Chapter 7. Series of Functions and Power Series

Introduction. Taylor's Theorem for the expansion of a function is the first step to expanding a function as a power series. Brooke Taylor derived the Theorem that bears his name in his "Methodos Incrementorum Directa et Inversa (1715)". Basically Taylor derived the well known formula from the Gregory-Newton formula. The expansion at the point x =0 is now known as Maclaurin's theorem. This was given in Colin Maclaurin's "Treatise of Fluxions (1742)". His proof uses his method of undetermined coefficients. However both men did not worry about convergence. The Lagrange form of the remainder for the Taylor's series is named after him who said that the series should not be used without taking into consideration of the remainder. However, convergence was considered much later by Cauchy who stressed that to obtain a convergent series, the remainder must tend to zero. Cauchy gave in his "Cours d'analyse", the Cauchy principle of convergence and also gave us the root test. Cauchy considered the question of $\Sigma\, u_n(x)$ is continuous if each $u_n(x)$ is. He also claimed that if $\Sigma\, u_n(x)$ is convergent, then one may integrate the series term by term. He overlooked the need for uniform convergence. Karl Weierstrass had the idea of uniform convergence as early as 1842. He used the notion of uniform convergence to give conditions for the integration of a series term by term. His approximation of continuous function on a closed and bounded interval by polynomial is now a powerful method in numerical mathematics.

We shall investigate many of these ideas from power series to power series functions: differentiation and integration of power series functions, uniform convergence of a sequence of functions. We shall revisit Taylor's Theorem, investigate its convergence statement and the Weierstrass Approximation Theorem.

7.1 Power series

Definition 1. A *power series* is a series of the form $\sum_{n=0}^{\infty} a_n x^n$.

It is a *real power series if* x, a_n are in **R** for all n and a *complex power series if x,* $a_n \in$ **C** *for all n.*

Most of the results about power series apply equally well to complex series.

Example 2.

(1) $Exp(x) = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots$. Here $a_n = \dfrac{1}{n!}$, x can be real or complex.

(2) $\cos(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots + (-1)^n \dfrac{x^n}{(2n)!} + \cdots$. Here, $a_{2n} = \dfrac{(-1)^n}{(2n)!}$, $n \geq 1$, a_0 =1 and $a_{2n-1} = 0$ for $n \geq 1$.

(3) $\sin(x) = x - \dfrac{x3}{3!} + \dfrac{x^5}{5!} + \cdots + (-1)^{n+1} \dfrac{x^{2n-1}}{(2n-1)!} + \cdots$. Here, $a_{2n-1} = \dfrac{(-1)^{n+1}}{(2n-1)!}$, $n \geq 1$, a_0 =0 and $a_{2n} = 0$ for $n \geq 1$.

Definition 3. If we define $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on the understanding only for x for which the series $\sum_{n=0}^{\infty} a_n x^n$ converges, then f is a function defined on some subset of **R** (or **C**

as appropriate). That is, $f : \mathcal{D} \to \mathbf{R}$ (or $\mathbf{C}$) is a function with domain $\mathcal{D} = \{x \in \mathbf{R}$ (respectively $\mathbf{C}$): $\sum_{n=0}^{\infty} a_n x^n$ is convergent.$\}$. The function f may be continuous, differentiable, integrable, etc. Thus in our Example 2 above, we have defined the exponential function exp, cosine and sine functions for complex argument as well. First we have a theorem about what the domain can be.

Theorem 4. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. There are three possibilities.

(1) $\sum_{n=0}^{\infty} a_n x^n$ converges only when $x = 0$ and diverges everywhere else.

(2) $\sum_{n=0}^{\infty} a_n x^n$ converges (absolutely) for all x, or

(3) There exists a real number $r > 0$ such that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < r$ and diverges for $|x| > r$. The power series $\sum_{n=0}^{\infty} a_n x^n$ may converge or diverge when $|x| = r$.

Our proof will make use of the Comparison Test for series. Let $\mathcal{D} = \{x \in \mathbf{R}$ (respectively $\mathbf{C}$): $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.$\}$. First we shall show that if $\mathcal{D}$ contains a point different from 0, then $\mathcal{D}$ contains an open *disk* with 0 as its centre. The proof is for both real and complex series. By a *disk* with centre x_0 and radius $r > 0$, we mean the generic term for the set $\{x : |x - x_0| < r\}$ in $\mathbf{R}$ or $\mathbf{C}$. For the complex case, it is the usual meaning of the 2-dimensional disk but for the real case, it is just the interval $(x_0 - r, x_0 + r)$. We shall use this terminology for both real and complex series unless otherwise specified. Before we proceed, we prove the following useful observation.

Proposition 5. If the power series $\sum_{n=0}^{\infty} a_n x^n$ is convergent at some point $x_0 \neq 0$, then $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all x such that $|x| < |x_0|$.

Proof. Suppose $x_0 \neq 0$ and $\sum_{n=0}^{\infty} a_n x_0^n$ is convergent. Then by Proposition 10 of Chapter 6, $a_n x_0^n \to 0$. Hence the sequence $(a_n x_0^n)$ is convergent and so the sequence $(a_n x_0^n)$ is bounded (see Theorem 11 Chapter 2 Sequences). That means there exists a real number $M > 0$ such that

$$|a_n x_0^n| \leq M \quad \text{for all integer } n \geq 0. \quad \text{--------------------} \quad (1)$$

Now for any x such that $|x| < |x_0|$,

$$\left|\frac{x}{x_0}\right| = \beta < 1. \quad \text{------------------------------} \quad (2)$$

Therefore,

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M\beta^n \qquad \text{by (1) and (2).}$$

Now since $\beta < 1$, $\sum_{n=0}^{\infty} M\beta^n = M\sum_{n=0}^{\infty} \beta^n$ is convergent as $\sum_{n=0}^{\infty} \beta^n$ is a convergent geometric series. (See Example 4 of Chapter 6.). Therefore, by the Comparison Test for Series

(Proposition 12 Chapter 6 Sequences), $\sum_{n=0}^{\infty}|a_n x^n|$ is convergent. Hence, $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all x such that $|x| < |x_0|$.

Now we return to the proof of Theorem 4.

Proof of Theorem 4. Observe that $\sum_{n=0}^{\infty} a_n x^n$ always converge when $x = 0$. Recall $\mathcal{D} = \{x : \sum_{n=0}^{\infty} a_n x^n$ converges absolutely.$\}$. Hence $0 \in \mathcal{D}$ and $\mathcal{D} \neq \varnothing$. If $\mathcal{D} = \{0\}$, then for any $x \neq 0$, $\sum_{n=0}^{\infty} a_n x^n$ is divergent. This is because if for some $x_0 \neq 0$, $\sum_{n=0}^{\infty} a_n x_0^n$ is convergent, then by Proposition 5, $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all x such that $|x| < |x_0|$. This would contradict that $\mathcal{D} = \{0\}$. Thus if $\mathcal{D} = \{0\}$, $\sum_{n=0}^{\infty} a_n x^n$ converges only when $x = 0$ and diverges everywhere else. This gives conclusion (1) of Theorem 4.

Now suppose $\mathcal{D} \neq \{0\}$. This means there exists $x_0 \neq 0$ such that $\sum_{n=0}^{\infty} a_n x_0^n$ converges absolutely. Then by Proposition 5, for all x such that $|x| < |x_0|$,, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. Hence we have

$$\{ x : |x| < |x_0| \} \subseteq \mathcal{D}. \quad \text{-------------------------} \quad (3)$$

We now investigate the diameter of $\mathcal{D}$.

