n \leq f$ for each integer $n \geq 1$,

$$|f_n(x)| \leq |f(x)| \leq K$$

for all x in [*a, b*] and all integer $n \geq 1$. If $f_n \to f$ pointwise on [*a, b*], then by Theorem 11, $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.

Remark. If we are interested in improper integral, then we note that if f is non-negative, then the improper integral of f is the same as the Lebesgue integral of f on [*a*, *b*]. We can rephrase Corollary 12 as follows: If (f_n) is a sequence of non-negative integrable functions such that $0 \leq f_n \leq f_{n+1} \leq f$ and $f_n \to f$ pointwise on [*a, b*] and if f has finite improper integral then

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

We cannot apply Theorem 11 in this case. So we will have to use the *Lebesgue Monotone Convergence Theorem*: Suppose (f_n) is a monotone increasing sequence of non-negative Lebesgue integrable functions and (f_n) converges pointwise to a Lebesgue integrable function f, then $\lim_{n\to\infty} Lebesgue \int_a^b f_n = Lebesgue \int_a^b f$.
We may of course replace the Lebesgue integrals by improper integrals to give the results for improper integrals, i.e., requiring f_n and f to have finite improper integrals.

Corollary 13. Suppose $(f_n : [a, b] \to \mathbf{R}, n = 1, 2, \;)$ is a sequence of non-negative and Riemann integrable functions. Suppose $\sum_{n=1}^{\infty} f_n$ converges pointwise on $[a, b]$ to a non-negative function $f : [a, b]: \to \mathbf{R}$. If f is Riemann integrable or if f has finite improper integral,

$$\sum_{n=1}^{\infty} \int_a^b f_n = \int_a^b f .$$

Remark. Corollary 13 follows from Corollary 12 if f is Riemann integrable and follows from the remark after Corollary 12 if f has finite improper integral. Corollary 13 is also true if f_n and f are required to have finite improper integrals.

Example 14.

1. For each integer $n \geq 1$, let $f_n(x) = e^{-nx^2} x^p$ for some integer $p > 0$. Plainly $f_n \to 0$ pointwise on $\mathbf{R}$. For each integer $n \geq 1$, $|f_n(x)| = \left|e^{-nx^2} x^p\right| \leq |x|^p \leq 1$ for $|x| \leq 1$. Therefore, (f_n) is uniformly bounded on $[-1, 1]$. Hence, by Theorem 11,
$$\int_0^1 f_n(x)dx = \int_0^1 e^{-nx^2} x^p dx \to 0.$$
2. Let $(f_n : [0, 2] \to \mathbf{R})$ be a sequence of functions, where $f_n(x) = \dfrac{e^x \sin(nx)}{n}$ for x in $[0, 2]$ and for integer $n \geq 1$. Then $f_n \to 0$ uniformly on $[0, 2]$ since for any x in $[0, 2]$, $|f_n(x)| \leq \dfrac{e^x}{n} \leq \dfrac{e^2}{n}$ and $\dfrac{e^2}{n} \to 0$ as $n \to \infty$. Note that for all x in $[0, 2]$ and for all integer $n \geq 1$, $|f_n(x)| \leq \dfrac{e^2}{n} \leq e^2$. This means $(f_n : [0, 2] \to \mathbf{R})$ is uniformly bounded on $[0, 2]$.
Therefore, we can either use Theorem 11 or Theorem 1 to conclude that
$$\int_0^2 \frac{e^x \sin(nx)}{n} dx \to \int_0^2 0dx = 0.$$
3. For each integer $n \geq 1$, let $f_n(x) = \dfrac{nx}{nx+1}$.
Then $f_n \to f$ pointwise on $[0, 1]$, where $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$.
By Theorem 14, $f_n \nrightarrow f$ uniformly since f is not continuous but f_n are continuous on $[0, 1]$.
However $|f_n(x)| = \left|\dfrac{nx}{nx+1}\right| \leq 1$ for all x in $[0, 1]$ and for all integer $n \geq 1$. Thus, (f_n) is uniformly bounded on $[0, 1]$. Therefore, by Theorem 11,
$$\int_0^1 f_n(x)dx = \int_0^1 \frac{nx}{nx+1} dx \to \int_0^1 f(x)dx = \int_0^1 1dx = 1.$$
4. For each integer $n \geq 1$, let $f_n : [0, 1] \to \mathbf{R}$ be defined by
$$f_n(x) = \begin{cases} 4n^2 x, & 0 \leq x \leq \frac{1}{2n} \\ 2n - 4n^2(x - \frac{1}{2n}), & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0, & \frac{1}{n} \leq x \leq 1 \end{cases} .$$
Then for each x in $[0, 1]$, $f_n(x) \to 0$ as $n \to \infty$. That is, $f_n \to f$ pointwise on $[0, 1]$, where $f(x) = 0$ for all x in $[0, 1]$. This is seen as follows. If $x = 0$ or 1, then $f_n(x) = 0$ for all integer $n \geq 1$ and so $f_n(x) \to 0$. If $0 < x < 1$, then by the Archimedean property of $\mathbf{R}$, there exists a positive integer N such that $\dfrac{1}{N} < x$.

Hence for all integer $n \geq N$, $\frac{1}{n} \leq \frac{1}{N} < x$ and so $f_n(x) = 0$ for all integer $n \geq N$. It follows that $f_n(x) \to 0$ as $n \to \infty$.
Note that $\sup\{f_n(x) : x \in [0,1]\} = f_n(\frac{1}{2n}) = 2n \to \infty$ and so (f_n) is not uniformly bounded on [0, 1]. Consequently $f_n \nrightarrow f$ uniformly on [0, 1].
Note also that for each integer $n \geq 1$, $\int_0^1 f_n(x)dx = 1$ and so

$$\int_0^1 f_n(x)dx \nrightarrow \int_0^1 f(x)dx = 0.$$

9.6 Monotone Sequence and Uniform Convergence

If a monotone sequence of continuous functions defined on [a, b] converges to a continuous function, then the integrals of the functions converges to the integral of the limiting function. This is a consequence of the compactness of the closed and bounded interval [a, b], phrased as follows.

Theorem 15. Suppose ($f_n : [a, b] \to \mathbf{R}$) is a monotone sequence of continuous functions, which converges pointwise on [a, b] to a function $f : [a, b] \to \mathbf{R}$. If f is continuous, then (f_n) converges uniformly to f, i.e., the convergence is uniform.

