Chapter 2 Sequences.

For most part of the discussion we shall confine to real sequences. However, most of the definitions and results apply equally well to complex sequences, i.e., sequences taking values in the complex numbers. Of course, we can have sequences whose values are lying in such mathematical objects as $\mathbf{R}^n$, metric spaces, abstract topological spaces, set of all functions on [0, 1], groups, etc. Sequences are used for very different purposes, presenting themselves as charts of movement of share values, temperature, distances, blood pressure, etc. The purpose can be to monitor movement or fluctuation of temperatures when extreme high or low can be critical. In a patient the attention is called when temperature rises above 37.5 C for further treatment or medical attention. This is an example of a sequence approaching a certain ‘critical’ value. Vibration problems can be approached via Fourier series, which is a sort of sequence. Number theory problems such as primality testing may use Lucasian sequence, which has its origin based on Fibonacci sequence. These sequences are used in the so called N+1 primality testing, where the knowledge of the prime factors of N+1 are required to begin a search for these sequences. The goal is to find a suitable Lucasian sequence such that its N+1 term is congruent to 0 modulo N. However our concern will be with the behaviour of the sequence at infinity, i.e., what happens to the values of the sequence (a_n), a_n as n gets larger and larger.

We shall look at the convergence of sequences. Through special sequences, i.e., monotone sequences, we shall study Cauchy sequences, formulate a convergence criterion (that presupposes the existence of the irrational numbers) and derive the Bolzano-Weierstrass Theorem. Although for $\mathbf{R}$, completeness is equivalent to the convergence of any Cauchy sequence, which is equivalent to that any bounded monotone sequence is convergent and which is also equivalent to the conclusion of the Bolzano Weierstrass Theorem, only the Cauchy principle of convergence is capable of generalization to $\mathbf{R}^n$ and beyond and leads to a definition of completeness for metric spaces.

Definition 1. Let $\boldsymbol{P}$ be the set of positive integers. A *sequence* is simply a function from P into the set of real numbers $\mathbf{R}$.
$\boldsymbol{P}$ is of course the set $\{1,2, \dots \}$. Thus a function a: $\boldsymbol{P} \to \mathbf{R}$ is a sequence.

The image $a(n)$ is called the n-th term of the sequence and is also written as a_n ,
We also write $(a_1 , a_2, \dots)$ or simply (a_n) for the sequence.
Here we use the round bracket for sequences. One should not confused the sequence $(a_1 , a_2, \dots)$ with a row vector. Note that $\{a_1 , a_2, \dots \}$ is a set and should not be confused with a sequence, likewise $\{a_n\}$ is a singleton set and not a sequence.

We are interested in the behaviour of the values or points of the sequences. We want to know if they are bunched together like a cluster or they become further and further apart or oscillatory. We focus on whether the points are bunched together or not. We have a technical term for this bunching together.

Remark. 1. More generally, if X is a set, a sequence in X is a function a: $P \to X$. For instance, X can be the complex numbers $\mathbf{C}$ or $\mathbf{R}^2$ or $\mathbf{R}^n$, $n \geq 3$, etc.

2. We can replace $\boldsymbol{P}$ by any set Q with an ordering - a generalization of a sequence called a *net* leading to the notion of Moore-Smith convergence of net.

Definition 2. Let (a_n) be a sequence in $\mathbf{R}$. We say (a_n) tends to a real number a in $\mathbf{R}$ if for any $\varepsilon > 0$, there exists a positive integer N_0 such that for all n in $\boldsymbol{P}$ with $n \geq N_0$, $|a_n - a| < \varepsilon$. That is,

$$n \geq N_0 \Rightarrow |a_n - a| < \varepsilon.$$

Notation:

If (a_n) tends to a, we write

$$a_n \to a \text{ as } n \to \infty$$

or $$\lim_{n\to\infty} a_n = a$$

Or just simply, $a_n \to a$.

Definition 3. We say (a_n) *converges* if there exists a real number a such that $a_n \to a$, otherwise (a_n) diverges or is divergent.

Remark.

1. The number N_0 in definition 2 depends on ε.
2. We may replace in definition 2 "$n \geq N_0$" by "$n > N_0$" without changing the sense of the definition. That is, it gives rise to equivalent definition.

Example 4.

1. $a_n = c$ for all n in $\boldsymbol{P}$. This is a constant sequence and obviously $a_n \to c$.

Given any $\varepsilon > 0$, take any positive integer N. Obviously for any $n \geq N$, $|a_n - c| = |c - c| = 0 < \varepsilon$.

2. $a_n = (-1)^n$. Then (a_n) is divergent. There is a quick way to see this. Observe that the value changes from 1 to -1 and so there is no way it can get close to any value.

If you like the following is a proof of this fact.

For any a in $\mathbf{R}$, by the triangle inequality,

$$|1-a| + |(-1)-a| \geq |1-a-((-1)-a)| = 2.$$

Hence, either $|1-a| \geq 1$ or $|(-1)-a| \geq 1$.

Take any positive integer N_0. If $|a-1| \geq 1$, then take any even $n > N_0$ and we have $|a_n - a| = |1-a| \geq 1$ and if $|(-1)-a| \geq 1$, then take any odd $n > N_0$ and we have

$$|a_n - a| = |(-1)-a| \geq 1.$$

Thus (a_n) cannot converge to any a and so is divergent.

3. $a_n = 1/n$. Then $a_n \to 0$.

For any $\varepsilon > 0$, there exists a positive integer N_0 such that $0 < \frac{1}{N_0} < \varepsilon$ (by the archimedean property of $\mathbf{R}$). Thus for $n \geq N_0$, $\frac{1}{n} \leq \frac{1}{N_0} < \varepsilon$ and this means $|\frac{1}{n} - 0| < \varepsilon$ and so by definition $a_n \to 0$.

Example 3 illustrates the notion of continuity of a function at the point 0 and the limit of a function at 0. We shall use this notion to derive the properties of the sequence.

We make the definition as follows:

Let A be a subset of **R** containing 0 and $f : A \to \mathbf{R}$ is a function. The function f is said to have a *limit* at 0 if there exists a real number L such that given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{for all } x \in A,\ 0 < |x - 0| < \delta \Rightarrow |f(x) - L| < \varepsilon .$$

We write $\lim_{x\to 0} f(x) = L$. If $L = f(0)$, we say f is *continuous* at 0. We define the left and right limit at 0 similarly as follows.

$\lim_{x\to 0^+} f(x) = L$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{for all } x \in A,\ 0 < x < \delta \Rightarrow |f(x) - L| < \varepsilon .$$

$\lim_{x\to 0^-} f(x) = L$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{for all } x \in A,\ -\delta < x < 0 \Rightarrow |f(x) - L| < \varepsilon .$$

In the next chapter we shall study limits and continuity in general and in more detail. For our purpose here we shall use only the notion of limit and continuity at the point 0.

Let $\boldsymbol{P}^{-1}$ denotes the set $\{1/n\ ;\ n \in \boldsymbol{P}\}$. That is $\boldsymbol{P}^{-1} = \{1, ½, 1/3, \ldots\}$.
Then let $K = \boldsymbol{P}^{-1} \cup \{0\} = \{0, 1, ½, 1/3, \ldots.\}$.

Here is an easy result:

Proposition 5. Let (a_n) be a sequence in **R**. Define a function

$$f : \mathrm{K} = \boldsymbol{P}^{-1} \cup \{0\} \to \mathbf{R}$$

by $f(1/n) = a_n$ for $n > 0$ and $f(0) = a$.
Then $a_n \to a$ if and only if f is continuous at 0.

Proof.
First suppose that f is continuous at 0. Recall the definition of continuity at 0:
Given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{for all } x \in \mathrm{K},\ |x - 0| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon . \quad \text{--------------} \quad (1)$$

We want to show that $a_n \to a$. We want to find an integer N_0 in $\boldsymbol{P}$ such that

$$n \geq N_0 \Rightarrow |f(x) - a| < \varepsilon.$$

Let N be the largest integer such that $N \leq 1/\delta$.
Then $n > N$ implies that $n > 1/\delta \geq N$. That means $1/n < \delta$. Then by (1) , taking x to be $1/n$,

$$|a_n - a| = |f(x) - a| = |f(x) - f(0)| < \varepsilon.$$

Let $N_0 = N + 1$, then $n \geq N_0 \Rightarrow 1/n < \delta \Rightarrow |a_n - a| < \varepsilon$. Therefore, $a_n \to a$.

Conversely suppose $a_n \to a$. Then given $\varepsilon > 0$, there exists a positive integer N_0 such that

$$n \geq N_0 \Rightarrow |a_n - a| < \varepsilon. \quad \text{---------------------------} \quad (2)$$

Then for any x in $(-1/N_0 , 1/N_0) \cap \mathrm{K}$, x is in $\boldsymbol{P}^{-1} \cup \{0\}$ and $|x| < 1/N_0$.
If $x = 0$, then $|f(x) - f(0)| = 0 < \varepsilon$.
If $x \neq 0$, then $x = 1/n$ for some positive integer n and $1/n < 1/N_0$. Therefore, $n > N_0$ and so by (2) $|a_n - a| < \varepsilon$. Hence $|f(x) - f(0)| = |f(1/n) - a| = |a_n - a| < \varepsilon$.
Therefore, f is continuous at $x = 0$.

