Chapter Six. Series

We have seen sequences in chapter 2. Starting from a sequence we can form a new sequence by taking its partial sums. Such a sequence is called a series and series are important enough to gain a special place of interest. We shall study series of real numbers and investigate criterion or tests for convergence or divergence. If a series increases extremely slowly, some of the tests will fail to determine its convergence or divergence and refined tests are required. We shall give some refined tests and other special tests in a later chapter.

6.1 Definition and Convergence

Definition 1. Suppose (a_n) is a sequence. We can form the *series*

$$a_1 + a_2 + a_3 + \ \dots$$

More specifically, an (infinite) *series* consists of

(1) a sequence (a_n),

(2) the sequence (s_n) of partial sums, where $s_n = \sum_{k=1}^{n} a_k$.

The term a_n is called the n-th *term* of the series and s_n the n-th *partial sum* of the series.

If (s_n) converges to a real number S, then we say the series *converges* to S and we write

$$\sum a_n = S \text{ or } \sum_{n=1}^{\infty} a_n = S \text{ or } a_1 + a_2 + \cdots = S.$$

If (s_n) is divergent, then we say the series is *divergent*. If (s_n) is divergent and s_n tends to $\pm\infty$, then we say the series is *properly divergent*.

We usually write $\sum a_n$ or $a_1 + a_2 + a_3 + \ \dots$ for the series.

Example 2. The series $c + c + c + \ \dots$ converges if and only if $c = 0$.

The series $c + c + c + \ \dots$ is of course given by $\sum a_n$ with $a_n = c$ for all integers n in $\boldsymbol{P}$. Therefore, the n-th partial sum $s_n = \sum_{k=1}^{n} a_k = nc$. If $c \neq 0$, then $(\boldsymbol{s_n})$ is divergent. We shall explain this below. If $c > 0$ and $s_n \to a$, we shall then deduce a contradiction. Since $s_n = nc \geq c > 0$, $a > 0$. If $s_n \to a$, then by definition given any $\varepsilon > 0$, there exists positive integer N such that for all n in $\boldsymbol{P}$,

$$n \geq N \Rightarrow |s_n - a| = |nc - a| < \varepsilon.$$

Take $\varepsilon = c > 0$. Then by the above argument, there is an integer N such that for all n in $\boldsymbol{P}$,

$$n \geq N \Rightarrow |s_n - a| = |nc - a| < c. \qquad \text{----------------} \quad (1)$$

Now by the Archimedean property of **R**, there exists an integer N_0 such that N_0 $c > a + c$.

Take any integer $n \geq \max(N, N_0)$. Then $nc \geq N_0\, c > a + c$. Hence $|s_n - a| = |nc - a| = nc - a > c$.

But since $n \ge N$, by (1), we have that $|s_n - a| = |nc - a| < c$ and this contradicts $|s_n - a| > c$. This means (s_n) is divergent. If $c < 0$, then as shown above $(-s_n) = (n(-c))$ is divergent. Hence (s_n) is divergent. We have thus shown that (s_n) is divergent if $c \ne 0$. Plainly, when $c = 0$, (s_n) is convergent since each s_n is equal to 0.

The above argument is trivial. It is more instructive to use the fact that any convergent sequence is bounded. (Reference: Theorem 11 of Chapter 2 Sequences.). Let this result work for you. Note that if $c \ne 0$, then the n-th partial sums $\{s_n : n \in \boldsymbol{P}\}$ is not bounded. This is deduced by invoking the Archimedean property of **R**. Take any real number $K > 0$. Then by the Archimedean property of **R**, there exists a positive integer N such that $|s_N| = |Nc| = N|c| > K$. Hence $\{s_n : n \in \boldsymbol{P}\}$ is not bounded. It follows that if $c \ne 0$, then (s_n) cannot be convergent for otherwise $\{s_n : n \in \boldsymbol{P}\}$ will be bounded.

Example 3. The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Here $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Thus the n-th partial sum,

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n$$
$$= (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1} \to 1 - 0 = 1.$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Remark. "=" sign here has meaning different from the usual equality as in " 2 + 2 = 4". It means the limit of the series is 1. Here is an example of a different use of the equality sign.

Example 4. Geometric Series.

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + \cdots$$

converges to $\frac{1}{1-a}$ if $|a| < 1$.

Here we deviate from the previous indexing convention, where the terms are indexed starting from the number 1. We define here for a sequence (a_n) indexed by non-negative integers, starting with a_0, the n-th partial sum s_n to be $\sum_{k=0}^{n-1} a_k = a_0 + a_1 + \cdots a_{n-1}$. This change in the starting or beginning of the index will not alter the theory at all. We can in effect transform this series to the usual form by letting $b_k = a_{k-1}$ for $k = 1, 2, \ldots$. Then $\sum_{k=0}^{n-1} a_k = \sum_{k=1}^{n} b_k$.

We begin by letting $c_k = a^k$ and $s_n = c_o + c_1 + \ldots + c_{n-1}$.

Then $s_n = 1 + a + a^2 + \cdots + a^{n-1} = \frac{(1 + a + a^2 + \cdots + a^{n-1})(1-a)}{1-a}$, if $a \ne 1$

$$= \frac{1 - a^n}{1-a} = \frac{1}{1-a} - \frac{a^n}{1-a} = \frac{1}{1-a} - \frac{c_n}{1-a} \text{ if } |a| < 1.$$

Now if $|a| < 1$, then the sequence (c_n) = (a^n) is convergent and $c_n = a^n \to 0$. Therefore, if $|a| < 1$, $s_n = \frac{1}{1-a} - \frac{c_n}{1-a} \to \frac{1}{1-a} - 0 = \frac{1}{1-a}$. Thus, we conclude that if $|a| < 1$, the geometric series $\sum_{n=0}^{\infty} a^n$ is convergent and converges to $\frac{1}{1-a}$.

If $|a| > 1$, then the sequence (c_n) = (a^n) is divergent as it is unbounded. Therefore, $(s_n) = \left(\frac{1}{1-a} - \frac{c_n}{1-a}\right)$ is also divergent. Hence if $|a| > 1$, the geometric series $\sum_{n=0}^{\infty} a^n$ is divergent. If $a = 1$, then $s_n = n$ and so the sequence (s_n) is divergent as again it is unbounded. Consequently, $\sum_{n=0}^{\infty} a^n$ is divergent when $a = 1$.

If $a = -1$, then $s_n = \begin{cases} 0, n \text{ even}, n \geq 2 \\ 1, \ n \text{ odd}, n \geq 1 \end{cases}$ and (s_n) is divergent. It is easily seen that (s_n) is not Cauchy. (For instance, for any positive integer N, just take any even $n > N$ and odd $m > N$, then $|s_n - s_m| = 1 > 1/2$. This shows that it is not Cauchy.) Therefore, by Cauchy Principle of Convergence, (s_n) is divergent. Thus, the geometric series $\sum_{n=0}^{\infty} a^n$ is convergent if and only if $|a| < 1$.

Remark 5. Series can start from any term of a sequence (a_n). For example,

$a_0 + a_1 + \ldots$ (Start with a_0 .)
$a_1 + a_2 + \ldots$ (Start with a_1.)
$a_2 + a_3 + \ldots$ (Start with a_2.)

are three series with terms from the same sequence but starting with different terms. We can write the above series in the summation notation, $\sum_{n=0}^{\infty} a_n$, $\sum_{n=1}^{\infty} a_n$, $\sum_{n=2}^{\infty} a_n$. It does not matter the series begins with which term of the sequence, it does not alter the theory in anyway. We can make some convention by requiring that s_n denotes the partial sum of n terms only. Hence, the n-th partial sum for the above three series will be given by

$s_n = a_0 + a_1 + \ldots + a_{n-1}$,
$s_n = a_1 + a_2 + \ldots + a_n$,
and $s_n = a_2 + a_3 + \ldots + a_{n+1}$.

If the series begins with the k-th term a_k , then $s_n = a_k + a_{k+1} + \ldots + a_{k+n-1}$. Plainly, if k and j are integers such that $k < j$, then $\sum_{n=k}^{\infty} a_n$ converges $\Leftrightarrow$ $\sum_{n=j}^{\infty} a_n$ converges.

Note that $\sum_{n=k}^{\infty} a_n = \sum_{n=k}^{j-1} a_n + \sum_{n=j}^{\infty} a_n$. Thus, if s_n is the n-th partial sum for $\sum_{n=k}^{\infty} a_n$ and t_n is the n-th partial sum for $\sum_{n=j}^{\infty} a_n$, for $n > j - k$,

$$s_n = a_k + a_{k+1} + \ldots + a_{k+n-1} = \sum_{q=k}^{k+n-1} a_q = \sum_{q=k}^{j-1} a_q + \sum_{q=j}^{j+(k+n-j-1)} a_q$$

$$= \sum_{q=k}^{j-1} a_q + a_j + a_{j+1} + \ldots + a_{j+(n-(j-k))-1} = \sum_{q=k}^{j-1} a_q + t_{n-(j-k)}. \quad \text{--------------} \quad (1)$$

Note that the sequence (s_n) converges $\Leftrightarrow$ (s_{n+j-k}) converges. But by (1) $s_{n+j-k} = \sum_{q=k}^{j} a_q + t_n$, and so (s_{n+j-k}) converges $\Leftrightarrow$ (t_n) converges. Therefore, (s_n) converges $\Leftrightarrow$ (t_n) converges. Thus, the behaviour regarding convergence does not depend on the beginning term of the series.

Since series can be regarded as a sequence, more specifically, its n-th partial sums form a sequence, properties for sequences can now be translated into properties for series .