Proof. We shall assume that (f_n) is increasing. Take $x \in [a, b]$. Since ($f_n(x)$) converges to $f(x)$, given any $\varepsilon > 0$, there exists a positive integer $N(x)$ (N depends on x) such that for all integer n,

$$n \geq N(x) \Rightarrow |f_n(x) - f(x)| < \varepsilon/2 . \quad \text{----------------------} \quad (1)$$

Because both f_n and f are continuous on [a, b], there exists a $\delta(x, N(x)) > 0$ ($\delta(x, N(x))$ depends on x and $N(x)$) such that for all y in [a, b],

$$|y - x| < \delta(x, N(x)) \Rightarrow |f_{N(x)}(y) - f(y) - (f_{N(x)}(x) - f(x))| < \varepsilon/2 . \quad \text{------} \ (2)$$

Let $B(x, \delta(x,N(x))) = (x - \delta(x, N(x)), x + \delta(x, N(x)))$. We have then that for any $y \in B(x, \delta(x,N(x))) \cap [a, b]$,

$$|f_{N(x)}(y) - f(y)| \leq |f_{N(x)}(y) - f(y) - (f_{N(x)}(x) - f(x))| + |f_{N(x)}(x) - f(x)|$$

by the triangle inequality

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ by (1) and (2).} \quad \text{----------------------------} \quad (3)$$

The collection $\mathcal{C} = \{B(x, \delta(x,N(x))) : x \in [a, b]\}$ is an open cover of [a, b] (by open intervals). Now [a, b] is countably compact by the Heine-Borel Theorem (Theorem 43 Chapter 2) and hence compact (see remark after the corollary). Therefore $\mathcal{C}$ has a finite subcover. We may, if we do not wish to invoke compactness, proceed as follows. By Theorem 30 Chapter 3, $\mathcal{C}$ has a countable subcover. Therefore, by countable compactness of [a, b], $\mathcal{C}$ has a finite subcover, say

$$B(x_1,\delta(x_1,N(x_1))) \cup B(x_2,\delta(x_2,N(x_2))) \cup \cdots \cup B(x_L,\delta(x_L,N(x_L)))$$

where L is some positive integer. Hence

$$[a,b] \subseteq B(x_1,\delta(x_1,N(x_1))) \cup B(x_2,\delta(x_2,N(x_2))) \cup \cdots \cup B(x_L,\delta(x_L,N(x_L))).$$

$$\text{--------------------------} \quad (4)$$

Now, let $N = \max(N(x_1), N(x_2), \ldots, N(x_L))$. Take *any* x in [a, b], then by (4),

$$x \in B(x_k, \delta(x_k, N(x_k))) \text{ for some integer } k \text{ such that } 1 \leq k \leq L.$$

It then follows by (3) that

$$\left| f_{N(x_k)}(x) - f(x) \right| < \varepsilon.$$

Because (f_n) is increasing, for any integer $n \geq N = \max(N(x_1), N(x_2), \ldots, N(x_L)) \geq N(x_k)$,

$$|f_n(x) - f(x)| = f(x) - f_n(x) \leq f(x) - f_{N(x_k)}(x) = |f_{N(x_k)}(x) - f(x)| < \varepsilon .$$

Since this is true for any x in $[a, b]$, (f_n) converges uniformly to f on $[a, b]$.
If (f_n) is decreasing, for any integer $n \geq N = \max(N(x_1), N(x_2), \ldots, N(x_L)) \geq N(x_k)$,

$$|f_n(x) - f(x)| = f_n(x) - f(x) \leq f_{N(x_k)}(x) - f(x)| = |f_{N(x_k)}(x) - f(x)| < \varepsilon .$$

We deduce in the same way that (f_n) converges uniformly to f on $[a, b]$.
This completes the proof.

Remark.
(1) In Example 14 (1), each f_n is continuous and (f_n) is decreasing. The pointwise limit f being the 0 constant function is also continuous. Therefore, by Theorem 15 (f_n) converges uniformly on [0, 1] to f. Hence we need only invoke Theorem 1 to conclude that $\int_0^1 f_n(x)dx = \int_0^1 e^{-nx^2} x^p dx \to 0$.

(2) Note that it is essential that (f_n) be monotone in Theorem 15. For instance in Example 14 (4), the sequence (f_n) is neither increasing nor decreasing and each f_n is continuous on [0,1]. The sequence (f_n) converges pointwise to a continuous zero constant function f. But the convergence is not uniform.

(3) However, if each $f_n : [a, b] \to \mathbf{R}$ is monotone and the sequence $(f_n : [a, b] \to \mathbf{R})$ converges to a continuous function $f : [a, b] \to \mathbf{R}$, then the convergence is uniform. We give a proof below. Since f is continuous on the closed and bounded interval $[a, b]$, f is uniformly continuous. Hence given $\varepsilon > 0$ there exists $\delta > 0$ such that for any x, y in $[a, b]$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/5. \text{ ---------------------------------- (1).}$$

We shall be using this inequality.
Take a partition $P : a = x_0 < x_1 < \ldots < x_k = b$, such that $\|P\| = \max\{x_i - x_{i-1} : i = 1, 2, \ldots, k\} < \delta/2$.
Since $(f_n(x_i))$ is convergent for each $i = 0,1,2, \ldots, k$, there exists an integer $N(x_i)$ depending on x_i, such that for all integers n,

$$n \geq N(x_i) \Rightarrow |f_n(x_i) - f(x_i)| < \varepsilon/5. \text{ ------------------------------- (2)}$$

Let $N = \max\{N(x_i) : i = 0, 1, 2, \ldots, k\}$. Take any y in $[a, b]$. Then
$y \in [x_{i-1}, x_i]$ for some i, $1 \leq i \leq k$. Take $n \geq N$. Then

$$\begin{aligned}
|f_n(y) - f(y)| &= |f_n(y) - f_n(x_{i-1}) + f_n(x_{i-1}) - f(x_{i-1}) + f(x_{i-1}) - f(y)| \\
&\leq |f_n(y) - f_n(x_{i-1})| + |f_n(x_{i-1}) - f(x_{i-1})| + |f(x_{i-1}) - f(y)| \\
&\leq |f_n(x_i) - f_n(x_{i-1})| + |f_n(x_{i-1}) - f(x_{i-1})| + |f(x_{i-1}) - f(y)| \\
&\qquad \text{since } f_n \text{ is monotone} \\
&\leq |f_n(x_i) - f(x_i)| + |f(x_i) - f(x_{i-1})| + |f(x_{i-1}) - f_n(x_{i-1})| \\
&\qquad + |f_n(x_{i-1}) - f(x_{i-1})| + |f(x_{i-1}) - f(y)| \\
&< \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = \varepsilon \\
&\qquad \text{since } n \geq N \geq N(x_i), N(x_{i-1}) \text{ and } |x_{i-1} - y| \leq |x_i - x_{i-1}| < \delta.
\end{aligned}$$

Hence for any $n \geq N$, $|f_n(y) - f(y)| < \varepsilon$ for all y in $[a, b]$. This shows that the convergence is uniform.