Proposition 5 allows us to formulate results about continuous functions into results about sequences. We can use the properties of continuity and limit (see Chapter 3) to dedeuce results about sequences. The following examples illustrate this amply.

Example 6.

1. $\frac{1}{n^k} \to 0$ as $n \to \infty$ for all $k > 0$.
   Consider $f : \mathrm{K} = \boldsymbol{P}^{-1} \cup \{0\} \to \mathbf{R}$ defined by $f(1/n) = a_n = \frac{1}{n^k} = (\frac{1}{n})^k$ and $f(0) = 0$. Thus the function is given by $f(x) = x^k$ for $x \geq 0$. Recall that as a real valued function defined on $[0, \infty)$, $f : [0, \infty) \to \mathbf{R}$ is defined by $f(x) = x^k$ for $x > 0$ and $f(0) = 0$.
   This function is continuous at $x = 0$ since its limit at 0 is $f(0)$.
   $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^k = \lim_{x\to 0^+} e^{k\ln(x)} = \lim_{x\to 0^+} \frac{1}{e^{-k\ln(x)}} = 0 = f(0)$ since $\lim_{x\to 0^+} e^{-k\ln(x)} = \infty$ as $\lim_{x\to 0^+} -k\ln(x) = \infty$. Hence our original function is continuous at 0. Therefore, (a_n) converges to 0.

2. Let $a_n = \frac{27n^2 + 3n - 1}{15n^2 - 2n - 13}$.
   Then $f : \mathrm{K} = \boldsymbol{P}^{-1} \cup \{0\} \to \mathbf{R}$ is given by
   $$f(1/n) = a_n = \frac{27n^2 + 3n - 1}{15n^2 - 2n - 13} = \frac{27 + \frac{3}{n} - \frac{1}{n^2}}{15 - \frac{2}{n} - \frac{13}{n^2}}.$$
   Thus $f(x) = \frac{27 + 3x - x^2}{15 - 2x - 13x^2}$. Since this function on $\mathbf{R}$ is a rational function whose domain contains 0. Therefore, f is continuous at 0 and $f(0) = \frac{27}{15} = \frac{9}{5}$. Therefore, $a_n \to \frac{9}{5}$.

Below we list the properties for sequences, which are easy consequences of continuity via Proposition 5.

Properties 7.

1. If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.

2. If $a_n \to a$, then $\lambda a_n \to \lambda a$ for any real number λ.

3. If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$

4. If $a_n \to a$ and $a \neq 0$, then $\frac{1}{a_n} \to \frac{1}{a}$

Thus,

5. If $a_n \to a$ and $b_n \to b$ with $b \neq 0$, then $\frac{a_n}{b_n} \to \frac{a}{b}$

Proof. By proposition 5 the functions $f : K = \boldsymbol{P}^{-1} \cup \{0\} \to \mathbf{R}$ and $g : \mathrm{K} = \boldsymbol{P}^{-1} \cup \{0\} \to \mathbf{R}$ defined by $f(1/n) = a_n$, $g(1/n) = b_n$, $f(0) = a$ and $g(0) = b$ are continuous at 0. Therefore, $f + g$ is continuous at $x = 0$ and $f(0) + g(0) = a + b$. Thus, by Proposition 5, $(a_n + b_n)$ is convergent and $a_n + b_n \to a + b$. This proves part 1.
The remaining statements follows similarly from the continuity at 0 of λf for part 2, of $f \cdot g$ for part 3, of $1/f$, if $a \neq 0$, for oart 4 and of f / g , if $b \neq 0$, for part 5.

Remark. 1. The above proof makes use of Proposition 5 using continuity. It makes a connection with the concept of limit of sequences to that of the continuity of a function. This is the first time the connection is encounter. If your are more comfortable with the notion of continuity, then Proposition 5 is very useful. But of course properties 1 to 5 can be proved directly from Definition 2. We shall see later (in the next chapter) a characterization of continuity of a function at a point by sequences.
2. Example 6 (2) can now be deduced using (5) of Properties 7 as follows:

$$\frac{27n^2+3n-1}{15n^2-2n-13}=\frac{27+\frac{3}{n}-\frac{1}{n^2}}{15-\frac{2}{n}-\frac{13}{n^2}}\to\frac{27+0-0}{15-0-0}=\frac{9}{5}\,.$$

Next let (a_n) be a real sequence. Let a be a real number. We want to know if $a_n \to a$. We consider the difference $a_n - a$ and if $a_n - a \to 0$, then we can conclude that $a_n \to a$. The question is then when do we know $a_n - a \to 0$. The following furnishes a simple criterion by way of another sequence.

Propositon 8. Comparison Test.
If there exists a sequence (b_n) such that
(1) $|b_n| \to 0$,
(2) $|a_n - a| \le |b_n|$
Then $a_n \to a$.
Proof. Given $\varepsilon > 0$, by (1), there exists an integer N such that $n \ge N \Rightarrow |b_n| < \varepsilon$. Therefore, for all $n \ge N$, $|a_n - a| \le |b_n| < \varepsilon$. This means $a_n \to a$.

Remark. Note that $b_n \to 0$ if and only if $|b_n| \to 0$. By the Squeeze Theorem, $a_n - a \to 0$ via Proposition 5. So this result is trivial but it is very useful.

Example 9.

If $|a| < 1$, then the sequence (a^n) converges to 0.
Since $|a| < 1$, $1/|a| > 1$. Then we can write $1/|a| = 1+ \beta$ and $\beta > 0$.
Hence $|a^n - 0| = \dfrac{1}{(1+\beta)^n} < \dfrac{1}{n\beta}$.
The last inequality follows from the inequality $(1+\beta)^n \ge 1+n\beta > n\beta$ for positive integer n. (Use the binomial expansion for $(1 + \beta)^n$ to deduce the above inequality.)
Since $\frac{1}{n} \to 0$, $\dfrac{1}{n\beta} \to 0$. By the Comparison Test, $a^n \to 0$.
If $a = 1$, then $a^n = 1$ and so $a^n \to 1$. If $a = -1$, then $a^n = (-1)^n$ diverges (see (2) of Example 4).

Exercise. Complex sequence
If $a = i$, the imaginary complex number $i = \sqrt{(-1)}$, then a^n diverges. Suppose a is a complex number. If $|a| > 1$, then (a^n) diverges, if $|a| < 1$, then $a^n \to 0$. Note that the Comparison Test applies equally well to complex sequences.

Definition 10. A sequence (a_n) is said to be *bounded* if its range is bounded, that is, there exists a positive real number K such that $|a_n| \le K$ for all n in $\boldsymbol{P}$. It is said to be *bounded above* if there exists a real number M such that $a_n \le M$ for all n in $\boldsymbol{P}$. M is called an upper bound of (a_n). It is said to be *bounded below* if there exists a real number L such that $L \le a_n$ for all n in $\boldsymbol{P}$. The number L is called a lower bound of (a_n). Plainly, (a_n) is bounded if and only if it is both bounded above and bounded below.

Theorem 11. If a sequence (a_n) converges, then (a_n) is bounded.

Proof.
(a_n) converges means there exists an element a such that $a_n \to a$. Thus, by the definition of convergence, taking $\varepsilon = 1$, there exists an integer N such that

$$n \ge N \Rightarrow |a_n - a| < \varepsilon = 1.$$

Since $||a_n| - |a|| \le |a_n - a|$, we have then $n \ge N \Rightarrow ||a_n| - |a|| \le |a_n - a| < 1$ which implies that $|a_n| < |a| + 1$. Let $M = \max\{|a_1|, |a_2|, \ldots , |a_{N-1}|, |a|+1\}$. Then obviously $|a_n| \le |a| + 1 \le M$ for all positive integer $n \ge N$. Plainly, for $1 \le j \le N-1$,

$$|a_j| \le \max\{|a_1|, |a_2|, \ldots , |a_{N-1}|\} \le M.$$

Therefore, $|a_n| \le M$ for all positive integer n. This means (a_n) is bounded.

Remark. The converse of Theorem 11 is false. That is to say, if (a_n) is bounded, it does not necessarily follow that (a_n) is convergent. For instance, take $a_n = (-1)^n$. Then (a_n) is bounded but not convergent.

Proposition 12. If $a_n \to a$ and $b_n \to b$ and there exists an integer N such that $a_n \le b_n$ for all $n \ge N$, then $a \le b$.

Proof. We can use continuity and results about limit of function here.
Let $f : \mathrm{K} = \boldsymbol{P}^{-1} \cup \{0\} \to \mathbf{R}$ and $g : \mathrm{K} = \boldsymbol{P}^{-1} \cup \{0\} \to \mathbf{R}$ be defined by $f(1/n) = a_n$, $g(1/n) = b_n$, $f(0) = a$ and $g(0) = b$. Then by Proposition 5 both f and g are continuous at 0. Then for all x in K and $|x| < 1/N$, $f(x) \le g(x)$ since $f(1/n) = a_n \le b_n = g(1/n)$ for all $n \ge N$ by the given condition.
Therefore, $a = f(0) = \lim_{x\to 0} f(x) \le \lim_{x\to 0} g(x) = b.$

This result involves ordering and is a result about real sequences.