Let $D_+ = \{|x| : x \in \mathcal{D}\}$. Obviously $0 \in D_+$ since $0 \in \mathcal{D}$. Since $\mathcal{D} \neq \{0\}$, by (3), $[0, |x_0|) \subseteq D_+$. Since, $x_0 \in \mathcal{D}$, $[0, |x_0|] \subseteq D_+$. Now if $m < n$ and $m, n \in D_+$, then by what we have just shown, $[0, n] \subseteq D_+$. Therefore, $[m, n] \subseteq D_+$. This means D_+ is an interval containing 0. Clearly D_+ is nontrivial as $\mathcal{D} \neq \{0\}$. Therefore, either D_+ is an unbounded interval or a bounded non trivial interval. Now we shall deduce the second conclusion as follows.

If D_+ is an unbounded interval, then since $D_+ \subseteq [0, \infty)$ and $0 \in D_+$, $D_+ = [0, \infty)$. Then we claim that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all x. This is deduced as follows. Since $D_+ = [0, \infty)$, for any x, there exists a real number K such that $|x| < K$. Since $K \in D_+ = [0, \infty)$, there exists x_0 such that $K = |x_0|$ and $x_0 \in \mathcal{D}$. Therefore, $\sum_{n=0}^{\infty} a_n x_0^n$ converges (absolutely) and so by Proposition 5, $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent. Therefore, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all x.

If D_+ is bounded, then there exists $y_0 \notin \mathcal{D}$ such that $\sum_{n=0}^{\infty}|a_n y_0^n|$ diverges. Then for any y with $|y| > |y_0|$, $\sum_{n=0}^{\infty} a_n y^n$ is divergent. This is because if $\sum_{n=0}^{\infty} a_n y^n$ is convergent, then by Proposition 5, $\sum_{n=0}^{\infty}|a_n y_0^n|$ is convergent since $|y| > |y_0|$. This contradicts that $\sum_{n=0}^{\infty}|a_n y_0^n|$ is divergent. Thus, for any y with $|y| > |y_0|$, $\sum_{n=0}^{\infty} a_n y^n$ is divergent and so $\sum_{n=0}^{\infty}|a_n y^n|$ is divergent. Therefore, D_+ is bounded above by $|y_0|$. (More easily, since D_+ is by definition bounded below by 0 and non trivial, it is also bounded above as it is bounded.) Now let

$$r = \sup D_+ = \sup\{|x| : \sum_{n=0}^{\infty} a_n x^n \text{ is absolutely convergent.}\}.$$

Note that since $\mathcal{D} \neq \{0\}$, D_+ is a nontrivial interval, bounded above by $|y_0| > 0$. Hence, $r > 0$. Let x be such that $|x| < r$. Then $|x|$ is not an upper bound for D_+. Hence, there exists x_0 in $\mathcal{D}$ such that $|x_0| \in D_+$ and

$$|x| < |x_0| \leq r.$$

Now, $\sum_{n=0}^{\infty} a_n x_0^n$ is convergent since x_0 is in $\mathcal{D}$. It follows by Proposition 5 that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. This proves the first assertion of part (3). Now take any y such that $|y| > r$. Then $|y|$ is an upper bound of D_+ but not the least upper bound of D_+. Then $\sum_{n=0}^{\infty} a_n y^n$ must diverge. This is because if $\sum_{n=0}^{\infty} a_n y^n$ is convergent, then by Proposition 5, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all x such that $|x| < |y|$. Then pick any x_0 such that $r < |x_0| < |y|$. For instance, we can take x_0 such that $|x_0| = \frac{1}{2}(r + |y|)$. ($|x_0|$ is the mid point of the interval [r, $|y_0|$] and so $r < |x_0| < |y|$.) Then $\sum_{n=0}^{\infty} a_n x_0^n$ is absolutely convergent and so $x_0 \in \mathcal{D}$ and $|x_0| \in D_+$. Hence $|x_0| \leq r$ contradicting $r < |x_0|$. Therefore, $\sum_{n=0}^{\infty} a_n y^n$ is divergent. This proves the second assertion of part (3). If $|x| = r$, we have no information about whether $\sum_{n=0}^{\infty} a_n x^n$ is convergent or divergent. Indeed it may do either, as the following example will show.

Example 6. (The Logarithmic series)

The logarithmic series is given by $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

Now $\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{xn}{n+1}\right| = |x|\left|\frac{n}{n+1}\right| \to |x|$. Therefore, by the Ratio Test (Theorem 21, Chapter 6 Series),

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \text{ is absolutely convergent if } |x| < 1,$$

and

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \text{ is divergent for } |x| > 1.$$

But for $x = 1$, $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \cdots$ is convergent by the Leibnitz's Alternating Series Test (Theorem 20 Chapter 6 Series), while for $x = -1$,

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} = -1 - \frac{1}{2} - \frac{1}{3} - \cdots$$

is divergent.

Definition 7. The number r in Theorem 4 is called the *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n x^n$. The set $D(0, r) = \{x : |x| < r\}$ is called the *disc of convergence*.

For a complex power series, this fits in with the geometric image of a disc. For a real power series $D(0, r) = (-r, r)$ is just the interval of convergence. Note that we shall also refer to the interval of convergence as the disc of convergence for the real poweer series as a generic disk of one dimension.

Remark.
We can almost always find the radius of convergence r by using the d'Alembert's Ratio Test. We shall give a formula later.

If $\sum_{n=0}^{\infty} a_n x^n$ converges for all x, we may by convention, say the radius of convergence is $+\infty$. Hence in this case $D(0,\infty) = \mathbf{R}$ for real power series and $D(0,\infty) = \mathbf{C}$ for complex power series.

Example 8.

(1) The power series $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n$.

For $x \neq 0$, $\left|\frac{x^{n+1}}{x^n}\right| = |x| \to |x|$ as $n \to\infty$. Therefore, by d'Alembert's Ratio Test, $\sum_{n=0}^{\infty} x^n$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. Hence the radius of convergence is 1.

If $|x| =1$, then $|x|^n \to 1 \neq 0$ as $n \to\infty$ and so $\sum_{n=0}^{\infty} a_n x^n$ diverges by Proposition 10 Chapter 6 Series.

(2) $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = Exp(x)$.

For $x \neq 0$, $\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \left|\frac{x}{n+1}\right| \to 0$ as $n \to\infty$. Hence, by d'Alembert's Ratio Test, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all x.

(3) $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} nx^n$.

For $x \neq 0$, $\left|\frac{(n+1)x^{n+1}}{nx^n}\right| = \frac{n+1}{n}|x| \to |x|$ as $n \to\infty$. Therefore, by d'Alembert's Ratio Test, $\sum_{n=0}^{\infty} x^n$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. Hence the radius of convergence is 1.

If $|x| =1$, then $|nx^n| \to\infty$ as $n \to\infty$ and so $\sum_{n=0}^{\infty} nx^n$ diverges by Proposition 10 Chapter 6 Series.

(4) $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n!x^n$.

For $x \neq 0$, $\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty$ as $n \to\infty$. Therefore, by d'Alembert's Ratio Test, $\sum_{n=0}^{\infty} n!x^n$ diverges if $x \neq 0$ and converges only for $x = 0$.

(5) $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

For $x \neq 0$, $\left| \frac{\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}}{\frac{(-1)^n x^{2n}}{(2n)!}} \right| = \frac{x^2}{(2n+2)(2n+1)} \to 0$ as $n \to \infty$. Therefore, by d'Alembert's Ratio Test, cos(x) converges absolutely for all x.

(6) $\sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n-1}$.

For $x \neq 0$, $\left| \frac{\frac{(-1)^{n+2}x^{2n+1}}{(2n+1)!}}{\frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}} \right| = \frac{x^2}{(2n+1)(2n)} \to 0$ as $n \to \infty$. Therefore, by d'Alembert's Ratio Test, sin(x) converges absolutely for all x.