9.7 Consequence of Uniform Convergence

We know from Example 7 that $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$ is uniformly convergent on [δ, 2π − δ], for 0 < δ < π. Therefore, $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$ is convergent on (0, 2π). Since for $x = 0$ or 2π, $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}=0$, the series $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$ is convergent on [0, 2π]. Since the sine function is periodic, $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$ is convergent for all x in **R**. The situation with the cosine series $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}$ is very different.

The n-th partial sum, $s_n(x)=\sum_{k=1}^{n}\cos(kx)=\frac{\sin(nx+\frac{1}{2}x)-\sin(\frac{1}{2}x)}{2\sin(\frac{1}{2}x)}$. Hence, for any x in [δ, 2π − δ], 0 < δ < π,

$$\left|\sum_{k=1}^{n}\cos(kx)\right|\leq\frac{1}{\left|\sin(\frac{1}{2}x)\right|}\leq\frac{1}{\left|\sin(\frac{1}{2}\delta)\right|}.$$

Therefore, by Dirichlet's Test (Theorem 5), $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}$ converges uniformly on [δ, 2π − δ] for 0 < δ < π. Hence, $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}$ is convergent on (0, 2π). When $x = 0$, or 2π, $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}=\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. Then by the periodicity of cosine function $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}$ is convergent on **R**, except for $x = 0$ or multiple of 2π, where it is divergent.

Note that in applying Dirichlet's test we only need the sequence $\left(\frac{1}{n}\right)$ to be non negative, decreasing and converges to 0. The same can be said about the sequence $\left(\frac{1}{n^s}\right)$ for $0 < s \leq 1$. Hence we have the following

Theorem 16. The series $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^s}$ and $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^s}$ for $0 < s \leq 1$ are uniformly convergent on [δ, 2π − δ], 0 < δ < π. Moreover, $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^s}$ converges pointwise on **R** and $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^s}$ converges pointwise on **R**, except for x of the form $2\pi k$, any integer k. For $s > 1$, both series converges absolutely and uniformly on **R**.

The proof of Theorem 16 for the case $0 < s \leq 1$ is exactly the same as in the discussion preceding Theorem 16 by using Dirichlet's Test. For the case of $s > 1$, it is a consequence of the Weierstrass M Test (Theorem 1 Chapter 8).

More generally we have,

Theorem 17. The series

$$\sum_{n=1}^{\infty}a_n\sin(nx)\text{ and }\sum_{n=1}^{\infty}a_n\cos(nx),$$

where (a_n) is a decreasing non-negative sequence converging to 0, are uniformly convergent on [δ, 2π − δ] for any δ such that 0 < δ < π.
Moreover, $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges pointwise on **R** and $\sum_{n=1}^{\infty} a_n \cos(nx)$ converges pointwise on **R**, except possibly for x of the form 2πk, any integer k.

The proof is exactly the same as described earlier.

Series of the form $\sum_{n=1}^{\infty} a_n \sin(nx) + b_n \cos(nx)$ are called trigonometric series. Theorem 17 thus gives a sufficient condition for its convergence. Note that when (a_n) and (b_n) are obtained in a special way then we get a special trigonometric series. Fourier series is an example of a special trigonometric series.

There is the question of whether the series $\sum_{n=1}^{\infty} a_n \sin(nx)$ should always be continuous at the end points of the interval [0, 2π]. Indeed if a function is representable by a Fourier series, then such a requirement may be too stringent and what we require is perhaps that the series be continuous where the function is continuous in the interior (0, 2π), where the convergence of the series is a requirement for continuity. Indeed, there are continuous functions whose Fourier series does not converge at a point in the interior of [0, 2π] and when it does, it may not converge to the value of the function that generated it. There are conditions for convergence and also for uniform convergence, particularly for Fourier series. We shall not go into this area here and it is outside the scope of the book. This is an area worked on by mathematicians such as Euler, d'Alembert, Lagrange, Riemann, Dirichlet, Heine, Du Bois-Reymond, Cantor, Jordan, Lebesgue and Fejèr. The question whether we can differentiate or integrate a trigonometric series or Fourier series is a delicate one. Indeed we might not even need the stringent uniform convergence as in the usual case. So we may carry out computation when we normally would not. Take the series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \qquad \text{--------------------- (A)}$$

It is the Fourier series for the function $f(x) = \frac{1}{2}(\pi - x)$ in the interval (0, 2π). The series converges to a function discontinuous at 0 and π and which is equal to $f(x)$ at all points in (0, 2π). This is easy to see for $f(0)$ and $f(2\pi)$ are non zero whereas the series converges to 0 there. However, we can still integrate term by term to get a Fourier series for the integral of the function f as the sine series do not have a constant term. For x in (0, 2π),

$$\begin{aligned}\int_0^x f(t)dt &= \int_0^x \frac{1}{2}(\pi - t)dt = \left[-\frac{(\pi - t)^2}{4}\right]_0^x = \frac{\pi^2}{4} - \frac{(\pi - x)^2}{4} \\ &= \sum_{n=1}^{\infty}\left[-\frac{\cos(nt)}{n^2}\right]_0^x = \sum_{n=1}^{\infty}\left(-\frac{\cos(nx)}{n^2} + \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} -\frac{\cos(nx)}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \sum_{n=1}^{\infty} -\frac{\cos(nx)}{n^2} + \frac{\pi^2}{6},\end{aligned}$$

since $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Note that the series converges at 0 to f (0) = 0 and at 2π to f (2π) = 0. Thus $\sum_{n=1}^{\infty} -\frac{\cos(nx)}{n^2} + \frac{\pi^2}{6}$ is the Fourier series for $\frac{\pi^2}{4} - \frac{(\pi - x)^2}{4}$ in the interval [0, 2π], and

converges to $\int_0^x f(t)dt$ at every point in [0, 2π]. This is an example of a *result of Lebesgue*, that the Fourier series of $\int_0^x f(t)dt$ the integral of a Lebesgue integrable function f can be obtained by term by term integration and the series so obtained converges uniformly to $\int_0^x f(t)dt$ on [0, 2π]. A very subtle result indeed.
We may also use the above expansion to obtain

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} = \frac{(\pi-x)^2}{4} - \frac{\pi^2}{12} \quad \text{for } x \text{ in } [0, 2\pi].$$

Note that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ converges uniformly on any subinterval [a, b] in (0, 2π) but not on [0, 2π].