Remark. 1. The proof makes use of what we know about properties of continuous function and limits and follows easily if familiarity with properties of continuous functions and limits for function is assumed. We can give an alternative proof using Definition 2 as follows.
$a_n \to a$ means that for any $\varepsilon > 0$, there exists a positive integer K such that

$$n \ge K \Rightarrow |a_n - a| < \varepsilon/2$$

and similarly, since $b_n \to b$, there exists also a positive integer L such that

$$n \ge L \Rightarrow |b_n - b| < \varepsilon/2.$$

Now take $M = \max(N, K, L)$. We have then that

$$n \ge M \Rightarrow a - \varepsilon/2 < a_n < a + \varepsilon/2 \text{ and } b - \varepsilon/2 < b_n < b + \varepsilon/2.$$

Consequently, fixing an integer $n \ge M$, we get

$$a < a_n + \varepsilon/2 \le b_n + \varepsilon/2 < b + \varepsilon/2 + \varepsilon/2 = b + \varepsilon.$$

Hence, $a < b + \varepsilon$. Since ε is arbitrary we have that $a \le b$.
2. If $a_n \to a$ and $b_n \to b$ and there exists an integer N such that $a_n < b_n$ for all $n \ge N$, then we do not necessarily get $a < b$.
Example: Let $a_n = 1/n^2$ and $b_n = 1/n$. Then $a_n = 1/n^2 < b_n = 1/n$ for $n \ge 2$. But obviously $a_n \to 0 = a$ and $b_n \to 0 = b$ and $a = b$.

Theorem 13. Squeeze Theorem.
If $a_n \to a$ and $b_n \to a$ and there exists an integer N such that for all $n \ge N$, $a_n \le c_n \le b_n$, then $c_n \to a$.

Theorem 13 follows easily from the Squeeze Theorem for functions. We shall give a proof by the definition.

Proof. Since $a_n \to a$ and $b_n \to a$, for any $\varepsilon > 0$, there exists a positive integer K such that

$$n \ge K \Rightarrow |a_n - a| < \varepsilon$$

and there exists also a positive integer L such that

$$n \ge L \Rightarrow |b_n - a| < \varepsilon.$$

Let $M = \max(N, K, L)$. We have thus, that $n \ge M \Rightarrow a - \varepsilon < a_n < a + \varepsilon$ and $a - \varepsilon < b_n < a + \varepsilon$. Consequently, for any $n \ge M$, we get $a - \varepsilon < a_n \le c_n \le b_n < a + \varepsilon$. Thus ,

$$n \ge M \Rightarrow |c_n - a| < \varepsilon.$$

This means $c_n \to a$.

Remark. Even though Squeeze Theorem is a Theorem for real sequences only, we can apply it to deduce the following results about complex sequences.
For a complex sequence (a_n), (a_n) converges if and only if the real part (Re a_n) and the imaginary part (Im a_n) converge, where $a_n = \text{Re } a_n + i \text{ Im } a_n$ and Re a_n and Im a_n are respectively the real and imaginary parts of a_n .
Note that for any complex number z,

$$|\text{Re } z|,\ |\text{Im } z| \le |z| \le |\text{Re } z| + |\text{Im } z|.$$

Thus for any complex number a,

$$|\text{Re } a_n - \text{Re } a| \le |a_n - a|$$

and

$$|\text{Im } a_n - \text{Im } a| \le |a_n - a|.$$

Therefore, if $a_n \to a$ and so $|a_n - a| \to 0$, then by the Squeeze Theorem,

$$|\text{Re } a_n - \text{Re } a| \to 0 \text{ and } |\text{Im } a_n - \text{Im } a| \to 0$$

and consequently Re $a_n \to$ Re a and Im $a_n \to$ Im a.
Conversely, since

$$|a_n - a| \le |\text{Re } a_n - \text{Re } a| + |\text{Im } a_n - \text{Im } a|,$$

by the Squeeze Theorem, if Re $a_n \to$ Re a and Im $a_n \to$ Im a, then $|a_n - a| \to 0$ and so $a_n \to a$.

Of course, we may use the Comparison test as well.

We shall now describe an important class of real sequences.

Definition 14. A real sequence (a_n) is *increasing* if $n > m \Rightarrow a_n \ge a_m$.
It is *decreasing* if $n > m \Rightarrow a_n \le a_m$.
It is *strictly increasing* if $n > m \Rightarrow a_n > a_m$.

It is *strictly decreasing* if $n > m \Rightarrow a_n < a_m$.
It is a *monotone sequence* if it is either increasing or decreasing.

Theorem 15. Monotone Convergence Theorem. Suppose (a_n) is a bounded monotone (real) sequence. Then (a_n) is convergent. In particular, if (a_n) is increasing and bounded above, then $a_n \to \sup \{a_n : n \in \boldsymbol{P}\}$ and if (a_n) is decreasing and bounded below, then $a_n \to \inf \{a_n : n \in \boldsymbol{P}\}$.

Proof . Actually the proposition is equivalent to the completeness of **R**.
Suppose (a_n) is a bounded monotone increasing sequence. Then the set $S = \{a_n : n \in \boldsymbol{P}\}$ is bounded above and obviously non-empty. Therefore, by the completeness property of **R**, S has a supremum or the least upper bound in **R**. Let $a = \sup S$. We claim that $a_n \to a$.
Now take any $\varepsilon > 0$, then $a - \varepsilon < a$. Hence $a - \varepsilon$ is not an upper bound of S. Therefore, there exists an integer N such that $a - \varepsilon < a_N \le a$. Therefore, since (a_n) is increasing for all $n \ge N$, $a_n \ge a_N$ and so we have

$$a - \varepsilon < a_N \le a_n \le a.$$

It follows that $n \ge N \Rightarrow |a_n - a| = a - a_n < \varepsilon$. Therefore, by the definition of convergence, $a_n \to a$.

The case when (a_n) is a bounded monotone decreasing sequence, is similar. This time the limit is the infimum of S. We may also just consider the sequence $(-a_n)$. This is an increasing sequence and also bounded above. Therefore, by what we have just proved $(-a_n)$ is convergent and converges to $\sup \{-a_n : n \in \boldsymbol{P}\}$. Hence (a_n) is convergent and converges to $-\sup \{-a_n : n \in \boldsymbol{P}\} = \inf \{a_n : n \in \boldsymbol{P}\}$.

Example. $\left(1 - \frac{1}{n}\right)$ is a bounded increasing sequence and so is convergent.

Remark. Theorem 15 says that any real bounded monotone sequence is convergent. The statement can be taken as the completeness axiom for **R**. Another equivalent definition for completeness of **R** is : every *Cauchy* sequence in **R** is convergent. Note that the essential step in the proof of Theorem 15 is the existence of supremum in the case of increasing sequence and the existence of infimum in the case of decreasing sequence. The existence is provided by the (order) completeness of **R**. We can use the property "that every bounded monotone sequence in **R** has a limit" to prove the completeness of **R**.

The following theorems give different ways of thinking about completeness for **R**. Under different conditions, it may be useful to know the different ways and to know which is the more efficient way to use.

Theorem 16. Every *Cauchy sequence* in **R** is convergent if and only if every bounded monotone sequence in **R** is convergent.

Theorem 17. Every bounded monotone sequence in **R** is convergent if and only if **R** is order complete, i.e., every bounded above subset of **R** has a supremum in **R**.

We shall prove these two theorems later in the chapter.

The notion of a Cauchy sequence expresses when a sequence is somehow "bunched" together. Thus Theorem 16 and Theorem 15 says that a Cauchy sequence is convergent. Now we shall find out what this notion is.

Definition 18. A sequence (a_n) is a Cauchy sequence if and only if given any $\varepsilon > 0$, there exists an integer N such that for all $n, m \geq N$, $|a_n - a_m| < \varepsilon$.

Definition 18 uses the distance function to express the "bunching" togetherness of a Cauchy sequence. Here the distance between two terms a_n and a_m is given by the modulus of the difference $a_n - a_m$. Hence, definition 18 says that a sequence is Cauchy if the distance between two terms, a_n and a_m is getting closer and closer as n and m get larger and larger. We shall see later that for **R**, this "bunching togetherness" produces a convergent sequence.

An easy consequence of the definition is:

Theorem 19. Any Cauchy sequence is bounded.

Proof. Suppose (a_n) is a Cauchy sequence. Then taking $\varepsilon = 1$, there exists an integer N such that for all $n, m \geq N$, $|a_n - a_m| < 1$. Hence, we have for all $n \geq N$, $|a_n - a_N| < 1$. It follows that for all $n \geq N$, $|a_n| < |a_N| + 1$. Let $M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, |a_N| + 1\}$. Then plainly $|a_n| \leq M$ for all positive integer n. Hence (a_n) is a bounded sequence.

Theorem 20. Cauchy Principle of Convergence.
A sequence (a_n) in **R** is convergent$\Leftrightarrow$ it is Cauchy.

Note that Theorem 20 is a consequence of Theorem 16 and 17 assuming that **R** is order complete.

We shall prove this by constructing two monotone sequences and invoking Theorem 15.