For Example (5) and (6) above as well as (2), $r = +\infty$. These examples also give us the question: if these power series are considered as a function, are they continuous?

7.2 Continuity of Power Series Functions

Definition 9. By Theorem 4, each power series $\sum a_n x^n$, if it is not just only convergent at $x = 0$, has radius of convergence $r > 0$, r may be finite or infinite. $\sum a_n x^n$ thus defines a function

$$f : D(0, r) \to \mathbf{R} \text{ (or } \mathbf{C})$$

called a *power series function.*

Theorem 10. If $f : D_r \to \mathbf{R}$ (or $\mathbf{C}$) is defined by a power series with the open disc $D_r = \{x : |x| < r\}$ as its disc of convergence, then f is continuous. That is, f is continuous at each point of D_r .

Remark.
Note that D_r is open. If $0 < r < \infty$, the power series may be convergent at some point on the boundary of D_r . If it does, then for real power series, if we extend the domain to include either one or two of the end points at which the power series is convergent, then the power series function is also continuous at these points. The proof is of course much harder.

Proof of Theorem 10. We shall show that for any x_0 in D_r , given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in D_r$,

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Take any point x_0 in D_r , then $|x_0| < r$. Choose a fixed point c such that $|x_0| < c < r$. Let $D_c = D(0, c)$, the disc of radius c centred at 0. We shall show that the restriction of f to D_c , i.e,

$$f\,|_{D_c} : D_c \to \mathbf{R(C)}$$

is continuous at x_0. A consequence of this is that f is continuous at x_0.

Note that since $0 < c < r$, the series $\sum_{n=0}^{\infty} |a_n c^n|$ is convergent and so it is Cauchy. Therefore, there exists an integer N_0 such that for all $n > m \geq N_0$,

$$\sum_{k=m}^{n} |a_k c^k| < \frac{\varepsilon}{3}. \qquad \text{--------------------} \quad (1)$$

Now we return to examine the values of $f(x)$ for x in D_c , *i.e.,* for $|x| < c,$ or more precisely, the tail end of the power series expansion of $f(x)$ by using (1).
For all x in D_c *(hence* $|x| < c$ *) and* for all $n > m \geq N_0$,

$$\left|\sum_{k=m}^{n} a_k x^k\right| \leq \sum_{k=m}^{n} |a_k x^k| = \sum_{k=m}^{n} |a_k c^k| \left|\frac{x^k}{c^k}\right| = \sum_{k=m}^{n} |a_k c^k| \left|\frac{x}{c}\right|^k$$

$$\leq \sum_{k=m}^{n} |a_k c^k| \cdot 1 \ \text{ since } \left|\frac{x}{c}\right| < 1$$

$$< \frac{\varepsilon}{3} \qquad \text{------------------------------} \quad (2)$$

by (1).
Therefore, for x in D_c ,

$$\left| f(x) - \sum_{k=0}^{N_0} a_k x^k \right| = \left| \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k - \sum_{k=0}^{N_0} a_k x^k \right| = \left| \lim_{n\to\infty} \sum_{k=N_0+1}^{n} a_k x^k \right|$$

$$\leq \lim_{n\to\infty} \left| \sum_{k=N_0+1}^{n} a_k x^k \right| \leq \frac{\varepsilon}{3} \qquad \text{---------------------} \quad (3)$$

by (2).
The next thing to note is that $\sum_{k=0}^{N_0} a_k x^k$ is a polynomial function in x and so is continuous everywhere and in particular, continuous at $x = x_0$. Now let $S_{N_o}(x) = \sum_{k=0}^{N_0} a_k x^k$. Then $S_{N_o}(x)$ is continuous at x for all x. By the continuity of $S_{N_o}(x)$ at x_0, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x in D_c ,

$$|x - x_0| < \delta \Rightarrow |S_{N_o}(x) - S_{N_o}(x_0)| < \frac{\varepsilon}{3}. \text{ -------------------} \quad (4)$$

Therefore, for all x in D_c ,

$$|x - x_0| < \delta \Rightarrow | f(x) - f(x_0)| = |f(x) - S_{N_o}(x) + S_{N_o}(x) - S_{N_o}(x_0) + S_{N_o}(x_0) - f(x_0)|$$

$$\leq |f(x) - S_{N_o}(x)| + |S_{N_o}(x) - S_{N_o}(x_0)| + |S_{N_o}(x_0) - f(x_0)|$$

$$\leq \frac{\varepsilon}{3} + |S_{N_o}(x) - S_{N_o}(x_0)| + \frac{\varepsilon}{3}$$

by (3)

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

by (4).
Therefore, $f\,|_{D_c} : D_c \to \mathbf{R}(\mathbf{C})$ is continuous at x_0 and so f is continuous at x_0 . It follows that $f : D_r \to \mathbf{R}$ (or $\mathbf{C}$) is continuous at x_0 for any x_0 in D_r and so is continuous.

Remark.
(1) Proof is exactly the same for complex power series functions.
(2) If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence r with $0 < r < \infty$. It may happen that $\sum_{n=0}^{\infty} a_n x^n$ is convergent at some point x on the boundary of the disc of convergence with $|x| = r$, the radius of convergence. For complex power series, it need not follow that $\sum_{n=0}^{\infty} a_n x^n$ is continuous at a boundary point if it is convergent there. However, for real power series , by using the notion of compactness and uniform convergence (Abel's Theorem) it is true that convergence at the end point of the interval of convergence implies continuity there.
(3) The proof of Theorem 10 actually uses the notion of uniform convergence of a sequence of functions.

7.3 Pointwise Convergence and Uniform Convergence of a Sequence of Functions

Note that a power series $\sum_{n=0}^{\infty} a_n x^n$ is convergent if and only if the n-th partial sum $\sum_{k=0}^{n} a_k x^k$ is convergent. If we let $f_n(x) = \sum_{k=0}^{n} a_k x^k$, then $\sum_{n=0}^{\infty} a_n x^n$ is convergent if and only if the sequence $(f_n(x))$ is convergent.

Now each $f_n(x)$ is a function, obviously well defined by a polynomial function. The convergence of $(f_n(x))$ for a fixed x is an example of the notion of *pointwise convergence*.

Let $D = \{x : \sum_{n=0}^{\infty} a_n x^n$ is convergent$\}$. Then defining $f : D \to \mathbf{R}$ by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ we obtain a function and obviously for each x in D, $f_n(x) \to f(x)$. Observe that each $f_n(x)$ is defined on D too. Therefore, $f_1, f_2, \ldots, f_n, \ldots$ is a sequence of functions defined on D such that for each x in D, $f_n(x) \to f(x)$. We say $f_n \to f$ *pointwise* on D. The emphasis is : for each x, $(f_n(x))$ is a sequence and that this sequence is convergent and converges to $f(x)$.

We now make the following formal definition.

Definition 11. Let $f_1, f_2, \ldots, f_n, \ldots : E \to \mathbf{R}$ ($\mathbf{C}$) and $f : E \to \mathbf{R}$ ($\mathbf{C}$) be functions defined on a non-empty set $E \subseteq \mathbf{R}$ ($\mathbf{C}$) . We say $f_n \to f$ *pointwise* on E *if* $f_n(x)$ converges to $f(x)$ at each point x of E. We also say f_n *converges pointwise to f on E.* That is to say, for each $x \in E$, given any $\varepsilon > 0$, there exists an integer $N_0(x)$ (depending on x *and* ε and may be different for different x) such that

$$n \geq N_0(x) \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Example 12.

(1) Let $f_n(x) = \dfrac{1}{n^2 + x^2}$, $n = 1, 2, 3\ldots$.

Then for each x, $0 \leq \dfrac{1}{n^2 + x^2} \leq \dfrac{1}{n^2}$ and so by the Squeeze Theorem (Theorem 13 Chapter 2 Sequences), since $\dfrac{1}{n^2} \to 0$, $f_n(x) \to 0$ for each x. Therefore, $f_n \to 0$, the 0 constant function pointwise.