9.8 Newton's Binomial Theorem

Now we turn to a well known series, the Newton's binomial series. It is easier to use Taylor's Theorem with Cauchy's integral form of the remainder to show that the binomial series converge.
We state the result below.

Theorem 18. Let I be an open interval containing the point x_0 and n be a non-negative integer. Suppose $f : I \to \mathbf{R}$ has $n+1$ derivatives. Then for any x in I,

$$f(x) = f(x_0) + \tfrac{1}{1!}(x-x_0)f'(x_0) + \cdots + \tfrac{1}{k!}(x-x_0)^k f^{(k)}(x_0)$$
$$+ \cdots + \tfrac{1}{n!}(x-x_0)^n f^{(n)}(x_0) + R_n(x),$$

where the remainder $R_n(x)$ is given by the following three forms:

$$R_n(x) = \frac{1}{(n+1)!}(x-x_0)^{n+1} f^{(n+1)}(\eta) \text{ for some } \eta \text{ between } x \text{ and } x_0 \text{ (Lagrange form)}$$

$$R_n(x) = (x-x_0)\frac{f^{(n+1)}(\eta)}{n!}(x-\eta)^n \text{ for some } \eta \text{ between } x \text{ and } x_0 \text{ (Cauchy form)}$$

and if $f^{(n+1)}(x)$ is Riemann integrable on $[x_0 , x]$ when $x_0 \le x$ and on $[x, x_0]$ when $x_0 > x$,

$$R_n(x) = \int_{x_0}^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt.$$

Proof. The proof of the theorem with the Lagrange form of the remainder is given in Theorem 44 Chapter 4.
Without loss of generality assuming $x_0 < x$, define $p_n : [x_0 , x] \to \mathbf{R}$ by

$$p_n(t) = f(x) - f(t) - \tfrac{1}{1!}(x-t)f'(t) - \cdots - \frac{1}{k!}(x-t)^k f^{(k)}(t) \cdots - \frac{1}{n!}(x-t)^n f^{(n)}(t) \quad \text{--- (1)}$$

for t in $[x_0 , x]$.
Then we have

$$p_n(x_0) = f(x) - f(x_0) - \tfrac{1}{1!}(x-x_0)f'(x_0) - \cdots - \tfrac{1}{k!}(x-x_0)^k f^{(k)}(x_0)$$
$$- \cdots - \tfrac{1}{n!}(x-x_0)^n f^{(n)}(x_0)$$
$$= R_n(x),$$

the remainder by the definition.
Differentiate $p_n(t)$, we get

$$p_n'(t) = -f'(t) + [f'(t) - (x-t)f''(t)] + [(x-t)f''(t) - \frac{(x-t)^2}{2!}f^{(3)}(t)] + \cdots$$
$$+ [\frac{(x-t)^{n-1}}{(n-1)!}f''(t) - \frac{(x-t)^n}{n!}f^{(n+1)}(t)]$$
$$= -\frac{(x-t)^n}{n!}f^{(n+1)}(t) \quad \text{----------------------------------} \quad (2)$$

Since p_n is differentiable on $[x, x_0]$, by the Mean Value Theorem, there exists $\eta \in (x_0, x)$ such that

$$\frac{p_n(x) - p_n(x_0)}{x - x_0} = p_n'(\eta) = -\frac{(x-\eta)^n}{n!} f^{(n+1)}(\eta).$$

It follows then that

$$p_n(x) - p_n(x_0) = -(x - x_0)\frac{(x-\eta)^n}{n!} f^{(n+1)}(\eta). \quad \text{---------------} \quad (3)$$

But by (1), $p_n(x) = 0$ and so from (3) we obtain,

$$R_n(x) = p_n(x_0) = (x - x_0)\frac{(x-\eta)^n}{n!} f^{(n+1)}(\eta).$$

This gives the Cauchy form of the remainder.

Now suppose $f^{(n+1)}$ is Riemann integrable on $[x_0, x]$, then we have

$$p_n'(t) == -\frac{(x-t)^n}{n!} f^{(n+1)}(t)$$

is Riemann integrable on $[x_0, x]$ because it is a product of two Riemann integrable functions (see Corollary 55 Chapter 5). Then by Darboux Fundamental Theorem of Calculus (Theorem 42 Chapter 5)

$$p_n(x) - p_n(x_0) = \int_{x_0}^{x} p_n'(t)dt = -\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

It then follows, since $p_n(x) = 0$, that

$$R_n(x) = p_n(x_0) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt$$

giving the Cauchy's integral form of the remainder.

Before we state the expansion for the binomial series, we recall the definition of the usual binomial coefficient and define the generalized binomial coefficient.

For a positive integer n, the binomial expansion for $(a + b)^n$ is given by

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where the binomial coefficient $\binom{n}{k} = \dfrac{n!}{(n-k)!k!} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$, $\binom{n}{0} = 1$. Thus, in a similar fashion, we define for any real number β and each integer $k > 0$, the generalized binomial coefficient by

$$\binom{\beta}{k} = \frac{\beta(\beta-1)(\beta-2)\cdots(\beta-k+1)}{k!} \text{ and } \binom{\beta}{0} = 1.$$

Theorem 19. For any real number β,

$$(1+x)^\beta = \sum_{k=0}^{\infty} \binom{\beta}{k} x^k$$

for $|x| < 1$.

Proof. Define $f(x) = (1 + x)^\beta$. We shall show that $\sum_{k=0}^{\infty} \binom{\beta}{k} x^k$ is the Taylor series expansion for f for $|x| < 1$. We need to show the convergence of the series on the right hand side.