Proof.
($\Rightarrow$) Suppose (a_n) in **R** is convergent and $a_n \to a$. Then given any $\varepsilon > 0$, there exists an integer N such that $n \geq N \Rightarrow |a_n - a| < \varepsilon/2$. Thus for all $n, m \geq N$, by the triangle inequality,

$$|a_n - a_m| \leq |a_n - a| + |a - a_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore, by Definition 18, (a_n) is Cauchy.
($\Leftarrow$) The converse is much harder. There are alternative proofs. One can use the Bolzano Weierstrass Theorem to prove the converse as well. (See later)
We shall introduce the notion of *limit superior* and *limit inferior* for sequences. Suppose (a_n) is Cauchy. Then by Theorem 19, (a_n) is bounded. In general, for a bounded sequence, we can perform the following construction for limit superior and limit inferior.
By boundedness, there exists a positive constant M such that $|a_n| \leq M$ for all n in $\boldsymbol{P}$. We shll invoke the (order) completeness property of **R**. For each n in $\boldsymbol{P}$, define the set

$$S_n = \{ a_n , a_{n+1}, \dots \} = \{a_k : k \geq n\}.$$

Then S_n is a subset of $\{ a_1 , a_2, \dots \}$ and is therefore also bounded and obviously non-empty. By the (order) completeness of **R**, the supremum and infimum of S_n exist. Let $x_n = \sup S_n$ and $y_n = \inf S_n$. Then

$$\text{for all } k \geq n, \quad y_n \leq a_k \leq x_n. \qquad \text{----------------------} \quad (1)$$

Now, since $-M \leq a_n \leq M$ for all n in P, M is also an upper bound for S_n and $-M$ a lower bound for S_n for each n in $\boldsymbol{P}$. Therefore, $-M \leq$ the least upper bound or supremum of $S_n = \sup S_n = x_n \leq M$. This means the set $\{ x_1 , x_2, \dots \}$ is bounded. Similarly, $-M \leq$ greatest lower bound of $S_n = \inf S_n = y_n \leq M$. This means $\{ y_1 , y_2, \dots \}$ is also bounded. Therefore, the sequence (x_n) is bounded below and the sequence (y_n) is bounded above. We shall next show that (x_n) is decreasing and (y_n) is increasing.

Suppose $m > n$. Then $S_m = \{ a_m , a_{m+1}, \dots \} \subseteq \{ a_n , a_{n+1}, \dots \} = S_n$. Therefore, $x_m = \sup S_m \leq \sup S_n = x_n$. Thus, (x_n) is decreasing. Similarly by the definition of infimum, $y_m = \inf S_m \geq \inf S_n = y_n$. (We can deduce this as follows: by definition, $y_n = \inf S_n$, a lower bound for S_n , is also a lower bound for S_m since $S_m \subseteq S_n$. Therefore, $y_n \leq$ the greatest lower bound or $\inf S_m = y_m$.) Hence (y_n) is increasing.

It follows by Theorem 15 that (x_n) is convergent and $x_n \to \inf\{x_1 , x_2, \dots \}$. This limit when it exists is defined to be the *limit superior* of (a_n). Let $a = \inf\{x_1 , x_2, \dots \}$. That is, $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\} = \lim_{n\to\infty} x_n = a.$

Now (y_n) is increasing and bounded above and so by Theorem 15, it is convergent and $y_n \to \sup\{y_1 , y_2, \dots \}$. Let $b = \sup\{y_1 , y_2, \dots \}$. Similarly, the *limit inferior* of (a_n) is defined to be the limit of y_n. Hence, $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = \lim_{n\to\infty} y_n = b.$

(Note that from the above argument the lim sup and lim inf of a bounded sequence always exist.)

If (a_n) is Cauchy, then $a = b$ and subsequently (a_n) is convergent and converges *to a.*

We shall proceed to prove this statement. First note that by (1) above $a = \lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n = b$. Suppose on the contrary that $a \neq b$. Then $a > b$. We shall derive a contradiction. We shall show that if $a > b$, then (a_n) is not Cauchy and hence the contradiction. How do we do this? We shall show that we can find a $\varepsilon > 0$ such that for any positive integer N we can find integers $n, m \geq N$ with $|a_n - a_m| \geq \varepsilon$.

Take $\varepsilon = (a - b)/3 > 0$. Remember that $a = \inf\{x_1 , x_2, \dots \}$ and so for each N in $\boldsymbol{P}$, $x_N \geq a > a - \varepsilon$. Therefore, $a - \varepsilon$ is not an upper bound for S_N . Thus there exists $n \geq N$ such that

$$a - \varepsilon < a_n \leq x_N \ (= \sup S_N) \ \text{--------------------------} \quad (2)$$

Similarly, since $b = \sup\{y_1 , y_2, \dots \}$, $b \geq y_N$ for any N in $\boldsymbol{P}$. Hence for any integer N in $\boldsymbol{P}$, $b + \varepsilon > b \geq y_N$. Thus $b + \varepsilon$ is not a lower bound for S_N . Therefore, we can find an integer $m \geq N$ such that

$$b + \varepsilon > a_m \geq y_N \ (= \inf S_N) \ \text{---------------------------} \ (3)$$

Thus from (2) and (3), we get

$$a_n - a_m > a - \varepsilon - (b + \varepsilon) = a - b - 2\varepsilon = 3\varepsilon - 2\varepsilon = \varepsilon > 0 \ \text{ since } a - b = 3\varepsilon.$$

Therefore, $|a_n - a_m| = a_n - a_m > \varepsilon$. Hence, for each integer N in $\boldsymbol{P}$, we can find integers $n, m \geq N$ with $|a_n - a_m| \geq \varepsilon$. Thus (a_n) is not Cauchy. This is the required contradiction and so $a = b$.

Therefore, by the inequality (1) and the Squeeze Theorem, (a_n) is convergent and converges to a.

This completes the proof.

This is the most important theorem. It expresses the most commonly stated result that every Cauchy sequence is convergent is equivalent to (order) completeness of **R**. There are a few characterization of completeness for **R** in terms that can be generalized to $\mathbf{R}^n$ **.** The next Theorem which is a very useful tool in analysis is one that can be generalized to $\mathbf{R}^n$. It is also equivalent to completeness for **R**.

Theorem 21. Bolzano Weierstrass Theorem.
Every bounded sequence in **R** has a convergent subsequence.

Remark.

1. In the proof of Theorem 20, we have actually proved the following useful fact:
   For a bounded real sequence (a_n), (a_n) is convergent if and only if
   $$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n .$$
   Note that for a bounded sequence (a_n), $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ both exist.

2. For **R**, every bounded sequence has a convergent subsequence is equivalent to every Cauchy sequence is convergent.

3. The basic conclusion of the theorem generalizes to *complete metric space*, which is like $\mathbf{R}^n$ with a distance function and in which every Cauchy sequence is convergent, but we needed to replace the notion of bounded sequence to the notion of a *totally bounded* sequence. It is indeed true that boundedness implies totally boundedness for $\mathbf{R}^n$. Therefore, Theorem 21 is true for $\mathbf{R}^n$.

4. A simple extension to complex sequences. Suppose (a_n) is a complex sequence. Then (a_n) is a Cauchy sequence $\Leftrightarrow$ (Re a_n) and (Im a_n) are (real) Cauchy sequences $\Leftrightarrow$ (Re a_n) and (Im a_n) are convergent (real) sequences$\Leftrightarrow$ (a_n) is convergent.

5. We say a *metric space* is Cauchy complete if every Cauchy sequence is convergent. A simple Cauchy sequence argument will show that **R** is Cauchy complete implies that $\mathbf{R}^n$ is Cauchy complete. Indeed product of Cauchy complete spaces is Cauchy complete.

6. If (a_n) is convergent then (a_n) is Cauchy. This is true in other situation when (a_n) is a rational sequence a: $\boldsymbol{P} \to Q$, where Q is the set of rational numbers or a normed vector space or a metric space. But the converse need not be true. (a_n) Cauchy does not necessarily imply that (a_n) is convergent. It is true if and only if Q is complete. For example, let a: $\boldsymbol{P} \to \mathbf{Q}$ = rational numbers be the sequence (a_n) = (3, 3.1, 3.14, …), where a_n = first n-digits of π. Then (a_n) does not converge in **Q** for if it did, π would be rational.

7. Note that for **R**, order complete (every bounded subset has a supremum and an infimum in **R**) is equivalent to Cauchy or metric complete (every cauchy sequence is convergent). The notion of Cauchy completeness can be generalized to $\mathbf{R}^n$ (and to metric spaces) but not order completeness. For instance the complex number cannot be totally ordered: it cannot have a positive cone. Indeed neither does $\mathbf{R}^n$, $n \geq 2$ has a positive cone and so it cannot be order complete.

We often need to use Theorem 15. But of course we need to obtain a monotone sequence. The next result will extract a monotone subsequence from a bounded sequence. We now formally define a subsequence of a sequence.

Definition 22. Suppose (a_n) is a sequence. Then a subsequence is given by (a_{n_k}), where $n_1 < n_2 < n_3 < \ldots$. More generally, if a: $\boldsymbol{P} \to Q$ is a sequence, then a subsequence of a is given by the composite $a \circ n : \boldsymbol{P} \xrightarrow{n} \boldsymbol{P} \xrightarrow{a} Q$, where n is a strictly increasing function $n : \boldsymbol{P} \to \boldsymbol{P}$. Then

$$a \circ n(k) = a(n(k)) = a_{n(k)} = a_{n_k}$$

using our convention for writing terms of a sequence ($n(k)$ is written n_k).

