Chapter 5. Integration

Introduction. The term "integration" has several meanings. It is usually met as the reverse process to differentiation, i.e. finding anti-derivative to a function. An anti-derivative of a function f is a function F such that its derivative F' satisfies $F' = f$ on some suitable domain. In mechanics the velocity of an object in motion may be determined from its acceleration by anti-differentiation. In Business, a revenue function may be computed from a given Marginal Revenue function by using anti-differentiation. In these and other examples, the question of how to find an anti-derivative of a particular function f leads to another meaning of the term "integration" related to area under a graph, definite integration. These two meanings under appropriate condition are linked by the so called "fundamental theorem of calculus". There are other subtle mathematical formulations also called integration some of which are generalizations of the definite integration.

5.1 Anti-derivative

Definition 1. Let I be an open interval. Suppose $F : I \to \mathbf{R}$ is differentiable on I and that $F' = f$. Then F is called an *anti-derivative, indefinite integral* or *primitive* of f.

Remark.

1. For a given $f : I \to \mathbf{R}$, there may be many or none such F with $F' = f$. If $F' = f$, then let $G = F + C$, where C is a constant. Then $G' = F' = f$. Thus any two anti-derivatives differ by a constant.
2. We can replace I in Definition 1 by any interval if we take the derivatives at the end points of I to be the appropriate one-sided limits. (See Chapter 3 for consequences.)

Theorem 2. If F is a particular anti-derivative of f on an open interval I, then every anti-derivative of f on I is given by $F(x) + C$, where C is a constant, that is, the set of all anti-derivatives of f on I is $\{ F(x) + C : C \in \mathbf{R}\}$.

Proof. Suppose G is another anti-derivative of f on I. Let $h = G - F$ on I. Then $h' = G' - F' = 0$ on I. Therefore, by Theorem 16 Chapter 4, h is a constant function, say $h = C$. Then $h = G - F = C$ and so $G = F + C$.

We normally write $\int f(x)dx = F(x) + C$ to denote finding an anti-derivative of f, whenever a function F with $F'(x) = f(x)$ or $\frac{d}{dx}F(x) = f(x)$ is found. Not every function has an anti-derivative as the following example will show.

Example 3. The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$ does not have an anti-derivative. This is seen as follows. Suppose f has an anti-derivative $F : \mathbf{R} \to \mathbf{R}$ with $F' = f$. Then for $x > 0$, $F'(x) = f(x) = 1$. Therefore, by Theorem 2, for $x > 0$, $F(x) = x + a$ for some constant a in $\mathbf{R}$. For $x < 0$, $F'(x) = f(x) = -1$. Thus, also by Theorem 2, for $x < 0$, $F(x) = -x + b$ for some constant b in $\mathbf{R}$. Thus, $\lim_{x \to 0^+} F(x) = a$ and $\lim_{x \to 0^-} F(x) = b$. Since F is differentiable, F is continuous and is therefore continuous at $x = 0$. Thus $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^-} F(x) = F(0)$ Hence $a = b = c$, say. Therefore, $F(x) = |x| + c$. But then this implies that F is not differentiable at $x = 0$ since the function $|x|$ is not differentiable at $x = 0$. This contradiction shows that f does not have an anti-derivative.

The function f in the above example is not continuous at $x = 0$. But not every discontinuous function does not have an anti-derivative as the following example shows.

Example 4. The function $f:\mathbf{R}\to\mathbf{R}$ defined by

$$f(x) = \begin{cases} 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

has an anti-derivative $F:\mathbf{R}\to\mathbf{R}$, defined by $F(x) = \begin{cases} x^2\sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$ even though f is not continuous at $x = 0$. We can take the sequence (a_n), where $a_n = \dfrac{1}{2n\pi}$. Then $a_n \to 0$. Then $f(a_n) = \dfrac{1}{n\pi}\sin(2n\pi) - \cos(2n\pi) = -1$ for each n in P. Therefore, $f(a_n) \to -1 \neq f(0)$. Hence the sequence $(f(a_n))$ does not converge to $f(0)$. Therefore, by Definition 2 of Chapter 3, f is not continuous at $x = 0$.

Theorem 5.

1. $\int 1dx = x + C$;
2. If f has an anti-derivative on an open interval I, then for any constant a, af also has an anti-derivative on I and $\int af(x)dx = a\int f(x)dx$.
3. If f_1 and f_2 have anti-derivatives on I, then for any real numbers a and b,
$$\int[af_1(x) + bf_2(x)]dx = a\int f_1(x)dx + b\int f_2(x)dx.$$

Theorem 5. is proved by direct verification and is left to the reader.

Theorem 6. For any rational number n, $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$, $n \neq -1$.

Proof. For $n \neq -1$, $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right) = (n+1)\cdot\frac{x^n}{n+1} = x^n$.

Remark. Using Theorem 5 and Theorem 6, we can find anti-derivative of any polynomial function

Example 7.

1. $\int(3x+5)dx = 3\int xdx + 5\int 1dx = 3(\frac{x^2}{2} + C_1) + 5(x + C_2)$
$= \frac{3}{2}x^2 + 5x + 3C_1 + 5C_2 = \frac{3}{2}x^2 + 5x + C$, where $C = 3C_1 + 5C_2$.
2. $\int(5x^4 - 8x^3 + 9x^2 + 7)dx = 5\cdot\frac{1}{5}x^5 - 8\cdot\frac{1}{4}x^4 + 9\cdot\frac{1}{3}x^3 + 7x + C$
$= x^5 - 2x^4 + 3x^3 + 7x + C.$
3. $\int(x^{\frac{2}{3}} + x^{\frac{1}{5}})dx = \frac{1}{\frac{2}{3}+1}x^{\frac{2}{3}+1} + \frac{1}{\frac{1}{5}+1}x^{\frac{1}{5}+1} + C = \frac{3}{5}x^{\frac{5}{3}} + \frac{5}{6}x^{\frac{6}{5}} + C.$
4. $\int\sqrt{x}\,(x + \frac{1}{x})dx = \int(x^{\frac{3}{2}} + x^{-\frac{1}{2}})dx = \frac{1}{\frac{3}{2}+1}x^{\frac{3}{2}+1} + \frac{1}{-\frac{1}{2}+1}x^{-\frac{1}{2}+1} + C = \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + C.$

We shall now extend our method of finding anti-derivative. This is an application of the *chain rule*. Suppose I and J are open intervals, $g: I \to \mathbf{R}$ and $F : J \to \mathbf{R}$ are differentiable functions with $g(I) \subseteq J$. Then F and g are composable. The *Chain*

Rule says $(F \circ g)'(x) = F'(g(x))g'(x)$ for every x in I (Reference: Theorem 8 Chapter 4). If we write f for F', we have $(F \circ g)'(x) = f(g(x))g'(x)$ for every x in I. This shows that $F \circ g$ is an anti-derivative of $f(g(x))g'(x)$. In this way, knowing the anti-derivative of f will enable us to determine the anti-derivative of a more complex function $f(g(x))g'(x)$. However, to apply this method of finding anti-derivative, we need to put our function in the form of $f(g(x))g'(x)$ for suitable differentiable function g and function f whose anti-derivative is known.

Hence, we have the following theorem.

Theorem 8 (Change of Variable, Substitution).
Suppose $g: I \to \mathbf{R}$ is differentiable and $f: J \to \mathbf{R}$ is such that f has an anti-derivative $F: J \to \mathbf{R}$ and $g(I) \subseteq J$. Then

$$\int f(g(x)) \cdot g'(x)dx = F(g(x)) + C.$$

Moreover, F can be chosen to be any anti-derivative of f. If we write $y = g(x)$, then the theorem is usually also remembered as

$$\int f(y)\frac{dy}{dx}\,dx = \int f(y)dy = F(y(x)) + C\ .$$

Remark. This is a theorem about anti-derivative. It gives us an extension to finding anti-derivatives by enlarging our "lookup" table through differentiation of composite functions.

Example 9.

1. Find $\int \sqrt{3x+4}\,dx$. Let $y = 3x + 4$. Then $\frac{dy}{dx} = 3$. Here we let $f(y) = \sqrt{y}$ and $g(x) = 3x + 4$. An anti-derivative of f is $F(y) = \dfrac{1}{\frac{1}{2}+1}y^{\frac{1}{2}+1} = \dfrac{2}{3}y^{\frac{3}{2}}$ and $g'(x) = 3$.
   $\int \sqrt{3x+4}\,dx = \frac{1}{3}\int \sqrt{3x+4} \cdot 3dx = \frac{1}{3}\int f(3x+4)\,g'(x)\,dx = \frac{1}{3}\int f(g(x))\,g'(x)\,dx$
   $= \frac{1}{3}\cdot\frac{2}{3}g(x)^{\frac{3}{2}} + C = \frac{2}{9}(3x+4)^{\frac{3}{2}} + C.$
2. Find $\int(2x^3+x)\sqrt{x^2+1}\,dx$. Let $g(x) = x^2+1$ and so $g'(x) = 2x$, $x^2 = g(x) - 1$.
   $\int(2x^3+x)\sqrt{x^2+1}\,dx = \int(2x^2+1)\sqrt{x^2+1}\ xdx = \frac{1}{2}\int(2x^2+1)\sqrt{x^2+1}\ 2xdx$
   $= \frac{1}{2}\int(2(g(x)-1)+1)\sqrt{g(x)}\,g'(x)dx = \int(g(x)-\frac{1}{2})\sqrt{g(x)}\,g'(x)dx$
   $= \int(y^{\frac{3}{2}} - \frac{1}{2}y^{\frac{1}{2}})dy$ by Theorem 8, where $y = g(x)$ is to be substituted,
   $= \dfrac{1}{\frac{3}{2}+1}y^{\frac{3}{2}+1} - \frac{1}{2}\cdot\dfrac{1}{\frac{1}{2}+1}y^{\frac{1}{2}+1} + C = \frac{2}{5}y^{\frac{5}{2}} - \frac{1}{3}y^{\frac{3}{2}} + C$
   $= \frac{2}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C.$
3. Find $\int x\sin(x^2)dx$. Let $g(x) = x^2$ and $f(x) = \sin(x)$. Then $g'(x) = 3x$. Therefore,
   $\int x\sin(x^2)dx = \frac{1}{2}\int \sin(x^2)2xdx = \frac{1}{2}\int \sin(g(x))g'(x)dx$
   $= \frac{1}{2}(-\cos(g(x))) + C = -\frac{1}{2}\cos(x^2) + C.$
   since an anti-derivative of $\sin(x)$ is $-\cos(x)$.

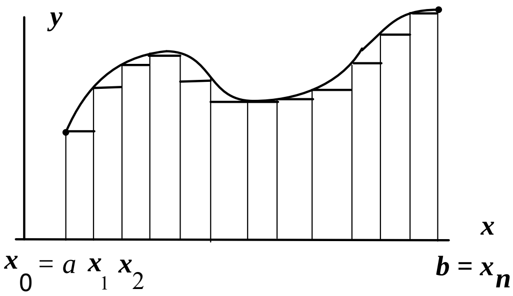
5.2 Riemann Integrals.

Area under a graph may be computed by Riemann integral. If the velocity of an object is known numerically over a time interval from time = 0 to say, time = T, the distance traveled in this time interval may be computed by determining the area under the velocity-time graph of the object using Riemann integration. This approach is different from using an anti-derivative of the velocity function, which may not be available. What is now called the Riemann integral of a function f is a mathematically rigorous formulation of the intuitive notion of the area under the graph of f.

Area under a curve.

We now give a description of a procedure to compute the area of a region under a curve.

Consider a continuous function f defined on $[a, b]$ whose graph is given as follows.



The area under the curve between $x = a$ and $x = b$ and $y = f(x)$ is approximated by the rectangles shown below the graph of f. We have partitioned $[a, b]$ into smaller intervals $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$ with P: $a = x_0 < x_1 < \ldots < x_n = b$.
We now define the *lower sum* with respect to the 'partition' P to be the quantity given by

$$\sum_{i=1}^{n}(x_i - x_{i-1})m_i, \text{ where } m_i = \min\{ f(x) : x \in [x_{i-1}, x_i]\}$$

By the *Extreme Value Theorem*, there exists c_i in $[x_{i-1}, x_i]$ such that $m_i = f(c_i)$. Now we let the partition be equally spaced, say $x_i - x_{i-1} = \frac{b-a}{n} = \Delta x$ for all i. Then the lower sum with respect to the partition is

$$\sum_{i=1}^{n} \frac{b-a}{n} f(c_i) = \sum_{i=1}^{n} f(c_i)\Delta x.$$

If we take the common width of the rectangles Δx to be as small as we wish we shall eventually approximate the area under the curve. That is to say, the area under the curve

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\frac{b-a}{n}.$$

I.e. Given $\varepsilon > 0$, there exists a positive integer N such that for any integer n

$$n > N \Rightarrow \left|\sum_{i=1}^{n} f(c_i)\Delta x - A\right| = \left|\sum_{i=1}^{n} f(c_i)\frac{b-a}{n} - A\right| < \varepsilon.$$

Note here that the c_i's are generally different for different n.
This gives an approach of finding area from below.
Similarly we can define the *upper sum* with respect to the 'partition' P to be the summation

$$\sum_{i=1}^{n}(x_i - x_{i-1})M_i,$$

where M_i = max{ $f(x)$: $x \in [x_{i-1}, x_i]$} = $f(d_i)$ for some d_i in $[x_{i-1}, x_i]$ by the *Extreme Value Theorem*. If we take the partition to be a regular partition as before, such that $x_i - x_{i-1} = \frac{b-a}{n} = \Delta x,$ the area under the curve may be given by

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(d_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(d_i)\frac{b-a}{n}.$$

This is an approach to finding area under a curve from above.

Example 10.

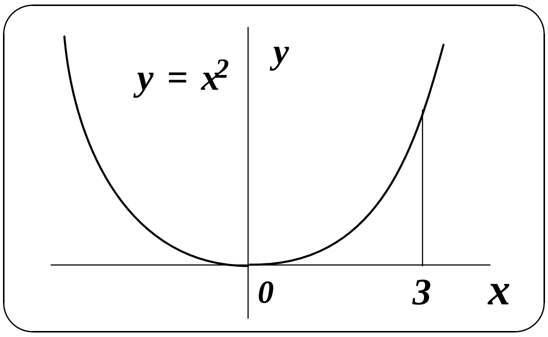
Find the area of the region bounded by $y = x^2$ and the x-axis and the line $x = 3$.
Let $f(x) = x^2$. Suppose n is a positive integer. Let $x_0 = 0$ and $x_n = 3$. [We are going to partition [0,3] into n equal parts.]
Let $\Delta x = \frac{3-0}{n}$, $x_i = x_0 + i \cdot \Delta x = \frac{3i}{n}$. Since x^2 is an increasing function on $[0, \infty)$,

$$m_i = \min\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_{i-1}) = x_{i-1}^2 = \tfrac{3^2}{n^2}(i-1)^2 \text{-----------------} \quad (1)$$

Therefore, $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3^2}{n^2}(i-1)^2 \cdot \frac{3}{n}$ by (1)

$$= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n}(i-1)^2 = \lim_{n\to\infty} \tfrac{27}{n^3} \sum_{i=1}^{n-1} i^2 = \lim_{n\to\infty} \frac{27}{n^3} \cdot \frac{(n-1).n.(2(n-1)+1)}{6}$$

$$= \frac{9}{2} \lim_{n\to\infty} \frac{(n-1)n(2n-1)}{n^3} = \frac{9}{2} \lim_{n\to\infty} (1 - \tfrac{1}{n})(2 - \tfrac{1}{n}) = \frac{9}{2} \cdot 2 = 9.$$



Typically Example 10 illustrates the method of finding area from below. Similarly, by taking the upper sum we should get the same value for the area of the region in Example 10. Now if instead of taking the minimum or maximum in each subinterval $[x_{i-1}, x_i]$, we simply choose any value e_i in $[x_{i-1}, x_i]$, then by the continuity of f , the sum $\sum_{i=1}^{n}(x_i - x_{i-1}) f(e_i)$ will lie between the upper sum and the lower sum. In this way, as the size of the subintervals, Δx tends to 0, the sum $\sum_{i=1}^{n}(x_i - x_{i-1}) f(e_i)$ would tend to the same value as the area under the curve. This approach has some advantages. Firstly we do not need to find the maximum nor the minimum in each subinterval. Secondly we do not need to confine to continuous functions so long as the function is defined. However we need to impose on having the sum $\sum_{i=1}^{n}(x_i - x_{i-1}) f(e_i)$ to be bounded and that for a meaningful definition of area, the sum should behave just like the case for the continuous function. Thus we want the sum $\sum_{i=1}^{n}(x_i - x_{i-1})\, f(e_i)$ called a *Riemann sum* to approach a finite value as the partition of the interval [*a*, *b*] gets finer and finer.

Riemann Sum and Riemann Integral

We now formalize the above discussion. We will even be extending the definition to discontinuous functions. The advantage of the following definition named after the German mathematician George Friedrich Bernhard Riemann (1826-1866), is that it avoids the need to find the maximum or minimum of the function over each subinterval. Let f be defined on the closed interval [*a*, *b*]. We shall subdivide [*a*, *b*] into *n subintervals* by choosing n-1 intermediate points between *a* and *b*.

Definition 11. A *partition* Δ of [*a*, *b*] is a finite set of numbers $\{x_0, x_1, ..., x_n\}$ such that

$$\Delta\colon a = x_0 < x_1 < \ldots x_{n-1} < x_n = b.$$

Let Δx_i be the length of the i-th subinterval $[x_{i-1}, x_i]$, i.e.$\Delta x_i = x_i - x_{i-1}$. The <u>*norm*</u> or <u>*mesh*</u> of the partition Δ is $\max\{\Delta x_i : i = 1, \ldots, n\}$ and is denoted by ||Δ||. Thus $\|\Delta\| = \max\{x_i - x_{i-1} : i = 1, \ldots, n\}$.

Suppose now $\Delta\colon a = x_0 < x_1 < \ldots x_{n-1} < x_n = b$ is a partition. Let ξ_ι be a point in $[x_{i-1}, x_i]$, for i = 1, 2, ..., n. The *Riemann sum S* of f with respect to Δ is defined by

$$S = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(\xi_i)\Delta x.$$

We denote S by $R(f, \Delta, \xi)$, where $\xi = (\xi_1, \xi_2, \ldots \xi_n)$ is the choice of ξ_i in $[x_{i-1}, x_i]$.

We say the function f is *Riemann integrable* on [*a, b*] if and only if there exists a number L such that given any ε > 0, there exists δ > 0 such that for all partition Δ of [*a*, *b*] with norm ||Δ|| < δ, we have for every Riemann sum S for Δ, $|S - L| < \varepsilon$, i.e.,

$$\left|\sum_{i=1}^{n} f(\xi_i)\Delta x_i - L\right| = |R(f, \Delta, \xi) - L| < \varepsilon\,,$$

where Δ is the partition $\Delta\colon a = x_0 < x_1 < \ldots x_{n-1} < x_n = b$ and $\xi = (\xi_1, \xi_2, \ldots \xi_n)$ is given by any choice of ξ_ι in $[x_{i-1}, x_i]$.

If f is Riemann integrable on [*a*,*b*], we then write $\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i = L$and this limit is denoted by $\int_a^b f(x)$.

Definition 12. If $f : [a, b] \to \mathbf{R}$ is integrable on $[a, b]$, then the *definite integral* of f or *Riemann integral* of f from a to b is denoted by $\int_a^b f(x)dx = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$, a is called the *lower limit*, b is called the *upper limit* and $f(x)$ is called the *integrand*.

Remark.

Though Definition 12 is a remarkable elegant definition, it is not easy to use it to prove the properties of the Riemann integral. Thus, we shall look at an approach by Darboux using his so called Darboux sums, which are generalization of the upper and lower sums. This approach is particularly useful for verification of the properties of the Riemann integral. We shall develop this after the following example.

Example 13 $\int_0^1 xdx = \frac{1}{2}$.