First note that the radius of convergence for the series $\sum_{k=0}^{\infty} \binom{\beta}{k} x^k$ is 1. This is because

$$\left| \frac{\binom{\beta}{n+1}}{\binom{\beta}{n}} \right| = \left| \frac{\dfrac{\beta(\beta-1)(\beta-2)\cdots(\beta-n)}{(n+1)!}}{\dfrac{\beta(\beta-1)(\beta-2)\cdots(\beta-n+1)}{n!}} \right| = \left| \frac{\beta-n}{n+1} \right| \to 1 \text{ as } n \to \infty .$$

(See Theorem 18 Chapter 7.)

Thus $\sum_{k=0}^{\infty}\binom{\beta}{k}x^k$ converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. We shall now show that the series converges to $f(x)$ for $-1 < x < 1$.

By Theorem 18, the integral form of the remainder of the Taylor expansion of degree n about $x = 0$ for f is given by

$$R_n(x) = \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt. \text{ ------------------------- (1)}$$

We shall now compute the derivatives of f. Observe that for $x > -1$,

$$f(x) = (1+x)^{\beta} = e^{\beta\ln(1+x)}.$$

Therefore, f is infinitely differentiable on $(-1, \infty)$ since the exponential function is infinitely differentiable on **R** and $\beta\ln(1+x)$ is infinitely differentiable on $(-1, \infty)$ and f is a composite of these two functions. Therefore, by the Chain Rule, for $x > -1$,

$$f'(x) = e^{\beta\ln(1+x)}\frac{\beta}{1+x} = \beta e^{\beta\ln(1+x)-\ln(1+x)} = \beta e^{(\beta-1)\ln(1+x)} = \beta(1+x)^{\beta-1}.$$

Hence, $f^{(2)}(x) = \beta(\beta-1)(1+x)^{\beta-2}$. Thus we have the formula,

$$f^{(k)}(x) = \beta(\beta-1)\cdots(\beta-(k-1))(1+x)^{\beta-k} \qquad \text{------------- (2)}$$

for integer $k \geq 1$ and for $x > -1$.

Therefore, for integer $k \geq 1$,

$$f^{(k)}(0) = \beta(\beta-1)\cdots(\beta-(k-1)).$$

Hence we have by Theorem 18,

$$\begin{aligned} f(x) &= f(0) + \tfrac{1}{1!}xf'(0) + \cdots + \tfrac{1}{k!}x^k f^{(k)}(0)\cdots + \tfrac{1}{n!}x^n f^{(n)}(0) + R_n(x) \\ &= 1 + \beta x + \cdots + \tfrac{\beta(\beta-1)\cdots(\beta-(k-1))1}{k!}x^k\cdots + \tfrac{\beta(\beta-1)\cdots(\beta-(n-1))}{n!}x^n + R_n(x) \\ &= 1 + \beta x + \cdots + \binom{\beta}{k}x^k\cdots + \binom{\beta}{n}x^n + R_n(x) \\ &= \sum_{k=0}^{n}\binom{\beta}{k}x^k + R_n(x). \end{aligned}$$

It follows that $\left|\sum_{k=0}^{n}\binom{\beta}{k}x^k - f(x)\right| = |R_n(x)|$. So if we can show that for $|x| < 1$, $R_n(x) \to 0$ as $n \to \infty$, then by the Comparison Test, $\sum_{k=0}^{n}\binom{\beta}{k}x^k \to f(x)$ as $n \to \infty$.

We shall show the convergence for $-1 < x < 0$.

Now by (1), for any integer $n \geq 1$ and for $x > -1$,

$$\begin{aligned} R_n(x) &= \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \\ &= \int_0^x \frac{(x-t)^n}{n!}\beta(\beta-1)\cdots(\beta-n)(1+t)^{\beta-n-1}\,dt \\ &= (n+1)\int_0^x (x-t)^n \frac{\beta(\beta-1)\cdots(\beta-n)}{(n+1)!}(1+t)^{\beta-n-1}dt \\ &= (n+1)\int_0^x (x-t)^n\binom{\beta}{n+1}(1+t)^{\beta-n-1}dt \\ &= -(n+1)\binom{\beta}{n+1}\int_x^0 (x-t)^n(1+t)^{\beta-n-1}dt \\ &= (-1)^{n+1}(n+1)\binom{\beta}{n+1}\int_x^0 (t-x)^n(1+t)^{\beta-n-1}dt \\ &= (-1)^{n+1}(n+1)\binom{\beta}{n+1}\int_x^0 \left(\frac{t-x}{1+t}\right)^n(1+t)^{\beta-1}dt. \end{aligned}$$

Hence for $-1 < x < 0$, and for any integer $n \geq 1$,

$$|R_n(x)| = (n+1)\left|\binom{\beta}{n+1}\right|\int_x^0 \left(\frac{t-x}{1+t}\right)^n(1+t)^{\beta-1}dt \text{ ---------------- (3)}$$

Now we examine the integrand in (3) and we shall show that it is bounded for each x in $(-1, 0)$.

Observe that for $-1 < x \le t \le 0$, $\left(\frac{t-x}{1+t}\right) \le |x| = -x$. This is because

$$\left(\frac{t-x}{1+t}\right) \le -x \Leftrightarrow t-x \le -x-xt \quad \Leftrightarrow t \le -xt \Leftrightarrow t(1+x) \le 0$$

and the last inequality is true since $t \le 0$ and $(1+x) > 0$.

Thus $$\int_x^0 \left(\frac{t-x}{1+t}\right)^n (1+t)^{\beta-1} dt \le \int_x^0 |x|^n (1+t)^{\beta-1} dt. \quad \text{-------------------- (4)}$$

Now since $0 < 1+x \le 1+t$ for $-1 < x \le t \le 0$,

if $\beta < 1$, i.e., $\beta-1 < 0$, then $(1+t)^{\beta-1} \le (1+x)^{\beta-1}$

and if $\beta \ge 1$, $(1+t)^{\beta-1} \le (1+0)^{\beta-1} = 1$.