Proposition 23. Every real bounded sequence has a monotone subsequence.

Proof. Let (a_n) be a bounded sequence. It is sufficient to pick the so called "peak" in the graph of the function giving the sequence. We say the sequence (a_n) has a peak at k if, for all $j \geq k$, $a_k \geq a_j$. a_k is called the *peak* and k the *peak index*. Now we shall find our subsequence by using these peaks. If there are infinite number of these peaks say having the peak indices, k_1 , k_2 , k_3 , … with $k_1 < k_2 < k_3 < \ldots$. Then by definition of the peak,

$$a_{k_1} \geq a_{k_2} \geq a_{k_3} \geq \ldots .$$

Thus the subsequence (a_{k_j}) is a monotone decreasing sequence. If there are only finite number of these peaks or no peak, then there is an index k, beyond which there are no peaks. Let $n_1 = k + 1$. Then since n_1 is not a peak index , there exists an index n_2 such that $n_2 > n_1$ but $a_{n_2} > a_{n_1}$. Similarly since a_{n_2} is not a peak, it means that it is not true that for all $j \geq n_2$, $a_j \leq a_{n_2}$. Hence there exists an index $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$. Thus, in this way we recursively define $n_{k+1} > n_k$ such that that $a_{n_{k+1}} > a_{n_k}$. Therefore, (a_{n_k}) is a monotone increasing sequence. We have thus constructed a monotone subsequence of (a_n).

24. Proof of Theorem 21 Bolzano Weierstrass Theorem

Suppose (a_n) is a bounded sequence. Then by Proposition 23 (a_n) has a monotone subsequence (a_{n_k}). Since (a_n) is bounded, (a_{n_k}) is also bounded. Therefore, by Theorem 15, (a_{n_k}) is convergent. Hence (a_n) has a convergent subsequence.

To proof Theorem 16 and 17 we shall need the following general observation.

Proposition 25. If (a_n) is a Cauchy sequence that has a convergent subsequence, then it is convergent.

Proof. Suppose (a_n) is Cauchy and (a_{n_k}) is a convergent subsequence. Given any $\varepsilon > 0$, since (a_n) is cauchy, there exists a positive integer M such that

$$n, m \geq M \Rightarrow |a_n - a_m| < \varepsilon/2 . \quad \text{------------------------------} \quad (1)$$

Suppose $a_{n_k} \to a$. Then there exists a positive integer L such that

$$k \geq L \Rightarrow |a_{n_k} - L| < \varepsilon/2. \quad \text{---------------------------------} \quad (2)$$

Let $N = \max(M,L)$. Then $k \geq N \Rightarrow k \geq L$ and $n_k \geq k \geq M$. Therefore, we have

$$n \geq N \Rightarrow |a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| \quad \text{for any } k \geq N$$

$$< \varepsilon/2 + |a_{n_k} - a| \qquad \text{by (1) since } n, n_k \geq M$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

by (2) since $k \geq L$.

Thus, by definition, (a_n) is convergent.

Remark. This is a result that applies to sequences in any metric space. The proof is exactly the same. It provides a convergence criterion for non complete metric spaces. The result is of course superfluous when the space is complete.

26. proof of Theorem 16.
Suppose that any bounded monotone sequence is convergent. We shall then show that any Cauchy sequence is convergent. Let (a_n) be a Cauchy sequence. Then by Theorem 19 (a_n) is bounded. By Proposition 23 (a_n) has a bounded monotone subsequence (a_{n_k}). By our assumption (a_{n_k}) is convergent. Suppose $a_{n_k} \to a$. Then by Proposition 25 (a_n) is convergent and converges to a.
Conversely suppose any Cauchy sequence is convergent. We shall show then that any bounded monotone sequence is convergent. This is a little harder to show.
Let (a_n) be a bounded monotone sequence. Without loss of generality we may assume that it is increasing. Suppose it is a finite sequence, that is, its image is finite. Then it has a constant subsequence (a_{n_k}). (Why? Consider the preimages of singleton subsets of the image of the sequence. One of them must be infinite and gives the required constant subsequence.) Then for any $n \geq n_1 = N$, $a_n = a_{n_1} = c$ for some constant c. We deduce this as follows. For any $n \geq n_1$, there exists a positive integer k such that $n_k \geq n \geq n_1$. Since (a_n) is increasing $a_{n_k} = c \geq a_n \geq a_{n_1} = c$ and so , $a_n = c$. Thus (a_n) is convergent and we have nothing to prove. Suppose that (a_n) is infinite, i.e., its range is infinite. We shall show then that (a_n) is Cauchy. The idea below uses the fact that any bounded subset in **R** is "*totally bounded*". Since (a_n) is increasing, for all n in ***P***, $a_n \geq a_1 = c$. Since (a_n) is bounded there exists d such that $c \leq a_n < d$ for all n in ***P.*** Now given any $\varepsilon > 0$, there exists an integer l such that $|d - c|/ 2^l < \varepsilon$. We now divide or partition the interval $[c, d]$ into 2^l subintervals each of length $(d - c)/ 2^l$. Then one of these subintervals must contain infinite number of the range of (a_n). Suppose it is the k-th subinterval I_k of $[c, d]$. Then I_k contains a subsequence (a_{n_k}). In particular $\{a_{n_k} : k \in \boldsymbol{P}\} \subseteq I_k$ and so for any k, j in ***P,*** $|a_{n_k} - a_{n_j}| < \varepsilon$. Let $N = n_1$. Then for any integers $m \geq n \geq N = n_1$, there exists an integer L such that $m \leq n_L$. Therefore, since (a_n) is increasing, $a_N = a_{n_1} \leq a_n \leq a_m \leq a_{n_L}$. It follows that $|a_n - a_m| \leq |a_{n_L} - a_{n_1}| < \varepsilon$. Thus (a_n) is Cauchy. Therefore, by assumption, (a_n) is convergent. Hence any bounded monotone increasing sequence is convergent. If (a_n) is bounded and decreasing, then $(-a_n)$ is bounded and increasing and so is convergent and it follows that (a_n) is convergent. Hence any bounded monotone sequence is convergent.

27. Proof of Theorem 17.
Recall the statement of the theorem: Every bounded monotone sequence in **R** is convergent if and only if **R** is order complete.
(if part) Suppose **R** is order complete. Then this is just Theorem 15. The proof is exactly the same.
(only if) Suppose every bounded monotone sequence is convergent. Now take a non-empty subset A of **R** which is bounded above. We shall show that it has a supremum. Since it is bounded above, there exists a real number K such that for all a in A, $a \leq K$. If there exists an element a in A such that $a = K$ then K = supremum of A and we have nothing to prove. Thus we assume that for all a in A, $a < K$. Pick an element L in A. Then $L < K$. Now we are going to define two monotone bounded

sequences in **R** inside the interval $[L, K]$, one always inside A and one always outside of A and they will both converge to the supremum of A. Let $a_1 = L$ and $b_1 = K$. Let c_1 be the mid point of $[L, K]$, i.e., $c_1 = (K + L)/2$. We shall define the sequence (a_n) and (b_n) as follows.
If c_1 is an upper bound of A, let $b_2 = c_1$ and $a_2 = a_1$. Then $|b_2 - a_2| \le (K - L)/2$. If c_1 is not an upper bound of A, then there exists an element a_2 in A such that $c_1 < a_2 < K$ and let $b_2 = K$. We have then $|b_2 - a_2| \le |K - c_1| \le (K - L)/2$. In either cases, we have $|b_2 - a_2| \le (K - L)/2$. We have also,

$$a_1 \le a_2 < b_2 \le b_1.$$

If $b_2 \in A$, then we are done because $b_2 = \sup A$ and the process of definition terminates. If $b_2 \notin A$ then we repeat the process using the interval $[a_2, b_2]$. Let $c_2 = (b_2 + a_2)/2$ the mid point of $[a_2, b_2]$. If c_2 is an upper bound of A, let $b_3 = c_2$ and $a_3 = a_2$. If c_2 is not an upper bound of A, then there exists an element a_3 in A such that $c_2 < a_3 < b_2$ and we let $b_3 = b_2$. Again if $b_3 \in A$, the process terminates with the supremum being b_3 and we are done. In particular $|b_3 - a_3| \le (b_2 - a_2)/2 \le (K - L)/2^2$ and $a_2 \le a_3 < b_3 \le b_2$. If this process terminates in a finite steps, then the set A has a supremum. If the process does not terminate, then we have a bounded increasing sequence (a_n) and a bounded decreasing sequence (b_n) such that $a_n \le b_n$ for all n in $\boldsymbol{P}$. (a_n) is bounded above by K and (b_n) is bounded below by L. Furthermore,

$$|b_n - a_n| \le (K - L)/2^{n-1}. \quad \text{-----------------------} \ (1)$$

Thus, by our assumption both (a_n) and (b_n) converges. Suppose $a_n \to a$ and $b_n \to b$ in **R** then $a \le b$ since $a_n \le b_n$ for all n in $\boldsymbol{P}$. We now claim $a = b$. Suppose $a \ne b$, then $a < b$. Note that for each n in $\boldsymbol{P}$, $a_n \le \lim_{k\to\infty} a_k = a$ since (a_n) is increasing and $\lim_{k\to\infty} b_k \le b_n$ because (b_n) is decreasing. Therefore, we have for each n in $\boldsymbol{P}$,

$$a_n \le a < b \le b_n. \quad \text{-----------------------} \ (2)$$

Let now $\varepsilon = |b - a|/2$. Choose a positive integer n such that $(K - L)/2^{n-1} < \varepsilon$. Then we have from (2), $|b - a| \le |b_n - a_n| \le (K - L)/2^{n-1} < \varepsilon = |b - a|/2$. This is absurd and so $a = b$. Now we show that $\sup A = a$. Since each b_n is an upper bound for A and $b = \lim_{k\to\infty} b_k$, b is an upper bound for A and so $a = b$ is an upper bound for A. We shall show that a is the least upper bound for A. We do this by showing that for any $\varepsilon > 0$, $a - \varepsilon$ cannot be an upper bound for A. Since $\lim_{k\to\infty} a_k = a$ for any $\varepsilon > 0$, there exists a positive integer N such that

$$n \ge N \Rightarrow \le |a_n - a| < \varepsilon \Rightarrow a - \varepsilon < a_n < a + \varepsilon.$$