Properties 7.
(1) If $\sum a_n$ converges then its sum is unique.
(2) If $\sum a_n = a$ and $\sum b_n = b$, then $\sum (a_n + b_n) = a + b$.
(3) If $\sum a_n = a$, then $\sum \lambda a_n = \lambda a$.
(4) For a complex series, $\sum a_n$ converges $\Leftrightarrow \sum Re\, a_n$ and $\sum Im\, a_n$ converges.
(Here for a complex number z, $z = Re\, z + i\, Im\, z$.)
If $\sum a_n = a$, then $\sum Re\, a_n = Re\, a$ and $\sum Im\, a_n = Im\, a$.

(1) is just an assertion about uniqueness of limit.
(2) follows from (1) of Properties 7 page 4 Chapter 2 Sequences.
(3) follows from (2) of Properties 7 page 4 Chapter 2 Sequences.
(4) follows from the remark after the Squeeze Theorem for sequences , Theorem 13 of Chapter 2 Sequences, page 6.

6.2 Cauchy Series.

We shall translate Cauchy principle of convergence for sequences to a principle of convergence for series.

Definition 8. $\sum a_n$ is a *Cauchy series* if the partial sum (s_n) is a Cauchy sequence. I.e., if given $\varepsilon > 0$, there exists an integer N such that

$$m>n \geq N \Rightarrow |s_n - s_m| < \varepsilon \Rightarrow \left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon,$$

This is equivalent to saying that there exists an integer N such that for all $n \geq N$ and for all positive integer p, $\left| \sum_{n+1}^{n+p} a_k \right| < \varepsilon$

Then we have the principle of convergence for series.

Theorem 9. $\sum a_n$ is convergent if and only if $\sum a_n$ is Cauchy.

Proof. This theorem is just a restatement of the Cauchy principle of convergence for the n-th partial sum sequence. This is also true of complex series, indeed for any *complete normed space* such as $\mathbf{R}^n$. The theorem for the real case follows from Theorem 20 of Chapter 2. For the complex case, just observe that by Property 7 (4), $\sum a_n$ is convergent if and only if its real and imaginary parts are convergent if and only if its real and imaginary parts are Cauchy if and only if $\sum a_n$ is Cauchy.

Remark. We use this theorem to prove most of the results about series. In practice, we rarely know what the sum of the series is.

The next result is a useful means of deciding when a series does not converge, specifically it gives a necessary condition for convergence in terms of the terms of the series.

Proposition 10. If $\sum a_n$ converges, then $a_n \to 0$.

Proof. If $\sum a_n$ converges, then $\sum a_n$ is Cauchy. Then by definition 8, given $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$ and for all positive integer p, $\left|\sum_{n+1}^{n+p} a_k\right| < \varepsilon$. Taking $p = 1$, we have then that for all $n \geq N$, $|a_{n+1}| < \varepsilon$. This means that $a_n \to 0$.

Remark. Thus if (a_n) does not converge to 0, then $\sum a_n$ diverges.

Example
$\sum a^n$ is divergent if $|a| \geq 1$ since (a^n) does not converge to 0.

Remark.
1. By proposition 10, if (a_n) does not converge to 0, then $\sum a_n$ diverges

2. The converse of Proposition 10 is false.
   That is to say that if $a_n \to 0$, it need not follow that $\sum a^n$ is convergent.
   The following is a counterexample,
   $$\sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent even though } \frac{1}{n} \to 0.$$
   The n-th partial sum $s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is obviously an increasing sequence. We shall show that it is unbounded and so it is divergent. This is because if (s_n) is convergent, then it is bounded (reference: Theorem 11 of Chapter 2 Sequences). We shall look at a subsequence of (s_n) form by successively doubling the number of terms. The first few terms of this subsequence are:

   $s_1 = 1$, $s_2 = 1+ 1/2$, $s_4 = 1 + 1/2 + (1/3+1/4) > 1 + 1/2 + 1/2$
   $s_8 = 1 + 1/2 + (1/3+1/4) + (1/5+1/6+1/7+1/8) > 1 + 1/2 + 1/2 + 1/2$
   ....

   This subsequence is (s_{2^n}). $s_{2^{n+1}}$ is obtained from (s_{2^n}) by adding the next 2^n terms. Therefore,
   $$\begin{aligned} s_{2^{n+1}} &= s_{2^n} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n} \\ &\geq s_{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}} = s_{2^n} + 2^n \cdot \frac{1}{2^{n+1}} = s_{2^n} + \frac{1}{2}. \end{aligned}$$
   Thus we have for each integer $n \geq 0$,
   $$s_{2^{n+1}} \geq s_{2^n} + \frac{1}{2}.$$
   Repeated use of this inequality yields,
   $$\begin{aligned} s_{2^{n+1}} &\geq s_{2^n} + \frac{1}{2} \\ &\geq s_{2^{n-1}} + \frac{1}{2} + \frac{1}{2} \geq \cdots \geq s_2 + (n-1)\frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2} + \frac{n}{2} = 1 + (n+1)\frac{1}{2}. \end{aligned}$$

Hence, for any positive integer n, $s_{2^n} \geq 1 + \frac{1}{2}n$. Take any real number $K > 0$. By the Archimedean property of **R**, there exists a positive integer N such that $N \cdot \frac{1}{2} > K$. Therefore, $s_{2^N} \geq 1 + \frac{1}{2}N > K$. Hence taking $M = 2^N$, $n > M \Rightarrow s_n > s_M \geq 1 + \frac{1}{2}N > K$. Thus, (s_n) is unbounded. Consequently, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

6.3 Series of Non-Negative Terms

Now we shall investigate a few tests for convergence.
We next state two results about series of *non-negative* terms. For such series, the n-th partial sums (s_n) is an increasing sequence. The first result is a consequence of the Monotone Convergence Theorem (Theorem 15 Chapter 2. Sequences).

Proposition 11. Suppose $\sum a_n$ is a series of real non-negative terms. Then $\sum a_n$ is convergent if and only if (s_n) is bounded.

Proof. Since for each integer $n \geq 1$, $a_n \geq 0$,$s_{n+1} = a_1 + a_2 + \ldots + a_n + a_{n+1} \geq a_1 + a_2 + \ldots + a_n = s_n$ for each positive integer n. Therefore, (s_n) is an increasing sequence. Therefore, by the Bounded Monotone Convergence Theorem (Theorem 15, Chapter 2. Sequences), if (s_n) is bounded, (s_n) is convergent and so the series $\sum a_n$ is convergent. Conversely, if (s_n) is convergent, then (s_n) is bounded (by Theorem 11 Chapter 2 Sequences) .

Proposition 12 (Comparison Test)
Let $\sum a_n$ and $\sum b_n$ be two series of real non-negative terms such that

$$a_n \leq \lambda\, b_n,$$

for all positive integer n and for some positive real number λ.
(1) If $\sum b_n$ is convergent, then $\sum a_n$ converges.
(2) If $\sum a_n$ is divergent, then $\sum b_n$ diverges.

Proof.
(1) By proposition 11, $\sum b_n$ converges if and only if its n-th partial sums (s_n) is bounded. Thus if $\sum b_n$ is convergent, then its n-th partial sums (s_n) is bounded, say by $K > 0$. I.e., $s_n \leq K$.
Now, for each positive integer n, $\sum_{k=1}^{n} a_k \leq \lambda \sum_{k=1}^{n} a_k = \lambda\, s_n$, since $a_n \leq \lambda\, b_n$. Therefore, if we let $t_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$ be the n-th partial sum for the series $\sum a_n$, $t_n = \sum_{k=1}^{n} a_k \leq \lambda\, s_n \leq \lambda K$ for all positive integer n. That means (t_n) is bounded and so by Proposition 11, $\sum a_n$ is convergent.
(2) If $\sum a_n$ is divergent, then by Proposition 11, (t_n) is unbounded. Note that, for each positive integer n, $s_n \geq \frac{1}{\lambda} t_n$. Thus since (t_n) is unbounded, given any $L > 0$, there exists an integer N such that $n \geq N \Rightarrow t_n > \lambda\, L$. Hence, $n \geq N \Rightarrow s_n \geq \frac{1}{\lambda} t_n > L$. Thus (s_n) is unbounded and so by Proposition 11, the series $\sum b_n$ is divergent.

Example 13. $\sum \frac{1}{n^2}$ is convergent.

Since $\frac{1}{(n+1)^2} \le \frac{1}{n(n+1)}$ and $\sum \frac{1}{n(n+1)}$ is convergent (see Example 3), so by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent . Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent.

Proposition 14. Suppose $\sum |a_n|$ is convergent. Then $\sum a_n$ converges.

Proof. We shall use Cauchy Convergence Principle for Series.
Proof is just simply observing that if $\sum |a_n|$ is Cauchy, then so is $\sum a_n$. If $\sum |a_n|$ is convergent, then $\sum |a_n|$ is Cauchy by Theorem 9. That means given any $\varepsilon > 0$, there exists an integer N, such that for all $n \ge N$ and for all positive integer p, $\sum_{n+1}^{n+p} |a_k| < \varepsilon$. Therefore, since $\left| \sum_{n+1}^{n+p} a_k \right| \le \sum_{n+1}^{n+p} |a_k|$, by the triangle inequality, for all $n \ge N$ and for all positive integer p, $\left| \sum_{n+1}^{n+p} a_k \right| \le \sum_{n+1}^{n+p} |a_k| < \varepsilon$. Hence $\sum a_n$ is Cauchy. Therefore, by Theorem 9, $\sum a_n$ is convergent.

Remark.
1. An useful equivalent statement for Proposition 14 is:

   If $\sum a_n$ diverges, then $\sum |a_n|$ is divergent.
2. The converse of Proposition 14 is not true (see Example 16 below).

Definition 15. We say the series $\sum a_n$ *converges absolutely* if $\sum |a_n|$ is convergent.