Hence for $-1 < x \le t \le 0$ and for any real number β,

$$(1+t)^{\beta-1} \le \max\{(1+x)^{\beta-1}, 1\} = M_x.$$

Thus, from (4), we have for $-1 < x < 0$ and any integer $n \ge 1$,

$$\int_x^0 \left(\frac{t-x}{1+t}\right)^n (1+t)^{\beta-1} dt \le \int_x^0 |x|^n M_x dt = |x|^{n+1} M_x.$$

Therefore, it follows from (3), for $-1 < x < 0$, and for any integer $n \ge 1$,

$$|R_n(x)| = (n+1)\left|\binom{\beta}{n+1}\right| \int_x^0 \left(\frac{t-x}{1+t}\right)^n (1+t)^{\beta-1} dt \le (n+1)\left|\binom{\beta}{n+1}\right| |x|^{n+1} M_x. \quad \text{---- (5)}$$

Next we claim that

Lemma 20. For any real number β, $n\binom{\beta}{n}x^n \to 0$ if $|x| < 1$.

Proof. If $x = 0$, plainly $n\binom{\beta}{n}x^n \to 0$. Now suppose $x \ne 0$ and $|x| < 1$.

Then $$\left|\frac{(n+1)\binom{\beta}{n+1}x^{n+1}}{n\binom{\beta}{n}x^n}\right| = \frac{n+1}{n}\left|\frac{\beta(\beta-1)\cdots(\beta-n)}{\beta(\beta-1)\cdots(\beta-n+1)}\frac{n!}{(n+1)!}x\right| = \tfrac{1}{n}|(\beta-n)x| \to |x|$$

Since $|x| < 1$, it follows that $n\binom{\beta}{n}x^n \to 0$.

We now continue with the proof of Theorem 19.

By Lemma 20, $(n+1)\left|\binom{\beta}{n+1}\right||x|^{n+1}M_x \to 0$ as $n \to \infty$. It then follows from (5) by the Comparison Test that $|R_n(x)| \to 0$ as $n \to \infty$ for $-1 < x < 0$. Therefore, for $-1 < x < 0$, $\sum_{k=0}^{n}\binom{\beta}{k}x^k \to f(x)$ as $n \to \infty$.

Now we consider the case of convergence for $0 < x < 1$. From (1) and (2) we obtain as above,

$$R_n(x) = (n+1)\int_0^x (x-t)^n \binom{\beta}{n+1}(1+t)^{\beta-n-1} dt \quad \text{for } x > -1.$$

Therefore, for $1 > x > 0$,

$$|R_n(x)| = (n+1)\left|\binom{\beta}{n+1}\right| \int_0^x (x-t)^n (1+t)^{\beta-n-1} dt$$

$$\le (n+1)\left|\binom{\beta}{n+1}\right| \int_0^x (x-t)^n dt \quad \text{when } n+1 > \beta$$

$$\leq (n+1)\left|\binom{\beta}{n+1}\right| \int_0^x x^n dt = \left|\binom{\beta}{n+1}\right| x^{n+1}$$
$$\leq (n+1)\left|\binom{\beta}{n+1}\right| x^{n+1}$$

This means $|R_n(x)| \leq (n+1)\left|\binom{\beta}{n+1}\right| x^{n+1}$ for any integer $n > \beta - 1$ and any x such that $0 < x < 1$. Thus, by Lemma 20 and the Comparison Test, $|R_n(x)| \to 0$ as $n \to \infty$ for $0 < x < 1$. Consequently, for $0 < x < 1$, $\sum_{k=0}^{n}\binom{\beta}{k}x^k \to f(x)$ as $n \to \infty$. Plainly, for $x = 0$, the series converges to $f(0) = 1$. This completes the proof of Theorem 19.

Remark.

1. Convergence of $\sum_{k=0}^{\infty}\binom{\beta}{k}x^k$ at the end points ± 1 is more delicate. We may use more specialized test such as Raabe's Test. We shall be dealing with this in Chapter 13.

2. We have seen that power series can be differentiated any number of times and integrated any number of time within its radius of convergence. Thus it plays a role in the series solution of differential equation. One particular effective way is the method of undetermined coefficients. One may obtain the cosine and the sine power series as the solutions to the differential equation

$$y'' + y = 0$$

with boundary condition $s(0) = 0$, $s'(0) = 1$, $c(0) = 1$, $c'(0) = 0$. Similarly, we obtain the Bessel function of order n as solution of the Bessel differential equation of order n,

$$y'' + \tfrac{1}{x}y' + (1 - \tfrac{n^2}{x^2})y = 0.$$

In this direction, there is also the method of majorants. Combine with formal power series technique and delicate handling of the convergence of solution, i.e., the radius of convergence, these methods are powerful tool in the solution of differential equations. There is also the method of Frobenius. These and related application do not come under the scope of the book. No doubt convergence plays an important role in the series solution.

3. The basic convergence for a sequence of functions is pointwise convergence. Uniform convergence is much more restrictive. Historically, the eighteenth and nineteenth century men have been using sum of series of functions without paying due regard to convergence so long as the method works, differentiating and integrating term by term so long as the application is plausible. We can now explain why some of the methods did work. For example, the Arzelà's Dominated Convergence Theorem dispenses with the need for uniform convergence. The Lebesgue Dominated Convergence Theorem further generalizes this, where we can include improperly integrable functions. The notion of convergence in the mean will not necessarily require even pointwise convergence. Fourier series or trigonometric series plays a very important role in the solution of partial differential equations. The estimation of the partial sums of a Fourier series in terms of the Dirichlet's kernel gives a criterion for pointwise convergence of Fourier series. Lebesgue (1906) showed that term by term integration of Fourier series representing a Lebesgue integrable function f is possible whether or not the original Fourier series is convergent. In particular, the new series obtained by term by term integration converges uniformly to the integral of the function within the domain $[-\pi, \pi]$. Even though the Fourier series of f may