By the construction of the sequence (a_n), each a_n is in A. Therefore, $a - \varepsilon < a_N$ and $a_N \in A$ and so $a - \varepsilon$ cannot be an upper bound for A. Hence a = least upper bound of $A = \sup A$. This completes the proof.

Remark. Note that the statement of the Bolzano Weierstrass Theorem (Theorem 21) does not involve ordering. Thus we can make the same statement for $\mathbf{R}^n$ and for general metric spaces. The theorem is true also in $\mathbf{R}^n$. However, it is not true for general (Cauchy) complete metric spaces. For **R** we can show that it is equivalent to the Monotone Convergence Theorem for **R.** Thus completeness for **R** has many interpretations: we can find a convergent subsequence for a bounded sequence, any Cauchy sequence is convergent, any bounded monotone sequence is convergent, any bounded above subset of **R** has a supremum in **R**, any bounded below subset of **R** has an infimum in **R**.

Definition 28. The notion of (a_n) tending to $+\infty$ means $\lim_{n\to\infty} a(n) = \infty$ regarding the limit as a function on $\boldsymbol{P}$. That is, given any real number $K > 0$, there exists an integer N such that for all n in $\boldsymbol{P}$, $n \geq N \Rightarrow a_n > K$. When this happens, we write $a_n \to +\infty$. Similarly, the sequence (a_n) is said to tend to $-\infty$ if given any real number $L < 0$, there exists an integer N such that for all n in $\boldsymbol{P}$, $n \geq N \Rightarrow a_n < L$ and we write $a_n \to -\infty$.

The rules for functions translate to the following useful results for computing limits involving infinity. Of course the following rules can be proved directly using Definition 28.

Proposition 29.
Suppose (a_n), (b_n) are two sequences in **R**.
1. If $a_n \to +\infty$ or $a_n \to -\infty$, then $\frac{1}{a_n} \to 0$.
2. If $a_n \to +\infty$ *and* $b_n \to a$, a finite, then $a_n + b_n \to +\infty$
3. If $a_n \to -\infty$ *and* $b_n \to a$, a finite, then $a_n + b_n \to -\infty$
4. If $a_n \to +\infty$ *and* $b_n \to a > 0$ a finite, then $a_n b_n \to +\infty$
5. If $a_n \to +\infty$ *and* $b_n \to a < 0$ a finite, then $a_n b_n \to -\infty$

These rules are particular useful when a_n is a rational function of n.

Example
1. $(n + 1/n)$ tends to $+\infty$
2. $5 - n + 1/2^n$ tends to $-\infty$
3. $\dfrac{n+1}{n^2+1} = \dfrac{\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n^2}} \to 0$
4. $\dfrac{n^2+1}{2n^2+n+1} = \dfrac{1+\frac{1}{n^2}}{2+\frac{1}{n}+\frac{1}{n^2}} \to \dfrac{1}{2}$

Subset of the real numbers

There is a class of subsets of **R** that is important in analysis. For example we know by Proposition 12 that if (a_n) is a sequence in the closed interval $[a, b]$ and if $a_n \to c$, then $a \leq c \leq b$. That is to say the limit stays in the interval $[a, b]$. There is also an important covering property of $[a, b]$ that is used in the proof of the *uniform continuity* of continuous function on $[a, b]$ and is also the essence of the Extreme Value Theorem.

Definition 30. Let S be a subset of **R**. Then an element a in **R** is said to be a *limit point* of S if for every $\varepsilon > o$, the open interval $(a - \varepsilon, a + \varepsilon)$ contains a point of S different from a. Hence, $(a - \varepsilon, a + \varepsilon) \cap (S - \{a\}) \neq \varnothing$.

Example. 0 is a limit point of the open interval (0, 1). It is also a limit point of the set $\{1/n : n \in \boldsymbol{P}\}$

There are equivalent definitions of a limit point.

Proposition 31. Let S be a subset of **R** and a an element in **R.** . The following statements are equivalent:
(1) a is a limit point of S.
(2) There is a sequence (a_n) in $S - \{a\}$ such that $a_n \to a$.
(3) Any open interval I containing a has infinite intersection with S.

Proof. (1) $\Rightarrow$ (2). For each positive integer n, take a_n in $(a - 1/n, a + 1/n) \cap (S - \{a\})$. Plainly $a_n \to a$.
(2) $\Rightarrow$ (3). Take an open interval I containing a. Then, there exists $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subseteq I$. By (2) there is a sequence (a_n) in $S - \{a\}$ such that $a_n \to a$. Therefore, there exists a positive integer N such that

$$n \geq N \Rightarrow |a_n - a| < \varepsilon \Rightarrow a - \varepsilon < a_n < a + \varepsilon \Rightarrow a_n \in (a - \varepsilon, a + \varepsilon).$$

Note that $\{a_n : n \geq N\}$ is infinite. If it were finite, then (a_n) would have a constant subsequence, which obviously converge to a point b in $S - \{a\}$. Therefore, since (a_n) is convergent, it has the same limit as any subsequence and hence $b = a$ and so $a \in S - \{a\}$. But $a \notin S - \{a\}$. This contradiction shows that $\{a_n : n \geq N\}$ is infinite and hence $(a - \varepsilon, a + \varepsilon) \cap S$ is infinite. Therefore, since $(a - \varepsilon, a + \varepsilon) \cap S \subseteq I \cap S$, $I \cap S$ is infinite.
(3) $\Rightarrow$ (1). For any $\varepsilon > 0$, $(a - \varepsilon, a + \varepsilon) \cap S$ is infinite and so it must contain a point in S different from a.

For a set S of **R**, let S' be the set of limit points of S.

Definition 32. A subset S of **R** is said to be *closed* if $S' \subseteq S$.

Example. **1.** Any singleton set $\{a\}$ in R is closed since S' is empty.
2. $[0, 1]$ is closed, $[0, \infty)$ is also closed.
3. The rational numbers **Q** in **R** is not closed in **R**. This is because taking an irrational number like $\sqrt{2}$, we can find a sequence of rational numbers converging to it and so $\sqrt{2}$ is a limit point of **Q** but $\sqrt{2}$ is not in **Q** as it is not rational.

The following property of closed set is very useful.

Proposition 33. A subset A of **R** is closed if and only if any convergent sequence in A converges to an element in A.

Proof. (Only if part) Suppose A is closed. Suppose on the contrary that there is a sequence in A which converges to an element a not in A. Then by proposition 31, a is a limit point of A. Since A is closed, $a \in A$. This contradicts $a \notin A$. Therefore, any convergent sequence in A must converge to an element in A.
(If part) Let a be a limit point of A. Then by Proposition 31, there is a sequence (a_n) in $S - \{a\}$ such that $a_n \to a$. By assumption, the limit of this sequence is in A. Hence $a \in A$. Therefore, the set of limit points $A' \subseteq A$. Hence A is closed.

Let S be a subset of **R**. Then the set $H = S \cup S'$ is a closed set in **R**. This is seen as follows. Let a be a limit point of H. Then take any open interval I containing a. Then by Proposition 31, I has infinite intersection with H. Since $I \cap H = (I \cap S) \cup$

$(I \cap S')$, $(I \cap S)$ must be infinite. This is because if $(I \cap S)$ were finite, then $(I \cap S')$ would be infinite and so I would contain a limit point of S and so I would have an infinite intersection with S by Proposition 31 thus contradicting the assumption of finiteness for $(I \cap S)$. Therefore, a is a limit point of S. Hence $H' \subseteq S' \subseteq H$. Therefore, H is closed. H is also the smallest closed subset containing S. We deduce this in the following way. Suppose there exists a smaller closed subset A containing S, i.e., $S \subseteq A \subsetneq H = S \cup S'$. We shall then derive a contradiction. Thus, there is an element h in S' such that $h \notin A$. Since $S' \subseteq A'$, $h \in A'$. Since, A is closed, $A' \subseteq A$ and so $h \in A$ contradicting $h \notin A$. Therefore, there does not exist a closed subset containing S smaller than H.