Example 16. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent but not absolutely. (We shall show later that it converges by *Alternating Series Test*.)

Remark. Most tests are tests for absolute convergence. Plainly any test for non-negative series gives a test for $\sum |a_n|$. The next result illustrates this point.

Proposition 17. Suppose (a_n) is a bounded sequence. Then $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converges.

Proof. The sequence (a_n) is bounded implies that there exists a real number $M > 0$ such that $|a_n| \le M$ for all positive integer n. Thus, we have for all positive integer n, $0 \le \left| \frac{a_n}{n^2} \right| \le \frac{M}{n^2}$. Therefore, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (see Example 13), by the Comparison Test (Proposition 12), $\sum_{n=1}^{\infty} \left| \frac{a_n}{n^2} \right|$ converges. It follows then by Proposition 14 that $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ is convergent.

Example 18. $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ is absolutely convergent (and therefore convergent) for any x by Proposition 17.

Definition 19. If the series $\sum a_n$ is such that $\sum a_n$ is convergent but $\sum |a_n|$ is divergent, we say the series $\sum a_n$ is *conditionally convergent*.

Hence the series in Example 16 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

6.4 Alternating Series Test

Now we come to the alternating series test which can be apply to the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

.

Theorem 20 (Alternating Series Test, Leibnitz's Test)
If (a_n) is a monotone decreasing, non-negative sequence and $a_n \to 0$, then $\sum (-1)^{n+1} a_n$ is convergent.

Proof. We shall show that $\sum (-1)^{n+1} a_n$ is Cauchy.
There is also a proof making use of the fact that $s_{2n} = s_{2n-1} - a_{2n}$, both (s_{2n}) and (s_{2n-1}) are bounded and monotone and so are convergent and that $a_{2n} \to 0$.
Suppose m and n are positive integers such that $m > n$.

$$\left| \sum_{k=n}^{m} (-1)^{k+1} a_k \right| = \left| (-1)^{n+1} a_n + (-1)^{n+2} a_{n+1} + \cdots + (-1)^{m+1} a_m \right|$$

$$= \left| a_n - a_{n+1} + \cdots \pm a_m \right|$$

$$\leq \begin{cases} |a_n - a_{n+1}| + |a_{n+2} - a_{n+3}| + \cdots + |a_{m-1} - a_m| \text{ if } m - n \text{ is odd} \\ |a_n - a_{n+1}| + |a_{n+2} - a_{n+3}| + \cdots + |a_{m-2} - a_{m-1}| + |a_m| \text{ if } m - n \text{ is even} \end{cases}$$

$$\leq \begin{cases} (a_n - a_{n+1}) + (a_{n+2} - a_{n+3}) + \cdots + (a_{m-1} - a_m) \text{ if } m - n \text{ is odd} \\ (a_n - a_{n+1}) + (a_{n+2} - a_{n+3}) + \cdots + (a_{m-2} - a_{m-1}) + a_m \text{ if } m - n \text{ is even} \end{cases}$$

$$= \begin{cases} a_n - (a_{n+1} - a_{n+2}) - (a_{n+3} - a_{n+4}) - \cdots - (a_{m-2} - a_{m-1}) - a_m \text{ if } m - n \text{ is odd} \\ a_n - (a_{n+1} - a_{n+2}) - (a_{n+3} - a_{n+4}) - \cdots - (a_{m-1} - a_m) \text{ if } m - n \text{ is even} \end{cases}$$

$$\leq a_n, \qquad \text{---} \quad (1)$$

since (a_n) is a monotone decreasing and non-negative sequence.
Now since $a_n \to 0$, given $\varepsilon > 0$ there exists integer N in $\boldsymbol{P}$ such that $n \geq N \Rightarrow |a_n| = a_n < \varepsilon$. Therefore, by (1), for $n \geq N$ and any $m > n$,

$$\left| \sum_{k=n}^{m} (-1)^{k+1} a_k \right| \leq a_n < \varepsilon.$$

Thus the series $\sum (-1)^{n+1} a_n$ is Cauchy and so is convergent by Theorem 9 (Cauchy Principle of Convergence for Series). This completes the proof.

Alternatively, we may use the following technique: if the subsequence formed by the terms of the sequence indexed by even integers is convergent and the subsequence formed by the terms indexed by the odd integers is also convergent and both subsequences converge to the same limit, then the sequence is convergent.

Let s_n denote the n-th partial sum of the series. So we shall look at the following subsequences (s_{2n}) and (s_{2n-1}). Now for each integer $n \geq 1$,

$$s_{2n+1} - s_{2n-1} = -a_{2n}+a_{2n+1} \leq 0.$$

Therefore, (s_{2n-1}) is a decreasing sequence. Observe that for each integer $n \geq 1$,

$$\begin{aligned} s_{2n-1} &= a_1 - a_2 + a_3 - \dots - a_{2n-2} + a_{2n-1} \\ &= (a_1 - a_2) + (a_3 - a_4) + \dots (a_{2n-3} - a_{2n-2}) + a_{2n-1} \\ &\geq a_{2n-1} \geq 0, \end{aligned}$$

since each of the bracketed terms are greater or equal to 0.
Hence (s_{2n-1}) is a decreasing sequence bounded below by 0. Therefore, by the Monotone Convergence Theorem (Theorem 15, Chapter 2 Sequences), (s_{2n-1}) is convergent. Similarly, for each integer $n \geq 1$,

$$s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0.$$

Thus, (s_{2n}) is an increasing sequence. Next we shall show that it is bounded above. Note that

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - \dots - a_{2n-2} + a_{2n-1} - a_{2n} \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2n-2} - a_{2n-1}) - a_{2n} \\ &\leq a_1, \end{aligned}$$

since each of the bracketed terms are greater or equal to 0. Therefore, (s_{2n}) is an increasing sequence bounded above by a_1. Hence by the Monotone Convergence Theorem (Theorem 15, Chapter 2 Sequences), (s_{2n}) is convergent.
Now

$$s_{2n} = s_{2n-1} - a_{2n}.$$

Therefore, $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n-1} - \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} s_{2n-1} - 0 = \lim_{n\to\infty} s_{2n-1}$, since $\lim_{n\to\infty} a_n = 0$.
(If $a_n \to 0$, then all subsequence of (a_n) also converges to 0. This can be proved easily. See Proposition 19 of Chapter 3.) Hence, both (s_{2n}) and (s_{2n-1}) converge to the same limit and so (s_n) is convergent and that means the series $\sum (-1)^{n+1} a_n$ is convergent.

6.5 The Ratio Test

The next test we shall give is one of the most important test for series. D'Alembert gave the absolute convergence part of the ratio test in 1768 in Opuscules mathématiques, 5. But it was Edward Waring (1734-98) who gave in 1776 the now well known ratio test for convergence and attributed to Cauchy. We shall give first a simplified version.

Theorem 21 (Ratio Test, D'Alembert's Test)
Let $\sum a_n$ be a series..
Then (i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \alpha < 1$, then $\sum a_n$ is absolutely convergent (hence convergent).
(ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \alpha > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then $\sum a_n$ is divergent.
(iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \alpha = 1$, then $\sum a_n$ may converge or diverge. No inference can be made. The convergence may be investigated by other methods.

Proof.
(i) Suppose $\alpha < 1$. Choose a real number c such that $\alpha < c < 1$. Let $\varepsilon = c - \alpha$. Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = a$, there exists a positive integer N such that for all integer n,

$$n \geq N \Rightarrow \alpha - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right| < \alpha + \varepsilon = c \qquad \text{-------------------- } (*)$$

Therefore, for all integer $n \geq N$,

$$|a_{n+1}| < |a_n|\, c. \qquad \text{------------------------------- } (1)$$

So let p be any positive integer. It then follows from (1) that

$$|a_{N+p}| < c\,|a_{N+p-1}| < c^2\,|a_{N+p-2}| < \ldots < c^{\,p}\,|a_N|.$$

Thus since $\sum_{p=1}^{\infty} c^p$ converges because $0 < c < 1$ (Example 4, Geometric Series) , by the Comparison Test (Proposition 12), $\sum_{p=1}^{\infty}|a_{N+p}|$ converges. Therefore, $\sum_{n=1}^{\infty}|a_n|$ is convergent. It follows that $\sum a_n$ is absolutely convergent (Reference: Proposition 14).

(ii) Suppose $\alpha > 1$. Choose a real number c such that $\alpha > c > 1$. Then as before, since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \alpha$, taking $\varepsilon = \alpha - c$, there exists a positive integer M such that for all integer n,

$$n \geq M \Rightarrow c = \alpha - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right| < \alpha + \varepsilon\,. \qquad \text{------------------- } (**)$$

It follows then that for all integer $n \geq M$,

$$|a_{n+1}| > c\,|a_n| \qquad \text{-------------------------------- } (2)$$

Thus for all positive integer p, using (2) we get,

$$|a_{M+p}| > c\,|a_{M+p-1}| > c^2\,|a_{M+p-2}| > \ldots > c^p\,|a_M|. \qquad \text{----------------- } (3)$$

Because $c > 1$, the sequence ($c^p\,|a_M|$) diverges since $|a_M| \neq 0$. In particular, $c^p\,|a_M| \to \infty$. Therefore, by (3), $|a_{M+p}| \to \infty$ as $p \to \infty$. Hence $a_{M+p} \nrightarrow 0$ as $p \to \infty$. Therefore, by Proposition 10, $\sum_{p=1}^{\infty} a_{M+p}$ diverges and it follows then that $\sum_{n=1}^{\infty} a_n$ is divergent.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then by definition, take any $c > 1$, there exists a positive integer M such that for all integer n, $..n \geq M \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| > c \Rightarrow |a_{n+1}| > c|a_n|$. We then proceed exactly as before using (2) to deduce that $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\alpha = 1$, no inference can be made. The example below will illustrate this point. $\sum 1/n$ is divergent and $\sum 1/n^2$ is convergent. Ratio test for both series gives α as 1.