not converge at a point in [−π, π] in the sense of Cauchy, for f a bounded integrable function or if unbounded the improper integral $\int_{-\pi}^{\pi} f$ is absolutely convergent, and at every point in the interval [−π, π] at which both limits *right limits* $f(x+)$ and *left limit* $f(x-)$ exist, Fejér (1880-1959) showed that the Fourier series converges in the sense of Frobenius or Cesàro 1 summable to $\frac{1}{2}(f(x+) + f(x-))$. If $\sum_{n=0}^{\infty} a_n$ is a series and $s_n = \sum_{k=0}^{n} a_k$ and if the limit $\lim_{n\to\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1}$ exists and equals C, then we say the series $\sum_{n=0}^{\infty} a_n$ is Cesàro 1 summable to C. This notion is *regular* in the sense that if $\sum_{n=0}^{\infty} a_n$ converges, then it is Cesàro 1 summable to the same limit. We may have $\sum_{n=0}^{\infty} a_n$ diverging in the usual sense, Cauchy's sense and be Cesàro 1 summable. For example the series 1 −1 +1 −1+1 −1 + … is divergent but Cesàro 1 summable to 1/2. (We make a distinction between convergent in the Cauchy sense and in any other sense of summation. Summability is used for sum other than the usual sum in the Cauchy's sense.) Thus this deviates from the normal sense of summability, which is Cauchy's sense of continually adding more and more terms in the ordinary sense. Thus it is easier to accept this notion of summability at situation where the ordinary sense of summation may not be physically significant. Thus divergent series in the sense of Cauchy can indeed be used. For instance, Frobenius showed that if the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1, and if $\sum_{n=0}^{\infty} a_n$ is Cesàro 1 summable to A, then $\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = A$. This of course extends Abel's Theorem (Corollary 19 Chapter 8). For instance, the series $\sum_{n=0}^{\infty} (-1)^n x^n$ which is the power series expansion for $f(x) = \frac{1}{1+x}$ has radius of convergence 1; $\sum_{n=0}^{\infty} (-1)^n$ is divergent but is Cesàro 1 summable to 1/2, which is the value of $f(1)$. One can see that the development of this area of infinite series did not follow a logical pattern or logical path and controversy abounds. The theory of Fourier series or of trigonometric series covers normally the following issues: the representability of a function by a Fourier or trigonometric series, the uniqueness of Fourier or trigonometric representation, pointwise and uniform convergence, differentiation and integration of Fourier series term by term. Often the lack of pointwise convergence or uniform convergence leads to deeper and far reaching results as discussed above. The uniqueness of the trigonometric series representation of a bounded function has to await the development of Lebesgue integration theory when in 1903 Lebesgue showed that if a function represented by a trigonometric series, i.e., $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(x))$, is bounded, then the a_n and b_n are Fourier coefficients, equivalently the series on the right hand side is actually the Fourier series of the function f. Fourier series are particularly useful in the solution of partial differential equation, for example, the wave equation and the heat equation and Dirichlet's problem. Despite the success and impact of the Fourier series solutions of partial differential equations, the computation of series solutions is not very manageable, for instance when the solution converges, it may converge too slowly for computation. Thus, the search for solution in closed form, that is, in terms of elementary functions and integrals of such functions, leads to an important method, the method of Fourier transform and in another direction to the method of Laplace transform. Both methods are now powerful technique in the solutions of partial differential equations and ordinary differential equations.

Exercises 21.

1. Test the following for convergence and uniform convergence, in the respective domain.

   (i) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$, $0 \le x \le 1$; (ii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+x}$, $0 \le x < \infty$;

   (iii) $\sum_{n=1}^{\infty} ne^{-nx} \sin(nx)$, $x > a > 0$.

2. Show, by establishing the uniform convergence of the series under the integral sign on the left of each of the following statements, that the equality hold in each case.

   (i) $\int_0^{\pi}\left(\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}\right)dx = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^3}$; (ii) $\int_1^2 \sum_{n=1}^{\infty}\left(\frac{\ln(nx)}{n^2}\right)dx = \sum_{n=1}^{\infty} \frac{\ln(4n)-1}{n^2}$;

   (iii) $\int_1^2\left(\sum_{n=1}^{\infty} ne^{-nx}\right)dx = \frac{e}{e^2-1}$

   (Hint: Show that each of the series under the integral sign is dominated by a convergent constant series and apply Weierstrass M-test)

3. Show, by establishing the uniform convergence of the term by term differentiated series, each of the following.

   (i) $\frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}\right) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$ for all real x;

   (ii) $\frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{n}{x^n}\right) = -\sum_{n=1}^{\infty} \frac{n^2}{x^{n+1}}$ for $|x| > 1$;

   (iii) $\frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{1}{n^3(1+nx^2)}\right) = -2x\sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}$ for all real x.

4. (i) Use Abel's Test to show that the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-nx}}{n}$ converges uniformly on $[0, \infty)$. Explain, why if $f(x) = \sum_{n=1}^{\infty}(-1)^n\frac{e^{-nx}}{n^2}$, then $f'(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-nx}}{n}$ for $x \ge 0$.

   (ii) Use Dirichlet's Test to show that $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+x^2}$ converges uniformly for all x on **R.**

   Explain why we can differentiate term by term to get $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2x}{(n+x^2)^2}$.

5. Knowing your theorem.

Suppose (f_n) is a sequence of differentiable functions defined on an interval $[a, b]$. Suppose that f_n converges pointwise to a function f. Suppose each f_n' is continuous on $[a, b]$ and f_n' converges uniformly to a function g on $[a, b]$. Give justifications or reasons for the following propositions.

(i) The function g is continuous on $[a, b]$.

(ii) Each function f_n' and the function g are integrable on $[a, b]$.

(iii) $\int_a^x g = \lim_{n\to\infty} \int_a^x f'_n$.

(iv) $\lim_{n\to\infty} \int_a^x f'_n = \lim_{n\to\infty} (f_n(x) - f_n(a))$.

(v) $\int_a^x g = \lim_{n\to\infty} (f_n(x) - f_n(a)) = f(x) - f(a)$.

(vi) $g = f'$.

(vii) f_n converges uniformly to f on $[a, b]$.

6. By using the binomial series expansion for $\frac{1}{\sqrt{1-x^2}}$, show that

$$\sin^{-1}(x) = x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\frac{x^{2n+1}}{2n+1}, \text{ for } |x| < 1 .$$

7. Using question 6 or otherwise, show that

$$\cos^{-1}(x) = \frac{\pi}{2} - x - \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\frac{x^{2n+1}}{2n+1}, \text{ for } |x| < 1$$

8. Let $f_1(x) = 1$ on [0, 1], i.e., f_1 is the constant function 1 on [0, 1]. For $n \geq 2$, define

$$f_n(x) = \begin{cases} nx, & 0 \le x < \frac{1}{n} \\ 2 - nx, & \frac{1}{n} \le x < \frac{2}{n} \\ 0, & \frac{2}{n} \le x \le 1 \end{cases} .$$

Show that $f_n(x)$ converges to some function $f(x)$ on [0, 1] but that the convergence is not uniform.