Take any partition: P: $x_0 = 0 < x_1 < \ldots < x_n = 1$. Let ξ_i be any point in $[x_{i-1}, x_i]$. Let $f(x) = x$. The Riemann sum,

$$R(f, P, \xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i \le \sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} x_i \Delta x_i = \sum_{i=1}^{n} x_i (x_i - x_{i-1})$$

$$\le \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2 + (x_i - x_{i-1})^2) \le \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2) + \frac{1}{2}\sum_{i=1}^{n}(x_i - x_{i-1})\|P\| = \frac{1}{2} + \frac{1}{2}\|P\|$$

Therefore,

$$R(f, P, \xi) - \frac{1}{2} \le \frac{1}{2}\|P\| \text{ ------------------ (1)}$$

Likewise,

$$R(f, P, \xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i \ge \sum_{i=1}^{n} f(x_{i-1})\Delta x_i = \sum_{i=1}^{n} x_{i-1}\Delta x_i = \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1})$$

$$\ge \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2 - (x_i - x_{i-1})^2) \ge \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2) - \frac{1}{2}\sum_{i=1}^{n}(x_i - x_{i-1})\|P\| = \frac{1}{2} - \frac{1}{2}\|P\|$$

It follows that

$$\frac{1}{2} - R(f, P, \xi) \le \frac{1}{2}\|P\|. \text{ ------------------- (2)}$$

These two inequalities (1) and (2) implies that

$$.\left|R(f, P, \xi) - \frac{1}{2}\right| \le \frac{1}{2}\|P\|.$$

Now given any $\varepsilon > 0$, just take $\delta = 2\varepsilon$. Then for any partition P with $\|P\| < \delta$, $\left|R(f, P, \xi) - \frac{1}{2}\right| \le \frac{1}{2}\|P\| < \varepsilon$. Hence, by Definition 11, f is Riemann integrable on $[0, 1]$ and $\int_0^1 f(x)dx = \frac{1}{2}$.

Let $f : [a, b] \to \mathbf{R}$ be a function. We recall that the function f is *bounded* if its range is bounded. This means there exist real numbers α and β such that $\alpha \le f(x) \le \beta$ for all x in $[a, b]$ or equivalently, there exists positive real number K such that $| f(x) | \le K$ for all x in $[a, b]$. If f is not bounded, then one can have arbitrarily large Riemann sum or arbitrarily negatively large Riemann sum and so f would not be Riemann integrable. (Reference: my article, "Riemann Integral and Bounded Function" on My Calculus Web). Thus any unbounded function is not Riemann integrable.

5.3 Upper and Lower Darboux Sums, Upper and Lower Integrals

Suppose now $f : [a, b] \to \mathbf{R}$ is a bounded function.

Let $P : a = x_0 < x_1 < \ldots < x_n = b$ be a partition for $[a, b]$.
The *upper Darboux sum* with respect to the partition P is defined by

$$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}),$$

where $M_i = \sup\{ f(x) : x \in [x_{i-1}, x_i] \}$. Note that since f is bounded on $[a, b]$, f is bounded on each $[x_{i-1}, x_i]$ and so the supremum M_i exists for each i. Like wise for each i, $m_i = \inf\{ f(x) : x \in [x_{i-1}, x_i] \}$ exists since f is bounded on each $[x_{i-1}, x_i]$. We define the *lower Darboux sum* with respect to the partition P by

$$L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}).$$

Because for each integer i such that $1 \le i \le n$, $m_i \le M_i$, $L(f, P) \le U(f, P)$.

Since f is bounded, there exist real numbers m and M such that $m \le f(x) \le M$ for all x in $[a, b]$. Hence $m \le M_i \le M$ and $m \le m_i \le M$ for $i = 1, 2, \ldots, n$. Therefore, for any partition P the upper Darboux sum

$$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \ge m \sum_{i=1}^{n} (x_i - x_{i-1}) = m(b-a).$$

Hence the set of all upper Darboux sums (over all partitions of $[a, b]$) is *bounded below* by $m(b - a)$. Likewise, the lower Darboux sum

$$L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \le M \sum_{i=1}^{n} (x_i - x_{i-1}) = M(b-a).$$

We conclude that the set of all lower Darboux sums (over all partitions of $[a, b]$) is *bounded above* by $M(b-a)$. We may now make the following definition following Darboux.

Definition 14. Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. Then the *upper Darboux integral* or *upper integral* is defined to be

$$U\int_a^b f = \inf\{U(f, P) : P \text{ a partition of } [a, b]\}.$$

The *lower Darboux integral* or *lower integral* is defined to be

$$L\int_a^b f = \sup\{L(f, P) : P \text{ a partition of } [a, b]\}.$$

Note that by the completeness property of the real numbers, the upper integral exists, because the set of all upper Darboux sum is bounded below and the lower integral exists because the set of all lower Darboux sum is bounded above.

We now observe some obvious properties of the Darboux sums with respect to partition.
Suppose Q and P are partitions of $[a, b]$. We say Q is a *refinement* of P if each partition point of P is also a partition of Q, more precisely if $P \subseteq Q$.

Lemma 15. The Refinement Lemma.
Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. Suppose Q and P are partitions of $[a, b]$ such that Q is a refinement of P. Then

$$L(f, P) \le L(f, Q) \text{ and } U(f, Q) \le U(f, P).$$

Proof. First we shall prove the lemma when Q contains just one additional point y than P. Let P be denoted by $P: \ a = x_0 < x_1 < \ldots < x_n = b$. Suppose $y \in (x_{j-1}, x_j)$ for some j between 1 and n. Then Q is the partition $Q: \ a = x_0 < x_1 < \ldots < x_{j-1} < y < x_j < \ldots < x_n = b$. Let $m_j' = \inf\{ f(x) : x \in [x_{j-1}, y]\}$, $m_j'' = \inf\{ f(x) : x \in [y, x_j]\}$. Then

$$m_j = \inf\{ f(x) : x \in [x_{j-1}, x_j]\} \le m_j', m_j''.$$

Therefore,

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{j-1} m_i(x_i - x_{i-1}) + m_j(y - x_{j-1}) + m_j(x_j - y) + \sum_{i=j+1}^{n} m_i(x_i - x_{i-1})$$

$$\le \sum_{i=1}^{j-1} m_i(x_i - x_{i-1}) + m_j'(y - x_{j-1}) + m_j''(x_j - y) + \sum_{i=j+1}^{n} m_i(x_i - x_{i-1}) = L(f,Q).$$

Let $M_j' = \sup\{ f(x) : x \in [x_{j-1}, y]\}$, $M_j'' = \sup\{ f(x) : x \in [y, x_j]\}$. Then

$$M_j = \sup\{ f(x) : x \in [x_{j-1}, x_j]\} \ge M_j', M_j''.$$

Therefore,

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\ &= \sum_{i=1}^{j-1} M_i(x_i - x_{i-1}) + M_j(y - x_{j-1}) + M_j(x_j - y) + \sum_{i=j+1}^{n} M_i(x_i - x_{i-1}) \\ &\ge \sum_{i=1}^{j-1} M_i(x_i - x_{i-1}) + M_j'(y - x_{j-1}) + M_j''(x_j - y) + \sum_{i=j+1}^{n} M_i(x_i - x_{i-1}) = U(f,Q). \end{aligned}$$

This proves the lemma for the case when Q has just one additional partition point than P.
For the general case, if Q contains k points not in P, then there is a sequence of partitions, $P = P_0, \ P_1, P_2, \ldots, P_k = Q$ where Q is obtained by adding one point at a time. That is P_{i+1}, is obtained by adding one point in Q not in P_i to P_i. Thus by the special case,

$$L(f, P) = L(f, P_0) \le L(f, P_1) \le L(f, P_2) \le \ldots \le (f, P_k) = L(f, Q)$$

and

$$U(f, P) = U(f, P_0) \ge U(f, P_1) \ge U(f, P_2) \ge \ldots \ge U(f, P_k) = U(f, Q).$$

This completes the proof.

The next result is an observation that any lower Darboux sum is less than or equal to any upper Darboux sum.

Theorem 16. Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. Suppose Q and P are partitions of $[a, b]$.

$$L(f, P) \le U(f, Q)$$

Proof. The partition $P\cup Q$ is a refinement of both P and Q. Therefore, by Lemma 15, $L(f, P) \le L(f, P\cup Q)$ and $U(f, P\cup Q) \le U(f, Q)$. Therefore,

$$L(f, P) \le L(f, P\cup Q) \le U(f, P\cup Q) \le U(f, Q).$$

Theorem 17. Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. Then

$$L\int_a^b f \le U\int_a^b f.$$

Proof. By Theorem 16 for any partition P of $[a, b]$, $L(f, P)$ is a lower bound of $\{U(f, Q): Q \text{ a partition of } [a, b]\}$. Therefore,

$$L(f,P) \le \inf\{U(f,Q) : Q \text{ a partition of } [a,b]\} = U\int_a^b f.$$

Thus the upper integral $U\int_a^b f$ is an upper bound of $\{L(f, P): P \text{ a partition of } [a, b]\}$. Therefore, $U\int_a^b f \ge \sup\{L(f,P) : P \text{ a partition of } [a,b]\} = L\int_a^b f$.

5.4 Darboux Integral

For a bounded function we have defined the upper and lower integrals. So instead of one integral we have two. Archimedes had devised a strategy to compute the area of a non-polygonal geometric object by constructing outer and inner polygonal approximations of the object. We have defined the upper and lower integrals, similar in philosophy to the approach of finding successive outer and inner polygonal approximations of a geometric object. Unlike the non-polygonal geometric objects, where the outer and inner polygonal approximations will tend to the same value, there is no guarantee that the two integrals are the same. When they are the same it gives meaning to a generalization of area.

Definition 18. Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. We say f is *Darboux integrable* if the lower and upper integrals are the same, i.e., $L\int_a^b f = U\int_a^b f$.

Remark. This is Darboux's version of the integrability of a bounded function completing the formulation of Riemann. We shall show that this is equivalent to Riemann integrability.

Example 19.

1. The Dirichlet function $h : [0, 1] \to \mathbf{R}$, defined by

$$h(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases},$$

is not *Darboux* integrable.

Let $P : 0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$ be any partition for the interval [0, 1]. By the density of the rational numbers and irrational numbers, in each of the subinterval $[x_{i-1}, x_i]$, $(i = 1,\ldots, n)$ we can always find a rational number and an irrational number. Hence for $i = 1,\ldots, n$, $\{h(x): x \in [x_{i-1}, x_i]\} = \{0, 1\}$. It follows that, for each $i = 1,\ldots, n$,

$M_i(h, P) = \sup\{h(x): x \in [x_{i-1}, x_i]\} = \sup\{0, 1\} = \max\{0, 1\} = 1$ and

$m_i(h, P) = \inf\{h(x): x \in [x_{i-1}, x_i]\} = \inf\{0, 1\} = \min\{0, 1\} = 0$.

Therefore, the upper Darboux sum with respect to the Partition P is

$$U(h,P) = \sum_{i=1}^{n} M_i(h,P)\Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1$$

and the lower Darboux sum with respect to the Partition P is

$$L(h,P) = \sum_{i=1}^{n} m_i(h,P)\Delta x_i = 0.$$

The above statement is true for any partition P for [0, 1]. Hence the lower Darboux integral of h,

$$L\int h = \sup\{L(h,P) : P \text{ is a partition for } [0,1]\} = \max\{0\} = 0$$

and the upper Darboux integral of h,

$$U\int h = \inf\{U(h,P) : P \text{ is a partition for } [0,1]\} = \min\{1\} = 1.$$

Therefore, the lower Darboux integral $L\int h$ is not equal to the upper Darboux integral $U\int h$ and so h is not Darboux integrable over [0, 1] by Definition 18. We observe also that by the above remark about density of the rational and irrational numbers, for any partition P with $\|P\| < \delta$, we can choose rational points ξ_i for each i and so the Riemann sum $R(h, P, \xi) = 0$ and if we choose irrational points ξ_i for each i, the Riemann sum $R(h, P, \xi) = 1$. Hence by Definition 11, h is not Riemann integrable.

2. The function $f : [0, 1]$ defined by $f(x) = x^2$ is Darboux integrable.

Let $P : 0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$ be any partition for the interval [0, 1]. For each $i = 1,\ldots, n$,

$M_i(f, P) = \sup\{f(x): x \in [x_{i-1}, x_i]\} = x_i^2$ and

$m_i(f, P) = \inf\{f(x): x \in [x_{i-1}, x_i]\} = x_{i-1}^2$

Therefore, the upper Darboux sum with respect to the Partition P is

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} M_i(f,P)(x_i - x_{i-1}) = \sum_{i=1}^{n} x_i^2(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} \tfrac{1}{3}(x_i^3 - x_{i-1}^3) + \tfrac{1}{6}(x_i - x_{i-1})^3 + \tfrac{1}{2}(x_i^2 - x_{i-1}^2)(x_i - x_{i-1}) \\ &\le \tfrac{1}{3} + \tfrac{1}{6}\sum_{i=1}^{n}(x_i - x_{i-1})\|P\|^2 + \tfrac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2)\|P\| \\ &\le \tfrac{1}{3} + \tfrac{1}{6}\|P\|^2 + \tfrac{1}{2}\|P\| \end{aligned}$$

because $\sum_{i=1}^{n}(x_i - x_{i-1}) = \sum_{i=1}^{n}(x_i^2 - x_{i-1}^2) = 1$.

Therefore, since we can choose the norm of the partition $\|P\|$ arbitrarily small,

$$U\int_0^1 f \le \frac{1}{3}.$$

Similarly the lower Darboux sum with respect to the partition P

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} m_i(f,P)(x_i - x_{i-1}) = \sum_{i=1}^{n} x_{i-1}^2(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} \tfrac{1}{3}(x_i^3 - x_{i-1}^3) + \tfrac{1}{6}(x_i - x_{i-1})^3 - \tfrac{1}{2}(x_i^2 - x_{i-1}^2)(x_i - x_{i-1}) \end{aligned}$$

$$\geq \tfrac{1}{3} + \tfrac{1}{6}\sum_{i=1}^{n}(x_i - x_{i-1})^3 - \tfrac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2)\|P\|$$
$$> \tfrac{1}{3} - \tfrac{1}{2}\|P\|.$$

Therefore, $L\int_0^1 f \geq L(f,P) > \dfrac{1}{3} - \dfrac{1}{2}\|P\|$. Since $\|P\|$ can be chosen to be arbitrarily small, $L\int_0^1 f \geq \dfrac{1}{3}$. It follows then from Theorem 17 that $U\int_0^1 f = L\int_0^1 f = \dfrac{1}{3}$ since $L\int_a^b f \leq U\int_a^b f$.

5.5 Integrability Criteria

Darboux integrability is equivalent to Riemann integrability as we shall show in this section. It is not always easy to compute the lower and upper integrals just to check whether they are the same to decide on integrability. This amounts to actually computing the integral. It is sufficient to check on the behaviour of the lower and upper sums to see if they are very close to some value or alternatively if their differences are getting smaller and smaller. This is crucial to formulate integrability criteria. We start by giving next a very useful characterization of the upper and lower integrals in terms of sequences.

Proposition 20. Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. Then there exists a sequence of partitions (P_k) of $[a, b]$ such that $P_k \subseteq P_{k+1}$, $\lim_{k\to\infty}\|P_k\| = 0$ and

$$L(f, P_k) \to L\int_a^b f \text{ and } U(f, P_k) \to U\int_a^b f.$$

Proof. By definition of the lower integral and upper integral, there exist partitions P_1' and P_1'' of $[a, b]$ such that

$$L\int_a^b f - 1 < L(f, P_1') \leq L\int_a^b f \quad\text{and}\quad U\int_a^b f \leq U(f, P_1'') < U\int_a^b f + 1\,.$$

Let P_1 be a common refinement of P_1' and P_1'' for which $\|P_1\| < 1$. Then by the Refinement Lemma (Lemma 15)

$$L\int_a^b f - 1 < L(f, P_1) \leq L\int_a^b f \quad\text{and}\quad U\int_a^b f \leq U(f, P_1) < U\int_a^b f + 1\,.$$

Similarly, there exist partitions P_2' and P_2'' of $[a, b]$ such that

$$L\int_a^b f - \frac{1}{2} < L(f, P_2') \leq L\int_a^b f \quad\text{and}\quad U\int_a^b f \leq U(f, P_2'') < U\int_a^b f + \frac{1}{2}\,.$$

Let P_2 be a common refinement of P_1, P_2' and P_2'' for which $\|P_2\| < 1/2$. Then

$$L\int_a^b f - \frac{1}{2} < L(f, P_2) \leq L\int_a^b f \quad\text{and}\quad U\int_a^b f \leq U(f, P_2) < U\int_a^b f + \frac{1}{2}.$$

We now define the sequence $\{P_k\}$ by repeating the above process. Suppose we have defined partition P_k such that $P_{k-1} \subseteq P_k$, $\|P_k\| < 1/k$. By the definition of the lower and upper integrals, there exist partitions P_{k+1}' and P_{k+1}'' of $[a, b]$ such that

$$L\int_a^b f - \frac{1}{k+1} < L(f, P'_{k+1}) \le L\int_a^b f \quad \text{and} \quad U\int_a^b f \le U(f, P''_{k+1}) < U\int_a^b f + \frac{1}{k+1}.$$

Let P_{k+1} be a common refinement of P_k, P_{k+1}' and P_{k+1}'' for which $\|P_{k+1}\| < 1/(k+1)$. Then by the Refinement Lemma,

$$L\int_a^b f - \frac{1}{k+1} < L(f, P_{k+1}) \le L\int_a^b f \quad \text{and}$$

$$U\int_a^b f \le U(f, P_{k+1}) < U\int_a^b f + \frac{1}{k+1} \quad \text{and} \quad P_k \subseteq P_{k+1}.$$

In this way we obtain the sequence of partitions $\{P_k\}$ of $[a, b]$ such that $P_k \subseteq P_{k+1}$ and $\lim_{k\to\infty} \|P_k\| = 0$. In particular, by the definition of convergence of sequence or by the Comparison Test (Proposition 8 Chapter 2), $L(f, P_k) \to L\int_a^b f$ and $U(f, P_k) \to U\int_a^b f$. This completes the proof.

Theorem 21. Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. The following statements are equivalent.

(1) f is Darboux integrable.

(2) There is a sequence (P_k) of partitions of $[a, b]$ such that $\lim_{k\to\infty} (U(f, P_k) - L(f, P_k)) = 0$. Furthermore, for any such sequence, $L(f, P_k) \to \int_a^b f$ and $U(f, P_k) \to \int_a^b f$.

(3) Given $\varepsilon > 0$, there exists a partition P for the interval $[a, b]$ such that the difference $U(f, P) - L(f, P) < \varepsilon$.

(4) f is Riemann integrable, i.e., there exists a number L such that given any $\varepsilon > 0$, we can find a $\delta > 0$ such that for any partition P for $[a, b]$ with norm $\| P \| < \delta$, and for any Riemann sum S with respect to P, $| S - L | < \varepsilon$.

Proof. (1) $\Rightarrow$ (2). Suppose f is Darboux integrable, that is, the lower and upper integrals are the same. Take the sequence of partitions (P_k) of $[a, b]$ given by Proposition 20 such that $L(f, P_k) \to L\int_a^b f$ and $U(f, P_k) \to U\int_a^b f$. Since $L\int_a^b f = U\int_a^b f$,

$$\lim_{k\to\infty} (U(f, P_k) - L(f, P_k)) = 0.$$

(2)$\Rightarrow$ (1). Suppose there is a sequence (P_k) of partitions of $[a, b]$ such that

$$\lim_{k\to\infty}(U(f,P_k)-L(f,P_k)) = 0.$$

For each integer $k \geq 1$, by Theorem 17, $L(f,P_k) \leq L\int_a^b f \leq U\int_a^b f \leq U(f,P_k)$.

Therefore,

$$0 \leq U\int_a^b f - L\int_a^b f \leq U(f,P_k) - L(f,P_k).$$

and so $$0 \leq U\int_a^b f - L\int_a^b f \leq \lim_{k\to\infty}(U(f,P_k)-L(f,P_k)) = 0.$$

Hence, $U\int_a^b f - L\int_a^b f = 0$ and so $L\int_a^b f = U\int_a^b f$ and f is Darboux integrable. Moreover, since $0 \leq U(f,P_k) - U\int_a^b f \leq U(f,P_k)-L(f,P_k)$ for each integer $k \geq 1$, and so by the Comparison Test, $U(f,P_k) \to U\int_a^b f = \int_a^b f$. Similarly, since $0 \leq L\int_a^b f - L(f,P_k) \leq U(f,P_k)-L(f,P_k)$ for each integer $k \geq 1$, by the Comparison Test for sequences, $L(f,P_k) \to L\int_a^b f = \int_a^b f$.

(2) $\Rightarrow$ (3) Suppose there is a sequence (P_k) of partitions of $[a, b]$ such that $\lim_{k\to\infty}(U(f,P_k)-L(f,P_k)) = 0$. Then, given $\varepsilon > 0$, there exists a positive integer N, such that $$n \geq N \Rightarrow U(f,P_n) - L(f,P_n) < \varepsilon.$$

Let $P = P_N$ and we have $U(f,P) - L(f,P) < \varepsilon$.