Remark. We have actually proved a more refined version of the test since we only use one side of the inequality (*) or (**).
Note that $\lim_{n\leftarrow\infty}\left|\frac{a_{n+1}}{a_n}\right| = \alpha \Leftrightarrow \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \alpha$. (This is a special case of the result that $\lim_{n\leftarrow\infty} b_n = \beta \Leftrightarrow \limsup_{n\to\infty} b_n = \liminf_{n\to\infty} b_n = \beta$. This result is proved in the proof for Theorem 20 (Cauchy Principle of Convergence) Chapter 2 Sequences.
We give the refined version of this test below.

Theorem 22 (Ratio Test, D'Alembert's Test, Refined Version)

Let $\sum_{n=1}^{\infty} a_n$ be a series.

(i) If $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ is absolutely convergent.

(ii) If $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ is divergent.

Proof.

(i) $S_n = \left\{\left|\frac{a_{n+1}}{a_n}\right|, \left|\frac{a_{n+2}}{a_{n+1,}}\right|, \cdots\right\} = \left\{\left|\frac{a_{j+1}}{a_j}\right| : j = n, n+1, \cdots\right\}$. Let $x_n = \sup S_n$.

Then $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} x_n$. Thus $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \alpha < 1$ means $\lim_{n\to\infty} x_n = \alpha$.
Hence, given any $\varepsilon > 0$, there exists a positive integer N such that for all integer n,

$$n \ge N \Rightarrow \alpha - \varepsilon < x_n < \alpha + \varepsilon. \quad \text{------------------------} \quad (1)$$

As in the proof of Theorem 21, take any c such that $\alpha < c < 1$ and choose $\varepsilon = c - \alpha$. Then for any integer n,

$$n \ge N \Rightarrow x_n < \alpha + \varepsilon = c.$$

Therefore, $x_N = \sup\left\{\left|\frac{a_{N+1}}{a_N}\right|, \left|\frac{a_{N+2}}{a_{N+1},}\right|, \cdots\right\} < c$. It follows by the definition of x_N that for any integer $n \ge N$, $\left|\frac{a_{n+1}}{a_n}\right| \le \sup\left\{\left|\frac{a_{N+1}}{a_N}\right|, \left|\frac{a_{N+2}}{a_{N+1},}\right|, \cdots\right\} < c$. Hence,

$$n \ge N \Rightarrow |a_{n+1}| < |a_n|\, c \quad \text{------------------------------} \quad (2)$$

Using this as in the proof of Theorem 21 part (i), we can show in exactly the same way that $\sum a_n$ is absolutely convergent.

(ii) Let $y_n = \inf S_n$. Then $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} y_n$. Therefore, $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \beta > 1$ means $\lim_{n\to\infty} y_n = \beta > 1$. Hence, given any $\varepsilon > 0$, there exists a positive integer N such that for all integer n,

$$n \ge N \Rightarrow \beta - \varepsilon < y_n < \beta + \varepsilon. \quad \text{------------------------} \quad (3)$$

Let c be any real number such that $\beta > c > 1$. Choose $\varepsilon = \beta - c > 0$. It follows from (3) that for any integer n, $n \ge N \Rightarrow y_n > \beta - \varepsilon = c > 1$. Thus,

$$y_N = \inf\left\{\left|\frac{a_{N+1}}{a_N}\right|, \left|\frac{a_{N+2}}{a_{N+1},}\right|, \cdots\right\} > c.$$

Therefore, by the definition of infimum, for any integer n,

$$n \ge N \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \ge \inf\left\{\left|\frac{a_{N+1}}{a_N}\right|, \left|\frac{a_{N+2}}{a_{N+1},}\right|, \cdots\right\} > c.$$

Hence, $$n \ge N \Rightarrow |a_{n+1}| > |a_n|\, c \quad \text{------------------------} \quad (4)$$

Then, using (4), in exactly the same way as in the proof of Theorem 21 part (ii), we show that $\sum a_n$ is divergent.

If $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then there exists a positive integer N such that for all integer n, $n \ge N \Rightarrow y_n > c > 1$. We can now proceed in exactly the same way as above to show that $\sum a_n$ is divergent.

Remark. Theorem 22 is a refined form of Theorem 21 in the sense that we do not need the limit of $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$ to apply the test but just the $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ for absolute convergence and $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ for divergence.

Example 23.

1. $\sum_{n=1}^{\infty}\frac{1}{n!}$ is convergent. Let $a_n = \frac{1}{n!}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1/(n+1)!}{1/n!} = \frac{1}{n+1} \to 0 < 1.$$

Therefore, by Theorem 21, the series is convergent.

2. $\sum_{n=1}^{\infty} n^2x^n$ for $x > 0$. Let $a_n = n^2x^n$. Then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2x^{n+1}}{n^2x^n} = x$.

Thus by Theorem 21, $\sum_{n=1}^{\infty} n^2x^n$ is convergent for $0 < x < 1$ and is divergent for $x > 1$. The series is divergent for $x = 1$.

Example 24. Conditionally convergent series.

1. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent by Leibnitz's Alternating Series Test (Theorem 20). We apply the test as follows. Write the series as $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$. Then $a_n = \frac{1}{n}$ for each *integer* $n \geq 1$. Plainly the sequence $(a_n) = \left(\frac{1}{n}\right)$ is decreasing and $a_n \to 0$. Thus by Theorem 20, $\sum_{n=1}^{\infty}(-1)^{n+1} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent. But $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and so $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.
2. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ is convergent by Leibnitz's Alternating Series Test (Theorem 20) as the sequence $(a_n) = \left(\frac{1}{2n-1}\right)$ is decreasing and $a_n \to 0$. Now $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ is divergent by the Comparison Test (Proposition 12) since $\frac{1}{2n-1} > \frac{1}{2n}$ and $\sum_{n=1}^{\infty} \frac{1}{2n}$ is divergent. Hence $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ converges conditionally. Note that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$.

Remark.
1. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges to ln(2). This can be shown later by rewriting its n-th partial sum, introducing an Eulerian sequence (that converges to the Euler constant γ).
2. Both series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ can be computed by an argument in infinite power series as expansion of $ln(1+x)$ via integration term by term for $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ and in the case of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ expansion of $tan^{-1}(x)$ via integration term by term. (See Chapter 9 Power Series and Integration.)

6.6 The Integral Test

Our next test will make use of the Riemann integral. The test is phrased in terms of continuous function, which is always Riemann integrable on any closed and bounded interval, or in terms of monotone decreasing function (and hence is Riemann integrable on any closed and bounded interval).

Theorem 25. (The Integral Test).
Suppose $\sum_{n=1}^{\infty} a_n$ is a series. Suppose the n-th term of the series a_n can be expressed as $a_n = f(n)$, where f is a function defined at least on the interval $[1, \infty)$ such that
1. f is non-negative and
2. f is monotone decreasing

(hence (a_n) is non-negative and decreasing). Then

(i) $\sum_{n=1}^{\infty} a_n$ converges if the sequence $\left(\int_1^n f(x)dx\right)$ tends to a finite limit L as $n \to\infty$. In particular, the sum $\sum_{n=1}^{\infty} a_n$ lies between L and $L + a_1$.

(ii) $\sum_{n=1}^{\infty} a_n$ diverges if $\int_1^n f(x)dx \to \infty$ as $n \to\infty$.

Remark. The function f in Theorem 25 is usually continuous in pratice.
Note that f is non-negative implies that the sequence $\left(\int_1^n f(x)dx\right)$ is monotonically increasing. Hence $\left(\int_1^n f(x)dx\right)$ is convergent if and only if it is bounded (hence bounded above). We could replace the condition in (i) by stating the equivalent condition that $\left(\int_1^n f(x)dx\right)$ be bounded and (i) and (ii) may be stated simply as

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Leftrightarrow \left(\int_1^n f(x)dx\right) \text{ is bounded.}$$

We have assumed that f is Riemann integrable on $[1, n]$ for each positive integer n. Note that if f is monotone on $[1, n]$, then f is Riemann integrable on $[1, n]$. (See Theorem 22 Chapter 5 Integration.) Thus we do not really require f to be continuous. However, in practice it is useful to know that f is continuous so that the integral $\int_1^n f(x)dx$ may be computed readily by using any anti-derivative of f since the Fundamental Theorem of Calculus can be applied.

Proof of Theorem 25. Since $f : [1, \infty) \to \mathbf{R}$ is non-negative and decreasing, f is bounded and Riemann integrable by Theorem 22 of Chapter 5 Integration. It follows that f is Riemann integrable on $[1, n]$ for each integer $n \geq 1$. Moreover for any integer $k \geq 1$ and for any x in $[k, k+1]$,

$$a_k = f(k) \geq f(x) \geq f(k+1) = a_{k+1} .$$

Hence, for any integer $k \geq 1$, $f(k+1) = a_{k+1} = \text{infimum}\,\{f(x) : x \in [k, k+1]\}$.
Therefore,

$$\sum_{k=2}^{n+1} a_k = \sum_{k=2}^{n+1} f(k) = \sum_{k=2}^{n+1} f(k)[k-(k-1)]$$

is a lower Darboux sum for the interval $[1, n+1]$, with respect to the partition
$P : 1 < 2 < 3 < \ldots < n < n+1$. Thus, since f is Riemann integrable on $[1, n+1]$, by the definition of the Riemann integral (more specifically Darboux integral, see Theorem 21 of Chapter 5),

$$\sum_{k=2}^{n+1} a_k \leq \int_1^{n+1} f . \qquad \text{------------------------------ (1)}$$

Note that for any integer $k \geq 1$, $f(k) = a_k = \text{supremum}\{f(x) : x \in [k, k+1]\}$. Hence,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} f(k)[(k+1)-k]$$

is an upper Darboux sum for the interval $[1, n+1]$, with respect to the partition P. By the definition of the Riemann integral (more specifically Darboux integral),

$$\int_1^{n+1} f \leq \sum_{k=1}^{n} a_k . \qquad \text{--------------------------------- (2)}$$

Now since f is non-negative, the sequence $\left(\int_1^{n+1} f\right)$ is a monotonically increasing sequence. Therefore, by the Monotone Convergence Theorem (Theorem 15 Chapter 2 Sequences), $\left(\int_1^{n+1} f\right)$ is convergent if and only if it is bounded above.