(2) Let $f_n(x) = \dfrac{\cos(x)}{n}$, $n = 1, 2, 3\ldots$.

For any x, $0 \leq \left|\dfrac{\cos(x)}{n}\right| \leq \dfrac{1}{n}$ and so for each x, $\lim_{n\to\infty} f_n(x) = 0$ by the Comparison Test (Proposition 8 Chapter 2 Sequences). Hence $f_n \to 0$, the 0 constant function pointwise.

(3) $f_n(x) = \dfrac{1}{1 + n^2 x^2}$, $n = 1, 2, 3\ldots$.

For each $x \neq 0$, $x^2 > 0$ and so $0 \leq \dfrac{1}{1 + n^2 x^2} \leq \dfrac{1}{n^2 x^2}$. Therefore, for $x \neq 0$, by the Squeeze Theorem, $\dfrac{1}{1 + n^2 x^2} \to 0$ since $\dfrac{1}{n^2 x^2} \to 0$. Hence, for $x \neq 0$, $f_n(x) \to 0$.

Now for $x = 0$, $f_n(x) = f_n(0) = 1$ for all n. It follows that $f_n(0) \to 1$.
Therefore, $f_n \to f$ pointwise , where $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$.

(4) Let $f_n : [0,1] \to \mathbf{R}$ be defined by $f_n(x) = x^n$, $n = 1, 2, 3\ldots$.
For each $0 \le x < 1$, $x^n \to 0$ and so for each $0 \le x < 1$, $f_n(x) \to 0$.
For each integer $n \ge 1$, $f_n(1) = 1$ and so $f_n(1) \to 1$. Therefore, $f_n \to f$ pointwise, where $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$.

(5) Let $f_n : [0,\infty) \to \mathbf{R}$ be defined by $f_n(x) = \dfrac{x^n - 1}{x^n + 1}$ for $n = 1, 2, 3\ldots$.
For each x such that $0 \le x < 1$, $x^n \to 0$ and so $f_n(x) = \dfrac{x^n - 1}{x^n + 1} \to \dfrac{0-1}{0+1} = -1$. For x $=1$, $f_n(x) = \dfrac{1^n - 1}{1^n + 1} = 0$ for each n. Therefore, $f_n(1) \to 0$. Now for $x > 1$,

$$f_n(x) = \frac{x^n - 1}{x^n + 1} = \frac{1 - \frac{1}{x^n}}{1 + \frac{1}{x^n}} \to \frac{1-0}{1+0} = 1 \text{ as } n \to \infty .$$

Hence, $f_n \to f$ pointwise, where $f(x) = \begin{cases} -1, & 0 \le x < 1 \\ 0, & x = 1 \\ 1, & x > 1 \end{cases}$.

If we remove the dependence of the integer N_0 in Definition 11 on the point x, *we then get the* notion of *uniform convergence* The examples above are examples of pointwise convergence. One will need some criterion to decide if the convergence is also uniform.

Definition 13. Let $f_1, f_2, \ldots, f_n, \ldots : E \to \mathbf{R}$ ($\mathbf{C}$) and $f : E \to \mathbf{R}$ ($\mathbf{C}$) be functions defined on a non-empty set $E \subseteq \mathbf{R}$ ($\mathbf{C}$) .
We say $f_n \to f$ *uniformly* on E *or* f_n converges uniformly to f if given any $\varepsilon > 0$, there exists an integer N_0 (depending only on ε) such that

$$\text{for all } n \ge N_0 \text{ and for all } x \in E,\quad | f_n(x) - f(x) | < \varepsilon.$$

Our proof of Theorem 10 actually uses the ideas of the following:

Theorem 14. Suppose $f_1, f_2, \ldots, f_n, \ldots : E \to \mathbf{R}$ ($\mathbf{C}$) and $f : E \to \mathbf{R}$ ($\mathbf{C}$) are functions defined on a non-empty set $E \subseteq \mathbf{R}$ ($\mathbf{C}$) . If $f_n \to f$ uniformly on E and if each f_n is continuous on E, then f is continuous on E.

Proof. The proof is a careful handling of the notion of uniform convergence.
Let $x_0 \in E$. We shall show that f is continuous at x_0 . Now $f_n \to f$ uniformly on E means there exists an integer N_0 such that

$$\text{for all } x \in E,\ n \ge N_0 \Rightarrow | f_n(x) - f(x) | < \varepsilon/3 . \quad \text{-------------------- (1)}$$

Let us examine what we need to show:
Given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x in E,

$$|x - x_0| < \delta \Rightarrow | f(x) - f(x_0) | < \varepsilon. \quad \text{----------------- } (*)$$

Now using the triangle inequality,

$$\begin{aligned} | f(x) - f(x_0) | &= | f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0) | \\ &\le | f(x) - f_n(x) | + | f_n(x) - f_n(x_0) | + | f_n(x_0) - f(x_0) |. \text{ --- } (2) \end{aligned}$$

The good thing about this inequality is that it is true for any n we care to choose. Therefore, taking $n = N_0$, which is given by (1), using (2) we get

$$\begin{aligned} |f(x) - f(x_0)| &\le |f(x) - f_{N_0}(x)| + |f_{N_0}(x) - f_{N_0}(x_0)| + |f_{N_0}(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + |f_{N_0}(x) - f_{N_0}(x_0)| + \frac{\varepsilon}{3} = |f_{N_0}(x) - f_{N_0}(x_0)| + \frac{2\varepsilon}{3} \quad \text{------ (3)} \end{aligned}$$

by (1).

Now we make use of the continuity of f_{N_0}. Since f_{N_0} is continuous at x_0, there exists $\delta > 0$ such that for all x in E,

$$|x - x_0| < \delta \Rightarrow |f_{N_0}(x) - f_{N_0}(x_0)| < \frac{\varepsilon}{3}. \quad \text{-------------------- (4)}$$

Therefore, it follows from (3) and (4) that for all x in E,

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < |f_{N_0}(x) - f_{N_0}(x_0)| + \frac{2\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

Thus, f is continuous at x_0. Since f is continuous at any $x_0 \in E$, f is continuous on E. This completes the proof.

We can give a proof of Theorem 10 using Theorem 14. All we need to show is uniform convergence. We shall, in the proof of uniform convergence, use a criterion that is named after Weierstrass.

Another Proof of Theorem 10.

Recall $f : D_r \to \mathbf{R}$ (or $\mathbf{C}$) is defined by $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Let $s_n(x) = \sum_{k=0}^{n} a_k x^k$. Then since D_r is the disc of convergence, $s_n \to f$ pointwise on D_r. Take $x_0 \in D_r$. Then $|x_0| < r$. Let c be a number such that $|x_0| < c < r$. Then we shall show that $s_n \to f$ uniformly on the closure of the disc D_c, $\overline{D_c} = \{x : |x| \le c\}$. It follows then that since $x_0 \in \overline{D_c}$, f is continuous at x_0 since each s_n is continuous on $\overline{D_c} \subseteq D_r$ by Theorem 14.
Now we proceed to show uniform convergence of (s_n) on $\overline{D_c}$. Since $c < r$, there exists $y_0 \in D_r$ such that $c < |y_0|$ and $\sum_{n=0}^{\infty} a_n y_0^n$ is convergent. Hence the sequence $(a_n y_0^n)$ is bounded and thus for all integer $n \ge 0$,

$$|a_n y_0^n| \le M,$$

for some real number $M > 0$. Now $c < |y_0|$ and so $\beta = \frac{c}{|y_0|} < 1$. Therefore, for all x in $\overline{D_c}$ and for all $n \ge 0$,

$$|a_n x^n| = |a_n y_0^n| \left|\frac{x}{y_0}\right|^n \le |a_n y_0^n| \left|\frac{c}{y_0}\right|^n \le M\beta^n \quad \text{---------------- (1)}$$