(3)$\Rightarrow$ (1) Suppose for any $\varepsilon > 0$, there exists a partition P for the interval $[a, b]$ such that the difference $U(f,P) - L(f,P) < \varepsilon$. Thus, for any $\varepsilon > 0$,

$$0 \leq U\int_a^b f - L\int_a^b f \leq U(f,P) - L(f,P) < \varepsilon.$$

Therefore, $U\int_a^b f \leq L\int_a^b f$. It follows then by Theorem 17 that $L\int_a^b f = U\int_a^b f$ and so f is Darboux integrable.

(4) $\Rightarrow$ (3). Assume f is Riemann integrable. Given $\varepsilon > 0$, then there exists $\delta > 0$ such that for any partition P for $[a, b]$ with norm $\| P \| < \delta$, and for any Riemann sum S with respect to P, $| S - L | < \varepsilon/4$. Let T ={ Riemann sum S: S has the same partition P}.
Then for any S in T,

$$L - \varepsilon/4 < S < L + \varepsilon/4 . \quad \text{------------------------} \quad (1)$$

Let P: $a = x_0 < x_1 < \ldots < x_n = b$ be a partition of $[a, b]$ with $\| P \| < \delta$. Let $M_i = \sup\{ f(x) : x \in [x_{i-1}, x_i] \}$ for $i = 1,\ldots, n$. By the definition of supremum, for each i, $1 \leq i \leq n$, there exists c_i in $[x_{i-1}, x_i]$ such that

$$f(c_i) > M_i - \frac{\varepsilon}{4(b-a)}.$$

Using this inequality, the Riemann sum

$$R(f,P,C) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) > \sum_{i=1}^{n}(M_i - \frac{\varepsilon}{4(b-a)})(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) - \frac{\varepsilon}{4} = U(f,P) - \frac{\varepsilon}{4},$$

where, $C=(c_1,\ldots,c_n)$. Therefore,

$$U(f,P) < R(f,P,C) + \frac{\varepsilon}{4} < L + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = L + \frac{\varepsilon}{2} \text{ --------------- (2)}$$

by inequality (1).

Now let $m_i = \inf\{ f(x) : x \in [x_{i-1}, x_i]\}$ for $i = 1,..., n$. By the definition of infimum, for each i such that $1 \le i \le n$, there exists d_i in $[x_{i-1}, x_i]$ satisfying

$$f(d_i) < m_i + \frac{\varepsilon}{4(b-a)}.$$

The Riemann sum

$$R(f,P,d) = \sum_{i=1}^{n} f(d_i)(x_i - x_{i-1}) < \sum_{i=1}^{n}\left(m_i + \frac{\varepsilon}{4(b-a)}\right)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) + \frac{\varepsilon}{4} = L(f,P) + \frac{\varepsilon}{4}$$

where $d=(d_1,\ldots,d_n)$. It follows that

$$L(f,P) > R(f,P,d) - \frac{\varepsilon}{4} > L - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = L - \frac{\varepsilon}{2} \text{ ------------- (3)}$$

by inequality (1). Inequality (2) and (3) implies that, $U(f, P) - L(f, P) < \varepsilon$. Hence (2) follows.

(2) $\Rightarrow$ (4) Suppose there is a sequence (P_k) of partitions of $[a, b]$ such that $\lim_{k\to\infty}(U(f,P_k) - L(f,P_k)) = 0$. This means given $\varepsilon > 0$, there exists a partition P_j : $a = x_0 < x_1 < ... < x_L = b$ with $L > 1$ such that

$$U(f,P_j) - L(f,P_j) < \frac{\varepsilon}{2}.$$

Denote the partition P_j by P. Let $K = \min\{x_i - x_{i-1} : i = 1, ..., L\}$. Since f is bounded, there exists $M > 0$ such that $|f(x)| < M$. First we shall specify our δ for the partition. Let R: $a = y_0 < y_1 < ... < y_N = b$ be any partition such that $\|R\| < \delta$, where

$$\delta = \min\left(K, \frac{\varepsilon}{2(2ML+1)}\right).$$

Then because $\|R\| < K$, the number of subintervals of R, N must be strictly bigger than L, which is the number of subintervals of P. Moreover because $K = \min\{x_i - x_{i-1} : i = 1, ..., L\}$ and for each $i = 1,2,.., N$, $y_i - y_{i-1} \le \|R\| < K$, each subinterval $[y_{i-1}, y_i]$ can contain at most one point from P. Therefore, each x_i for $i = 1,2,.., L-1$ must belong to one and only one subinterval $[y_{j_i-1}, y_{j_i}]$ for some j_i, $1 \le j_i \le N-1$, i.e., for $i = 1,2,.., L-1$, $y_{j_i-1} \le x_i \le y_{j_i}$. Let $I = \{j_i : i = 1, \ldots, L-1\}$ $I = \{j_i : i = 1,..., L-1\}$. Then for any Riemann sum with respect to the partition R,

$$R(f,R,\xi) = \sum_{i=1}^{N} f(\xi_i)(y_i - y_{i-1}),$$

where $\xi_i \in [y_{i-1}, y_i]$, $\xi = (\xi_1, \xi_2, \ldots \xi_n)$

$$= \sum_{i\notin I} f(\xi_i)(y_i - y_{i-1}) + \sum_{i=1}^{L-1} f(\xi_{j_i})(y_{j_i} - y_{j_i-1})$$

$$= \sum_{i\notin I} f(\xi_i)(y_i - y_{i-1}) + \sum_{i=1}^{L-1} f(x_i)(y_{j_i} - y_{j_i-1}) + \sum_{i=1}^{L-1} (f(\xi_{j_i}) - f(x_i))(y_{j_i} - y_{j_i-1})$$

$$= \left(\sum_{i\notin I} f(\xi_i)(y_i - y_{i-1}) + \sum_{i=1}^{L-1} f(x_i)(y_{j_i} - x_i) + \sum_{i=1}^{L-1} f(x_i)(x_i - y_{j_i-1})\right)$$

$$+ \sum_{i=1}^{L-1} (f(\xi_{j_i}) - f(x_i))(y_{j_i} - y_{j_i-1})$$

Note that the bracketed term is a Riemann sum S for the partition $R \cup P$. Thus

$$R(f,R,\xi) = S + \sum_{i=1}^{L-1} (f(\xi_{j_i}) - f(x_i))(y_{j_i} - y_{j_i-1})$$

$\le U(f,R\cup P)+\ 2\sum_{i=1}^{L-1} M(y_{j_i}-y_{j_i-1}) \le U(f,P)+\ 2\sum_{i=1}^{L-1} M(y_{j_i}-y_{j_i-1})$

by the Refinement Lemma

$< L(f,P)+\frac{\varepsilon}{2}+\ 2\sum_{i=1}^{L-1} M\|R\| < L(f,P)+\frac{\varepsilon}{2}+2ML\|R\|$

$< L\int_a^b f+\frac{\varepsilon}{2}+2ML\frac{\varepsilon}{2(2ML+1)} \le L\int_a^b f+\varepsilon$ since $\|R\|<\delta\le\frac{\varepsilon}{2(2ML+1)}$.

We have thus proved that

$$R(f,R,\xi) < L\int_a^b f+\varepsilon \qquad \text{------------------} \quad (4)$$

Similarly, $R(f,R,\xi)=\sum_{i=1}^{N} f(\xi_i)(y_i-y_{i-1}) = S+\sum_{i=1}^{L-1}\ (f(\xi_{j_i})-f(x_i))(y_{j_i}-y_{j_i-1})$

$\ge L(f,R\cup P)+\sum_{i=1}^{L-1}\ (f(\xi_{j_i})-f(x_i))(y_{j_i}-y_{j_i-1})$

$\ge L(f,P)+\sum_{i=1}^{L-1}\ (f(\xi_{j_i})-f(x_i))(y_{j_i}-y_{j_i-1})$

$> U(f,P)-\frac{\varepsilon}{2}-\sum_{i=1}^{L-1} 2M(y_{j_i}-y_{j_i-1}) > U(P,f)-\frac{\varepsilon}{2}-2\sum_{i=1}^{L-1} M\|R\|$

$> U\int_a^b f-\frac{\varepsilon}{2}-2ML\|R\|$

$> U\int_a^b f-\frac{\varepsilon}{2}-2ML\frac{\varepsilon}{2(2ML+1)} > U\int_a^b f-\varepsilon$,

i.e.

$$R(f,R,\xi) > U\int_a^b f-\varepsilon \qquad \text{----------------------} \ (5)$$

Since $L\int_a^b f = U\int_a^b f = C$ (because (2) $\Rightarrow$ (1)), it follows from inequalities (4) and (5) that $|R(f,R,\xi)-L|<\varepsilon$. We have thus shown that there exists a real number C such that for any partition R with norm $\|R\| < \delta$ and for any Riemann sum $S = R(f, R, \xi)$ with respect to R, $|S-C|<\varepsilon$. Hence, f is Riemann integrable .

Remark.

1. Note that we have given the proof of Theorem 21 in more than one ways. Most oftenly used part of the theorem will be the equivalence of statements (1) to (3). Note that (4) $\Rightarrow$ (3) $\Rightarrow$ (1) $\Leftrightarrow$ (2) $\Rightarrow$ (4).
2. Theorem 21 is a very useful tool for further development of the integral.
3. In view of Theorem 21, we shall simply say a function is integrable whenever anyone of the equivalent conditions in Theorem 21 is met.

Integrability of Monotone Function

Theorem 22. Suppose $f : [a, b] \to \mathbf{R}$ is a monotone function. Then f is Riemann integrable.

Proof. Since f is monotone on the closed and bounded interval $[a, b]$, f is bounded. Suppose that f is increasing. Let $P: \ a = x_0 < x_1 < \dots < x_n = b$ be a partition for $[a, b]$. Then

$$U(f,P)-L(f,P)=\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1}) \le \|P\| \sum_{i=1}^{n}(M_i-m_i)$$

Since f is increasing, $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Therefore,

$$U(f,P)-L(f,P) \le \|P\| \sum_{i=1}^{n}(f(x_i)-f(x_{i-1})) = \|P\|(f(b)-f(a))$$

If $f(b) = f(a)$, then given any $\varepsilon > 0$, for any partition P, $U(f, P) - L(f, P) = 0 < \varepsilon$.
If $f(b) \neq f(a)$, take any partition P such that $\|P\| < \varepsilon/(f(b) - f(a))$. Then

$$U(f,P) - L(f,P) \leq \|P\|(f(b) - f(a)) < \varepsilon.$$

Therefore, by Theorem 21, f is Riemann integrable. The proof is similar when f is decreasing.

Integrability of Continuous Function

Next we have a theorem which confirms that definite integrals do exist at least on continuous functions.

Theorem 23. Every function which is continuous on the closed interval $[a, b]$ is (*Riemann*) *integrable* on $[a, b]$.

Proof. If $f : [a, b] \to \mathbf{R}$ is continuous, then it is also uniformly continuous. (Reference: Theorem 29 Chapter 3.). Therefore, given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, y in $[a, b]$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/(b-a). \qquad \text{------------} \quad (1)$$

Let $P : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be a partition with norm $\|P\| < \delta$, which is given by (1) above. For $i = 1, \ldots, n$, let $M_i = \sup\{ f(x) : x \in [x_{i-1}, x_i]\}$. Since f is continuous on $[x_{i-1}, x_i]$, for each i, by the Extreme Value Theorem, $M_i = f(c_i)$ for some c_i *in* $[x_{i-1}, x_i]$.
Similarly for each $i = 1, \ldots, n$, let $m_i = \inf\{ f(x) : x \in [x_{i-1}, x_i]\}$. Again by the Extreme Value Theorem, for each $i = 1, \ldots, n$, there exists d_i *in* $[x_{i-1}, x_i]$ such that $m_i = f(d_i)$. The upper Darboux sum with respect to P is

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

and the lower Darboux sum with respect to P is

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} f(d_i)(x_i - x_{i-1}).$$

It follows that the difference

$$U(f,P) - L(f,P) = \sum_{i=1}^{n}(f(c_i) - f(d_i))(x_i - x_{i-1}) = \sum_{i=1}^{n} |f(c_i) - f(d_i)|(x_i - x_{i-1})$$

$$< \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(x_i - x_{i-1}) \qquad \text{by (1) since } |c_i - d_i| \leq \|P\| < \delta,\ 1 \leq i \leq n.$$

Therefore, $U(f,P) - L(f,P) < \dfrac{\varepsilon}{b-a} \sum_{i=1}^{n}(x_i - x_{i-1}) = \dfrac{\varepsilon}{b-a}(x_n - x_0) = \varepsilon.$
Thus, by Theorem 21, f is Riemann integrable. This completes the proof.

Remark. More generally, suppose $f : [a, b] \to \mathbf{R}$ is a bounded function, that is, for some real numbers M and L, $M < f(x) < L$ for all x in $[a, b]$. Suppose f is continuous except possibly on a subset of "measure zero". Then f is integrable. Indeed the converse is also true: f is integrable implies that f is continuous except on a subset of "measure zero". This result is stated by Darboux but is often referred to as Lebesgue Theorem. By a set of measure zero we mean that the points of discontinuity can be enclosed by at most countably infinite set of intervals whose total length is arbitrarily small. Any finite or countable set is of measure zero. The set of rational numbers is of measure zero. (See Chapter 14 for a definition of the Lebesgue measure of a set.) This gives another way of proving Theorem 22 since any monotone function on $[a, b]$ can have at most countable number of discontinuities.

Example 24. $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$. f is not continuous at 0 but f is integrable on $[-1,1]$.

Let $P : -1 = x_0 < x_1 < x_2 < \ldots < x_n = 1$ be a partition of $[-1,1]$. Let $\Delta x_i = x_i - x_{i-1}$.
Let

$$R(f, P, \xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i \text{ ----------------------- } (1)$$

be a Riemann sum with respect to the partition P and $\xi = (\xi_1, \xi_2, \ldots \xi_n)$, where ξ_i is in $[x_{i-1}, x_i]$.
If $\xi_i \neq 0$ for $i = 1, \ldots, n$, then $f(\xi_i) = 0$ for all $i = 1, \ldots, n$, and so $R(f, P, \xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = 0$. If $\xi_j = 0$ for some $1 \leq j \leq n$, then $f(\xi_i) = 0$ fo $i \neq j, j-1$ or $i \neq j, j+1$, whichever makes sense.
The Riemann sum

$$R(f, P, \xi) = f(\xi_j)\Delta x_j + f(\xi_{j+1})\Delta x_{j+1} \text{ or } R(f, P, \xi) = f(\xi_{j-1})\Delta x_{j-1} + f(\xi_j)\Delta x_j.$$

Since $|f(x)| \leq 1$ for all x in $[-1, 1]$,

$$|R(f, P, \xi)| \leq \begin{cases} |\Delta x_j| + |\Delta x_{j+1}| \\ \text{or } |\Delta x_{j-1}| + |\Delta x_j| \end{cases} \leq 2\|P\|.$$

Given any $\varepsilon > 0$, we can take any partition P with $\|P\| < \delta = \frac{\varepsilon}{2}$ and $|R(f, P, \xi) - 0| \leq 2\|P\| < \varepsilon$. Therefore, f is Riemann integrable on $[-1, 1]$ and $\int_{-1}^{1} f(x) = 0$.

Example 25 $\int_a^b 1dx = (b - a)$.

Proof. Let $f(x) = 1$, Let $P : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be a partition of $[a, b]$. Since for any constant function, maximum and minimum are the same and so the upper Darboux sum and the lower Darboux sum for the partition P are the same and it follows that any constant function is Riemann integrable. Therefore, f is Riemann integrable and the integral is given by the limit of a sequence of lower sums of a sequence of partitions by Theorem 21. Any lower sum $L(f,P)=\sum_{i=1}^{n} m_i(x_i-x_{i-1})=\sum_{i=1}^{n}(x_i-x_{i-1})=x_n-x_0=b-a$. Take (P_k) to be a sequence of partitions such that $\|P_k\|\to 0$. Then $L(f, P_k)\to(b-a)$ since $L(f, P_k)$ is a constant sequence and so $\int_a^b 1dx=(b-a)$.

5.6 Properties of the Riemann Darboux integral

Theorem 26. Suppose $f:[a, b]\to \mathbf{R}$ is a bounded function. If f is bounded and integrable on the closed intervals $[a, c]$ and $[c, b]$, where $a < c < b$, then f is integrable on $[a, b]$ and $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$.

Proof. By Theorem 21, there exists a sequence (P_n) of partitions of $[a, c]$ such that

$$\lim_{n\to\infty}(U(f,P_n)-L(f,P_n)) = 0 \text{ and } L(f,P_n)\to\int_a^c f .$$

Similarly, there exists a sequence (Q_n) of partitions of $[c, b]$ such that

$$\lim_{n\to\infty}(U(f,Q_n)-L(f,Q_n)) = 0 \text{ and } L(f,Q_n)\to\int_c^b f .$$

Then $(P_n\cup Q_n)$ is a sequence of partitions of $[a, b]$. Note that

$$U(f, P_n\cup Q_n)=U(f, P_n)+U(f, Q_n)$$

and $$L(f, P_n\cup Q_n) = L(f, P_n)+L(f, Q_n).$$

Therefore,

$$\begin{aligned}&\lim_{n\to\infty}(U(f,P_n\cup Q_n)-L(f,P_n\cup Q_n))\\&=\lim_{n\to\infty}(U(f,P_n)+U(f,Q_n)-L(f,P_n)-L(f,Q_n))\\&=\lim_{n\to\infty}(U(f,P_n)-L(f,P_n))+\lim_{n\to\infty}(U(f,Q_n)-L(f,Q_n)) = 0+0=0.\end{aligned}$$

It follows by Theorem 21 that f is Riemann integrable on $[a, b]$. Moreover,

$$L(f,P_n\cup Q_n)=L(f,P_n)+L(f,Q_n)\to\int_a^c f+\int_c^b f .$$

Hence $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$.

Theorem 27. Suppose $f:[a, b]\to \mathbf{R}$ is a bounded function. If f is integrable on $[a, b]$, then for any c in (a, b), f is integrable on $[a, c]$ and on $[c, b]$ and

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx .$$

Proof. By Theorem 21, since f is Riemann integrable, there is a sequence (P_k) of partitions of $[a, b]$ such that

$$\lim_{k\to\infty}(U(f,P_k)-L(f,P_k)) = 0.$$

We may assume that the point c belongs to each partition P_k . We explain this as follows. Let $Q_k = P_k \cup \{c\}$, then by the Refinement Lemma, for each positive integer n,

$$U(f, Q_k) - L(f, Q_k) \le U(f, P_k) - L(f, P_k).$$

Therefore, by the Comparison Test for sequences, $\lim_{k\to\infty}(U(f,Q_k)-L(f,Q_k)) = 0$. If $c \notin P_k$, we replace P_k by Q_k. Hence we may assume that c belongs to partition P_k for each positive integer k. Therefore, each partition P_k is a union of a partition P_k' for $[a, c]$ and a partition P_k'' for $[c, b]$. In particular,

$$U(f, P_k) = U(f, P_k') + U(f, P_k'')$$

and
$$L(f, P_k) = L(f, P_k') + L(f, P_k'').$$

Thus, $U(f, P_k) - L(f, P_k) = U(f, P_k') - L(f, P_k') + U(f, P_k'') - L(f, P_k'')$

and so

$$U(f, P_k') - L(f, P_k') \le U(f, P_k) - L(f, P_k)$$

and
$$U(f, P_k'') - L(f, P_k'') \le U(f, P_k) - L(f, P_k).$$

Therefore, by the Comparison Test for sequences,

$$\lim_{k\to\infty}(U(f,P_k')-L(f,P_k')) = 0$$

and
$$\lim_{k\to\infty}(U(f,P_k'')-L(f,P_k'')) = 0$$

It follows by Theorem 21 that f is Riemann integrable on $[a, c]$ and on $[c, b]$. Moreover,

$$L(f,P_k) = L(f,P_k') + L(f,P_k'') \to \int_a^c f + \int_c^b f.$$

Thus, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Next we shall list the properties of the Darboux sums which we shall use later.