(i) If $\left(\int_1^{n+1} f\right)$ is convergent and $\lim_{n\to\infty}\int_1^{n+1} f = L$, then $\left(\int_1^{n+1} f\right)$ is bounded above by L. By the inequality (1), for each integer $n \geq 1$, $\sum_{k=1}^{n+1} a_k \leq \int_1^{n+1} f + a_1$ and so $\sum_{k=1}^{n+1} a_k \leq \int_1^{n+1} f + a_1 \leq L + a_1$. Hence the partial sum $\{ s_{n+1} : n \in \boldsymbol{P} \}$ is bounded above by $L + a_1$. Therefore by Proposition 11 or Monotone Convergence Theorem (Theorem 15 Chapter 2 Sequences), $\sum_{k=1}^{\infty} a_k$ is convergent and

$$\sum_{k=1}^{\infty} a_k \leq L + a_1.$$

Since $\sum_{k=1}^{\infty} a_k$ is convergent, it follows from (2), that

$$\lim_{n\to\infty}\int_1^{n+1} f = L \leq \sum_{k=1}^{\infty} a_k.$$

Hence, $L \leq \sum_{k=1}^{\infty} a_k \leq L + a_1$.

(ii) $\left(\int_1^{n+1} f\right)$ is divergent if and only if it is not bounded above. Hence if $\left(\int_1^{n+1} f\right)$ is not bounded above, then since the sequence is increasing, $\lim_{n\to\infty}\int_1^{n+1} f = \infty$. It follows by (2) that the sequence of n-th partial sums (s_n) is not bounded above. This may be deduced as follows. Since $\left(\int_1^{n+1} f\right)$ is not bounded above, for any real number K, there exists an integer N such that $n \geq N \Rightarrow \int_1^{n+1} f > K$. Therefore, by (2), $n \geq N \Rightarrow s_n = \sum_{k=1}^{n} a_k \geq \int_1^{n+1} f > K$ and so (s_n) is not bounded above and hence not bounded. Therefore, $\sum_{k=1}^{\infty} a_k$ is divergent and indeed $\sum_{k=1}^{\infty} a_k = \infty$.

Example 26.

(1) The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. We may show that it is convergent by using Theorem 25.
Let $f(x) = \frac{1}{x^2}$. Then $f(n) = \frac{1}{n^2}$ and f is non-negative and decreasing. Moreover, $\int_1^n \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^n = 1 - \frac{1}{n} \to 1$. Therefore, by Theorem 25 (i), $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and $1 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + \frac{1}{1^2} = 2$. Actually Euler showed that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ in 1748 and the problem of the evaluation of this series is known as the Basel problem. There are now numerous ways of showing this, from complex variable theory, Fourier series method, double integral and elementary method using trigonometric series, integration by parts to "purely" arithmetical method.

(2) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Here we give another proof of this fact. Let $f(x) = \frac{1}{x}$. then f is non-negative and decreasing on $[1, \infty)$. The integral $\int_1^n \frac{1}{x}dx = [\ln(x)]_1^n = \ln(n) \to \infty$. Hence by Theorem 25 (ii), $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

(3) More generally, the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ $(s > 0)$ converges if $s > 1$ and diverges if $s \leq 1$.

The series converges if $s > 1$, diverges if $s \le 1$.
Let $f(x) = \frac{1}{x^s}$ for x in $[1, \infty)$. If $s = 1$ we know from example (2) above that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. We now assume that $0 < s < 1$ or $s > 1$. Then the integral

$$\int_1^n \frac{1}{x^s}dx = \left[\frac{1}{-s+1}x^{-s+1}\right]_1^n = \frac{1}{1-s}[n^{1-s} - 1]. \text{ ---------------- } (1)$$

Note that the anti-derivative of $\frac{1}{x^s}$ for $0 < s < 1$ or $s > 1$ for $x > 0$ is $\frac{1}{1-s}x^{1-s}$ even for irrational s. (This can be easily verified by using the definition of power $a^x = \exp(x \ln(a))$ for positive a.)
If $s > 1$, then $1-s < 0$ and so $n^{1-s} \to 0$ as $n \to \infty$. Thus, by (1) $\int_1^n \frac{1}{x^s}dx \to \frac{1}{s-1}$.
Note that the function f is plainly non-negative. Observe that

$$f(x) = \frac{1}{x^s} = e^{-s\ln(x)} = \frac{1}{e^{s\ln(x)}}$$

is decreasing since $e^{s\ln(x)}$ is increasing on $[1, \infty)$ for $s > 0$. Therefore, by Theorem 25 (i), $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent for $s > 1$ and $\frac{1}{s-1} \le \sum_{n=1}^{\infty} \frac{1}{n^s} \le \frac{1}{s-1} + 1$.
If $0 < s < 1$, then $1-s > 0$ and so $n^{1-s} \to \infty$ as $n \to \infty$. It follows from (1) that the sequence $\left(\int_1^n \frac{1}{x^s}dx\right)$ is divergent and $\int_1^n \frac{1}{x^s}dx \to \infty$. Hence by Theorem 25 (ii) $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is divergent and $\sum_{n=1}^{\infty} \frac{1}{n^s} = \infty$.

Example 27. $\sum_{n=1}^{\infty} \frac{n}{e^n}$ is convergent .
This is because for $n > 0$, $e^n > 1 + n + n^2/2 + n^3/6 > n^3/6$ and so

$$\frac{n}{e^n} < \frac{6}{n^2}.$$

Therefore, by the Comparison Test (Proposition 12), $\sum_{n=1}^{\infty} \frac{n}{e^n}$ is convergent because $\sum_{n=1}^{\infty} \frac{6}{n^2}$ is convergent by Example 26 (3) above.

6.7 The Cauchy Root Test

The next test is called the root test for obvious reason. It is also a consequence of the Comparison Test.

We shall state two versions, a simpler looking version and a slightly weaker version involving lim sup.

Theorem 28 (Cauchy Root Test)
Let $\sum_{n=1}^{\infty} a_n$ be a series.
(i) Suppose there is a umber r with $0 \le r < 1$ and there exists an integer N such that

$$n \ge N \Rightarrow |a_n|^{\frac{1}{n}} \le r.$$

Then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If there exists an integer N such that $n \geq N \Rightarrow |a_n|^{\frac{1}{n}} \geq 1$ *or* there exists infinite number of n for which $|a_n|^{\frac{1}{n}} \geq 1$, then the series is divergent.

Proof. The proof is similar to that of the D'Alembert Ratio Test.
(i) By assumption, $n \geq N \Rightarrow |a_n|^{\frac{1}{n}} \leq r \Rightarrow |a_n| < r^n$. Since, $0 \leq r < 1$, $\sum_{n=1}^{\infty} r^n$ is convergent and so by the Comparison Test (Proposition 12), $\sum_{n=1}^{\infty} |a_n|$ is convergent and so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
(ii) If there exists an integer N such that $n \geq N \Rightarrow |a_n|^{\frac{1}{n}} \geq 1$, then$n \geq N \Rightarrow |a_n| \geq 1$ and so $|a_n| \nrightarrow 0$, which means$a_n \nrightarrow 0$. It follows then by Proposition 10 that $\sum_{n=1}^{\infty} a_n$ is divergent. Similarly if there exists infinite number of n for which $|a_n|^{\frac{1}{n}} \geq 1$, then there are infinite number of n, for which $|a_n| \geq 1$. Therefore, $|a_n| \nrightarrow 0$ and consequently $a_n \nrightarrow 0$. We then deduce similarly that $\sum_{n=1}^{\infty} a_n$ is divergent.

We state next a slightly weaker version of Theorem 28.

Theorem 29 (Cauchy Root Test, lim sup version.)
Let $\sum_{n=1}^{\infty} a_n$ be a series.
(i) Suppose $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$. Then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
(ii) If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
(iii) If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1$, then the test gives no information. $\sum_{n=1}^{\infty} a_n$ may converge or diverge.

Proof.
(i) If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = r < 1$, then taking $\varepsilon = (1-r)/2$, there exists an integer N such that

$$n \geq N \Rightarrow r - \varepsilon < x_n < r + \varepsilon = \frac{1+r}{2} < 1,$$

where $x_n = \sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, \cdots\} = \sup\{|a_j|^{\frac{1}{j}} : j \geq n\}$. Therefore, for all $n \geq N$, $|a_n|^{\frac{1}{n}} \leq x_N < R = \frac{1+r}{2} < 1$. It then follows by Theorem 28 (i), $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. (Note that the condition in Theorem 28 (i) states that there is a umber r with $0 \leq r < 1$ and that there exists an integer N such that $n \geq N \Rightarrow |a_n|^{\frac{1}{n}} \leq r$ implies that $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq r < 1$. Hence Theorem 28 (i) and Theorem 29 (i) are equivalent.)