Thus, for all $m > n$ and for all x such that $|x| \le c$,

$$\left|\sum_{k=n}^{m} a_k x^k\right| \le M\left|\sum_{k=n}^{m} \beta^k\right| = M\beta^n(1 + \beta + \beta^2 + \cdots + \beta^{m-n}) = M\beta^n \frac{1 - \beta^{m-n+1}}{1 - \beta}$$

$$\le M\frac{\beta^n}{1 - \beta}. \quad \text{----------------------- (2)}$$

Since $\beta^n \to 0$ as $n \to \infty$ (because $|\beta| < 1$), given $\varepsilon > 0$ there exists an integer N such that

$$n \ge N \Rightarrow \beta^n < \frac{\varepsilon}{2M}(1 - \beta). \quad \text{----------------------- (3)}$$

Hence by (2) and (3), for any $n, m \ge N$ with $m > n \ge N$ and for any x such that $|x| \le c$,

$$\left|\sum_{k=n}^{m} a_k x^k\right| \le M\frac{\beta^n}{1 - \beta} < \frac{M}{1 - \beta}\frac{\varepsilon}{2M}(1 - \beta) = \frac{\varepsilon}{2} \quad \text{---------- (4)}$$

So we now examine the "distance" $|f(x) - s_n(x)|$ for any x in $\overline{D_c}$.

For any x in $\overline{D_c}$ and for any $n \ge N$,

$$|f(x) - s_n(x)| = |f(x) - \sum_{k=0}^{n} a_k x^k| = |\lim_{m\to\infty} s_m(x) - \sum_{k=0}^{n} a_k x^k|$$

$$= \left|\lim_{m\to\infty}\left(s_m(x) - \sum_{k=0}^{n} a_k x^k\right)\right| = \left|\lim_{m\to\infty}\left(\sum_{k=n+1}^{m} a_k x^k\right)\right|$$

$$= \lim_{m\to\infty}\left|\sum_{k=n+1}^{m} a_k x^k\right|$$

$$\le \frac{\varepsilon}{2} < \varepsilon$$

by (4).

This means $s_n \to f$ uniformly on $\overline{D_c}$. (This argument is usually summarized as Weierstrass M-Test.) Therefore, by Theorem 14, f is continuous on $\overline{D_c}$ .Similarly for any d such that $c < d < r$, $s_n \to f$ uniformly on $\overline{D_d}$ and $\overline{D_c} \subseteq \overline{D_d}$. Therefore, by Theorem 14, f is also continuous on $\overline{D_d}$. Since $D_r = \cup\{\overline{D_c} : 0 \le c < r\}$, f is continuous on D_r . This is deduced as follows. Take any x in D_r. Then $x \in D_c$ for some c such that $0 < c < r$. Since f is continuous on $\overline{D_c}$, f is continuous on D_c which is *open*. Therefore, f is continuous at x.

Example 15.

(1) Let $f_n(x) = \dfrac{1}{n^2 + x^2}$, $n = 1, 2, 3\ldots$. $f_n \to f$, the 0 constant function uniformly. Why?

Observe that $|f_n(x) - f(x)| = \left|\dfrac{1}{n^2+x^2} - 0\right| \le \dfrac{1}{n^2}$ for all x.

Now since $\dfrac{1}{n^2} \to 0$, given $\varepsilon > 0$, there exists an integer N_0 such that

$$n \ge N_0 \Rightarrow \frac{1}{n^2} < \varepsilon.$$

Hence, for all integer $n \ge N_0$ and for all x,

$$|f_n(x) - f(x)| \le \frac{1}{n^2} < \varepsilon.$$

Thus, $f_n \to f$ uniformly on **R**. This is the function of Example (1) in Example 12.

(2) Similarly the sequence of functions in Example (2) of Example 12 is uniformly convergent on **R** to the 0 constant function.

(3) The sequence of functions in Example (3) of Example 12 , converges pointwise on **R** to f , where $f(x) = \begin{cases} 0, & x \ne 0 \\ 1, & x = 0 \end{cases}$. This function f is obviously not continuous at $x = 0$. Note that each term f_n of the sequence is continuous on **R**, and so by Theorem 14, (f_n) does not converge uniformly on **R** to f (for if it did, f would be continuous on **R**) .

We can also deduce the non-uniform convergence of f_n to f by examining the difference $|f_n(x) - f(x)|$ for all possible values of x. The idea is to introduce a kind of distance function for functions, introducing the technique of metric spaces.

Now $$|f_n(x) - f(x)| = \begin{cases} \dfrac{1}{1+n^2x^2}, & x \ne 0 \\ 0, & x = 0 \end{cases}.$$

Therefore, $\sup\{|f_n(x) - f(x)| : x \in \mathbf{R}\} = 1$ for each n, since $\dfrac{1}{1+n^2x^2} \to 1$ as $x \to 0$ and

$$0 \le \frac{1}{1+n^2x^2} \le 1.$$

Now we shall show non-uniform convergence as follows.
Take ε = 1/2. Then for each integer N, since $\varepsilon < \sup\{|f_N(x) - f(x)| : x \in \mathbf{R}\} = 1$, there exists a point x_N (*depending on N*) such that

$$\varepsilon = 1/2 < |f_N(x_N) - f(x_N)| \le \sup\{|f_N(x) - f(x)| : x \in \mathbf{R}\} = 1$$

by the definition of supremum or least upper bound.
Therefore, (f_n) cannot converge uniformly on $\mathbf{R}$ to f

(4) The sequences of functions in (4) and (5) of Example 12 do not converge uniformly since the limiting functions are not continuous functions. This conclusion is again an application of Theorem 14. We make some observation here regarding the examples in Example 12. Note that all the functions are bounded functions defined on their respective domains and the pointwise limits are also bounded functions. Therefore, the above use of the supremum $\sup\{|f_n(x) - f(x)| : x \in E\}$, where E is the respective domain gives rise to the notion of convergence *in metric*. The distance or metric between two bounded functions f and g on E is defined to be $d(f, g) = \sup\{|f(x) - g(x)| : x \in E\}$. Indeed with this metric $f_n \to f$ uniformly on $\boldsymbol{E}$ is equivalent to $d(f_n, f) \to 0$. The proof above uses this idea to show non-convergence. Let us now apply this to (4) and (5) of Example 12.

For (4) of Example 12, recall that $f_n(x) = x^n$ for x in [0, 1] and $f_n \to f$ pointwise , where $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$. Then

$$|f_n(x) - f(x)| = \begin{cases} x^n, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}.$$

Hence for each $n = 1,2,3, \ldots$, $\sup\{|f_n(x) - f(x)| : x \in [0, 1]\} = 1$, because $0 \le \{|f_n(x) - f(x)| \le 1$ and $\lim_{x\to 1} x^n = 1$. Thus $d(f_n, f) \nrightarrow 0$. So we can conclude that f_n cannot converge uniformly to f. To explain this further, we proceed as in (3) above. Take ε = 1/2. Then for each integer N, since $\varepsilon < \sup\{|f_N(x) - f(x)| : x \in [0,1]\} = 1$, there exists a point x_N such that

$$\varepsilon = 1/2 < |f_N(x_N) - f(x_N)| \le \sup\{|f_N(x) - f(x)| : x \in [0,1]\} = 1$$

by the definition of supremum or least upper bound. So by definition, (f_n) cannot converge uniformly on [0, 1] to f.