Lemma 28. Suppose $f : [a, b] \to \mathbf{R}$ and $g : [a, b] \to \mathbf{R}$ are bounded functions. Let $P : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be a partition of $[a, b]$. Then

$$L(f, P) + L(g, P) \le L(f + g, P)$$

and
$$U(f, P) + U(g, P) \ge U(f + g, P). \qquad \text{---------------------------- (A)}$$

Furthermore, for any real number k,

$L(kf, P) = k\,L(f, P)$ and $U(kf, P) = k\,U(f, P)$ if $k \ge 0$

$U(kf, P) = k\,L(f, P)$ and $L(kf, P) = k\,U(f, P)$ if $k < 0$ --------------------- (B).

Proof. These are observation regarding properties of infimum and supremum.
First we set up some notation. Suppose $h : [a, b] \to \mathbf{R}$ is a bounded function and $P : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ is a partition of $[a, b]$. For each i =1, …, n, let $M_i(h, P) = \sup\{h(x): x \in [x_{i-1}, x_i]\}$ and $m_i(h, P) = \inf\{h(x): x \in [x_{i-1}, x_i]\}$. We begin by examining the components of the lower sums.

$m_i(f, P) = \inf\{f(x): x \in [x_{i-1}, x_i]\}$ is a lower bound of $\{f(x): x \in [x_{i-1}, x_i]\}$ and $m_i(g, P) = \inf\{g(x): x \in [x_{i-1}, x_i]\}$ is a lower bound of $\{g(x): x \in [x_{i-1}, x_i]\}$. Therefore,

$$m_i(f, P) + m_i(g, P) \le f(x) + g(x) \text{ for all } x \in [x_{i-1}, x_i].$$

It follows that $m_i(f, P) + m_i(g, P)$ is a lower bound of $\{f(x) + g(x): x \in [x_{i-1}, x_i]\}$. Therefore,

$$m_i(f, P) + m_i(g, P) \le \inf\{(f + g)(x) : x \in [x_{i-1}, x_i]\} = m_i(f + g, P) \text{ --------- (1)}$$

by the definition of infimum,

and so

$$L(f, P) + L(g, P) = \sum_{i=1}^{n} m_i(f,P)(x_i - x_{i-1}) + \sum_{i=1}^{n} m_i(g,P))(x_i - x_{i-1})$$
$$\leq \sum_{i=1}^{n} m_i(f+g,P)(x_i - x_{i-1}) = L(f+g,P).$$

This proves the first inequality of (A). We proceed with the second inequality of (A) similarly. By definition $M_i(f, P)$ is an upper bound of $\{f(x): x \in [x_{i-1}, x_i]\}$ and $M_i(g, P)$ is an upper bound of $\{g(x): x \in [x_{i-1}, x_i]\}$. Therefore, for all $x \in [x_{i-1}, x_i]$, $M_i(f, P) + M_i(g, P) \geq f(x) + g(x)$. It follows that $M_i(f, P) + M_i(g, P)$ is an upper bound for the set $\{(f+g)(x) : x \in [x_{i-1}, x_i]\}$. Hence, by the definition of supremum,

$$M_i(f, P) + M_i(g, P) \geq \sup\{(f+g)(x) : x \in [x_{i-1}, x_i]\} = M_i(f+g, P) \text{ ------- (2)}$$

Then

$$U(f, P) + U(g, P) = \sum_{i=1}^{n} M_i(f,P)(x_i - x_{i-1}) + \sum_{i=1}^{n} M_i(g,P))(x_i - x_{i-1})$$
$$\geq \sum_{i=1}^{n} M_i(f+g,P)(x_i - x_{i-1}) = U(f+g,P)$$

by (2).

This proves the second inequality of (A). To prove the inequality (B), note that for a bounded set of real numbers, A, for any $k \geq 0$, $\inf(kA) = k\inf(A)$, $\sup(kA) = k\sup(A)$. Also for $k < 0$, $\inf(kA) = k\sup(A)$, $\sup(kA) = k\inf(A)$. It follows that for each $i = 1, \ldots, n$,

$m_i(kf, P) = k\, m_i(f, P)$ and $M_i(kf, P) = k\, M_i(f, P)$ if $k \geq 0$ and
$m_i(kf, P) = k\, M_i(f, P)$ and $M_i(kf, P) = k\, m_i(f, P)$ if $k < 0$.

Therefore, it follows from this set of equalities that
$L(kf, P) = k\,L(f, P)$ and $U(kf, P) = k\,U(f, P)$ if $k \geq 0$;
$U(kf, P) = k\,L(f, P)$ and $L(kf, P) = k\,U(f, P)$ if $k < 0$.

Theorem 29. Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. If f is Riemann integrable, then for any real number k, kf is Riemann integrable and

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx.$$

Proof. By Theorem 21, since f is integrable, there is a sequence (P_n) of partitions of $[a, b]$ such that $\lim_{n\to\infty}(U(f,P_n) - L(f,P_n)) = 0$ and that $L(f,P_n) \to \int_a^b f$ and $U(f,P_n) \to \int_a^b f$.

If $k \geq 0$, then

$$\lim_{n\to\infty}(U(kf,P_n) - L(kf,P_n)) = \lim_{n\to\infty}(kU(f,P_n) - kL(f,P_n))$$
$$= k\lim_{n\to\infty}(U(f,P_n) - L(f,P_n)) = 0$$

If $k < 0$, then

$$\lim_{n\to\infty}(U(kf,P_n) - L(kf,P_n)) = \lim_{n\to\infty}(kL(f,P_n) - kU(f,P_n))$$
$$= -k\lim_{n\to\infty}(U(f,P_n) - L(f,P_n)) = 0.$$

Therefore, by Theorem 21, kf is integrable and $L(kf,P_n) = kL(f,P_n) \to k\int_a^b f$ if $k \geq 0$ and $L(kf,P_n) = kU(f,P_n) \to k\int_a^b f$ if $k < 0$ and so $\int_a^b kf = k\int_a^b f$.

Theorem 30. Suppose $f : [a, b] \to \mathbf{R}$ and $g : [a, b] \to \mathbf{R}$ are bounded functions. If f and g are integrable, then $f + g$ is integrable and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Proof. Since f and g are integrable, by Theorem 21 there exist a sequence (P_n) of partitions of $[a, b]$ such that $\lim_{n\to\infty}(U(f,P_n)-L(f,P_n)) = 0$ and a sequence (Q_n) of partitions of $[a, b]$ such that $\lim_{n\to\infty}(U(g,Q_n)-L(g,Q_n)) = 0$. Take the sequence of partition (R_n), where $R_n = P_n \cup Q_n$ for each positive integer n. By the Refinement Lemma, for each positive integer n,

$$L(f, P_n) \le L(f, R_n) \le U(f, R_n) \le U(f, P_n)$$

and

$$L(g, Q_n) \le L(g, R_n) \le U(g, R_n) \le U(g, Q_n).$$

It follows that $0 \le U(f, R_n) - L(f, R_n) \le U(f, P_n) - L(f, P_n)$ -------------------- (1)

and $\quad 0 \le U(g, R_n) - L(g, R_n) \le U(g, Q_n) - L(g, Q_n)$. ------------------ (2)

Because of inequalities (1) and (2), $\lim_{n\to\infty}(U(f,P_n)-L(f,P_n)) = 0$ and that $\lim_{n\to\infty}(U(g,Q_n)-L(g,Q_n)) = 0$, by the Comparison Test for sequences, $\lim_{n\to\infty}(U(f,R_n)-L(f,R_n)) = 0$ and $\lim_{n\to\infty}(U(g,R_n)-L(g,R_n)) = 0$. Consequently, (see for example Theorem 21 (2)), we have that

$$L(f,R_n) \to \int_a^b f\ ,\ U(f,R_n) \to \int_a^b f\ \ L(g,R_n) \to \int_a^b g\ ,\ U(g,R_n) \to \int_a^b g. \text{ ------ (3)}$$

Now by Lemma 28, for each positive integer n,

$$L(f, R_n) + L(g, R_n) \le L(f+g, R_n) \le U(f+g, R_n) \le U(f, R_n) + U(g, R_n) \text{ ----- (4)}$$

Therefore,

$$0 \le U(f+g, R_n) - L(f+g, R_n) \le U(f, R_n) - L(f, R_n) + U(g, R_n) - L(g, R_n) \text{ ---- (5)}$$

Since

$$\lim_{n\to\infty}(U(f,R_n)-L(f,R_n)+U(g,R_n)-L(g,R_n))$$
$$= \lim_{n\to\infty}(U(f,R_n)-L(f,R_n)) + \lim_{n\to\infty}(U(g,R_n)-L(g,R_n)) = 0,$$

it follows from the inequality (5) and the Comparison Test for sequences,

$$\lim_{n\to\infty}(U(f+g,R_n)-L(f+g,R_n)) = 0.$$

Therefore, by Theorem 21 (2), $f + g$ is integrable. By the inequality (4), the limits in (3) and the Squeeze Theorem for sequences, $L(f+g,R_n) \to \int_a^b f + \int_a^b g$. But $L(f+g,R_n) \to \int_a^b (f+g)$ and so $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Remark. A proof using Theorem 21(4) is presented in Ng Tze Beng's "Calculus, an introduction", 9.4.3.

Theorem 31 Suppose $f : [a, b] \to \mathbf{R}$ is a bounded functions. If $f \ge 0$ and integrable, then $\int_a^b f \ge 0$.

Proof. Since f is integrable, by Theorem 21 (2), there exists a sequence (P_n) of partitions of $[a, b]$ such that $L(f,P_n) \to \int_a^b f$. Since $f \ge 0$, $L(f, P_n) \ge 0$ for each positive integer n. $\int_a^b f$ being the limit of the sequence $(L(f, P_n))$ is therefore greater than or equal to 0.

Theorem 32. Suppose $f : [a, b] \to \mathbf{R}$ and $g : [a, b] \to \mathbf{R}$ are bounded functions. If f and g are integrable on $[a, b]$ and $f(x) \ge g(x)$ for all x in $[a, b]$, then

$$\int_a^b f \geq \int_a^b g.$$

Proof . Since $f(x) \geq g(x)$ for all x in $[a, b]$, $f - g \geq 0$ and so by Theorem 31 $\int_a^b (f - g) \geq 0$. But by Theorem 29 and Theorem 30,

$$\int_a^b (f - g) = \int_a^b f + \int_a^b (-g) = \int_a^b f - \int_a^b g.$$

It follows that $\int_a^b f - \int_a^b g \geq 0$ and so $\int_a^b f \geq \int_a^b g$.

Remark. One can give a proof using either the upper Darboux sum or the lower Darboux sum and Theorem 21 (2).

We next state a useful observation.

Proposition 33. Suppose $f : [a, b] \to \mathbf{R}$ is bounded and integrable. If $g : [a, b] \to \mathbf{R}$ is a function such that $g = f$ except on a finite set of points in $[a, b]$, then g is integrable and $\int_a^b g = \int_a^b f$.

Proof. $g(x) - f(x) = 0$ for all x but a finite set of points in $[a, b]$. This means $g - f$ is a linear combination of functions of the type considered in example 24. Therefore, by Example 24, Theorem 29 and 30, $g - f$ is integrable and $\int_a^b (g - f) = 0$. Since $g = (g - f) + f$ and both $(g - f)$ and f are integrable, by Theorem 30, g is integrable and

$$\int_a^b g = \int_a^b (f + (g - f)) = \int_a^b f + \int_a^b (g - f) = \int_a^b f + 0 = \int_a^b f.$$

This completes the proof.

Remark. In fact much more is true. By Lebesgue Theorem, if $g = f$ except on a set of "measure" 0 in $[a, b]$, and if f is integrable, then g is integrable and $\int_a^b g = \int_a^b f$.

Theorem 34. Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. Suppose f is continuous on (a, b). Then f is integrable and the integral does not depend on the values at the end points of the interval.

Proof. We shall use the equivalent condition (3) for integrability in Theorem 21. That is, we shall show that given any $\varepsilon > 0$, there is a partition P for the interval $[a, b]$ such that the difference $U(f, P) - L(f, P) < \varepsilon$.
Since f is bounded, there exists $K > 0$ such that $|f(x)| \leq K$ for all x in $[a, b]$.
Let (a_n) be a decreasing sequence in $(a, (a+b)/2)$ such that $a_n \to a$. Let (b_n) be an increasing sequence in $((a+b)/2, b)$ such that $b_n \to b$. Given any $\varepsilon > 0$, there exists a positive integer N such that $n \geq N$ implies that $0 < a_n - a < \varepsilon / (8K)$ and $0 < b - b_n < \varepsilon / (8K)$. Let m be any fixed integer $\geq N$. Since f is continuous, f is continuous on the closed and bounded subinterval $[a_m, b_m]$. Therefore, by Theorem 23, the restriction of f to $[a_m, b_m]$ is integrable. By Theorem 21 (3), there exists a partition P for the interval $[a_m, b_m]$ such that

$$U(f|_{[a_m, b_m]}, P) - L(f|_{[a_m, b_m]}, P) < \varepsilon/2.$$

Adding in the two end points a and b we get a partition Q for $[a, b]$. That is, $Q = P \cup \{a, b\}$. Then the upper Darboux sum with respect to Q is

$$U(f, Q) = \sup\{f(x) : x \in [a, a_m]\}(a_m - a) + U(f|_{[a_m, b_m]}, P)$$

$$+\sup\{f(x): x\in[b_m,b]\}(b-b_m)$$
$$\le K(a_m-a)+K(b-b_m)+U(f\mid_{[a_m,b_m]},P)<\varepsilon/4+U(f\mid_{[a_m,b_m]},P)$$

Likewise, the lower Darboux sum with respect to Q is

$$L(f,Q)=\inf\{f(x): x\in[a,a_m]\}(a_m-a)+L(f\mid_{[a_m,b_m]},P)$$
$$+\inf\{f(x): x\in[b_m,b]\}(b-b_m)$$
$$\ge -K(a_m-a)+L(f\mid_{[a_m,b_m]},P)-K(b-b_m)>-\varepsilon/4+L(f\mid_{[a_m,b_m]},P).$$

Therefore,

$$U(f,\ Q)-L(f,Q)<\varepsilon/2+U(f\mid_{[a_m,b_m]},P)-L(f\mid_{[a_m,b_m]}\ ,P).$$
$$<\varepsilon/2+\varepsilon/2=\varepsilon$$

It follows by Theorem 21 (3) that f is integrable. By Proposition 33 the integral is independent of the values of f at the end points of the interval.

We can show much more.

Given any integer k in $\boldsymbol{P}$, there exist integers n_k and m_k such that $0<a_{n_k}-a<\dfrac{1}{8kK}$ and $0<b-b_{m_k}<\dfrac{1}{8kK}$. By Theorem 21 (3), there exists a partition P_k for the interval $[a_{n_k},b_{m_k}]$ such that

$$U(f\mid[a_{n_k},b_{m_k}],P_k)-L(f\mid[a_{n_k},b_{m_k}],P_k)<\frac{1}{2k}.$$

Without loss of generality we may assume that $n_k < n_{k+1}$ and $m_k < m_{k+1}$ for each integer k in **P**.

Let $Q_k = P_k\cup\{a,b\}$. Then, we can show similarly as above that

$$U(f\ ,Q_k)-L(f\ ,Q_k)<\frac{1}{k}.$$

Hence $\lim\limits_{k\to\infty}(U(f,Q_k)-L(f,Q_k)) =0$. Therefore, by Theorem 21 (2), $U(f,Q_k)\to\int_a^b f$.

But $$U(f,Q_k)=\sup\{f(x): x\in[a,a_{n_k}]\}(a_{n_k}-a)+U(f\mid[a_{n_k},b_{m_k}],P_k)$$
$$+\sup\{f(x): x\in[b_{m_k},b]\}(b-b_{m_k})$$
and $$|\sup\{f(x): x\in[a,a_{n_k}]\}(a_{n_k}-a)+\sup\{f(x): x\in[b_{m_k},b]\}(b-b_{m_k})|$$
$$<\frac{1}{4k}.$$

Since $\lim\limits_{k\to\infty}\dfrac{1}{4k}=0$, by the Comparison Test,

$$\lim_{k\to\infty}(\sup\{f(x): x\in[a,a_{n_k}]\}(a_{n_k}-a)+\sup\{f(x): x\in[b_{m_k},b]\}(b-b_{m_k}))=0.$$

It follows that $\lim\limits_{k\to\infty}U(f\mid[a_{n_k},b_{m_k}],P_k)=\lim\limits_{k\to\infty}U(f,Q_k)=\int_a^b f$. It is clear that $U(f\mid[a_{n_k},b_{m_k}],P_k)$ is independent of the end points of the interval $[a, b]$.

Similarly, we can show that $L(f\mid[a_{n_k},b_{m_k}],P_k)\to\int_a^b f$.

Since $L(f\mid[a_{n_k},b_{m_k}],P_k)\le\int_{a_{n_k}}^{b_{m_k}} f\le U(f\mid[a_{n_k},b_{m_k}],P_k)$, by the Squeeze Theorem for sequences, $\lim\limits_{k\to\infty}\int_{a_{n_k}}^{b_{m_k}} f=\int_a^b f$.

Remark. Note that the only use of the continuity of f on (a, b) in the proof above is to deduce the integrability of f on the subintervals $[a_m , b_m]$.

Example 35.

(1) Let $f(x)=\begin{cases} x\ , x\neq 0\\ 2\ , x=0\end{cases}$.

By Proposition 33, $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} xdx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$ assuming the *Fundamental Theorem of Calculus*.

(2) Let $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 2x^3 - 1, & 1 \le x \le 2 \end{cases}$.

$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$ by Theorem 34 and Theorem 26

$= \int_0^1 x^2 dx + \int_1^2 (2x^3 - 1)dx$ and assuming the *Fundamental Theorem of Calculus*

$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^4}{2} - x\right]_1^2 = \frac{1}{3} + (8-2) - (\frac{1}{2} - 1) = \frac{1}{3} + 6 + \frac{1}{2} = 6\frac{5}{6}$.

(3) $\int_0^3 [x]dx = \int_0^1 [x]dx + \int_1^2 [x]dx + \int_2^3 [x]dx = \int_0^1 0dx + \int_1^2 1dx + \int_2^3 2dx$ by Proposition 33

$= 0 + 1 \cdot (2-1) + 2 \cdot (3-2) = 0 + 1 + 2 = 3$.

Riemann Sums Convergence Theorem

We have seen in Theorem 21 (2) that if $f : [a, b] \to \mathbf{R}$ is integrable, then the integral is the limit of a sequence of lower Darboux sums with respect to a sequence of partitions of $[a, b]$. We shall show next that the integral is the limit of any sequence of Riemann sums with respect to a sequence of partitions $\{P_n\}$ of $[a, b]$ so long as $\|P_n\| \to 0$.

Theorem 36. Suppose $f : [a, b] \to \mathbf{R}$ is Riemann integrable on $[a, b]$ and (P_n) is a sequence of Partition of $[a, b]$. Let $R(f, P_n, C_n)$ be a Riemann sum with respect to the partition P_n with C_n being any choice of points in the subintervals of P_n. If $\|P_n\|$ converges to 0 as n tends to infinity, then the sequence of Riemann sums

$$(R(f, P_n, C_n))$$

tends to $\int_a^b f$.

Proof. Since f is integrable, by Theorem 21 (4) given any $\varepsilon > 0$, we can find a $\delta > 0$ such that for any partition P for $[a, b]$ with norm $\| P \| < \delta$, and for any Riemann sum S with respect to P, $|S - \int_a^b f| < \varepsilon$. Since $\|P_n\|$ converges to 0, we can find an integer N such that for any $n \ge N$, $\|P_n\| < \delta$. Therefore, because each $R(f, P_n, C_n)$ is a Riemann sum with respect to P_n, $|R(f, P_n, C_n) - \int_a^b f| < \varepsilon$ for all $n \ge N$. This means the sequence $(R(f, P_n, C_n))$ converges to $\int_a^b f$ as n tends to infinity.

Remark. The usual application of the above theorem is to a sequence of partitions (P_n), where $\|P_n\| = (b - a)/n$ as is the case when P_n is a regular partition and the Riemann sum $R(f, P_n, C_n)$ with respect to P_n is chosen such that C_n the choice of points in the subintervals of P_n. is chosen either to be the end point or beginning point of the subinterval. In more detail, the application is stated below.