(ii) If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = r > 1$, then take $\varepsilon = (r-1)/2$. Since (x_n) is decreasing, $x_n \geq r$. Since $x_n = \sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, \cdots\} = \sup\{|a_j|^{\frac{1}{j}} : j \geq n\}$, by the definition of supremum, for each positive integer n, there exists an integer $k_n \geq n$ such that $|a_{k_n}|^{\frac{1}{k_n}} > x_n - \varepsilon \geq r - \varepsilon = \frac{1+r}{2} > 1$. Hence there are infinite number of n such that

$|a_n|^{\frac{1}{n}} > 1$ since $\{k_n : n \geq 1\}$ is infinite. Therefore, by Theorem 28 (ii), $\sum_{n=1}^{\infty} a_n$ is divergent. Alternatively just observe that for each integer $n \geq 1$, there exists $k_n \geq n$ such that $|a_{k_n}|^{\frac{1}{k_n}} > 1$, i.e., $|a_{k_n}| > 1$. Therefore, $|a_n| \nrightarrow 0$ and consequently $a_n \nrightarrow 0$. Hence, by Proposition 10, $\sum_{n=1}^{\infty} a_n$ is divergent.

If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \infty$, then each $x_n = \infty$. This means $S_n = \{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, \cdots\}$ is unbounded and so, there exists $k_n \geq n$ such that $|a_{k_n}|^{\frac{1}{k_n}} > 1$, i.e., $|a_{k_n}| > 1$. We then proceed in exactly the same way to show that $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$, then $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1$ because $n^{\frac{1}{n}} \to 1$. We know $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, then $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1$. But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. (See Example 26 (3).) Thus the test gives no information when $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1$.

Remark.

1. Theorem 29 is a slightly weaker form of Theorem 28. In terms of $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$, it allows the technique of finding limit to be readily used. For instance if $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists, then $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$. Indeed, we may state another weaker form of Theorem 29 with $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ replaced by $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$. Observe that if there exists infinite number of n for which $|a_n|^{\frac{1}{n}} \geq 1$, then $x_n = \sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, \cdots\} = \sup\{|a_j|^{\frac{1}{j}} : j \geq n\} \geq 1$ for each positive integer *n, if any one of x_n exists*. Thus $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ need not necessary be strictly greater than 1. So in this respect Theorem 28 (ii) is a little stronger than Theorem 29 (ii). If x_n does not exist, then the sequence $\left(|a_n|^{\frac{1}{n}}\right)$ is unbounded and $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \infty$.
2. Neither the Ratio Test nor the Cauchy Root Test says anything about the convergence or divergence of series like $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Thus we need more delicate test for this kind of series. We shall deal with this in Chapter 13.

Example 30.

(1) The series $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.

For each integer $n \geq 1$, let $a_n = \left(\frac{n}{n+1}\right)^{n^2}$. Then

$$|a_n|^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e} < 1.$$

This uses knowledge of

$$\lim_{n\to\infty} (1+\tfrac{1}{n})^n = e. \text{ Thus, } \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{e} < 1.$$

Therefore, by Theorem 29 (i), $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ is absolutely convergent. Actually, Theorem 28 (i) may be used. But we use an estimate of $|a_n|^{\frac{1}{n}}$. By a binomial expansion of $(1+\frac{1}{n})^n$, we deduce easily that $(1+\frac{1}{n})^n \geq 2$ for any integer $n \geq 1$.

Therefore, for any integer $n \geq 1$, $|a_n|^{\frac{1}{n}} = \dfrac{1}{(1+\frac{1}{n})^n} \leq \dfrac{1}{2} < 1$ and so by Theorem 28 (i), $\sum_{n=1}^{\infty}\left(\dfrac{n}{n+1}\right)^{n^2}$ is absolutely convergent.

We can use the ratio test too but it involves some manipulation such as seen below:

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left(\frac{n+1}{n+2}\right)^{(n+1)^2}\left(\frac{n+1}{n}\right)^{n^2} \\
&= \frac{(n+1)^{2n^2}}{(n+2)^{n^2+2n+1}}\frac{(n+1)^{2n+1}}{n^{n^2}} = \frac{((n+1)^2)^{n^2}}{((n+2)n)^{n^2}}\frac{(n+1)^{2n+1}}{(n+2)^{2n+1}} \\
&= \frac{(n^2+2n+1)^{n^2}}{(n^2+2n)^{n^2}}\frac{(n+1)^{2n+1}}{(n+2)^{2n+1}} \\
&= \frac{(n^2+2n+1)^{n^2+2n}}{(n^2+2n)^{n^2+2n}}\frac{(n^2+2n)^{2n}}{(n^2+2n+1)^{2n}}\frac{(n+1)^{2n+1}}{(n+2)^{2n+1}} \\
&= \frac{(n^2+2n+1)^{n^2+2n}}{(n^2+2n)^{n^2+2n}}\frac{n^{2n}}{(n+1)^{2n}}\frac{(n+1)}{(n+2)} \\
&= \left(1+\frac{1}{(n^2+2n)}\right)^{n^2+2n}\frac{1}{(1+\frac{1}{n})^{2n}}\frac{(n+1)}{(n+2)} \to e\cdot\frac{1}{e^2}\cdot 1 = \frac{1}{e} < 1
\end{aligned}$$

(2) Let $\sum_{n=0}^{\infty} a_n$ be series where the term a_n is defined by

$$a_{2k-1} = (\tfrac{1}{2})^{2k-1} \text{ for } k \geq 1 \text{ and } a_{2k} = (\tfrac{1}{4})^{2k}.$$

Then for each integer $n \geq 1$,

$$\frac{a_{n+1}}{a_n} = \begin{cases} \dfrac{(\frac{1}{4})^{n+1}}{(\frac{1}{2})^n} \text{ if } n \text{ is odd} \\ \dfrac{(\frac{1}{2})^{n+1}}{(\frac{1}{4})^n} \text{ if } n \text{ is even} \end{cases} = \begin{cases} \frac{1}{4}(\frac{1}{2})^n \text{ if } n \text{ is odd} \\ \frac{1}{2}(2)^n \text{ if } n \text{ is even} \end{cases}.$$

Plainly, for each integer $n \geq 1$,

$$\inf\{\frac{a_{n+1}}{a_n}, \frac{a_{n+2}}{a_{n+1}}, \ldots\} = 0 \text{ and } \sup\{\frac{a_{n+1}}{a_n}, \frac{a_{n+2}}{a_{n+1}}, \ldots\} = \infty$$

and so

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty \text{ and } \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = 0.$$

Thus $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist. We cannot apply the two ratio tests (Theorem 21 and Theorem 22). However, $(a_n)^{\frac{1}{n}} = \begin{cases} \frac{1}{2} \text{ if } n \text{ is odd} \\ \frac{1}{4} \text{ if } n \text{ is even} \end{cases}$. Thus $\sup\{(a_n)^{\frac{1}{n}}, (a_{n+1})^{\frac{1}{n+1}}, \ldots\} = \frac{1}{2}$. Therefore, $\limsup_{n\to\infty} (a_n)^{\frac{1}{n}} = \frac{1}{2} < 1$. Hence by Theorem 29, the series is absolutely convergent.

(3) The series $\sum_{n=2}^{\infty} \dfrac{n+1}{n^3 \ln(n)}$.

For each integer $n \geq 2$, let $a_n = \dfrac{n+1}{n^3\ln(n)}$. Then it can be shown that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \text{ and } \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = 1.$$

Hence we cannot infer anything from both the Ratio test and Cauchy Root Test.

However, note that for n > 2, $\ln(n) > 1$ and so for integer $n > 2$,

$$a_n = \frac{n+1}{n^3 \ln(n)} < \frac{1}{n^2} + \frac{1}{n^3}$$

Therefore, by the Comparison Test (Theorem 12), since we know $\sum_{n=2}^{\infty} \frac{1}{n^2} + \frac{1}{n^3}$ is convergent, the series $\sum_{n=2}^{\infty} \frac{n+1}{n^3 \ln(n)}$ is convergent.

6.8 The Euler Constant γ .

Though the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, the difference of the n-th partial sum of the series and $ln(n)$ actually converges. We shall use integral estimation to show the convergence. Firstly note that $\frac{1}{a}$ and $\frac{1}{b}$ are respectively the maximum and minimum of the function $f(t) = \frac{1}{t}$ on the interval $[a, b]$ for $1 \le a < b$.

Thus, for each integer $n \ge 1$,

$$\int_n^{n+1} f = \int_n^{n+\frac{1}{2}} f + \int_{n+\frac{1}{2}}^{n+1} f \ge \int_n^{n+\frac{1}{2}} \frac{1}{n+\frac{1}{2}} + \int_{n+\frac{1}{2}}^{n+1} \frac{1}{n+1} = \frac{1}{2}\frac{1}{n+\frac{1}{2}} + \frac{1}{2}\frac{1}{n+1}$$

$$> \frac{1}{2}\frac{1}{n+1} + \frac{1}{2}\frac{1}{n+1} = \frac{1}{n+1} \qquad \text{------------------------------------} \quad (*)$$

and

$$\int_n^{n+1} f = \int_n^{n+\frac{1}{2}} f + \int_{n+\frac{1}{2}}^{n+1} f \le \int_n^{n+\frac{1}{2}} \frac{1}{n} + \int_{n+\frac{1}{2}}^{n+1} \frac{1}{n+\frac{1}{2}} = \frac{1}{2}\frac{1}{n} + \frac{1}{2}\frac{1}{n+\frac{1}{2}}$$

$$< \frac{1}{2}\frac{1}{n} + \frac{1}{2}\frac{1}{n} = \frac{1}{n}.$$

Hence, for each integer $n \ge 1$,

$$\frac{1}{n+1} < \int_n^{n+1} f < \frac{1}{n} \qquad \text{----------------------------} \quad (1)$$

Therefore, using (1) we get for integer $n \ge 2$,

$$\sum_{k=1}^{n-1} \frac{1}{k+1} < \int_1^2 f + \int_2^3 f + \cdots + \int_{n-1}^{n} f < \sum_{k=1}^{n-1} \frac{1}{k}.$$

Thus,

$$\sum_{k=1}^{n-1} \frac{1}{k+1} < \int_1^n f = \ln(n) < \sum_{k=1}^{n-1} \frac{1}{k}. \quad \text{--------------------------} \ (2)$$

Now let $s_n = \sum_{k=1}^{n} \frac{1}{k}$ for integer $n \ge 1$. Then from (2) we obtain for integer $n \ge 2$,

$$s_n - \ln(n) = 1 + \sum_{k=1}^{n-1} \frac{1}{k+1} - \ln(n) < 1 + 0 = 1$$

and

$$s_n - \ln(n) = \frac{1}{n} + \sum_{k=1}^{n-1} \frac{1}{k} - \ln(n) > \frac{1}{n} + 0 = \frac{1}{n}.$$

For integer $n \ge 1$, let $d_n = s_n - \ln(n)$. Then for integer $n \ge 2$,

$$0 < \frac{1}{n} < d_n < 1$$

and $d_1 = s_1 - \ln(1) = 1 > 0$. Hence, we have that $0 < d_n \le 1$ for all integer $n \ge 1$. Therefore, the sequence (d_n) is a bounded sequence. We shall show that it is a decreasing sequence.