For Example (5) of Example 12, ,$|f_n(x) - f(x)| = \begin{cases} \dfrac{2x^n}{x^n+1}, & 0 \le x < 1 \\ 0, & x = 1 \\ \dfrac{2}{x^n+1}, & x > 1 \end{cases}$ and so

$\sup\{|f_n(x) - f(x)| : x \in [0,\infty)\} = 1$. By exactly the same argument as above we can show that the convergence is not uniform.

Remark 16.

(1) We have proved the following:

If D_r is the disc of convergence for the power series, $\sum_{n=0}^{\infty} a_n x^n$ and r its radius of convergence, then $s_n(x) = \sum_{k=0}^{n} a_k x^k$ converges uniformly on $\overline{D_c}$, the closed disc of radius $0 < c < r$. Therefore, by Theorem 14, the power series function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous on $\overline{D_c}$, for any $0 < c < r$.

(2) It is awkward to say anything about the continuity of $\sum_{n=0}^{\infty} a_n x^n$ at the boundary point of the disc of convergence for complex power series. However, for real power series function, convergence at the end point of the interval of convergence implies continuity there. This fact is usually referred to as Abel's Theorem. The proof is somewhat subtle and uses a technique due to Abel involving Abel's summation formula and we shall give a proof in the next chapter when we deal with uniform convergence and differentiability.

17. Commutation of two different kinds of limiting processes.

We now give a re-interpretation of Theorem 14 in the light of power series function. Suppose ($f_n : E \to \mathbf{R}$) is a sequence of continuous function. Then the n-th partial sum $s_n = \sum_{k=1}^{n} f_k$ is also a continuous function. Then if s_n converges to f uniformly on E, f is also continuous on E by Theorem 14. Now look at this statement from the point of view of limiting process. The function f is continuous at x_0 in E means

$$\lim_{x \to x_0} f(x) = f(x_0).$$

That is,

$$\lim_{x \to x_0} \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} f_k(x_0) = \sum_{k=1}^{\infty} \lim_{x \to x_0} f_k(x_0)$$

or

$$\lim_{x \to x_0} \Big(\big(\lim_{n \to \infty} \sum_{k=1}^{n} f_k\big)(x)\Big) = \lim_{n \to \infty} \Big(\sum_{k=1}^{n} \lim_{x \to x_0} f_k(x_0)\Big) = \lim_{n \to \infty} \Big(\lim_{x \to x_0} \sum_{k=1}^{n} f_k(x_0)\Big).$$

This is an example of two kinds of limit operations commuting.

The question we can ask is this: Is taking the limit of the function $\sum_{k=1}^{\infty} f_k$ given by taking the limit term by term? Theorem 14 says if f is the uniform limit of $\sum_{k=1}^{n} f_k$ and each f_k is continuous, then the answer is "yes". We can ask other kind of question. Presently we are looking at the case of each $f_k(x) = a_k x^k$, which is always continuous and differentiable. Then we have new questions.

(1) If each f_k is differentiable on E, and $s_n = \sum_{k=1}^{n} f_k$ converges uniformly to f on E, is f differentiable?

(2) If f in (1) is differentiable, then is $f'(x) = \sum_{k=1}^{\infty} f'_k(x)$ for x in E?

(3) If $s_n = \sum_{k=1}^{n} f_k$ converges uniformly to f on E, and each f_k is Riemann integrable on $[a, b] \subseteq E$, is f Riemann integrable on $[a, b]$?

(4) If f in (3) is Riemann integrable on $[a, b] \subseteq E$, is $\int_a^b f(x) = \sum_{k=1}^{\infty} \int_a^b f_k$?

We shall in the next two chapters answer some, if not all of the above questions.

We close this chapter with the following criteria for determining the radius of convergence of a power series.

7.4 Formula for Radius of Convergence, The Cauchy-Hadamard Formula

The first is an application of the Ratio Test for series.

Theorem 18. Suppose $\sum_{n=0}^{\infty} a_n x^n$ is a power series.
Suppose $\left|\frac{a_{n+1}}{a_n}\right| \to q > 0$. Then the radius of convergence for the power series is $\frac{1}{q}$.
If $\left|\frac{a_{n+1}}{a_n}\right| \to 0$, then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all x and so the radius of convergence is ∞.
If $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$, then $\sum_{n=0}^{\infty} a_n x^n$ diverges for all x except at $x = 0$, where it is convergent and so the radius of convergence is zero.

Proof. This is a simple direct application of the Ratio Test.
If $\left|\frac{a_{n+1}}{a_n}\right| \to q > 0$, for $x \neq 0$, the hypothesis of the theorem means that

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x| \to q|x| .$$

Therefore, by the Ratio Test for Series (Theorem 21 Chapter 6 Series), the series is absolutely convergent for $q\,|x| < 1$, i.e., $|x| < \frac{1}{q}$ and is divergent if $q\,|x| > 1$, i.e., $|x| > \frac{1}{q}$. Plainly, the series converges at $x = 0$. Hence the radius of convergence of the power series is $\frac{1}{q}$.
If $\left|\frac{a_{n+1}}{a_n}\right| \to 0$, for $x \neq 0$, then $\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x| \to 0$. Therefore, by the Ratio Test for Series (Theorem 21 Chapter 6), the series is absolutely convergent for *all* $x \neq 0$. Since the series is convergent at $x = 0$, it is absolutely convergent for all x. Hence the radius of convergence is ∞.
If $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$, then for $x \neq 0$, $\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x| \to \infty$. Therefore, by the Ratio Test for Series (Theorem 21 Chapter 6), the series is divergent for *all* $x \neq 0$. Hence it converges only for $x = 0$. Thus the radius of convergence is 0.

We can actually give a slightly better formula for the radius of convergence in that when the above ratio test is inapplicable in the sense that the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ does not exist but we can employ another limit which gives us the radius of convergence. This formula, which is sharper than the above formula of Theorem 8 may be more difficult to compute. This formula is known as the Cauchy-Hadamard formula.

Theorem 19 (Cauchy- Hadamard Formula) For any power series $\sum_{n=0}^{\infty} a_n x^n$, the radius of convergence r is given by
(i) $r = 0$ if $\limsup |a_n|^{\frac{1}{n}} = +\infty$,
(ii) $r = \infty$ if $\limsup |a_n|^{\frac{1}{n}} = 0$, and
(iii) $r = \dfrac{1}{\limsup |a_n|^{\frac{1}{n}}}$ if $0 < \limsup |a_n|^{\frac{1}{n}} < \infty$.

[$r = 0$ corresponds to the case when $\sum_{n=0}^{\infty} a_n x^n$ converges only for $x = 0$ and no where else. $r = \infty$ corresponds to the case when $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all x.]

Proof. We test the series by using the Cauchy root test. Note that the sequence $\left(|a_n|^{\frac{1}{n}}\right)$ is either bounded or unbounded. It is obviously bounded below by 0. If it is unbounded, then it is not bounded above and so for each n,

$$\sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, \cdots\} = \infty.$$

If $\left(|a_n|^{\frac{1}{n}}\right)$ is bounded, then for each n, $y_n = \sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, \cdots\}$ exists by the completeness property of **R** and the sequence $\left(y_n = \sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, \cdots\}\right)$ is decreasing and bounded below by 0 and hence it is convergent by the Monotone Convergence Theorem. The limit of the sequence (y_n) is defined to be

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} y_n .$$

We shall apply the refined Cauchy Root Test to $\sum_{n=0}^{\infty} a_n x^n$. Now observe that

$$|a_n x^n|^{\frac{1}{n}} = |x||a_n|^{\frac{1}{n}}.$$

Therefore, $\limsup_{n\to\infty} |a_n x^n|^{\frac{1}{n}} = \limsup_{n\to\infty} |x|\,|a_n|^{\frac{1}{n}} = |x| \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ for $x \neq 0$.
We thus proceed according to whether the limit $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ is 0, finite or infinite.
Case (i). $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \infty$.
If $x \neq 0$, then $\limsup_{n\to\infty} |a_n x^n|^{\frac{1}{n}} = \infty$. It follows by Theorem 29 (ii) (Root test) of Chapter 6 , that $\sum_{n=0}^{\infty} a_n x^n$ is divergent. Hence $\sum_{n=0}^{\infty} a_n x^n$ is divergent for all x except x = 0. Therefore. the radius of convergence r is 0.
Case (ii). $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$.
Then $\limsup_{n\to\infty} |a_n x^n|^{\frac{1}{n}} = \limsup_{n\to\infty} |x|\,|a_n|^{\frac{1}{n}} = |x| \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 0 < 1$. Therefore, by Theorem 29 (Root test) Chapter 6 Series, $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for any x. Hence, by convention, the radius of convergence $r = \infty$.
Case (iii). $0 < \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = q < \infty$.
Then $\limsup_{n\to\infty} |a_n x^n|^{\frac{1}{n}} = \limsup_{n\to\infty} |x|\,|a_n|^{\frac{1}{n}} = |x|q$.