Corollary 37. Suppose we are given a limit of the form $\lim_{n\to\infty} \sum_{i=1}^{n} g(i)$. If we can write $g(i) = f(x_i)\frac{b-a}{n}$, where either $x_i = a + i\frac{b-a}{n}$ or $x_i = a + (i-1)\frac{b-a}{n}$, then

$$\lim_{n\to\infty}\sum_{i=1}^{n} g(i) = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{b-a}{n} = \int_a^b f(x)dx\,,$$

if f is Riemann integrable on [a, b]. Therefore, if we can find an anti-derivative F of f, then $\lim_{n\to\infty}\sum_{i=1}^{n} g(i) = F(b) - F(a)$ by the *Fundamental Theorem of Calculus.*

Example 38.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{5n-2i}{n^3}} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{5-2\frac{i}{n}}$$

$$= \int_0^1 \sqrt{5-2x}\,dx = -\tfrac{1}{3}[(5-2x)^{3/2}]_0^1 = -\tfrac{1}{3}(3^{3/2}-5^{3/2})$$

$$= \frac{5\sqrt{5}}{3} - \sqrt{3}\ .$$

Darboux Sums Convergence Theorem

It is a natural question to ask that if the function f is integrable on [a, b], then does the same conclusion as in Theorem 36 holds true for either the upper Darboux sums or lower Darboux sums, that is, does the sequence of either upper Darboux sum or lower Darboux with respect to any sequence of partitions converge to the integral of f if the norm of the partitions tend to 0.

Theorem 39. Suppose $f : [a, b] \to \mathbf{R}$ is integrable and (P_n) is a sequence of partitions of [a, b]. If $\|P_n\|$ converges to 0 as n tends to infinity, then $\lim_{n\to\infty}(U(f,P_n) - L(f,P_n)) = 0$ and consequently $L(f,P_n) \to \int_a^b f$ and $U(f,P_n) \to \int_a^b f$ **.**

Proof. Since f is integrable, by Theorem 21(4), given $\varepsilon > 0$, then there exists $\delta > 0$ such that for any partition P for [a, b] with norm $\| P \| < \delta$, and for any Riemann sum S with respect to P,

$$| S - L | < \varepsilon/4\,.$$

Let $T = \{$ Riemann sum $S : S$ has the same partition $P\}$.
Then for any S in T,

$$L - \varepsilon/4 < S < L + \varepsilon/4\,. \qquad \text{----------------------------- (1)}$$

Let P: $a = x_0 < x_1 < \ldots < x_n = b$ be a partition of [a, b] with $\| P \| < \delta$. Let $M_i = \sup\{ f(x) : x \in [x_{i-1}, x_i]\}$ for $i = 1,\ldots, n$. By the definition of supremum, for each i, $1 \le i \le n$, there exists c_i in [x_{i-1} , x_i] such that

$$f(c_i) > M_i - \frac{\varepsilon}{4(b-a)}.$$

Then, using this inequality the Riemann sum

$$R(f,P,C) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

$$> \sum_{i=1}^{n}\Big(M_i - \frac{\varepsilon}{4(b-a)}\Big)(x_i - x_{i-1}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) - \frac{\varepsilon}{4} = U(f,P) - \frac{\varepsilon}{4},$$

where, $C=(c_1,\ldots,c_n)$.
Therefore,

$$U(f,P) < R(f,P,C) + \frac{\varepsilon}{4} < L + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = L + \frac{\varepsilon}{2}. \quad \text{---------------------- (2)}$$

by inequality (1).

Now let $m_i = \inf\{ f(x) : x \in [x_{i-1}, x_i] \}$ for $i = 1,\ldots, n$. By the definition of infimum, for each i, $1 \le i \le n$, there exists d_i in $[x_{i-1}, x_i]$ such that

$$f(d_i) < m_i + \frac{\varepsilon}{4(b-a)}.$$

Then, using this inequality, the Riemann sum

$$R(f,P,d) = \sum_{i=1}^{n} f(d_i)(x_i - x_{i-1})$$
$$< \sum_{i=1}^{n} (m_i + \frac{\varepsilon}{4(b-a)})(x_i - x_{i-1}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) + \frac{\varepsilon}{4} = L(f,P) + \frac{\varepsilon}{4},$$

where $d=(d_1,\ldots,d_n)$.

Therefore,

$$L(f,P) > R(f,P,d) - \frac{\varepsilon}{4} > L - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = L - \frac{\varepsilon}{2}. \qquad \text{------------------- (3)}$$

It follows from (2) and (3) that if $\|P_n\| < \delta$, *then* $U(f, P) - L(f, P) < \varepsilon$.

Now since $\|P_n\| \to 0$, there exists an integer N such that $n \ge N \Rightarrow \|P_n\| < \delta$. Therefore, by what we have just proved,

$$n \ge N \Rightarrow U(f, P_n) - L(f, P_n) < \varepsilon.$$

This means $\lim_{n\to\infty} (U(f,P_n) - L(f,P_n)) = 0$.

Remark. The argument in the proof of Theorem 39 is in the proof of Theorem 21 (4) $\Rightarrow$ (3) and is reproduced here for convenience.

Mean Value Theorem

Theorem 40 Mean Value Theorem for Integrals.
Suppose f is continuous on the closed interval $[a, b]$ with $a < b$. Then there exists a point χ in $[a, b]$ such that $\int_a^b f = f(\chi)(b-a)$.

Proof. Since f is continuous on $[a, b]$, by the *Extreme Value Theorem*, there exists c, d in $[a, b]$ such that $m = f(c) \le f(x) \le f(d) = M$ for all x in $[a, b]$. Therefore, by Theorem 32,

$$\int_a^b m \le \int_a^b f \le \int_a^b M$$

and so $f(c)(b-a) \le \int_a^b f \le f(d)(b-a)$.

Dividing the above inequality by $(b - a)$, we get

$$f(c) \le \frac{\int_a^b f}{b-a} \le f(d).$$

Therefore, since f is continuous, by the *Intermediate Value Theorem*, there exists χ between c and d , i.e. χ in $[a, b]$ such that $\frac{\int_a^b f}{b-a} = f(\chi)$. (The left hand side of the equation above is known as the mean value.) Hence $\int_a^b f = f(\chi)(b-a)$.

Example 41. We can estimate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^8(x)dx$ using the *Mean Value Theorem for Integrals*. It says there exists c in $[\frac{\pi}{6}, \frac{\pi}{3}]$such that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^8(x)dx = \sin^8(c)(\frac{\pi}{3} - \frac{\pi}{6})$ $= \frac{\pi}{6}\sin^8(c)$. Since the function sine is increasing on $[\frac{\pi}{6}, \frac{\pi}{3}]$,
$\frac{1}{2} = \sin(\frac{\pi}{6}) \le \sin(c) \le \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$. It follows that $\frac{1}{2^8} \le \sin^8(c) \le \frac{3^4}{2^8}$ and so
$\frac{\pi}{6}\cdot\frac{1}{2^8} \le \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^8(x)dx \le \frac{\pi}{6}\cdot\frac{3^4}{2^8}$. Therefore, $\frac{\pi}{1536} \le \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^8(x)dx \le \pi\frac{27}{512}$.

5.7 Fundamental Theorem of Calculus

The most important theorem in Calculus is the fundamental theorem because it provides the link between differentiation and Riemann integration. Our next result is a theorem of Darboux which gives the fundamental theorem without the assumption of the continuity of the derived function f' of f.

Theorem 42. Darboux Fundamental Theorem of Calculus.

Suppose $F : [a, b] \to \mathbf{R}$ is a function continuous on $[a, b]$ and differentiable on (a, b). Suppose the derived function $F' : (a, b) \to \mathbf{R}$ is integrable in the sense that any extension of F' to the end points a and b is integrable. Then $\int_a^b F' = F(b) - F(a)$.

Proof. For each integer $n \geq 1$ let the regular partition P_n for $[a, b]$ be

$$P_n : \ a = x_0 < x_1 < \ldots < x_n = b ,$$

with $\| P_n \| = (b-a)/n$, $x_k = a + k\frac{b-a}{n}$ for $k = 0, 1, \ldots, n$. For each $k = 1, \ldots, n$, since F is continuous on $[x_{k-1}, x_k]$ and differentiable on (x_{k-1}, x_k), by the Mean Value Theorem (Theorem 15 Chapter 4), there exist ξ_k in (x_{k-1}, x_k) such that

$$F'(\xi_k)(x_k - x_{k-1}) = F(x_k) - F(x_{k-1}). \quad \text{---------------------------} \quad (1)$$

Let $R(F', P_n, C_n)$ be the Riemann sum with respect to the partition P_n , where $C_n = (\xi_1, \xi_2, \ldots \xi_n)$ is the choice of ξ_i in $[x_{i-1}, x_i]$ given by (1).
Then

$$R(F', P_n, C_n) = \sum_{k=1}^{n} F'(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} F(x_k) - F(x_{k-1}) = F(x_n) - F(x_0) = F(b) - F(a)$$

Since $\| P_n \| = (b-a)/n \to 0$, by Theorem 36, the sequence of Riemann sums $(R(F', P_n, C_n))$ tends to $\int_a^b F'$. But the sequence is constant and so $R(F', P_n, C_n) \to F(b) - F(a)$. Therefore, $\int_a^b F' = F(b) - F(a)$.

The following is the more familiar form of the fundamental theorem of calculus.

Theorem 43 (First Fundamental Theorem of Calculus)

Suppose $F : [a, b] \to \mathbf{R}$ is a function continuous on $[a, b]$ and differentiable on (a, b). Suppose $F' : (a, b) \to \mathbf{R}$ is continuous and bounded. Then $\int_a^b F' = F(b) - F(a)$.

Proof. $F' : (a, b) \to \mathbf{R}$ is continuous and bounded implies that F' is integrable by Theorem 34 and that the integral does not depend on the values of the extension of F' at the end points. Hence, the theorem follows from Theorem 42.

Remark.

Suppose $f : [a, b] \to \mathbf{R}$ is integrable and has an "anti-derivative" $F : [a, b] \to \mathbf{R}$, which is continuous on $[a, b]$ and $F'(x) = f(x)$ for all x in (a, b). Note that F need not be differentiable at the end points of the interval. Then by Theorem 42, $\int_a^b f = F(b) - F(a)$.

The significance of Theorem 43 is that we can use any "anti-derivative" to compute definite integral. Let f be a function continuous on $[a, b]$ and so integrable on $[a, b]$. For any continuous function F defined on $[a, b]$ such that $F'(x) = f(x)$ for all x in (a, b), (i.e. F is an "anti-derivative" of f in the sense given above), $\int_a^b f = F(b) - F(a) = [F(x)]_a^b$. This is the usual form that the theorem is used.

Example 44 $\int_0^1 (x^3 - 6x^2 + 2x + 1)dx$

Let $f(x) = x^3 - 6x^2 + 2x + 1$. Then f is continuous on $[0, 1]$. We recognize immediately that $F(x) = \frac{x^4}{4} - 2x^3 + x^2 + x$ is an anti-derivative of f. Thus

$$\int_0^1 (x^3 - 6x^2 + 2x + 1)dx = [\tfrac{x^4}{4} - 2x^3 + x^2 + x]_0^1 = \tfrac{1}{4} - 2 + 1 + 1 - 0 = \tfrac{1}{4}.$$

Remark.

Not all continuous functions have anti-derivatives expressible in terms of elementary functions. By elementary function, we mean any function formed from real constant, the identity function, the exponential function, the logarithmic function, the trigonometric functions and the inverse trigonometric functions, by adding, multiplying, dividing or forming composition. Not all integrable functions have anti-derivatives. For example, the function $\dfrac{\sqrt{1-x^2}}{\sqrt{1-x^2/4}}$ does not have an anti-derivative in terms of elementary functions on [-1, 1] and the function $f(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$ does not have an anti-derivative on [-1, 1] but is integrable. We can still compute $\int_{-1}^1 f$ which is 0.

Theorem 45 (Second Fundamental Theorem of Calcullus).

Suppose $f : [a, b] \to \mathbf{R}$ is integrable. Define $F: [a, b] \to \mathbf{R}$ by $F(x) = \int_a^x f$ for x in $[a, b]$. *Then*

1. F is continuous on $[a, b]$.
2. If f is continuous at x in $[a, b]$, then F is differentiable at x and $F'(x) = f(x)$.

Proof. Since f is integrable, f is bounded on $[a, b]$**.** Thus, there exists a positive real number K such that $-K \leq f(x) \leq K$ for all x in $[a, b]$. We shall show that F is uniformly continuous and hence continuous. For any $x < y$ such that $a \leq x < y \leq b$ we have that $-K \leq f(t) \leq K$ for all t in $[x, y]$. Hence, taking integrals, by Theorem 32,

$$-K(y-x) \leq \int_x^y f \leq K(y-x)$$

and so

$$\left|\int_x^y f\right| \leq K(y-x) = K|y-x|. \quad \text{----------------------------- (1)}$$

For any $x < y$ such that $a \leq x < y \leq b$,

$$F(y) = \int_a^y f = \int_a^x f + \int_x^y f = F(x) + \int_x^y f.$$

Therefore,

$$F(y) - F(x) = \int_x^y f. \quad \text{-------------------------------- (2)}$$

Similarly, if $a \leq y < x \leq b$, by interchanging the role of x and y above, we get,

$$F(x)-F(y)=\int_y^x f \ . \qquad \text{------------------------------- (3)}$$

It follows from (1), (2) and (3) that, for any $a \le x \le y \le b$,

$$|F(x)-F(y)| \le K|y-x|$$

Hence for any $\varepsilon > 0$, take δ to be any real number such that $0 < \delta < \varepsilon/K$. Then for all x, y in $[a, b]$,

$$|x-y| < \delta \Rightarrow |F(x)-F(y)| \le K|x-y| < K\,\varepsilon/K = \varepsilon.$$

This means F is uniformly continuous on $[a, b]$ and so continuous on $[a, b]$.

Part (2).

Suppose f is continuous at $x_0 \in [a, b]$. We shall show that $F(x)=\int_a^x f(t)dt$ is differentiable at x_0. For $x > x_0$,

$$F(x)-F(x_0)=\int_a^x f-\int_a^{x_0} f=\int_a^{x_0} f+\int_{x_0}^x f-\int_a^{x_0} f=\int_{x_0}^x f$$

and for $x < x_0$,

$$F(x)-F(x_0)=\int_a^x f-\int_a^{x_0} f=\int_a^x f-\left(\int_a^x f+\int_x^{x_0} f\right)=-\int_x^{x_0} f=\int_{x_0}^x f$$

where we use the convention that when $c < d$ in $[a, b]$, $\int_d^c f=-\int_c^d f$.

For $x \ne x_0$, let $H(x)=\dfrac{F(x)-F(x_0)}{x-x_0}=\dfrac{\int_{x_0}^x f}{x-x_0}$. Then by the definition of the derivative, $\lim_{x\to x_0} H(x)=F'(x_0)$, where the limit is taken to be the appropriate left or right limit when $x_0 = a$ or b. We shall show that $H(x)$ tends to $f(x_0)$ as x tends to x_0.

Since f is continuous at x_0, given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon \Rightarrow f(x_0)-\varepsilon < f(x) < f(x_0)+\varepsilon\ .$$

That is to say, for all x in $[a, b]$ with $x \in (x_0-\delta, x_0+\delta)$, we have

$$f(x_0)-\varepsilon < f(x) < f(x_0)+\varepsilon \quad \text{------------------------------------ (4)}$$

Now we consider the right limit first. For x in $(x_0, x_0+\delta)$, using (4) and Theorem 32, we obtain

$$\int_{x_0}^x (f(x_0)-\varepsilon) \le \int_{x_0}^x f \le \int_{x_0}^x (f(x_0)+\varepsilon)\ .$$

That is to say, for any x in $(x_0, x_0+\delta)$,

$$(f(x_0)-\varepsilon)(x-x_0) \le \int_{x_0}^x f(t)dt \le (f(x_0)+\varepsilon)(x-x_0)\ .$$

Dividing the above inequality by $(x - x_0) > 0$, we obtain

$$f(x_0)-\varepsilon \le \frac{\int_{x_0}^x f(t)dt}{x-x_0}=H(x) \le f(x_0)+\varepsilon\ .$$

Therefore, $|H(x)-f(x_0)| \le \varepsilon$. This shows that $\lim_{x\to x_0^+} H(x)=f(x_0)$.

Similarly, for the left limit, we take x in $(x_0-\delta, x_0)$. Then from (4) and Theorem 32, we obtain for any x in $(x_0-\delta, x_0)$,

$$\int_x^{x_0} (f(x_0)-\varepsilon) \le \int_x^{x_0} f \le \int_x^{x_0} (f(x_0)+\varepsilon).$$

As before we get $f(x_0)-\varepsilon \le \dfrac{\int_x^{x_0} f(t)dt}{x_0-x}=\dfrac{\int_{x_0}^x f(t)dt}{x-x_0}=H(x) \le f(x_0)+\varepsilon$. This shows that the left limit $\lim_{x\to x_0^-} H(x)=f(x_0)$. Therefore, if x_0, is in (a, b), then $\lim_{x\to x_0^+} H(x)=\lim_{x\to x_0^-} H(x)=f(x_0)$ and so $F'(x_0)=\lim_{x\to x_0} H(x)=f(x_0)$. By the above argument $F'(a)=\lim_{x\to a^+} H(x)=f(a)$ and $F'(b)=\lim_{x\to b^-} H(x)=f(b)$. This shows that for all x in $[a, b]$, $F'(x)=f(x)$.

This completes the proof.

Corollary 46. Suppose $f : [a, b] \to \mathbf{R}$ is continuous. Define $F: [a, b] \to \mathbf{R}$ by $F(x) = \int_a^x f$ for x in $[a, b]$. Then F is differentiable and $F'(x) = f(x)$ for all x in $[a, b]$, i.e., F is an anti-derivative of f.

Definition 47. Suppose $f : [a, b] \to \mathbf{R}$ is integrable. Suppose c and d are two points in $[a, b]$ such that $c < d$. We define $\int_d^c f = -\int_c^d f$. With this definition, we have then for any x, y in $[a, b]$, $-\int_x^y f = \int_y^x f$. In particular, for any x, y and z in $[a, b]$, $\int_x^z f = \int_x^y f + \int_y^z f$

Corollary 48. Suppose $f : [a, b] \to \mathbf{R}$ is continuous. Let c be in $(a, b]$. Define $F: [a, b] \to \mathbf{R}$ by $F(x) = \int_c^x f$ for x in $[a, b]$. Then F is differentiable and $F'(x) = f(x)$ for all x in $[a, b]$, i.e., F is an anti-derivative of f.

Proof. Note that $F(x) = \int_c^x f = \int_a^x f - \int_a^c f$. Therefore, by Corollary 46, $F'(x) = \frac{d}{dx}\int_c^x f = \frac{d}{dx}\int_a^x f = f(x)$ for all x in $[a, b]$. Hence $F' = f$.

Corollary 49. Suppose I is an open interval and $f : I \to \mathbf{R}$ is continuous. Let x_0 be in I and define $F: I \to \mathbf{R}$ by $F(x) = \int_{x_0}^x f$ for x in I. Then $F'(x) = f(x)$ for each x in I.

Example 50

(1) Let $g: \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = \int_1^x \sqrt{2+t^2}\, dt$. Let $f(t) = \sqrt{2+t^2}$. Then $g(x) = \int_1^x f(t)dt = F(x)$. Thus by Corollary 49, $g'(x) = F'(x) = f(x) = \sqrt{2+x^2}$.

(2) Let $h: \mathbf{R} \to \mathbf{R}$ be defined by $h(x) = \int_1^{x^2} \sqrt{2+t^2}\, dt$.
Therefore, $h(x) = F(x^2)$, where F is defined in example (1). By the *Chain Rule* for differentiation, $h'(x) = F'(x^2)\cdot 2x = f(x^2)\cdot 2x$, where the function f is defined in Example (1). Hence $h'(x) = 2x\sqrt{2+x^4}$

Corollary 49 provides a way to find the anti-derivative of a continuous function. It is then a natural question to ask whether if we relax the continuity condition to one of integrability, then the conclusion of Corollary 49 holds. The answer is no in general. A counterexample is is provided by the function in Example 51.

Example 51. Let $g : [0, 1] \to \mathbf{R}$, be a function defined by

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } 0 < x < 1 \text{ is rational and } x = \frac{p}{q} \text{ in its lowest terms} \\ 1 & \text{if } x = 0 \text{ or } 1 \end{cases} .$$

Then g is bounded, integrable and $\int_0^1 g = 0$.

We shall now show that g is integrable by using one of the equivalent conditions for integrability in Theorem 21.