Definition 34. We define the closure of S to be $Cl(S) = S \cup S'$.

Example. The closure of (a, b) is $[a, b]$. The closure of **Q** is **R**.

Definition 35. A subset U of **R** is said to be *open* if it is an arbitrary union of open intervals.

Example. $(0, 1) \cup (2,3)$ is open but $(0,1) \cup [2, 3)$ is not.

The following characterization of open set is particularly useful.

Proposition 36. A subset U is open in **R** if and only if for each x in U there exists an open interval I such that $x \in I \subseteq U$.

Proof. If U is open in **R**, then by definition, U is an arbitrary union of open intervals. Take any x in U, x must belong to one of these open interval I which is obviously a subset of U. Conversely suppose for each x in U, there exists an open interval I_x such that $x \in I_x \subseteq U$. Then, $U = \cup \{I_x : x \in U\}$ and so U is an arbitrary union of open intervals. Therefore, U is open.

Our next definition is one of several equivalent meaning of compactness for subsets of **R**.

Definition 37. A subset S of **R** is said to be *sequentially compact* if any sequence in S has a convergent subsequence converging to a point in S.

Hence we have

Proposition 38. The closed interval $[a, b]$, where $a < b$ are real numbers, is sequentially compact.

Proof. Any sequence in $[a, b]$ is bounded above by b and bounded below by a and so is bounded. Therefore, by the Bolzano Weierstrass Theorem (Theorem 21), it has a convergent subsequence. By Proposition 33, the subsequence converges to a point in $[a, b]$.

More generally we have:

Theorem 39. Suppose S is a subset of **R**. Then S is sequentially compact $\Leftrightarrow$ S is closed and bounded.

Proof.
($\Leftarrow$) Suppose S is closed and bounded. Take a sequence (a_n) in S. Then (a_n) is bounded since S is. It then follows by the Bolzano Weierstrass Theorem that (a_n) has a convergent subsequence converging to a in **R**. If a is in S, then we are done. If $a \notin S$, then $a \in S'$. But S is closed so that $S' \subseteq S$ and so $a \in S$. Hence we have arrived at a contradiction. Therefore, $a \in S$. Thus, S is sequentially compact.
($\Rightarrow$) Suppose S is sequentially compact. Let s be a limit point of S. By Proposition 31, there is a sequence (a_n) in $S - \{s\}$ such that $a_n \to s$. Since S is sequentially compact, (a_n) has a subsequence converging to a point a in S. Since (a_n) is already convergent both the sequence and any subsequence must have the same limit. Therefore, $s = a$ and so $s \in S$. Hence all the limit points of S are in S. Therefore, S is closed. S must be bounded for other wise there exists a sequence in S that does not converge in **R.**

Now we proceed to investigate the covering property of a sequentially compact sets. We shall describe a simpler form of compactness or what is called *countable compactness*.

Definition 40. A family F of subsets of **R** is said to cover S if $S \subseteq \cup \{U: U \in F\}$. F is said to be a *cover* for S. If F is countable, then F is a countable cover for S. If each member of F is open, then we say the cover is an open cover. A *subcover* of F is a subset $G \subseteq F$ which is also a cover.

Example. $\{(n-1, n+1): n \in \mathbf{Z}\}$ is a countable open cover of **R**.

Definition 41. A subset S of **R** is said to be (countably) compact if any open cover of S has a finite subcover. More precisely, if $\{U_n : n \in \boldsymbol{P}\}$ is a countable cover for S, then there exist a finite number of members of $\{U_n : n \in \boldsymbol{P}\}$, $\{U_{n_1}, U_{n_2} \ldots, U_{n_k}\}$ for some integer k, such that $S \subseteq U_{n_1} \cup U_{n_2} \cup \ldots \cup U_{n_k}$.

Theorem 42. A subset S in **R** is (countably) compact $\Leftrightarrow$ S is sequentially compact.

Proof. **($\Rightarrow$)** Suppose S is (countably) compact. If S is finite, then any sequence in S has finite image and so has a constant subsequence and we are done. Suppose S is infinite and suppose on the contrary that S is not sequentially compact. Then there exists a sequence (a_n) in S that does not have a convergent subsequence converging to a point in S. Then the set $A = \{a_n : n \in \boldsymbol{P}\}$ must be infinite, for otherwise it would have a convergent constant subsequence. Then the set A cannot have a limit point in S. This is because if A has a limit point a in S, then we can construct a subsequence of (a_n) converging to a as follows. $(a-1, a+1) \cap A$ has a point in A different from a. Then there exists a positive integer n_1 such that $a_{n_1} \in (a-1, a+1) \cap A$ and $a_{n_1} \neq a$. Then by Proposition 31(3) $(a-1/2, a+1/2) \cap (A-\{a\})$ is infinite. Therefore, there exists an integer $n_2 > n_1$ such that $a_{n_2} \in (a-1/2, a+1/2) \cap (A-\{a\})$. In this way, we define $a_{n_k} \neq a$ recursively. If we have already defined $a_{n_k} \neq a$, then since (a–1/(k+1), a+1/(k+1)) ∩ (A-{a}) is infinite, there exists $n_{k+1} > n_k$ such that

$$a_{n_{k+1}} \in (a - 1/(k+1), a + 1/(k+1)) \cap (A - \{a\}).$$

Plainly the subsequence (a_{n_k}) converges to a. This contradicts that (a_n) has no convergent subsequence converging to a point in S. Therefore, A has no limit points in S. Thus, for each x in S and $x \notin A$, there exists an open interval U_x containing x such that $U_x \cap A = \varnothing$. Then the set $U = \cup\{U_x : x \in S - A\}$ is open, being arbitrary union of open intervals. Also, we have for each $x \in A$, there exists an open interval V_x such that $V_x \cap A = \{x\}$. Thus the family $F = \{V_x : x \in A\} \cup \{U\}$ is an open cover for S since $A \subseteq \cup\{V_x : x \in A\}$ and $S - A \subseteq U$. Now since A is countably infinite, the family is a countably infinite open cover of S. Plainly it cannot have a finite subcover**.** This is because if it did say, $S \subseteq V_{x_1} \cup V_{x_2} \cup \ldots \cup V_{x_N} \cup U$ for some positive integer N, then A = $A \cap S$ = $A \cap (V_{x_1} \cup V_{x_2} \cup \ldots \cup V_{x_N}) = (A \cap V_{x_1}) \cup (A \cap V_{x_2}) \cup \ldots \cup (A \cap V_{x_N})$ = $\{x_1 , x_2, \ldots, x_N\}$ since $A \cap U = \varnothing$. Therefore, A is finite, thus contradicting that A is countably infinite. Hence, S is sequentially compact.

(⇐) Conversely suppose S is sequentially compact. Suppose on the contrary that S is not countably compact. Then there exists an open cover of S by countably infinite number of open sets, say $F = \{U_i : i \in \boldsymbol{P}\}$ such that no finite subsets of F covers S. We shall derive a contradiction. Since no finite subset covers S, for each k,

$$S \cap (U_1 \cup U_2 \cup \ldots \cup U_k) \neq S.$$

We shall now construct a sequence (a_n) in S as follows. We can find a_1 in S such that $a_1 \notin S \cap U_1$ since $S \cap U_1 \neq S$. For each integer k >1 in P, we can find an element a_k in S such that $a_k \notin S \cap (U_1 \cup U_2 \cup \ldots \cup U_k)$ since $S \cap (U_1 \cup U_2 \cup \ldots \cup U_k) \neq S$. Then we claim that the sequence (a_n) cannot have a convergent subsequence that converges to a point in S. Wre deduce this as follows. Suppose on the contrary that (a_n) has a convergent subsequence (a_{n_k}) that converges to a point s in S. Then $s \in S \cap U_N$ for some positive integer N since $S \subseteq \cup \{U_i : i \in \boldsymbol{P}\}$. Then, for any $k \geq N$, by definition of the sequence (a_n),

$$a_k \notin S \cap (U_1 \cup U_2 \cup \ldots \cup U_k) \text{ implies that } a_k \notin S \cap (U_1 \cup U_2 \cup \ldots \cup U_N)$$

and hence that $a_k \notin S \cap U_N$. Thus, for any ε > 0 such that $(s - \varepsilon, s + \varepsilon) \subseteq U_N$, there dose not exist an integer L such that $k \geq L$ implies that $a_k \in (s - \varepsilon, s + \varepsilon)$. This is because $a_k \in (s - \varepsilon, s + \varepsilon) \Rightarrow a_k \in (s - \varepsilon, s + \varepsilon) \cap S \subseteq S \cap U_N$ but $a_k \notin S \cap U_N$ for $k \geq N$. Therefore, there does not exist an integer K such that

$$k \geq K \Rightarrow a_{n_k} \in (s - \varepsilon, s + \varepsilon) \Leftrightarrow |a_{n_k} - s| < \varepsilon.$$

Hence (a_{n_k}) cannot be convergent. But this contradicts our assumption that S is sequentially compact. Therefore, S is (countably) compact.

Remark. The proof of Theorem 42 can be easily adapted to $\mathbf{R}^n$ and metric spaces.

Theorem 43. (Heine Borel). A subset S of **R** is (countably) compact if and only if S is closed and bounded.