Therefore, by the refined Cauchy root test (Theorem 29 Chapter 6 Series), $|x|\ q < 1$ implies that $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent. That is, $|x| < 1/q$ implies that $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent. Also by the refined Cauchy root test, $|x|\ q > 1$ implies that $\sum_{n=0}^{\infty} a_n x^n$ is divergent. That means $|x| > 1/q$ implies that $\sum_{n=0}^{\infty} a_n x^n$ is divergent. Hence, the radius of convergence is $1/q$. This completes the proof.

We next present a reason why the Cauchy Hadamard formula is a sharper test then the ratio test.

Proposition 20. Suppose (a_n) is a positive sequence, i.e., a sequence of positive terms. Then

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf_{n\to\infty} (a_n)^{\frac{1}{n}} \le \limsup_{n\to\infty} (a_n)^{\frac{1}{n}} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}\ . \text{ ------ (A)}$$

Proof. Note that $\liminf_{n\to\infty} (a_n)^{\frac{1}{n}} \le \limsup_{n\to\infty} (a_n)^{\frac{1}{n}}$ and so we need only prove the remaining two inequalities,

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf_{n\to\infty} (a_n)^{\frac{1}{n}} \text{ and } \limsup_{n\to\infty} (a_n)^{\frac{1}{n}} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Consider the set $S_n = \left\{ \frac{a_{n+1}}{a_n}, \frac{a_{n+2}}{a_{n+1}}, \cdots \right\} = \left\{ \frac{a_{j+1}}{a_j} : j = n, n+1, \cdots \right\}$.

If $S_1 = \left\{ \frac{a_2}{a_1}, \frac{a_3}{a_2}, \cdots \right\} = \left\{ \frac{a_{j+1}}{a_j} : j = 1, 2, \cdots \right\}$ is not bounded above, then S_n is not bounded above for all integer $n \ge 1$. Hence sup S_n does not exist for each integer $n \ge 1$. By definition, $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$. Plainly we have $\limsup_{n\to\infty} (a_n)^{\frac{1}{n}} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$.

If S_1 is bounded above, then S_n is bounded above for all integer $n \ge 1$. Thus, by the completeness property of **R**, sup S_n exists for all integer $n \ge 1$. In this case we let $x_n = \sup S_n$. Since $S_n \supseteq S_{n+1}$, $x_n = \sup S_n \ge \sup S_{n+1} = x_{n+1}$ for each integer $n \ge 1$. Therefore, (x_n) is a decreasing sequence, which is obviously bounded below by 0. It follows by the Monotone Convergence Theorem, that (x_n) is convergent and converges to inf $\{x_1, x_2, \ldots \}$. This limit is defined to be $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$. That is $\lim_{n\to\infty} x_n = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = k$. Then $k \ge 0$. So given any $\varepsilon > 0$, there exists an integer N in $\boldsymbol{P}$, such that for any integer n, $n \ge N \Rightarrow |x_n - k| < \varepsilon$. I.e., $x_n < k + \varepsilon$ for all $n \ge N$. Therefore, for all $n \ge N$,

$$\frac{a_{n+1}}{a_n} \le \sup S_n = x_n < k + \varepsilon.$$

This means $a_{n+1} \le (k + \varepsilon)a_n$ for integer $n \ge N$. Iterated application of this inequality gives

$$a_n \le (k+\varepsilon)^{n-N} a_N$$

for all integer $n \ge N$. It then follows that for all integer $n \ge N$.

$$(a_n)^{\frac{1}{n}} \le (k+\varepsilon)^{1-\frac{N}{n}} (a_N)^{\frac{1}{n}}.$$

Note that both $\left\{ (a_N)^{\frac{1}{j}} : j \in \boldsymbol{P} \right\}$ and $\left\{ (k+\varepsilon)^{1-\frac{N}{j}} : j \in \boldsymbol{P} \right\}$ are bounded since the terms form two convergent sequences. Thus, for all integer $n \ge N$,

$$\sup\{(a_n)^{\frac{1}{n}}, (a_{n+1})^{\frac{1}{n+1}}, \ldots\} \le \sup_{j \ge n} (k+\varepsilon)^{1-\frac{N}{j}} (a_N)^{\frac{1}{j}}.$$

Therefore,

$$\limsup_{n\to\infty}(a_n)^{\frac{1}{n}} \le \limsup_{j\to\infty}(k+\varepsilon)^{1-\frac{N}{j}}(a_N)^{\frac{1}{j}} = \lim_{j\to\infty}(k+\varepsilon)^{1-\frac{N}{j}}(a_N)^{\frac{1}{j}} = k+\varepsilon, \text{ --- } (1)$$

since $(a_N)^{\frac{1}{j}} \to 1$ as $j\to\infty$ as $a_N > 0$ and $(k+\varepsilon)^{1-\frac{N}{j}} \to (k+\varepsilon)^1$ as $j\to\infty$.
Since (1) is true for any ε > 0, we have then

$$\limsup_{n\to\infty}(a_n)^{\frac{1}{n}} \le k = \limsup_{n\to\infty}\frac{a_{n+1}}{a_n}.$$

Now we turn our attention to the remaining inequality. Plainly S_n is bounded below. Thus, by the completeness property of the real numbers, infimum of S_n exists. Let $y_n = \inf S_n$, the infimum of S_n. Observe that (y_n) is an increasing sequence. Note that by definition, $\liminf_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty} y_n$.

If the limit does not exist, then $\lim_{n\to\infty} y_n = \infty$, and { y_n : $n \in$ ***P***) is unbounded, more specifically not bounded above. It follows that S_n is not bounded above for each integer n in ***P***. Hence given any $K > 0$, there exists an integer M such that $n \ge M \Rightarrow \frac{a_{n+1}}{a_n} \ge y_n > K$. Therefore, for $n \ge M$,

$$a_n \ge a_M K^{n-M}.$$

Thus, for all $j \ge n \ge M$,

$$(a_j)^{\frac{1}{j}} \ge (a_M)^{\frac{1}{j}}(K)^{1-\frac{M}{j}} \ge \inf_{j\ge n}(a_M)^{\frac{1}{j}}(K)^{1-\frac{M}{j}}.$$

This means

$$\inf\{(a_n)^{\frac{1}{n}}, (a_{n+1})^{\frac{1}{n+1}}, \ldots\} \ge \inf\{a_M^{\frac{1}{n}}(K)^{1-\frac{M}{n}}, (a_M)^{\frac{1}{n+1}}(K)^{1-\frac{M}{n+1}}, \ldots\}.$$