Given any $\varepsilon > 0$, by the Archimedean property of **R**, there exists a positive integer $m > 1$ such that $1/m < \varepsilon/2$.
Observe that there can only be a finite number of reciprocals of integers that are greater than or equal to $1/m$:
Take any rational number p/q with p/q in its lowest terms and $0 < p/q \le 1$. This means that $1 \le p \le q$ and the greatest common divisor of p and q is 1. $g(p/q) \ge 1/m$ if and only if $1/q \ge 1/m$ if and only if $1 \le q \le m$. Thus, the number of rational numbers in [0, 1] that have values greater than or equal to $1/m$ is finite and this set includes the point 0 since $g(0) = 1$. That is the finite set

$$S_m = \{ p/q: \ q = 1, \ldots, m; p = 1, \ldots, q\} \cup \{0\}$$

is precisely the set on which the values of g are greater than or equal to $1/m$.
We shall show that g satisfies condition (3) of Theorem 21. Recall that the finite set S_m is precisely the set $\{y \in [0, 1]: g(y) \ge 1/m\}$. Note that $0, 1 \in S_m$. Let the number of points of S_m be $k+1$. Order the elements $y_0, y_1, y_2, \ldots, y_k$ of S_m.as follows:

$$0 = y_0 < y_1 < y_2 < \ldots y_{k-1} < y_k = 1.$$

Choose $k-1$ pair of points, each pair constitute an interval containing each y_i in its interior and of length $< \varepsilon/2$ for $i = 1, 2, \ldots, k-1$ and such that they are all mutually disjoint. That is, we choose $x_1 < x_2 < \ldots < x_{2k-2}$ such that

$$0 = y_0 < x_1 < y_1 < x_2 < x_3 < y_2 < x_4 < x_5 < \ldots x_{2k-4} < x_{2k-3} < y_{k-1} < x_{2k-2} < y_k = 1.$$

We choose further two more points, x_0 and x_{2k-1} and name y_0 as x_{-1}, y_k as x_{2k} such that $0 = x_{-1} = y_0 < x_0 < x_1$ and $x_{2k-2} < x_{2k-1} < x_{2k} = y_k = 1$. We further require that

$$\sum_{j=0}^{k} (x_{2j} - x_{2j-1}) < \frac{\varepsilon}{2}. \quad \text{------------------------------------} \quad (1)$$

Obviously,

$$P\!: \ 0 = x_{-1} < x_0 < x_1 < x_2 < x_3 < x_4 < x_5 < \ldots x_{2k-4} < x_{2k-3} < x_{2k-2} < x_{2k-1} < x_{2k} = 1$$

forms a partition for [0, 1].
Now by the density of the irrational numbers in any interval, for $i = 0, 1, \ldots, 2k$

$$m_i(g, P) = \inf\{g(x): x \in [x_{i-1}, x_i]\} = 0. \quad \text{-----------------------} \quad (2)$$

Now since for each $j = 0, 1, 2, \ldots, k$, $y_j \in [x_{2j-1}, x_{2j}]$ and so by the definition of S_m,

$$M_{2j}(g, P) = \sup\{g(x): x \in [x_{2j-1}, x_{2j}]\} = g(y_j) \text{ ----------------------} (3).$$

for $j = 0, 1, 2, \ldots, k$.
Now because for $j = 1, 2, \ldots, k$, $[x_{2j-2}, x_{2j-1}] \cap S_m = \varnothing$,

$$M_{2j-1}(g, P) = \sup\{ g(x): x \in [x_{2j-2}, x_{2j-1}]\} < 1/m \text{ ----------------} (4).$$

for $j = 1, 2, \ldots, k$.

$$\begin{aligned} U(g, P) - L(g, P) &= \sum_{i=0}^{2k} M_i(g, P)\Delta x_i - \sum_{i=0}^{2k} m_i(g, P)\Delta x_i, \\ &= \sum_{i=0}^{2k} M_i(g, P)\Delta x_i \ \text{ by (2)} \\ &= \sum_{j=0}^{k} M_{2j}(g, P)\Delta x_{2j} + \sum_{j=0}^{k} M_{2j-1}(g, P)\Delta x_{2j-1} \\ &\le \sum_{j=0}^{k} g(y_j)\Delta x_{2j} + \sum_{j=0}^{k} \frac{1}{m}\Delta x_{2j-1} \ \text{ by (3) and (4)} \\ &\le \sum_{j=0}^{k} \Delta x_{2j} + \frac{1}{m}\sum_{j=0}^{k} \Delta x_{2j-1} \ \text{ since } g(y_j) \le 1 \\ &< \varepsilon/2 + \frac{1}{m}\sum_{j=0}^{2k} \Delta x_j \ \text{ by (1)} \end{aligned}$$

$< \varepsilon/2 + 1/m \quad \text{since } \sum_{j=0}^{2k} \Delta x_j = x_{2k} - x_{-1} = 1$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$

Therefore, given any $\varepsilon > 0$, we can find a partition P such that, $U(g, P) - L(g, P) < \varepsilon$. By Theorem 21 (2), g is integrable. Furthermore by Theorem 39, for any sequence of partitions (P_n) of [0, 1] with $\|P_n\| \to 0$ (take (P_n) to be the sequence of regular partitions), $L(g, P_n) \to \int_0^1 g$. But for each positive integer n, by the density of the irrational numbers, $L(g, P_n) = 0$. Therefore, $\int_0^1 g = 0$.

Remark.

1. Since g is non-negative and $\int_0^1 g = 0$, the function $G : [0,1] \to \mathbf{R}$ defined by $G(x) = \int_0^x g = 0$ for each x in [0,1]. Hence, $G'(x) = 0$ and so $G'(x) \neq g(x)$ for all rational x in [0,1].
   Thus G is not an anti-derivative of g. This gives a counterexample to relaxing the continuity condition in Corollary 50.
2. The function g in Example 51 is continuous at every irrational points and discontinuous at every rational points in [0, 1]. Therefore, we can apply Lebesgue Theorem (reference: Theorem 33 Chapter 14) to conclude that g is integrable because the rational points constitute a set of measure zero.
   We show below that g is continuous at irrational points.
   Given any $\varepsilon > 0$, choose a positive integer m such that $1/m < \varepsilon$.
   Take an irrational number x in [0, 1]. Let $\delta = \min\{|x - y| : y \in S_m\}$. Then $\delta > 0$ since x is irrational. Obviously the open interval $(x - \delta, x + \delta)$ do not meet S_m. This is, because if $(x - \delta, x + \delta) \cap S_m \neq \emptyset$, then there exists p/q in S_m such that $|x - p/q| < \delta$ contradicting that $|x - p/q| \geq \min\{|x - y| : y \in S_m\} = \delta$. That means for all y in $(x - \delta, x + \delta) \cap [0, 1]$, $y \notin S_m$ and consequently, $g(y) < 1/m < \varepsilon$. Therefore,
   $$|g(y) - g(x)| = |g(y) - 0| = g(y) < \varepsilon.$$
   We have thus shown that for all y in [0, 1] such that $|y - x| < \delta$ we have $|g(y) - g(x)| < \varepsilon$.
   It follows by definition that g is continuous at x. Since x is arbitrary, g is continuous at every irrational point x in [0, 1]. Now we shall show that g is discontinuous at any rational point x. For any rational point x in [0, 1], by definition $g(x) > 0$. Let $\varepsilon = g(x)/2 > 0$. For any $\delta > 0$, by the density of the irrational numbers in any interval, there exists an irrational number y_δ in $(x - \delta, x + \delta) \cap [0, 1]$. Then $|g(y_\delta) - g(x)| = |0 - g(x)| = g(x) > g(x)/2 = \varepsilon$. We have thus shown that for any $\delta > 0$, we can find a y_δ in $(x - \delta, x + \delta) \cap [0, 1]$ such that $|g(y_\delta) - g(x)| > g(x)/2 = \varepsilon$. This means g is not continuous at x, if x is rational. Hence g is not continuous at any rational number in [0,1]. This completes the proof of the assertion.

Example 51 shows that differentiation is not an inverse to Riemann integration as provided by the construction in Corollary 49, even on bounded function with at most countably infinite points of discontinuity. Also Riemann integration is not an inverse to differentiation on the class of differentiable functions with bounded derivatives modulo constant functions with domain a closed and bounded interval. This is witnessed by a very difficult example of a function f defined on [0, 1], which is differentiable, has bounded derivative but the derivative f' is discontinuous at every point of a set of "positive measure" in [0,1] . Therefore f' is not Riemann integrable by Lebesgue Theorem. The construction of such a function is in two stages. The first part is to construct a set of 'positive measure' . Such a set would necessarily be uncountably infinite. The next stage is to make use of this set and its complement together with the function $x^2\sin(\frac{1}{x})$ to make the definition. The details are found in [Chapter 8 Example 35, *Counterexamples in Analysis*, by Bernard R. Gelbaum and John M.H. Olmsted, published by Holdan-Day, Inc].

Example 52

(1) Let $f:[0,2]\to\mathbf{R}$ be defined by $f(x)=\begin{cases} x, & 0\le x\le 1\\ 2x-1, & 1\le x\le 2\end{cases}$.

The function f is plainly continuous. The function $F:[0,2]\to\mathbf{R}$ defined by

$$F(x)=\begin{cases}\int_0^x f=\int_0^x tdt, & x\le 1\\ \int_0^1 f+\int_1^x f, & x>1\end{cases}=\begin{cases}\int_0^x tdt, & x\le 1\\ \int_0^1 tdt+\int_1^x(2t-1)dt, & x>1\end{cases}$$

$$=\begin{cases}[\frac{t^2}{2}]_0^x=\frac{x^2}{2}, & x\le 1\\ [\frac{t^2}{2}]_0^1+[t^2-t]_1^x, & x>1\end{cases}=\begin{cases}\frac{x^2}{2}, & x\le 1\\ \frac{1}{2}+x^2-x, & x>1\end{cases},$$

is differentiable on [0,2] and $F'=f$ by Corollary 48.

(2) $$\frac{d}{dx}\int_x^{x^2}\frac{1}{1+t^2}dt=\frac{d}{dx}\left[\int_x^0\frac{1}{1+t^2}dt+\int_0^{x^2}\frac{1}{1+t^2}dt\right]=\frac{d}{dx}\int_0^{x^2}\frac{1}{1+t^2}dt+\frac{d}{dx}\left(-\int_0^x\frac{1}{1+t^2}dt\right)$$

$$=\frac{d}{dx}F(x^2)-\frac{d}{dx}F(x),\qquad \text{where } F(x)=\int_0^x\frac{1}{1+t^2}dt$$

$$=F'(x^2)\cdot 2x-F'(x)\qquad \text{by the } \textit{Chain Rule} \text{ and Corollary 48}$$

$$=\frac{1}{1+(x^2)^2}\cdot 2x-\frac{1}{1+x^2}$$

by the *First Fundamental Theorem of Calculus*,

$$=\frac{2x}{1+x^4}-\frac{1}{1+x^2}.$$

5.8 Products and Modulus of Integrable Functions

If f is integrable on $[a, b]$, does it follow that the modulus $|f|$ is also integrable over $[a, b]$?
And if f and g are integrable on $[a, b]$, is the product fg also integrable on $[a, b]$?
We shall answer these questions in the affirmative.

Theorem 53. Suppose $f:[a,b]\to\mathbf{R}$ is a real valued function. If f is Riemann integrable on $[a, b]$, then $|f|$ is also Riemann integrable on $[a, b]$ and

$$\left|\int_a^b f(x)dx\right|\le\int_a^b|f(x)|dx.$$

Proof. We need to work with the properties of supremum and infimum. The property that for any bounded subset A of $\mathbf{R}$, $\inf A=-\sup(-A)$ will be used. Since f is integrable, there exists a partition $P: a=x_0<x_1<\dots<x_n=b$ of $[a, b]$ such that the difference of the upper and lower Darboux sums with respect to P for f,

$$U(f,P)-L(f,P)=\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})<\varepsilon, \quad \text{------------------------- (1)}$$

where $M_i = \sup\{ f(x) : x \in [x_{i-1}, x_i]\}$, and $m_i = \inf\{ f(x) : x \in [x_{i-1}, x_i]\} = -\sup\{- f(x) : x \in [x_{i-1}, x_i]\}$.
Therefore, for integer $1 \le i \le n$,

$$M_i - m_i = \sup\{ f(x) : x \in [x_{i-1}, x_i]\} + \sup\{- f(x) : x \in [x_{i-1}, x_i]\}$$
$$= \sup\{ f(x) - f(y) : x, y \in [x_{i-1}, x_i]\},$$

using the fact that for two bounded sets A and B, $\sup(A+B) = \sup A + \sup B$.
Also the difference of the upper and lower Darboux sums with respect to P for $|f|$ *is*

$$U(|f|, P) - L(|f|, P) = \sum_{i=1}^{n}(M_i' - m_i')(x_i - x_{i-1}) < \varepsilon,$$

where $M_i' = \sup\{| f(x)| : x \in [x_{i-1}, x_i]\}$, and $m_i' = \inf\{| f(x)| : x \in [x_{i-1}, x_i]\} = -\sup\{-| f(x)| : x \in [x_{i-1}, x_i]\}$.
Therefore, for integer $1 \le i \le n$,

$$M_i' - m_i' = \sup\{| f(x)| : x \in [x_{i-1}, x_i]\} + \sup\{-| f(x)| : x \in [x_{i-1}, x_i]\}$$
$$= \sup\{| f(x)| - | f(y)| : x, y \in [x_{i-1} - x_i]\}.$$

In fact since $a \in \{ f(x) - f(y) : x, y \in [x_{i-1}, x_i]\}$ if and only if $-a \in \{ f(x) - f(y) : x, y \in [x_{i-1}, x_i]\}$, for integer $1 \le i \le n$,

$$M_i - m_i = \sup\{| f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\}.$$

Similarly, for integer $1 \le i \le n$,

$$M_i' - m_i' = \sup\{|| f(x)| - | f(y)|| : x, y \in [x_{i-1}, x_i]\}.$$

It follows from the inequality for any a, b in **R**, $||a| - |b|| \le |a - b|$, that

$$M_i' - m_i' \le M_i - m_i \text{ for } i = 1, \ldots, n. \quad \text{-----------------------} \quad (2)$$

The difference of the upper and lower Darboux sums,

$$U(|f|, P) - L(|f|, P) = \sum_{i=1}^{n}(M_i' - m_i')(x_i - x_{i-1})$$
$$\le \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \le U(P, f) - L(P, f) < \varepsilon$$

by (2) and (1).

Therefore, by Theorem 21(3), $|f|$ is integrable.
By Theorem 29, $-|f|$ is also integrable. Since we have the inequality $-|f| \le f \le |f|$, by Theorem 32,

$$-\int_a^b | f(x)| dx \le \int_a^b f(x)dx \le \int_a^b | f(x)| dx.$$

This means that $\left|\int_a^b f(x)dx\right| \le \int_a^b | f(x)| dx$. This completes the proof of Theorem 53.

We can use the above argument to prove the following.

Theorem 54. Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. If f is (Riemann) integrable on $[a, b]$, then $f^2 = f \times f$ is also (Riemann) integrable on $[a, b]$.

Proof. Let $K = \sup\{| f(x)| : x \in [a, b]\}$. If $K = 0$, we have nothing to prove since f would be the zero constant function. Assume $K > 0$. Take any $\varepsilon > 0$. Since f is integrable, there exists a partition $P: a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$ such that the difference of the upper and lower Darboux sums with respect to P for f,

$$U(f, P) - L(f, P) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) < \varepsilon/(2K),$$

For $i = 1, \ldots, n$, let $M_i' = \sup\{ f^2(x) : x \in [x_{i-1}, x_i]\}$, and $m_i' = \inf\{ f^2(x) : x \in [x_{i-1}, x_i]\} = -\sup\{- f^2(x) : x \in [x_{i-1}, x_i]\}$. Then for $i = 1, \ldots, n$,

$$M_i' - m_i' = \sup\{ f^2(x) : x \in [x_{i-1}, x_i]\} + \sup\{- f^2(x) : x \in [x_{i-1}, x_i]\}$$
$$= \sup\{ f^2(x) - f^2(y) : x, y \in [x_{i-1}, x_i]\}$$

$= \sup\{| f^2(x) - f^2(y)|: x, y \in [x_{i-1}, x_i]\}$
$= \sup\{| f(x) - f(y)||f(x) + f(y)|: x, y \in [x_{i-1}, x_i]\}$
$\leq 2K \sup\{| f(x) - f(y)|: x, y \in [x_{i-1}, x_i]\}$
$= 2K(M_i - m_i)$.

Therefore, the difference of the upper Darboux sum and the lower Darboux sum with respect to P for f^2, is

$$U(f^2, P) - L(f^2, P) = \sum_{i=1}^{n}(M_i' - m_i')(x_i - x_{i-1}) \leq 2K\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1})$$
$$< 2K\,\varepsilon/(2K) = \varepsilon.$$

This inequality implies by Theorem 21 (3), that f^2 is integrable on $[a, b]$.

An easy consequence of the above theorem is the following.

Corollary 55. Suppose f and g are integrable on $[a, b]$, then $fg = f \times g$ is also integrable on $[a, b]$,

Proof. Note that $fg = 1/2\,[(f+g)^2 - f^2 - g^2]$. By Theorem 54, f^2, g^2 and $(f+g)^2$ are integrable. Therefore, by Theorem 30, $[(f+g)^2 - f^2 - g^2]$ is integrable. The result then follows from Theorem 29..

Theorem 56. Suppose $f : [a, b] \to \mathbf{R}$ is integrable. If there exists $K > 0$ such that for all x in $[a, b]$, $|f(x)| \geq K$, then $1/f$ is also integrable on $[a, b]$.

Proof. Fix an $\varepsilon > 0$. Since f is integrable on $[a, b]$, by Theorem 21 (3), there exists a partition $P : a = x_0 < x_1 < \ldots < x_n = b$ such that the difference of the upper and lower Riemann sum,

$$U(f, P) - L(f, P) < \varepsilon K^2, \quad \text{-------------------------} \quad (1)$$

where $U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$, $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, and

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Then

$$U(f, P) - L(f, P) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}). \quad \text{--------------------} \ (2)$$

Let $M_i' = \sup\{1/f(x) : x \in [x_{i-1}, x_i]\}$, and $m_i' = \inf\{1/f(x) : x \in [x_{i-1}, x_i]\} = -\sup\{-1/f(x) : x \in [x_{i-1}, x_i]\}$. Then for integer $i = 1, \ldots, n$,

$M_i' - m_i' = \sup\{1/f(x) : x \in [x_{i-1}, x_i]\} + \sup\{-1/f(x): x \in [x_{i-1}, x_i]\}$
$= \sup\{1/f(x) - 1/f(y): x, y \in [x_{i-1}, x_i]\}$
$= \sup\{|1/f(x) - 1/f(y)|: x, y \in [x_{i-1}, x_i]\}$
$= \sup\{| f(x) - f(y)| / | f(x) f(y)| : x, y \in [x_{i-1}, x_i]\}$
$\leq \sup\{| f(x) - f(y)| / K^2 : x, y \in [x_{i-1}, x_i]\}$
because for any x in $[a, b]$, $1/|f(x)| \leq 1/K$,
$= \sup\{| f(x) - f(y)|: x, y \in [x_{i-1}, x_i]\}/K^2$
$= (M_i - m_i)/K^2$ -- (3)

Since $U(1/f, P) = \sum_{i=1}^{n} M_i'(x_i - x_{i-1})$ and $L(1/f, P) = \sum_{i=1}^{n} m_i'(x_i - x_{i-1})$, the difference of the upper and lower Darboux sums with respect to P for $1/f$,

$$U(1/f, P) - L(1/f, P) = \sum_{i=1}^{n}(M_i' - m_i')(x_i - x_{i-1}) \leq 1/K^2 \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \text{ by (3)}$$

$= (1/K^2)(U(f, P) - L(f, P))$ by (2)
$< (1/K^2)\varepsilon K^2 = \varepsilon$ by (18).

Hence, condition (3) of Theorem 21 is satisfied by $1/f$ and so $1/f$ is integrable on $[a, b]$. This completes the proof.