Proof. By Theorem 42, S is (countably) compact ⇔ S is sequentially compact. By Theorem 39 S is sequentially compact ⇔ S is closed and bounded.

Remark.

The usual definition of compactness for a subset S of **R** is: any open cover has a finite subcover. In general, for subset of a topological space, this is the definition. However, for metric spaces like **R** or $\mathbf{R}^n$, any open cover has a countable subcover (a result attributed to Lindelöf). Therefore, for subsets of **R** or for that matter of $\mathbf{R}^n$, countably compactness is equivalent to compactness. Thus, for subsets of **R**, the following statements are equivalent.
1. S is compact.
2. S is sequentially compact.
3. S is (countably) compact.
4. S is closed and bounded.

Exercises 44

Sequences

1. Let $a_n = \begin{cases} 1+\frac{1}{n}, & n \text{ odd} \\ 1-\frac{1}{2^n}, & n \text{ even} \end{cases}$.
   (i) Find positive integer N_1 such that $n > N_1 \Rightarrow |a_n - 1| < 0.01$.
   (ii) Find positive integer N_2 such that $n > N_2 \Rightarrow |a_n - 1| < 0.000016$.
   (iii) Given ε in R, ε > 0, find positive integer N such that
   $$n > N \Rightarrow |a_n - 1| < \varepsilon.$$
   (Hint: Prove $2^n > n$ for any n in **N**. Let $[\, x \,]$ be the greatest integer $\le x$, i.e. $[\, x \,] = n \in \mathbf{N}$ and $n \le x < n + 1$. Then take $N = [\, 1/\varepsilon \,]$.)

2. Prove that a sequence cannot converge to two different limits.

3. Let (a_n) be a real sequence. Suppose the subsequences (a_{2n}) and (a_{2n-1}) are convergent and converges to the same value a. Prove that $a_n \to a$.

4. Prove that if $a_n \to a$, then $|a_n| \to |a|$. If $(|a_n|)$ converges , show by a counter example that (a_n) need not converge.

5. (Existence of n-th root.). Suppose $a \ge 0$ and $n \in \mathbf{N}$, prove that there is a unique b in **R**, $b \ge 0$ such that $b^n = a$. (Use the completeness property of **R**.)
   Prove by induction or otherwise, that $h > 0 \Rightarrow (1 + h)^n \ge 1 + nh$ and deduce that $a > 1 \Rightarrow 1 < a^{1/n} \le 1 + \dfrac{a-1}{n}$ and conclude that $a^{1/n} \to 1$.
   Show that $a > 1 \Rightarrow \underset{n\to\infty}{Lim}\ a^n = +\infty$. I.e., for any $K > 0$, there exists an integer N such that $n \ge N \Rightarrow a_n > K$.
   Show that if $a_n \to +\infty$ and $a_n \ne 0$ for all n, then $1/a_n \to 0$.
   Using these results find $\underset{n\to\infty}{Lim}\ a^n$ and $\underset{n\to\infty}{Lim}\ a^{1/n}$ for $a = 1$, $0 < a < 1$ and $a = 0$.

6. Prove the following
   (i) $\underset{n\to\infty}{Lim}\ \dfrac{n}{n+1} = 1$ (ii) $\underset{n\to\infty}{Lim}\ \dfrac{n+1}{n^3+4} = 0.$

7. Use Squeeze Theorem or the Comparison test to prove
   (i) $\underset{n\to\infty}{Lim}\ \dfrac{\sin(n)}{n} = 0$ (ii) $\underset{n\to\infty}{Lim}\ \dfrac{n!}{n^n} = 0$
   (iii) $\underset{n\to\infty}{Lim}\ n^{1/n} = 1$

[Hint: write let $h_n = n^{1/n} - 1$ and show that $n = (1 + h_n)^n \geq 1 + \dfrac{n(n-1)}{2} h_n^2$]

(iv) $\lim\limits_{n\to\infty} \dfrac{\alpha(n)}{n} = 0$, where $\alpha(n)$ = number of primes dividing n. [Hint: show $\alpha(n) \leq \sqrt{n}$.]

8. Show that if (a_n) converges to 0 and (b_n) is a bounded sequence, then $(a_n b_n)$ converges to 0. Hence, or otherwise, show that $\lim\limits_{n\to\infty} \left(\dfrac{2n-1}{3n+1}\right)^n = 0$.

9. Find the limit of the following sequences.
(i) $\left(n\left(1-(1-\frac{a}{n})^{1/3}\right)\right)$, $a \leq 1$ (ii) (a_n), where $a_n = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n}$
[Hint : $1 - k^3 = (1-k)(1+k+k^2)$. Use Squeeze Theorem.]

Monotone Convergence Theorem

10. Show that $((1+1/n)^n)$ is an increasing sequence. Show that $(1+1/n)^n < 3$. Hence deduce that it is convergent.

11. (i) Suppose (a_n) is a decreasing (respectively increasing) sequence. Show that the sequence (U_n), where $U_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ is also a decreasing (respectively increasing) sequence. Hence deduce that the sequence $\left(\frac{1}{n}(1+\frac{1}{2}+\cdots+\frac{1}{n})\right)$ is convergent.
(ii) Prove that $a_n \to a \Rightarrow U_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n} \to a$. Show that the converse is false.

12. Suppose (a_n) is a monotone sequence. Prove that (a_n) is convergent if and only if (a_n^2) is convergent.

13. Suppose $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = q$ and $|q| < 1$. Show that $\lim\limits_{n\to\infty} a_n = 0$. Hence deduce that $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$ for any real number x.

14. Suppose (a_n) is a sequence defined by $a_1 = 1$, $a_{n+1} = 2(2a_n + 1)/(a_n + 3)$ for $n \geq 1$. Prove that $1 \leq a_n < 2$ and that (a_n) is increasing. Hence deduce that (a_n) is convergent and $a_n \to 2$.

15. Show that if for a sequence (a_n), the subsequence (a_{2n}) and (a_{2n-1}) both converges to a, then $a_n \to a$.

16. Suppose $a_1 > 0$. For $n \geq 1$, define $a_{n+1} = \dfrac{1}{1+a_n}$. Show that the sequence is convergent and find its limit. [Hint: first show that (a_{2n}) and (a_{2n-1}) are bounded monotone sequences converging to the same limit.]

17. Suppose (a_n) is a sequence of positive terms defined by $a_{n+1} = 3(a_n + 1)/(a_n + 3)$ for $n \geq 1$. Prove that if $a_1 < \sqrt{3}$ then (a_n) is strictly increasing and if $a_1 > \sqrt{3}$ then (a_n) is strictly decreasing. Hence deduce that (a_n) is convergent and determine its limit.

18. Suppose $a_n \to a$ and that $|a| < 1$. Show then that $a_n^n \to 0$.

19. Show that if a sequence is convergent and converges to a, then any subsequence is also convergent and converges to a .

20. If (a_n) is a Cauchy sequence and it has a convergent subsequence, then (a_n) is convergent and has the same limit as the subsequence.

21. Prove that every Cauchy sequence in **R** has a convergent subsequence. Deduce Cauchy principle of convergence for real sequencces.
[Hint: Bolzano-Weierstrass Theorem.]

Subsets of the real numbers.

22. For each of the following statements, determine whether it is true or false and justify your answer.
(a) Every bounded sequence converges.
(b) A convergent positive sequence of positive numbers has a positive limit.
(c) A convergent sequence of rational numbers has a rational limit.
(d) The limit of a convergent sequence in the interval (a, b) also belongs to (a, b).
(e) The set of irrational numbers is a closed subset of **R**.
(f) The set of rational numbers in the interval [0, 1] is (countably) compact.
(g) A subset of a (countably) compact set is also (countably) compact.
(h) Every closed set is compact.
(i) Every bounded set in **R** is a closed subset of **R**.
(j) Every sequence of rational numbers has a convergent subsequence.
(k) Every sequence in (0, 1) has a convergent subsequence.

23. Let S be the interval [1, 5).
(a) Using the definition of sequential compactness, show that S is not sequentially compact.
(b) Using the definition of countably compactness, show that S is not countably compact.
(c) Using the definition of closedness, show that S is not closed.

24. Let S be the set of rational numbers in [0,1].
(a) Using the definition of sequential compactness, show that S is not sequentially compact.
(b) Using the definition of countably compactness, show that S is not countably compact.
(c) Using the definition of closedness, show that S is not closed.

25. For $c > 0$, consider the quadratic equation $x^2 - x - c = 0$, $x > 0$.
Define the sequence (x_n) recursively by fixing $x_1 > 0$ and then, if n is an index for which x_n is defined, defining $x_{n+1} = \sqrt{c + x_n}$.
Prove that the sequence (x_n) converges monotonically to the solution of the above equation.

26. Suppose (b_n) is a bounded sequence of non-negative numbers and r is a number such that $0 \le r < 1$. Define $s_n = b_1 r + b_2 r^2 + \dots + b_n r^n$ for each n in **P**. Prove that (s_n) is convergent.

27. Show that $[2, 3] \cup [4,5]$ is (countably) compact.

28. Let A and B be compact subsets of **R**. Show that $A \cup B$ and $A \cap B$ are (countably) compact.

29. If $A \cup B$ is (countably) compact, does it follow that both A and B are (countably) compact?