Since, $\lim_{n\to\infty}\inf\{a_M^{\frac{1}{n}}(K)^{1-\frac{M}{n}}, (a_M)^{\frac{1}{n+1}}(K)^{1-\frac{M}{n+1}}, \ldots\} = \lim_{n\to\infty} a_M^{\frac{1}{n}}(K)^{1-\frac{M}{n}} = K > 0$, there exists an integer N such that for $j \ge N$, $\inf\{a_M^{\frac{1}{j}}(K)^{1-\frac{M}{j}}, (a_M)^{\frac{1}{j+1}}(K)^{1-\frac{M}{j+1}}, \ldots\} > K/2$. It follows that, for $j \ge N$, $\inf\{(a_j)^{\frac{1}{j}}, (a_{j+1})^{\frac{1}{j+1}}, \ldots\} > K/2$.
Hence, $\lim_{n\to\infty}\inf\{(a_n)^{\frac{1}{n}}, (a_{n+1})^{\frac{1}{n+1}}, \ldots\} = \infty$.
Therefore, $\liminf_{n\to\infty}\frac{a_{n+1}}{a_n} = \liminf_{n\to\infty}(a_n)^{\frac{1}{n}} = \limsup_{n\to\infty}(a_n)^{\frac{1}{n}} = \limsup_{n\to\infty}\frac{a_{n+1}}{a_n} = \infty$.
Suppose now that (y_n) is convergent, or equivalently that it is bounded above, let $\lim_{n\to\infty} y_n = q$. Then $q \ge 0$. If $q = 0$, we have nothing to prove. Assume now $q > 0$. Then for any $\varepsilon > 0$, there exists an integer N such that for all integer n in ***P***, $n \ge N \Rightarrow |y_n - q| < \varepsilon$. Thus for any $\varepsilon > 0$ such that $0 < \varepsilon < q/2$, $n \ge N \Rightarrow q + \varepsilon > y_n > q - \varepsilon > q/2 > 0$. Thus,

$$n \ge N \Rightarrow \frac{a_{n+1}}{a_n} \ge \inf\{\frac{a_{n+1}}{a_n}, \frac{a_{n+2}}{a_{n+1}}, \ldots\} = y_n > q-\varepsilon.$$

Hence, $n \ge N \Rightarrow a_{n+1} > (q-\varepsilon)a_n$. Therefore, $n \ge N \Rightarrow a_n \ge a_N(q-\varepsilon)^{n-N}$. So taking n-th root we get

$$n \ge N \Rightarrow (a_n)^{\frac{1}{n}} \ge (a_N)^{\frac{1}{n}}(q-\varepsilon)^{1-\frac{N}{n}}. \text{ ------------------ } (2)$$

From (2) we deduce that for $n \ge N$,

$$j \ge n \Rightarrow (a_j)^{\frac{1}{j}} \ge (a_N)^{\frac{1}{j}}(q-\varepsilon)^{1-\frac{N}{j}} \ge \inf\{(a_N)^{\frac{1}{n}}(q-\varepsilon)^{1-\frac{N}{n}}, (a_N)^{\frac{1}{n+1}}(q-\varepsilon)^{1-\frac{N}{n+1}}, \ldots\}.$$

It follows that for $n \ge N$,

$$\inf\{(a_n)^{\frac{1}{n}}, (a_{n+1})^{\frac{1}{n+1}}, \ldots\} \ge \inf\{(a_N)^{\frac{1}{n}}(q-\varepsilon)^{1-\frac{N}{n}}, (a_N)^{\frac{1}{n+1}}(q-\varepsilon)^{1-\frac{N}{n+1}}, \ldots\}.$$

Therefore,

$$\liminf_{n\to\infty}(a_n)^{\frac{1}{n}} \ge \liminf_{n\to\infty}(a_N)^{\frac{1}{n}}(q-\varepsilon)^{1-\frac{N}{n}} = q-\varepsilon$$

since $(a_N)^{\frac{1}{n}} \to 1$ as $n\to\infty$ and $(q-\varepsilon)^{1-\frac{N}{n}} \to q-\varepsilon$ as $n\to\infty$. Since this is true for arbitrary small $\varepsilon > 0$, $\liminf_{n\to\infty}(a_n)^{\frac{1}{n}} \ge q$.

Therefore, $\liminf_{n\to\infty} \dfrac{a_{n+1}}{a_n} = q \le \liminf_{n\to\infty} (a_n)^{\frac{1}{n}}$. This completes the proof.

This inequality shows that the Cauchy Hadamard formula is a sharper formula for the computation of the radius of convergence. It is possible that $\limsup_{n\to\infty} (|a_n|)^{\frac{1}{n}}$ exists but $\limsup_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ or $\liminf_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ does not exist, as the next example shows.

Example 21. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series where the terms a_n is defined by $a_{2k-1} = (\frac{1}{2})^{2k-1}$ for $k \ge 1$ and $a_{2k} = (\frac{1}{4})^{2k}$. Note that $a_n > 0$ for all integer $n \ge 0$.

Then $\dfrac{a_{n+1}}{a_n} = \begin{cases} \dfrac{(\frac{1}{4})^{n+1}}{(\frac{1}{2})^n} \text{ if } n \text{ is odd} \\ \dfrac{(\frac{1}{2})^{n+1}}{(\frac{1}{4})^n} \text{ if } n \text{ is even} \end{cases} = \begin{cases} \frac{1}{4}(\frac{1}{2})^n \text{ if } n \text{ is odd} \\ \frac{1}{2}(2)^n \text{ if } n \text{ is even} \end{cases}$.

Plainly, $\inf\{\dfrac{a_{n+1}}{a_n}, \dfrac{a_{n+2}}{a_{n+1}}, \dots\} = 0$ and $\sup\{\dfrac{a_{n+1}}{a_n}, \dfrac{a_{n+2}}{a_{n+1}}, \dots\} = \infty$ and so $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ does not exist. However, $(a_n)^{\frac{1}{n}} = \begin{cases} \frac{1}{2} \text{ if } n \text{ is odd} \\ \frac{1}{4} \text{ if } n \text{ is even} \end{cases}$. Thus $\sup\{(a_n)^{\frac{1}{n}}, (a_{n+1})^{\frac{1}{n+1}}, \dots\} = \dfrac{1}{2}$. Therefore, $\limsup_{n\to\infty} (|a_n|)^{\frac{1}{n}} = \limsup_{n\to\infty} (a_n)^{\frac{1}{n}} = \dfrac{1}{2}$. Hence the radius of convergence of the power series is 2 by the Cauchy Hadamard formula. Observe that $\limsup_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \infty$ and $\liminf_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 0$ so that Theorem 22 (Refined Ratio Test) of Chapter 6 cannot be applied to give any conclusion.

Exercises 22.

1. Find the radius of convergence of the following power series $\sum_{n=1}^{\infty} a_n x^n$, where $a_n =$ (i) n^2 ; (ii) $1/n$; (iii) $1/n^2$; (iv) 2^n ; (v) $2^n/n$; (vi) $1/3^n$; (vii) $\dfrac{n+1}{2^n+n}$; (viii) $\dfrac{(2n)!}{(n!)^2}$.

2. Use the Cauchy Hadamard formula to show that the three series
$$\sum_{n=1}^{\infty} a_n x^n \quad , \quad \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ and } \sum_{n=1}^{\infty} a_n \frac{x^{n+1}}{n+1}$$
have the same radius of convergence.

3. Compare the regions of convergence of the three (real) power series (say if they are the same and state the precise region of convergence).
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

$$\frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - 5x^4 + \cdots$$

4. (i) Give an example of a power series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence 1, which is divergent at each point on the circle of convergence (i.e., the boundary of the disk of convergence)..

(ii) Give an example of a power series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence 1, which is divergent at some points on the circle of convergence and divergent at other points.

(iii) Give an example of a power series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence 1, which is convergent at each point on the circle of convergence.