5.9 Integration By Parts And The Change of Variable Formula.

Integration By Parts

Integration by parts is a formula often used for computing Riemann integrals or definite integrals of the Lebesgue type. We shall start from the usual anti-derivative form of the formula. Let I be an interval. Suppose $F : I \to \mathbf{R}$ is an anti-derivative of $f : I \to \mathbf{R}$ and $G: I \to \mathbf{R}$ is an anti-derivative of $g: I \to \mathbf{R}$. Then we have that $F' = f$ and $G' = g$. The product formula for differentiation tells us that

$$(F\,G)' = F'\,G + F\,G' = f\,G + F\,g. \quad \text{------------------- (1)}$$

From here we consider additional condition on f and g to give a formula for the relation between anti-derivative of $f\,G$ and that of $F\,g$. At this point, if we assume that f and g are continuous, then both $f\,G$ and $F\,g$ are continuous. Therefore, $f\,G$ and $F\,g$ have anti-derivatives by Corollary 54. Using (1) we can write

$$f\,G = (F\,G)' - F\,g.$$

Thus any anti-derivative of $f\,G$ is of the form $F\,G$ – anti-derivative of $F\,g$. This is seen as follows. Suppose H is an anti-derivative of $F\,g$. Then $H' = F\,g = (F\,G)' - f\,G$ so that $(F\,G)' - H' = f\,G$. This means $K = F\,G - H$ is an anti-derivative of $f\,G$. On the other hand if K is an anti-derivative of $f\,G$, then $K' = f\,G = (F\,G)' - F\,g$ so that $(FG - K)' = (F\,G)' - K' = F\,g$ and so $H = FG - K$ is an anti-derivative of $F\,g$. Therefore, $K = FG - H$. This means we have

$$\int f(x)G(x)dx = F(x)G(x) - \int F(x)g(x)dx.$$

We have thus proved the following theorem.

Theorem 57. Let I be an interval. Suppose $F : I \to \mathbf{R}$ is an anti-derivative of $f : I \to \mathbf{R}$ and $G: I \to \mathbf{R}$ is an anti-derivative of $g: I \to \mathbf{R}$. If f and g are continuous, then we have the following formula for anti-derivatives:

$$\int f(x)G(x)dx = F(x)G(x) - \int F(x)g(x)dx.$$

Theorem 58. Suppose $F : [a, b] \to \mathbf{R}$ and $G : [a, b] \to \mathbf{R}$ are continuous. Suppose that F and G have continuous and bounded derivatives on the open interval (a, b). Then

$$\int_a^b F'(x)G(x)dx = [F(x)G(x)]_a^b - \int_a^b F(x)G'(x)dx. \quad \text{--------------- (C)}$$

Proof. Since $F' :(a, b) \to \mathbf{R}$ is continuous and bounded, by Theorem 34 F' is integrable on $[a, b]$. Since G is continuous, G is integrable**.** Therefore, by Corollary 55, the product $F'\,G$ is integrable on $[a, b]$. We deduce similarly that $F\,G'$ is integrable on $[a, b]$. Hence the sum $F'\,G + F\,G'$ is integrable on $[a, b]$. Since FG is an anti-derivative of $F'\,G + F\,G'$ on (a, b), by the First Fundamental Theorem of Calculus (Theorem 43),

$$\int_a^b (F'(x)G(x) + F(x)G'(x))dx = [F(x)G(x)]_a^b.$$

Then by the linearity of the integral (Theorem 30),

$$\int_a^b F'(x)G(x)dx + \int_a^b F(x)G'(x)dx = [F(x)G(x)]_a^b.$$

Formula (C) then follows.

We can improve the theorem a little.

Theorem 59. Suppose $F : [a, b] \to \mathbf{R}$ and $G : [a, b] \to \mathbf{R}$ are continuous. Suppose that F and G are differentiable on (a, b) and F' and G' are integrable. Then

$$\int_a^b F'(x)G(x)dx = [F(x)G(x)]_a^b - \int_a^b F(x)G'(x)dx.$$

Proof. The proof is almost exactly the same as that of Theorem 58. By Corollary 55 , $F' G$ and $F G'$ are integrable on $[a, b]$. By linearity, $F' G + F G'$ is integrable on $[a, b]$. Note that by the product rule for differentiation, $(F G)' = F' G + F G'$. Therefore, by Theorem 42 (Darboux Fundamental Theorem of Calculus),

$$\int_a^b (F'(x)G(x) + F(x)G'(x))dx = \int_a^b (FG)' = [F(x)G(x)]_a^b$$

The formula then follows by linearity of the integral.

Change of Variable Formula

Suppose $g:[a, b] \to [c, d]$ is a differentiable function and $f : [c, d] \to \mathbf{R}$ is a bounded function. A necessary condition for the change of variable formula

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx \qquad \text{---------------------- (D)}$$

to hold is that 1. $f(g(x))\, g'(x)$ is Riemann integrable on $[a, b]$ and

2. f is Riemann integrable over a domain containing the range of g.

Theorem 60. Suppose g: $[a, b] \to \mathbf{R}$ is a continuous function and $f : [c, d] \to \mathbf{R}$ is integrable and has an anti-derivative. Suppose the range of $g \subseteq [c, d]$ and g is differentiable on (a, b). If $(f \circ g)\, g'$ is integrable on $[a, b]$, then formula (D) holds.

Proof. Let F be an anti-derivative of f. Note that since F is differentiable on $[c, d]$, F is continuous and therefore, $F \circ g$ is continuous on $[a, b]$ because g: $[a, b] \to \mathbf{R}$ is continuous. Then $F' = f$. By the Chain rule for differentiation, $F \circ g$ is differentiable on (a, b) and

$$(F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))\, g'(x) = (f \circ g)(x)\, g'(x)$$

for all x in (a, b). If $(f \circ g)\, g'$ is integrable on $[a, b]$, then $(F \circ g)'$ is integrable on $[a, b]$. Therefore, by Darboux Fundamental Theorem of Calculus (Theorem 42),

$$\int_a^b f(g(x))g'(x)dx = \int_a^b (F \circ g)' = F \circ g(b) - F \circ g(a) = F(g(b)) - F(g(a)).$$

But f is integrable with antiderivative F, again by Darboux Fundamental Theorem of Calculus, $\int_{g(a)}^{g(b)} f = F(g(b)) - F(g(a))$.

It follows then that $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx$. This completes the proof.

Remark. If the function f is continuous on its domain then f is Riemann integrable and has an antiderivative given by the Fundamental Theorem of Calculus. Thus, we often replace the condition on f in Theorem 60 by continuity as in Theorem 61 below.

Theorem 61. If g: $[a, b] \to [c, d]$ is a continuous function differentiable on the open interval (a, b) and suppose that $g' : (a, b) \to \mathbf{R}$ is continuous and bounded and if f : $[c, d] \to \mathbf{R}$ is continuous, then formula (D) holds.

Proof. By assumption $f \circ g$ is continuous on $[a, b]$ and so is bounded on (a, b). Since $g' : (a, b) \to \mathbf{R}$ is continuous and bounded, $(f \circ g)\, g'$ is continuous and bounded on (a, b). Therefore, by Theorem 34, $(f \circ g)\, g'$ is integrable on $[a, b]$. Since f is continuous on $[c, d]$, by the Corollary to the Second Fundamental theorem of Calculus (Corollary 46), f has an anti-derivative on $[c, d]$, given by . $F(x) = \int_c^x f(t)dt$. Therefore, by Theorem 60, formula (D) holds.

Theorem 60 supposes that $(f \circ g)\, g'$ is integrable on $[a, b]$. Under what condition can we guarantee the Riemann integrability of $(f \circ g)\, g'$ on $[a, b]$. Formula (D) requires that f be Riemann integrable on a domain containing the range of g. If we impose sufficient condition on g' we can deduce that $(f \circ g)\, g'$ is Riemann integrable on $[a, b]$. It is, however, not true in general that if f is Riemann integrable and g is continuous, then $f \circ g$ is Riemann integrable. For a counter example see Example 5 of *Composition and Riemann Integrability* on My Calculus web.

Theorem 62. Suppose $f : [c, d] \to \mathbf{R}$ is Riemann integrable and g: $[a, b] \to [c, d]$ is a continuously differentiable strictly increasing function mapping $[a, b]$ onto $[c, d]$. Then $(f \circ g)\, g'$ is Riemann integrable on $[a, b]$ and formula (D) holds, that is,

$$\int_a^b f(g(x))g'(x)dx = \int_c^d f(x)dx .$$

Proof . Since f is integrable on $[c, d]$, by Theorem 21 (3), given any $\varepsilon > 0$, there exists a partition W for $[c, d]$,

$$W: c = l_0 < l_1 < \ldots < l_n = d$$

such that

$$U(f, W) - L(f, W) < \varepsilon/2 \qquad \text{--------------------------- (1)}$$

Since g: $[a, b] \to [c, d]$ is a strictly monotonic increasing bijective map, its inverse is also a strictly monotonic increasing bijection and so

$$g^{-1}W: a = z_0 < z_1 < \ldots < z_n = b$$

where $z_i = g^{-1}\, l_i$, $i = 1, 2, \ldots, n$, is a partition for $[a, b]$.

Because f is integrable, f is bounded on $[c, d]$, that is, there exists a real number $M > 0$ such that $|f(x)| < M$ for all x in $[c, d]$. Since $g' :[a, b] \to \mathbf{R}$ is continuous on $[a, b]$, g' is uniformly continuous. Therefore, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |g'(x) - g'(y)| < \varepsilon/(2(M+1)(b-a)) \qquad \text{-------------- (2)}$$

Now refine the partition $g^{-1}W$, by adding points if need be, to get a partition Q

$$Q:\ a = x_0 < x_1 < \ldots < x_k = b \text{ , where } k \ge n,$$

with $\| Q \| =$ max $\{x_i - x_{i-1} : i = 1, 2, \ldots, k\} < \delta$. This means $g^{-1}(W) \subseteq Q$. Then $g(Q) = P$ is a refinement of W since $W \subseteq g(Q) = P$. Let

$$P:\ c = y_0 < y_1 < \ldots < y_k = d\,,$$

then $y_i = g(x_i)$, $i = 1, 2, \ldots, k$. Therefore, by the definition of upper and lower Darboux sums,

$$L(f, W) \le L(f, P) \le U(f, P) \le U(f, W)$$

and so we have, $U(f, P) - L(f, P) \le U(f, W) - L(f, W) < \varepsilon/2$, that is,

$$U(f, P) - L(f, P) < \varepsilon/2. \text{ ------------------------------ } (3)$$

Since g is differentiable, by the Mean Value Theorem, there exists $\xi_i \in [x_{i-1}, x_i]$ such that

$$\Delta y_i = y_i - y_{i-1} = g(x_i) - g(x_{i-1}) = g'(\xi_i)(x_i - x_{i-1}) = g'(\xi_i)\Delta x_i\,. \text{ ------ } (4)$$

Note that $g'(\xi_i) \ge 0$ for $i = 1, 2, \ldots, k$ since g is increasing on $[a, b]$.

We shall use what we have just proved. Note that the upper Riemann sum of $(f \circ g)\, g'$ with respect to the partition Q (as given before with $\|Q\| < \delta$) is defined to be

$$U((f \circ g)\, g', Q) = \sum_{i=1}^{k} M_i((f \circ g)g', Q)\Delta x_i, \text{ ------------------------------ } (5)$$

where $M_i((f \circ g)\, g', Q) = \sup\{ f \circ g(x)\, g'(x) : x \in [x_{i-1}, x_i]\}$, $i = 1, 2, \ldots, k$.
For each integer $i = 1, 2, \ldots, k$,

$$\begin{aligned} M_i((f \circ g)\, g', Q) &= \sup\{ f \circ g(x)\, g'(x) : x \in [x_{i-1}, x_i]\} \\ &= \sup\{ f \circ g(x)\,(g'(x) - g'(\xi_i)) + f \circ g(x)\, g'(\xi_i) : x \in [x_{i-1}, x_i]\} \end{aligned}$$

But for each $i = 1, 2, \ldots, k$, and for any $x \in [x_{i-1}, x_i]$,

$$\begin{aligned} f \circ g(x)\,(g'(x) - g'(\xi_i)) &\le | f \circ g(x)||g'(x) - g'(\xi_i)| \le | f(g(x))|\, \varepsilon/(2(M+1)(b-a)) \\ &\qquad \text{by (2) since } |x - \xi_i| \le |x_i - x_{i-1}| \le \|Q\| < \delta \\ &\le M\, \varepsilon/(2(M+1)(b-a)) \\ &\qquad \text{since } | f(y)| < M \text{ for all } y \text{ in } [c, d] \\ &< \varepsilon/(2(b-a)). \end{aligned}$$

Therefore, for each $i = 1, 2, \ldots, k$,

$$\begin{aligned} M_i((f \circ g)\, g', Q) &= \sup\{ f \circ g(x)\,(g'(x) - g'(\xi_i)) + f \circ g(x)\, g'(\xi_i) : x \in [x_{i-1}, x_i]\} \\ &\le \varepsilon/(2(b-a)) + \sup\{ f \circ g(x)\, g'(\xi_i) : x \in [x_{i-1}, x_i]\} \\ &\le \varepsilon/(2(b-a)) + \sup\{ f \circ g(x) : x \in [x_{i-1}, x_i]\} g'(\xi_i) \quad \text{since } g'(\xi_i) \ge 0 \\ &= \varepsilon/(2(b-a)) + \sup\{ f(x) : x \in [y_{i-1}, y_i]\} g'(\xi_i), \quad \text{since } g \text{ is a bijection.} \end{aligned}$$

It follows from (5) and the above inequality that

$$\begin{aligned} U((f \circ g)\, g', Q) &\le \sum_{i=1}^{k} \frac{\varepsilon}{2(b-a)}\Delta x_i + \sum_{i=1}^{k} \sup\{ f(x) : x \in [y_{i-1}, y_i]\} g'(\xi_i)\Delta x_i \\ &= \frac{\varepsilon}{2} + \sum_{i=1}^{k} \sup\{ f(x) : x \in [y_{i-1}, y_i]\}\Delta y_i = \varepsilon/2 + U(f, P) \quad \text{by (4)} \\ &< \varepsilon/2 + L(P, f) + \varepsilon/2 \qquad \text{by (3)} \\ &= L(P, f) + \varepsilon \le L\int_c^d f + \varepsilon \end{aligned}$$

Therefore, the upper integral

$$U\int_a^b (f \circ g)g' \le U((f \circ g)g', Q) < L\int_c^d f + \varepsilon.$$

that is, for any $\varepsilon > 0$,

$$U\int_a^b (f \circ g)g' < L\int_c^d f + \varepsilon.$$

Therefore, since $\varepsilon > 0$ is arbitrary,

$$U\int_a^b (f \circ g)g' \le L\int_c^d f. \quad \text{----------------------- } (6)$$

Recall $L((f\circ g)\,g', Q) = \sum_{i=1}^{k} m_i(f\circ g)g', Q)\Delta x_i$,
where $m_i((f\circ g)\,g', Q) = \inf\{f\circ g\,(x)\,g'(x): x \in [x_{i-1}, x_i]\}$, $i = 1, 2,\ldots, k$.

But for integer $i = 1, 2,\ldots, k$,

$$
\begin{aligned}
m_i((f\circ g)\,g', Q) &= \inf\{f\circ g\,(x)\,g'(x): x \in [x_{i-1}, x_i]\}\\
&= \inf\{f\circ g\,(x)\,(g'(x) - g'(\xi_i)) + f\circ g\,(x)\,g'(\xi_i): x \in [x_{i-1}, x_i]\}\\
&\geq \inf\{-M\,\varepsilon/(2(M+1)(b-a)) + f\circ g\,(x)\,g'(\xi_i): x \in [x_{i-1}, x_i]\} \quad \text{by (2)}\\
&\geq -\varepsilon/(2(b-a)) + \inf\{f\circ g\,(x) : x \in [x_{i-1}, x_i]\}g'(\xi_i) \quad \text{since } g'(\xi_i) \geq 0\\
&= -\varepsilon/(2(b-a)) + \inf\{f(x) : x \in [y_{i-1}, y_i]\}g'(\xi_i) \quad \text{since } g \text{ is a bijection.}
\end{aligned}
$$

Therefore, the lower sum of $(f\circ g)\,g'$ wuth respect to the partition Q,

$$
\begin{aligned}
\backslash L((f\circ g)\,g', Q) &= \sum_{i=1}^{k} m_i(f\circ g)g', Q)\Delta x_i\\
&\geq \sum_{i=1}^{k} -\frac{\varepsilon}{2(b-a)}\Delta x_i + \sum_{i=1}^{k} \inf\{f(x) : x\in[y_{i-1}, y_i]\}g'(\xi_i)\Delta x_i\\
&= -\frac{\varepsilon}{2} + \sum_{i=1}^{k} \inf\{f(x) : x\in[y_{i-1}, y_i]\}\Delta y_i = -\varepsilon/2 + L(f, P) \quad \text{by (4).}
\end{aligned}
$$

It follows that

$$L(f, P) \leq L((f\circ g)\,g', Q) + \varepsilon/2 \qquad \text{------------------------ (7)}$$

Now

$$
\begin{aligned}
U\int_c^d f &\leq U(f, P) < L(f, P) + \varepsilon/2 \quad \text{by (3)}\\
&\leq L((f\circ g)\,g', Q) + \varepsilon/2 + \varepsilon/2 \qquad \text{by (7) above}\\
&\leq L\int_a^b (f\circ g)g' + \varepsilon
\end{aligned}
$$

using the fact that the upper integral of $f \leq U(f, P)$ and that $L((f\circ g)g', Q) \leq$ the lower integral of $(f\circ g)\,g'$.
Therefore, $U\int_c^d f < L\int_a^b (f\circ g)g' + \varepsilon$.
Since this is true for any $\varepsilon > 0$,

$$U\int_c^d f \leq L\int_a^b (f\circ g)g' \qquad \text{------------------------ (8)}$$

Combining (6) and (8) we get

$$U\int_c^d f \leq L\int_a^b (f\circ g)g' \leq U\int_a^b (f\circ g)g' \leq L\int_c^d f$$

But $L\int_c^d f \leq U\int_c^d f$ and so $U\int_c^d f = L\int_a^b (f\circ g)g' = U\int_a^b (f\circ g)g' = L\int_c^d f$.
Therefore, by Theorem 21 (1), $(f\circ g)\,g'$ is integrable on $[a, b]$ and

$$\int_a^b (f\circ g)g' = U\int_a^b (f\circ g)g' = L\int_c^d f = \int_c^d f$$

This completes the proof.

Remark.

1. Note that Theorem 62 holds true when g is continuously differentiable and increasing on $[a, b]$. To prove this, in place of $g^{-1}W$, just pick a point in the pre-image of each of the points in W. Then we refine this partition to Q. $P = g(Q)$ is a partition of $[c, d]$ but the points $y_i = g(x_i)$, $i = 1, 2,\ldots, k$ need not be distinct. Inequality (5) in the proof of Theorem 62 above still holds true, i.e., $\Delta y_i = g'(\xi_i)\Delta x_i$, for $i = 1, 2,\ldots, k$. Note that if $y_i = g(x_i) = y_{i-1} = g(x_{i-1})$, then, since g is increasing, g is constant on $[x_{i-1}, x_i]$ and $g'(x) = 0$ for all x in $[x_{i-1}, x_i]$ and we can choose any x for ξ_i. The rest of the proof is exactly the same as in the proof of Theorem 62.

2. In Theorem 62 we do not require that f has an antiderivative. This caters to a larger class of functions including step functions.

The case when g is strictly decreasing is given as follows.

Theorem 63. Suppose $f : [c, d] \to \mathbf{R}$ is integrable and g: $[a, b] \to [c, d]$ is a continuously differentiable strictly decreasing function mapping $[a, b]$ onto $[c, d]$. Then $(f \circ g)\, g'$ is integrable on $[a, b]$ and

$$\int_a^b f(g(x))g'(x)dx = \int_d^c f(x)dx = \int_c^d -f(x)dx$$

The proof of Theorem 63 is exactly the same as the proof for Theorem 62 except that we use the function $-(f \circ g)\, g'$ instead of $(f \circ g)\, g'$ and taking care that this time round, $g'(\xi_i) < 0$ *and in the equation*

$$\Delta y_i = g'(\xi_i)\Delta x_i$$

the points y_i *are oriented in the opposite direction, that is*

$$d = y_0 > y_1 > \ldots > y_k = c\ ,$$

where $y_i = g(x_i)$, $i = 1, 2, \ldots, k$. Care should be exercised when taking supremum or infimum, use $-g'(\xi_i) \geq 0$, for instance the following may be used to derive the supremum

$\sup\{-f \circ g(x)\, g'(x) : x \in [x_{i-1}, x_i]\}$

$= \sup\{-f \circ g(x)(g'(x) - g'(\xi_i)) - f \circ g(x)\, g'(\xi_i) : x \in [x_{i-1}, x_i]\}$

$\leq \sup\{M\varepsilon/(2(M+1)(b-a)) + f \circ g(x)(-g'(\xi_i)) : x \in [x_{i-1}, x_i]\}$

$\leq \varepsilon/(2(b-a)) + \sup\{f \circ g(x) : x \in [x_{i-1}, x_i]\}(-g'(\xi_i))$ since $-g'(\xi_i) \geq 0$.