9. Show that $f(x) = \sum_{n=0}^{\infty} e^{-nx}\cos(nx)$ converges uniformly on any subset of **R**, which is bounded below by a positive constant. Show that

$$f'(x) = -\sum_{n=0}^{\infty} ne^{-nx}[\cos(nx) + \sin(nx)] \text{ for all } x > 0.$$

10. Prove the following Ratio Test for uniform convergence.

Suppose $u_n(x)$ are bounded non-zero functions on the set S and that there exists $r < 1$ such that $\left|\frac{u_{n+1}(x)}{u_n(x)}\right| \le r$ for all $n \geq N$, some integer N and all x in S.

Then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on S.

11. If $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence $R > 0$, denote its sum by $f(x)$, then show that $a_k = \frac{f^{(k)}(0)}{k!}$ for each integer $k > 0$.

12. Show that the series $\sum_{n=1}^{\infty} \frac{1}{2^n - n\sin(nx)}$ is uniformly convergent on **R**.

13. (Optional). Show that if f is continuous on [0, 1], then there is a sequence of polynomial functions $p_n(x)$ such that $f(x) = \sum_{n=1}^{\infty} p_n(x)$. [Hint: use Weierstrass Approximation Theorem.]

14. Suppose $\Sigma\, a_n$ is absolutely convergent. Let $\Sigma\, b_n$ be a rearrangement of the same series. Let

$U_n = ½\,(|a_n|+a_n)$, $V_n = ½\,(|a_n| - a_n)$, $r_n = ½\,(|b_n|+b_n)$ and $s_n = ½\,(|b_n| - b_n)$. Verify that these are non-negative sequences.

(i) Show that $\Sigma\, U_n$ and $\Sigma\, V_n$ are convergent series with non-negative terms and that $a_n = U_n - V_n$ and $b_n = r_n - s_n$.

(ii) Note that $\Sigma\, r_n$ is a rearrangement of $\Sigma\, U_n$ and $\Sigma\, s_n$ is a rearrangement of $\Sigma\, V_n$. Use this, or otherwise, prove that $\Sigma\, r_n = \Sigma\, U_n$ and $\Sigma\, s_n = \Sigma\, V_n$.

(iii) Deduce that $\Sigma\, b_n = \Sigma\, a_n$.

15. The "error function" is defined by $\text{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}}\, dt$.

(i) Show that erf(x) can be represented by a power series $\sum_{n=0}^{\infty} a_n x^n$ valid for all x and compute a_0, a_1, a_2, a_3, a_4 and a_5.

(ii) Use part (i) to estimate the value of $\frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-\frac{t^2}{2}}\, dt$.

16. Prove that $\sum_{n=1}^{\infty}\left(\frac{\sin(nx)}{n^3}\right)x^3$ defines a continuous function on **R**.

17. Let $f_n(x) = \frac{x^2}{x^2 + (1-nx)^2}$ for x in [0, 1]. Show that (f_n) converges pointwise but not uniformly.

18. Can we differentiate $x = \sum_{n=1}^{\infty}\left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right)$, for x in [0, 1] term by term ?

19. Show that $\sum_{n=1}^{\infty}\left(\frac{\pi}{n} - \sin(\frac{\pi}{n})\right)$ converges.

20. Prove that for $|x| \le 1$, $\int_0^1 \frac{1-t}{1-xt^3}\, dt = \frac{1}{1\cdot 2} + \frac{x}{4\cdot 5} + \frac{x^2}{7\cdot 8} + \cdots$.
Hence deduce that
(i) $\frac{1}{1\cdot 2} + \frac{1}{4\cdot 5} + \frac{1}{7\cdot 8} + \cdots = \frac{\pi}{3\sqrt{3}}$,
(ii) $\frac{1}{1\cdot 2} + \frac{1}{7\cdot 8} + \frac{1}{13\cdot 14} + \cdots = \frac{\pi}{6\sqrt{3}} + \frac{1}{3}\ln(2)$.

21. Show that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges pointwise on $(-1, 1)$ but is not uniformly convergent. [Hint: Partial sums are not uniformly bounded.]

22. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n}\sin(\sqrt{n}\,x\pi)$ converges uniformly on the interval $[k, K]$ for any constant $k > 0$ and any $K > k$. Deduce that $\sum_{n=1}^{\infty} \frac{1}{n}\sin(\sqrt{n}\,x\pi)$ converges pointwise on **R**.
(Hint: For a fixed $x > 0$ consider a bracketing of the series by the sum $S_m = \sum_{n \in N_m} \frac{1}{n}\sin(\sqrt{n}\,x\pi)$, where $N_m = \left\{n : n \text{ an integer and } \left(\frac{m}{x}\right)^2 \le n \le \left(\frac{m+1}{x}\right)^2\right\}$, i.e., $\sum_{n=1}^{\infty} \frac{1}{n}\sin(\sqrt{n}\,x\pi) = \sum_{m=1}^{\infty} S_m$. For $N_m = \varnothing$, define $S_m = 0$.)

23. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\cos\left(\sqrt{n}\,x\frac{\pi}{2}\right)$ diverges for every x in **R**. (Hint: For a fixed $x > 0$ in **R**, consider a similar bracketing of the series as in question 22, $\sum_{n=1}^{\infty} \frac{1}{n}\cos\left(\sqrt{n}\,x\frac{\pi}{2}\right) = \sum_{m=1}^{\infty} T_m$ and show that $T_m \nrightarrow 0$.)

24. Prove that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\sin(\sqrt{n}\,x\pi)$ diverges for every $x \neq 0$.

25. (Hard) However, prove that $\sum_{n=1}^{\infty} \frac{1}{n^{\beta}}\sin(\sqrt{n}\,x\pi)$ converges at every x in **R** for $1/2 < \beta < 1$ and that the convergence is uniform on $[k, K]$, any $0 < k < K$ but not uniform on $[0, K]$.

26. (Hard) Suppose that a_n is positive for each integer $n \ge 1$ and that (a_n) is a decreasing sequence. Prove that $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly on any bounded interval if and only if $n\,a_n \to 0$ or equivalently $a_n = o(\frac{1}{n})$. Hence deduce that the series converges uniformly on **R** if and only if $n\,a_n \to 0$.

27. Prove that $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}\sin(nx)$ converges uniformly on **R**.