For any integer $n \ge 1$,

$$d_{n+1} - d_n = s_{n+1} - \ln(n+1) - (s_n - \ln(n)) = s_{n+1} - s_n - (\ln(n+1) - \ln(n))$$

$$= \frac{1}{n+1} - \int_n^{n+1} f < 0 \text{ by } (*).$$

Hence $d_{n+1} < d_n$ for any integer $n \geq 1$. This means the sequence (d_n) is a decreasing sequence bounded below by 0. Therefore, by the Monotone Convergence Theorem (Theorem 15 Chapter 2 Sequences), (d_n) is convergent and converges to the Euler constant γ .

Example 31.

Now we shall use this constant to evaluate the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$.

Let $a_n = (-1)^{n+1}\frac{1}{n}$ and $t_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k}$ be the n-th partial sum of the series.

By Leibnitz's Alternating Series Test (Theorem 20), this series is convergent. Hence (t_n) is a convergent sequence and so any subsequence of it should converge to the same limit.

Now
$$t_{2n} = \sum_{k=1}^{2n}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$
$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right).$$

Thus, for each integer $n \geq 1$,
$$t_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{2n} - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$
$$= s_{2n} - s_n = d_{2n} + \ln(2n) - (d_n + \ln(n))$$
$$= d_{2n} - d_n + \ln(2n) - \ln(n)$$
$$= \ln(2) + d_{2n} - d_n \ .$$

Since (d_n) is convergent and converges to the Euler constant γ, $\lim_{n\to\infty} d_{2n} = \lim_{n\to\infty} d_n = \gamma$.

Hence, $t_{2n} \to \ln(2) + \gamma - \gamma = \ln(2)$. Therefore, since (t_n) is convergent, $t_n \to \ln(2)$.

This means $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln(2)$. We shall see in a later chapter another proof of this using Abel's Theorem.

6.9 Dirichlet's Test

In order to deal with series such as trigonometric series we may use the following test due to Dirichlet which is a specialization of the Dirichlet's Test for uniform convergence of a series of functions.

Theorem 32 (Dirichlet's Test for Series).

Suppose (a_n) and (b_n) are two sequences satisfying

(1) the n-th partial sums, $s_n = \sum_{k=1}^{n} b_k$, are bounded. i.e., there exists some real number $K > 0$ such that $|s_n| \leq K$;

(2) (a_n) is decreasing and

(3) $a_n \to 0$.

Then the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Proof. We shall show that the series $\sum_{n=1}^{\infty} a_n b_n$ is a Cauchy series.

Note that for each $k \geq 2$,
$$b_k = s_k - s_{k-1}$$
and $b_1 = s_1$.

Hence we have for each integer $n \geq 1$ and any integer $p \geq 1$,

$$\sum_{k=n+1}^{n+p} a_k b_k = \sum_{k=n+1}^{n+p} a_k(s_k - s_{k-1}) = \sum_{k=n+1}^{n+p} a_k s_k - \sum_{k=n+1}^{n+p} a_k s_{k-1}$$
$$= \sum_{k=n+1}^{n+p} a_k s_k - \sum_{k=n}^{n+p-1} a_{k+1} s_k = \sum_{k=n+1}^{n+p-1} a_k s_k - \sum_{k=n+1}^{n+p-1} a_{k+1} s_k + a_{n+p} s_{n+p} - a_{n+1} s_n$$
$$= \sum_{k=n+1}^{n+p-1} (a_k - a_{k+1}) s_k + a_{n+p} s_{n+p} - a_{n+1} s_n \quad \text{-------------------------- (1)}$$

Then from (1) by the triangle inequality, for each integer $n \geq 1$ and any integer $p \geq 1$,

$$\left|\sum_{k=n+1}^{n+p} a_k b_k\right| \leq \sum_{k=n+1}^{n+p-1} |a_k - a_{k+1}||s_k| + |a_{n+p}|\ |s_{n+p}| + |a_{n+1}||s_n|.$$

Thus, by condition (1), for each integer $n \geq 1$ and any integer $p \geq 1$,

$$\left|\sum_{k=n+1}^{n+p} a_k b_k\right| \leq \sum_{k=n+1}^{n+p-1} |a_k - a_{k+1}| K + (|a_{n+p}| + |a_{n+1}|) K$$
$$\leq K \sum_{k=n+1}^{n+p-1} (a_k - a_{k+1}) + (|a_{n+p}| + |a_{n+1}|) K \quad \text{---------------------- (2)}$$

because (a_n) is decreasing by condition (2) of the theorem so that $|a_k - a_{k+1}| = a_k - a_{k+1}$ for each $k \geq 1$.

It then follows from (2) since $\sum_{k=n+1}^{n+p-1} (a_k - a_{k+1})$ is a telescopic sum,

$$\left|\sum_{k=n+1}^{n+p} a_k b_k\right| \leq K(a_{n+1} - a_{n+p}) + K(|a_{n+p}| + |a_{n+1}|) \quad \text{-------------- (3)}$$

for each integer $n \geq 1$ and any integer $p \geq 1$.

Note that condition (2) and (3) of the theorem says that (a_n) is decreasing and $a_n \to 0$. Therefore, $a_n \geq 0$. (This is because if some integer j, $a_j < 0$, then for all $n > j$, $a_n \leq a_j < 0$ and so $a_n \nrightarrow 0$, contradicting $a_n \to 0$.)

Thus, we obtain from (3), that for each integer $n \geq 1$ and any integer $p \geq 1$.

$$\left|\sum_{k=n+1}^{n+p} a_k b_k\right| \leq K(a_{n+1} - a_{n+p}) + K(a_{n+p} + a_{n+1}) = 2K a_{n+1} \quad \text{---------- (4)}$$

Now, since $a_n \to 0$, given any $\varepsilon > 0$, there exists a positive integer N such that for all n in $\boldsymbol{P}$,

$$n \geq N \Rightarrow |a_n| = a_n < \frac{\varepsilon}{2K}. \quad \text{--------------------------------- (5)}$$

Therefore, from (4) and (5), we have that for any integer $p \geq 1$,

$$n \geq N \Rightarrow \left|\sum_{k=n+1}^{n+p} a_k b_k\right| \leq 2K a_{n+1} < 2K \frac{\varepsilon}{2K} = \varepsilon.$$

Hence, $\sum_{n=1}^{\infty} a_n b_n$ is a Cauchy series and by Theorem 9, $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Example 33.

The series $\sum_{n=1}^{\infty} a_n \sin(nx)$ and $\sum_{n=1}^{\infty} a_n \cos(nx)$.

Note the following cosine formulae,

$$\cos(kx - \tfrac{1}{2}x) = \cos(kx)\cos(\tfrac{1}{2}x) + \sin(kx)\sin(\tfrac{1}{2}x) \quad \text{------------------ (1)}$$

and

$$\cos(kx + \tfrac{1}{2}x) = \cos(kx)\cos(\tfrac{1}{2}x) - \sin(kx)\sin(\tfrac{1}{2}x). \quad \text{------------------ (2)}$$

Subtracting (2) from (1), we get for any integer n and any x,

$$2\sin(kx)\sin(\tfrac{1}{2}x) = \cos(kx - \tfrac{1}{2}x) - \cos(kx + \tfrac{1}{2}x)$$
$$= \cos\left((k-1)x + \tfrac{1}{2}x\right) - \cos\left(kx + \tfrac{1}{2}x\right). \quad \text{-------------------- (3)}$$

Similarly using the sine formulae,

$$\sin(kx - \tfrac{1}{2}x) = \sin(kx)\cos(\tfrac{1}{2}x) - \cos(kx)\sin(\tfrac{1}{2}x) \quad \text{-------------------} \ (4)$$

and $$\sin(kx + \tfrac{1}{2}x) = \sin(kx)\cos(\tfrac{1}{2}x) + \cos(kx)\sin(\tfrac{1}{2}x). \quad \text{-------------------} \ (5)$$

Then, subtracting (4) from (5), we get

$$\begin{aligned} 2\cos(kx)\sin(\tfrac{1}{2}x) &= \sin(kx + \tfrac{1}{2}x) - \sin(kx - \tfrac{1}{2}x) \\ &= \sin\left(kx + \tfrac{1}{2}x\right) - \sin\left((k-1)x + \tfrac{1}{2}x\right). \quad \text{-----------------} \ (6) \end{aligned}$$

Thus, for any integer $n \geq 1$, using (6), we get

$$2\sin(\tfrac{1}{2}x)\sum_{k=1}^{n}\sin(kx) = \cos(\tfrac{1}{2}x) - \cos(nx + \tfrac{1}{2}x)$$

and so $$\sum_{k=1}^{n}\sin(kx) = \frac{\cos(\frac{1}{2}x) - \cos(nx + \frac{1}{2}x)}{2\sin(\frac{1}{2}x)}$$

if x is not a multiple of 2π.