$= \varepsilon/(2(b-a)) + \sup\{f(x) : x \in [y_i, y_{i-1}]\}(-g'(\xi_i))$, since g is onto on $[x_{i-1}, x_i]$.

Remark. Theorem 63 holds true when g is continuously differentiable and decreasing on $[a, b]$.

Example 64

In the integral $\int_0^1 x^3(2x^4-3)^{1/3}dx$, let $g(x) = 2x^4 - 3$, $g'(x) = 8x^3$ and $f(y) = y^{1/3}$.

Then, f is continuous and g is continuously differentiable and so by Theorem 61

$$\int_0^1 x^3(2x^4-3)^{1/3}dx = \frac{1}{8}\int_0^1 f(g(x))g'(x)dx = \frac{1}{8}\int_{g(0)}^{g(1)} f(y)dy = \frac{1}{8}\int_{-3}^{-1} y^{1/3}dy$$

$$= \frac{1}{8}\left[\frac{3}{4}y^{4/3}\right]_{-3}^{-1} = \frac{3}{32}(1 - 3\sqrt[3]{3}),$$

by the First Fundamental Theorem for Calculus.

5.10 The Second Mean Value Theorem for Integrals

Theorem 65. Suppose $F: [a, b] \to \mathbf{R}$ is continuous and $g : [a, b] \to \mathbf{R}$ is increasing and differentiable on $[a, b]$ and its derivative g' is integrable on $[a, b]$. Then there exists c in $[a, b]$ such that

$$\int_a^b F\, g' = F(c)(g(b) - g(a))$$

Proof. Since F is continuous on $[a, b]$, by the Extreme Value Theorem, there exists d and e in $[a, b]$ such that for all x in $[a, b]$,

$$F(d) \le F(x) \le F(e). \qquad \text{-----------------} \quad (1)$$

I.e., $F(d)$ and $F(e)$ are respectively the minimum and maximum of F.
Next since g is increasing and differentiable on $[a, b]$, $g'(x) \ge 0$ for all x in $[a, b]$. Thus multiplying (1) by $g'(x)$ we get for all x in $[a, b]$,

$$F(d)\, g'(x) \le F(x)\, g'(x) \le F(e) g'(x).$$

Note that since F is continuous, F is integrable. Since g' is integrable, by Corollary 55, the product $F\, g'$ is integrable on $[a, b]$. Therefore, by Theorem 32,

$$F(d) \int_a^b g'(x) \le \int_a^b F\, g' \le F(e) \int_a^b g'(x) dx.$$

By the Intermediate Value Theorem, there exists c between d and e and hence in $[a, b]$ such that

$$\int_a^b F\, g' = F(c) \int_c^b g'.$$

Since g' is integrable on $[a, b]$, by the First Fundamental Theorem of Calculus,

$$\int_a^b g' = g(b) - g(a).$$

It follows that $\int_a^b F\, g' = F(c)(g(b) - g(a))$.

Theorem 66. Suppose $f : [a, b] \to \mathbf{R}$ is increasing and differentiable on $[a, b]$ and its derivative $f\,'$ is Riemann integrable on $[a, b]$, and suppose that $g:[a, b] \to \mathbf{R}$ is continuous on $[a, b]$. Then there exists c in $[a, b]$ such that

$$\int_a^b f g = f(a) \int_a^c g + f(b) \int_c^b g.$$

Proof. Since g is continuous on $[a, b]$, we can define for each x in $[a, b]$ $G(x) = \int_a^x g(t)dt$. By the Second Fundamental Theorem of Calculus, G is an anti-derivative of g.
Then using integration by parts (Theorem 58),

$$\begin{aligned}\int_a^b f(x)g(x)dx &= [f(x)G(x)]_a^b - \int_a^b G(x) f\,'(x)dx \\ &= f(b)G(b) - \int_a^b G(x) f\,'(x)dx, \qquad \text{---------------} \ (1)\end{aligned}$$

since $G(a) = 0$.

By Theorem 65, since G is continuous and f is increasing with derivative $f\,'$ integrable, there exists c in $[a, b]$ such that

$$\int_a^b G(x) f\,'(x)dx = G(c)(f(b) - f(a)). \qquad \text{-----------------} \ (2)$$

Thus substituting (2) in (1) we obtain,

$$\begin{aligned}\int_a^b f g &= f(b)G(b) - \int_a^b G(x) f\,'(x)dx = f(b)G(b) - G(c)(f(b) - f(a)) \\ &= f(b)(G(b) - G(c)) + f(a)G(c) \\ &= f(b)\left[\int_a^b g(t)dt - \int_a^c g(t)dt\right] + f(a) \int_a^c g(t)dt \\ &= f(b) \int_c^b g(x)dx + f(a) \int_a^c g(x)dx = f(a) \int_a^c g + f(b) \int_c^b g\end{aligned}$$

This completes the proof.

Remark.

1. Theorem 66 is also known as the second mean value theorem or Bonnet's theorem and Theorem 65 is known as the mean value theorem for weighted means.

2. It is clear that Theorem 66 holds true when $f:[a, b] \to \mathbf{R}$ is decreasing and differentiable on $[a, b]$ and its derivative f' is integrable on $[a, b]$. Just apply Theorem 66 to $-f$.
3. Actually the conclusion of Theorem 66 is valid if f is merely monotone and g is integrable. As you might expect, the proof is much harder. A proof would require a Riemann- Stieltjes version of Theorem 65 and a Riemann-Stieltjes version of integration by parts.

Exercises 67.

1. Consider the partition $P = \{0, 1/4, 1/2, 1\}$ of the interval $[0, 1]$. Compute the lower and upper Darboux sums, i.e., $L(f, P)$ and $U(f, P)$ for the following three choices of function $f: [0,1] \to \mathbf{R}$:
(i) $f(x) = x$ for all x in $[0,1]$.
(ii) $f(x) = 19$ for all x in $[0, 1]$.
(iii) $f(x) = -x^2$ for all x in $[0, 1]$.

2. Suppose that the bounded function $f: [a, b] \to \mathbf{R}$ is such that $f(x) = 0$ for rational x in $[a, b]$. Prove that $L\int_a^b f \le 0 \le U\int_a^b f$.

3. Suppose $f: \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = \begin{cases} -\frac{1}{2}x^2, & x \le 0 \\ \frac{1}{2}x^2, & x \ge 0 \end{cases}$. Determine $f'(x)$ for each x in $\mathbf{R}$. What is the most general anti-derivative of the function $g(x) = |x|$?

4. Show that a necessary and sufficient condition for $f: [a, b] \to \mathbf{R}$ to be integrable is:

   For any $\varepsilon > 0$, there exist integrable functions g and h on $[a, b]$ such that

   $$g \le f \le h$$

   and $$\int_a^b h - \int_a^b g < \varepsilon$$

   [Hint: use one of the equivalent condition in Theorem 21.]

5. Use only the definition or equivalent definition of the integral to prove that

   $$\int_0^x t^3 dx = \frac{1}{4}x^4 .$$

6. Show that $\int_{-1}^1 \mu(x)dx = 1$ if $\mu(x) = |x|$.

7. Let $[x]$ be the largest integer $\le x$. Do the following functions have anti-derivatives on the whole of $\mathbf{R}$?

   (i) $f(x) = x, x \ne 0, f(0) = 1$.

   (ii) $f(x) = [x]$.

(iii) $f(x) = \dfrac{|x|}{1/2 + [x]}$.

8. Prove that the function $f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is integrable on [0, 1].

9. Suppose $f : [a, b] \to \mathbf{R}$ is a Lipschitz function, i.e., there exists a constant C such that $|f(x) - f(y)| \leq C|x - y|$. Prove that f is integrable.

10. Suppose that S is a non-empty bounded subset of **R**. For a real number k, define $kS = \{ ks : s \in S\}$.
Prove that
(i) $\sup kS = k \sup S$ and $\inf kS = k \inf S$ if $k \geq 0$
(ii) $\sup kS = k \inf S$ and $\inf kS = k \sup S$ if $k < 0$.

11. Suppose $f : [0, 1] \to \mathbf{R}$ is defined by $f(x) = \begin{cases} \frac{1}{2}x^2, & x \text{ rational} \\ -\frac{1}{2}x^2, & x \text{ irrational} \end{cases}$. Prove that f is not integrable.

12. Suppose $f : [a, b] \to \mathbf{R}$ is continuous and such that $\int_a^b f = 0$. Prove that there is a point c in $[a, b]$ such that $f(c) = 0$.

13. Suppose $f : [1, 2] \to \mathbf{R}$ is defined by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ \frac{1}{n}, & x \text{ rational and } x = \frac{m}{n} \text{ in its lowest term} \end{cases}.$$

Prove that f is integrable.

14. Suppose $f : [a, b] \to \mathbf{R}$ and g : $[a, b] \to \mathbf{R}$ are integrable. Prove the following cauchy Schwarz inequality:

$$\int_a^b fg \leq \sqrt{\int_a^b f^2}\ \sqrt{\int_a^b g^2}\ .$$

[Hint: For each number λ, define $p(\lambda) = \int_a^b (f - \lambda g)^2$. Then $p(\lambda)$ is a quadratic function always ≥ 0.]

15. Suppose $f : [a, b] \to \mathbf{R}$ is bounded and continuous except at one point x_0 in the interior (a, b). Prove that f is integrable.

16. Suppose $f : [a, b] \to \mathbf{R}$ and g : $[a, b] \to \mathbf{R}$ are integrable. Prove that

$$\int_a^b |f + g| \leq \int_a^b |f| + \int_a^b |g|.$$

17. Suppose that the function $f : \mathbf{R} \to \mathbf{R}$ is differentiable. Define the function H: $\mathbf{R} \to \mathbf{R}$ by

$$H(x) = \int_{-x}^{x} (f(t) + f(-t))dt \quad \text{for all } x \text{ in } \mathbf{R}.$$

Find $H''(x)$.

18. Suppose that the function $f: \mathbf{R} \to \mathbf{R}$ has a continuous second derivative. Prove that $f(x) = f(0) + f'(0)x + \int_0^x (x-t) f''(t)dt$ for all x in $\mathbf{R}$.

19. Suppose that the function $f: \mathbf{R} \to \mathbf{R}$ is continuous. Define
$$G(x) = \int_0^x (x-t) f(t)dt$$
for all x in $\mathbf{R}$.
Prove that $G''(x) = f(x)$ for all x in $\mathbf{R}$.

20. Show that the conclusion of the Mean Value Theorem for Integrals (Theorem 40) can be strengthened so that we can choose the point χ to be in (a, b), not just in $[a, b]$.

21. Suppose that the function g: $\mathbf{R} \to \mathbf{R}$ is continuous and that $g(x) > 0$ for all x. Define $h(x) = \int_0^x \frac{1}{g(t)}dt$ for all x in $\mathbf{R}$ and let $J = h(\mathbf{R})$. Prove that if $f: J \to \mathbf{R}$ is the inverse of h: $\mathbf{R} \to \mathbf{R}$, then $f: J \to \mathbf{R}$ is a solution of the non-linear differential equation
$$\begin{cases} f'(x) = g(f(x)) \text{ for all } x \text{ in } J \\ f(0) = 0 \end{cases}.$$

22. Suppose that the function $f: [a, b] \to \mathbf{R}$ is continuous and let P be any partition of its domain $[a, b]$. Show that there is a Riemann sum $R(f, P, C)$ that equals $\int_a^b f$. [Hint: Use the Mean Value Theorem for Integrals.]

23. (i) Let p and n be counting numbers in $\mathbf{P}$ with $n \geq 2$. Prove by induction that
$$\sum_{k=1}^{n-1} k^p \leq \tfrac{n^{p+1}}{p+1} \leq \sum_{k=1}^{n} k^p .$$
(ii) Use (i) to prove that for a counting number p,
$$\int_0^1 x^p dx = \frac{1}{p+1}.$$

24. Suppose $f: [0, \infty) \to \mathbf{R}$ is continuous and that $\lim_{x\to\infty} f(x) = a$, where a is a real number. Prove that $\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)dt = a$.

25. Suppose that f is continuous on $[a, b]$ and that $\int_a^b f(x)g(x)dx = 0$ for any continuous function g on $[a, b]$ such that $g(a) = g(b) = 0$. Prove that $f = 0$ the zero constant function.

26. Prove that for any two counting numbers n and m in $\boldsymbol{P}$,
$$\int_0^1 x^m(1-x)^n dx = \int_0^1 (1-x)^m x^n dx.$$

27. Suppose that the function $f: \mathbf{R} \to \mathbf{R}$ has a continuous second derivative. Prove that for any two numbers a and b,
$$\int_a^b x f''(x)dx = b f'(b) + f(a) - af'(a) - f(b).$$

28. Suppose that the function $f: \mathbf{R} \to \mathbf{R}$ has a continuous second derivative. Fix a number a. Prove that
$$\int_a^x f''(t)(x-t)dt = -(x-a) f'(a) + f(x) - f(a) \text{ for all } x.$$

29. Suppose that the function $f: [0, \infty) \to \mathbf{R}$ is continuous and strictly increasing and that f is differentiable on $(0, \infty)$. Suppose $f(0) = 0$. Consider the formula

$$\int_0^x f + \int_0^{f(x)} f^{-1} = x f(x) \quad \text{for all } x \geq 0.$$

Provide a geometric interpretation of this formula in terms of areas. Then prove this formula.

30. Suppose that the function $f: [0, \infty) \to \mathbf{R}$ is continuous and strictly increasing with $f(0) = 0$ and $f([0, \infty)) = [0, \infty)$. Then define

$$F(x) = \int_0^x f \qquad \text{and} \qquad G(x) = \int_0^x f^{-1} \quad \text{for all } x \geq 0.$$

(i) Prove Young's Inequality:

$$ab \leq F(a) + G(b) \quad \text{for all } a \geq 0 \text{ and } b \geq \text{o}.$$

(ii) Use Young's Inequality with $f(x) = x^{p-1}$ for all $x \geq 0$ and $p > 1$ fixed, to prove that if the number q is chosen to have the property $1/p + 1/q = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for all } a \geq 0 \text{ and } b \geq 0.$$

[Hint: The formula in question 29 holds without assuming the differentiability of f. Use this formula. See question 46 for further details.]

31. (a) Suppose that the function $f: \mathbf{R} \to \mathbf{R}$ is differentiable and $f' = c f$ for some constant c. Prove that there is a constant k such that $f(x) = k e^{cx}$ for all x in $\mathbf{R}$.
(b) Show that if $f(x) = \int_0^x f(t)dt$ for all real number x, then $f = 0$, the 0 constant function.

32. Suppose $f: [a, b] \to \mathbf{R}$ is continuously differentiable, i.e., f is differentiable and $f': [a, b] \to \mathbf{R}$ is continuous. Use integration by parts to prove that

$$\lim_{x\to\infty} \int_a^b f(t)\sin(xt)dt = 0.$$

33. Use the Second Mean Value Theorem for Integrals (Theorem 66) to prove that

$$\left| \int_1^{10} \frac{\sin(x)}{x} dx \right| < 2 \quad .$$

34. Suppose that the function $f: \mathbf{R} \to \mathbf{R}$ is continuous. Prove that

$$\int_0^x f(u)(x-u)du = \int_0^x \left(\int_0^u f(t)dt \right) du.$$

[Hint: Differentiate both sides.]

35. Evaluate (a) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n} \sin(\frac{k\pi}{2n})$ (b) $\int_1^e (\ln(x))^2 dx$ (c) $\int_2^\pi x^2 \cos(x) dx$.

36. Let the function f be continuous on $[0, \infty)$. Show that

$$\int_0^\pi x f(\sin(x))dx = \frac{\pi}{2} \int_0^\pi f(\sin(x))dx \; .$$

Hence, or otherwise, compute $\int_0^\pi x \sin^4(x)dx$.
(Hint: Use the substitution $x = \pi - t$ and the identity $\sin(\pi - t) = \sin(t)$.)

37. Find the following derivatives.

a. $\frac{d}{dx} \int_1^x \frac{1}{1 + t^3 + t^6} dt.$ b. $\frac{d}{dx} \int_1^{\sin(2x)} \frac{1}{1 + t^5 + t^{10}} dt.$

c. $\dfrac{d}{dx}\int_{-3x}^{\sin(x^2)} \dfrac{1}{1+t^3+t^6}dt.$

38. Find the following integrals.
a. $\int x^2 17^{x^3} dx.$ b. $\int x\, e^{3+2x^2} dx.$ c. $\int (x+1)e^x 13^{xe^x} dx.$

39. Find the following integrals.
a. $\int \dfrac{1}{(x^2+5)(6+x^2)}dx.$ b. $\int_4^6 \dfrac{1}{\sqrt{x}\,\sqrt{8-x}}dx.$ c. $\int_0^1 \dfrac{x}{1+x^4}dx.$

40. Use integration by parts to find the following integrals.
a. $\int \sec^{-1}(x)dx.$ b. $\int x^2\cos(2x)dx.$ c. $\int \sin(\ln(x^2))dx.$

41. Evaluate : a. $\int_{\frac{1}{3}}^{3} \sqrt{x}\tan^{-1}(\sqrt{x})dx.$ b. $\int_0^4 \ln(x^2+1)dx.$ c. $\int_0^1 \dfrac{\ln(1+x)}{1+x^2}dx.$
d. $\int_0^{\pi/4} \ln(1+\tan(x))dx.$ e. $\int_0^{\pi/2} \ln\left(\frac{1+\sin(x)}{1+\cos(x)}\right)dx$. f. $\int_0^{\pi/2} \frac{\sin(2x)}{\sqrt{1-k\sin(x)}}dx,\ 0<k<1.$

42. Find
a. $\int \sin^6(3x)\cos(3x)dx;$ b $\int \sin^6(x)\cos^4(x)dx.$

43. Find the following integrals.
a. $\int \dfrac{x^2}{\sqrt{25-x^2}}dx.$ b. $\int \dfrac{x^2}{\sqrt{9+x^2}}dx.$ c. $\int \dfrac{\sqrt{x^2-25}}{x}dx.$
d. $\int \dfrac{x^5+x^4+x^3+x^2+x+1}{x^3-x}dx.$ e. $\int \dfrac{x^2+3}{x(x^2+x+1)}dx.$
f. $\int \dfrac{x^2+3x+5}{(x^2+2x+5)^2}dx.$ g. $\int \sqrt{9-\sqrt{x}}\ dx.$

44. Suppose f is continuous on [0, 8] and twice differentiable on (0, 8) such that $f''(x) \ge 0$. Then show that $\int_0^2 f(x^3)dx \ge 2f(2)$. (Hint: Jensen's inequality.)

45. (i) Suppose f is continuous on $[a, b]$ and twice differentiable on (a, b) with $f''(x) \le 0$ for all x in (a, b). Suppose g : $[c, d] \to \mathbf{R}$ is a function that maps $[c, d]$ into $[a, b]$. Suppose g is integrable on $[c, d]$. Let $J = \dfrac{\int_c^d g(x)dx}{d-c}$.
Then prove that $\int_c^d f(g(x))dx \le (d-c)\,f(J)$.

(ii) More generally, suppose f is convex on (a, b). Suppose $g:[c,d]\to(a,b)$ is integrable. Then

$$f\left(\frac{\int_c^d g}{d-c}\right) \le \frac{1}{d-c}\int_c^d f(g(x))dx . \quad \text{------------------ (*)}$$

This is known as Jensen's inequality. In particular, if f is twice differentiable on (a, b) and $f''(x) \ge 0$ for all x in (a, b), then the inequality (*) holds.

46. Let $h : [a,b] \to \mathbf{R}$ be any continuous and increasing function.

(i) Prove that $\int_a^x h(t)dt + \int_{h(a)}^{h(x)} h^{-1}(s)ds = xh(x) + C$ for some constant C.

Hence, or otherwise, deduce that

$$\int_a^b h(x)dx = bh(b) - ah(a) - \int_{h(a)}^{h(b)} h^{-1}(s)ds$$

(ii) Using part (i) or otherwise, evaluate $\int_0^1 \sqrt{1 + (x-1)^{\frac{1}{3}}}\, dx$.

[Hint: Take a partition for $[a, x]$ and induce a partition for $[h(a), h(x)]$. Consider lower and upper Darboux sums for h and h^{-1}.]

47. Suppose $f :[a, b] \to \mathbf{R}$ is Riemann integrable. Prove the following generalization of question 32,

(a) $\lim_{x\to\infty} \int_a^b f(t)\sin(xt)dt = 0$ and

(b) $\lim_{x\to\infty} \int_a^b f(t)\cos(xt)dt = 0.$