Hence $\left|\sum_{k=1}^{n}\sin(kx)\right| = \left|\frac{\cos(\frac{1}{2}x) - \cos(nx + \frac{1}{2}x)}{2\sin(\frac{1}{2}x)}\right| \leq \frac{1}{\left|\sin(\frac{1}{2}x)\right|}$ if x is not a multiple of 2π.

Thus the partial sum $\sum_{k=1}^{n}\sin(kx)$ is bounded. Therefore, if (a_n) is decreasing and $a_n \to 0$, then by the Dirichlet's Test, $\sum_{n=1}^{\infty} a_n \sin(nx)$ is convergent for any x not a multiple of 2π.

However, if x is a multiple of 2π, the series $\sum_{n=1}^{\infty} a_n \sin(nx) = 0$. Therefore, $\sum_{n=1}^{\infty} a_n \sin(nx)$ is convergent for any x *which is* a multiple of 2π, .

Now summing over (6), we obtain

$$2\sin(\tfrac{1}{2}x)\sum_{k=1}^{n}\cos(kx) = \sin(nx + \tfrac{1}{2}x) - \sin(\tfrac{1}{2}x)$$

and if x is not a multiple of 2π,

$$\sum_{k=1}^{n}\cos(kx) = \frac{\sin(nx + \frac{1}{2}x) - \sin(x\frac{1}{2})}{2\sin(\frac{1}{2}x)}$$

and $\left|\sum_{k=1}^{n}\cos(kx)\right| \leq \frac{1}{\left|\sin(\frac{1}{2}x)\right|}$. Therefore, if x is not a multiple of 2π, the n-th partial sum $\sum_{k=1}^{n}\cos(kx)$ is bounded. Consequently, if (a_n) is a decreasing sequence tending to 0, then the series $\sum_{n=1}^{\infty} a_n \cos(nx)$ is convergent. If x is a multiple of 2π, then the series becomes $\sum_{n=1}^{\infty} a_n$ and this may or may not converge.

Thus, $\sum_{n=1}^{\infty}\frac{1}{n}\sin(nx)$ is always convergent and $\sum_{n=1}^{\infty}\frac{1}{n}\cos(nx)$ converges only for x not a multiple of 2π .

Exercises 34.

1. Suppose $\sum a_n$ and $\sum b_n$ are two series of positive terms. Suppose $a_n / b_n \to k$ and $k > 0$. Prove that $\sum a_n$ is convergent if and only $\sum b_n$ is convergent.
   Suppose $a_n / b_n \to \infty$. Prove that if $\sum b_n$ is divergent, then $\sum a_n$ is divergent.

2. Suppose $\sum a_n$ and $\sum b_n$ are absolutely convergent , then $\sum a_n b_n$ is also absolutely convergent. Hence deduce that $\sum a_n$ converges absolutely implies that $\sum a_n^2$ is convergent. Is the converse true?

3. Using question 1 or otherwise, test the convergence of the following series:
   (i) $\sum_1^\infty \frac{1}{1+n^2}$. (ii) $\sum_1^\infty \frac{n+1}{n(n+2)}$. (iii) $\sum_1^\infty \frac{1}{n}\sin(\frac{1}{n})$.
   (iv) $\sum_1^\infty \frac{1}{2n+5}$. (v) $\sum_1^\infty \left(\frac{n+1}{n^2+1}\right)^3$. (vi) $\sum_1^\infty \frac{\ln(n)}{\sqrt{n+1}}$.

4. Test the following series for convergence.
   (i) $\sum_1^\infty \frac{(n!)^2}{(2n)!}$. (ii) $\sum_1^\infty \frac{n!}{n^n}$. (iii) $\sum_1^\infty \frac{n^2}{2^n}$. (iv) $\sum_1^\infty \frac{2^n}{n!}$. (v) $\sum_1^\infty \frac{1}{n\ln(n)}$.5

5. Determine the convergence of the following series, say whether the convergence is absolute or conditional.
   (i) $\sum_1^\infty (-1)^{n+1}\frac{n+3}{n(n+1)}$. (ii) $\sum_1^\infty (-1)^n\frac{\ln(n)}{n}$. (iii) $\sum_1^\infty (-1)^{n+1}\frac{n}{2^n}$.
   (iv) $\sum_1^\infty (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$, any x.

6. Investigate the convergence of $\sum_1^\infty \frac{\sin(nx)}{n^p}$ for $p > 0$.

7. Test the following series for convergence.
   (i) $\sum_1^\infty \frac{2^n+n}{3^n-n}$. (ii) $\sum_1^\infty \frac{e^{-n}}{\sqrt{n+1}}$. (iii) $\sum_1^\infty \sin\left(\frac{1}{n^p}\right)$, p real $p > 0$.
   (iv) $\sum_1^\infty \frac{\ln(n+1)-\ln(n)}{\tan^{-1}(\frac{2}{n})}$.

8. Determine the conditional convergence, absolute convergence or divergence of the following series.
   (i) $\sum_1^\infty (-1)^n e^{-n^2}$. (ii) $\sum_1^\infty \frac{(-1)^n}{\ln(\cosh(n))}$. (iii) $\sum_1^\infty (-1)^n\frac{n^2}{2+n^2}$.

9. (i) Prove that $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2} = \frac{\pi}{4}$.

   (ii) Let S_n ($n \geq 1$) be the n-th partial sum of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$. Show by induction that $S_{2n} = \sum_{k=1}^{n}\frac{1}{n+k}$. Use integral calculus to deduce that $\sum_1^\infty (-1)^{n+1}\frac{1}{n} = \ln(2)$.

10. Prove that if $\sum a_n^2$ and $\sum b_n^2$ are absolutely convergent , then $\sum a_n b_n$ is also absolutely convergent. [Hint: $|a_n b_n| \leq (a_n^2 + b_n^2)/2$]. Hence deduce that if $\sum a_n^2$ is absolutely convergent, then so is $\sum_{n=1}^\infty \frac{a_n}{n}$.

11. Given that $\sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$, show that (i) $\sum_1^\infty \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$ and

(ii) $\sum_{1}^{\infty}(-1)^n \frac{1}{n^4} = -\frac{7\pi^4}{720}$.

12. Suppose $\sum a_n$ is convergent. Show that $\sum \frac{n+1}{n} a_n$ is also convergent.

13. Prove that $\sum_{1}^{\infty} \frac{1}{n(2n-1)} = 2\ln(2)$ and that $\sum_{1}^{\infty}(-1)^{n-1}\frac{2n+1}{n(n+1)} = 1$.

14. Show that $\tan(\frac{x}{2}) = \cot(\frac{x}{2}) - 2\cot(x)$. Using this relation find the sum of the series

$$\sum_{1}^{\infty} \frac{1}{2^n}\tan\left(\frac{x}{2^n}\right).$$

15. Suppose the series of positive terms $\sum_{n=1}^{\infty} B_n$ is divergent, i.e., its limit is ∞. Suppose (a_n) is a sequence such that $a_n \to A$. Show that $\lim_{n\to\infty} \frac{B_1 a_1 + B_2 a_2 + \cdots + B_n a_n}{B_1 + B_2 + \cdots + B_n} = A$.

    Hence or otherwise, show that

    (i) $\lim_{n\to\infty} \frac{\sin(\theta) + \sin\left(\frac{\theta}{2}\right) + \cdots + \sin\left(\frac{\theta}{n}\right)}{1 + \frac{1}{2} + \cdots + \frac{1}{n}} = \theta$.

    (ii) $\frac{1}{n^2}\left\{1^2\sin(\theta) + 2^2\sin\left(\frac{\theta}{2}\right) + \cdots + n^2\sin\left(\frac{\theta}{n}\right)\right\} \to \frac{\theta}{2}$.

16. If $a_1 = \cos(\theta)$, $0 < \theta < 2\pi$, $b_1 = 1$ and $a_{n+1} = (a_n + b_n)/2$, $b_{n+1} = \sqrt{(a_{n+1} b_n)}$ for integers $n \geq 1$, show that (a_n) and (b_n) are both convergent and converges to the common limit $\sin(\theta)/\theta$.

17. Follow the methods of 6.8, prove that the sequence

$$(\gamma_n) = \left(\frac{1}{2\ln(2)} + \frac{1}{3\ln(3)} + \cdots + \frac{1}{n\ln(n)} - \ln(\ln(n))\right),\ n \geq 2,$$

is convergent. Hence, or otherwise, show that if p is a positive integer,

$$\lim_{n\to\infty} \sum_{k=n}^{n^p} \frac{1}{k\ln(k)} = \ln(p).$$

18. By using a partial fraction decomposition, or otherwise show that

$$\sum_{n=1}^{\infty} \frac{1}{n(4n^2-1)} = 2\ln(2) - 1.$$

19. Let $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Show that $1 + \frac{n}{2} \leq S_{2^n} \leq n + \left(\frac{1}{2}\right)^n$. Hence deduce that

    $(S_n)^{1/n} \to 1$.

20. Show that (i) $\lim_{n\to\infty} \frac{n+1}{(n!)^{1/n}} = e$ (ii) $\lim_{n\to\infty} \frac{((n+1)(n+2)\ldots(n+n))^{1/n}}{n} = \frac{4}{e}$.