Chapter 13 Special Tests for Convergence

Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms. Suppose applying the d'Alembert's Ratio test and we obtain $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. We cannot then infer convergence or divergence from the test. We may use a comparison test (Proposition 12 Chapter 6) or the integral test (Theorem 25 Chapter 6) depending on the series. Then we may use some more delicate tests, tests which are useful to study series $\sum_{n=1}^{\infty} a_n$ with $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Some series may fail these tests too and we may need even more delicate test to deduce convergence or divergence. There is a general theory of convergence and divergence introduced by Pringsheim in an article of 100 pages (Mathematische Annalen, vol 35 (1889)). All the tests we give in this chapter may be deduced from his general theory. However, it does not give a practical method of testing any series for convergence or divergence. As we shall see the tests are most effective for series of certain forms.

13.1 Kummer's Test

Kummer's test is a general test for convergence. We can use it to derive the other tests.

Theorem 1. (Kummer's test , 1835 Journ. für die reine und angewandte Mathematik 13. p 172).
Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.
(i) If there is a positive sequence $\{ b_n \}$, a positive constant A, and a positive integer N such that for all integer $n \geq N$,

$$\frac{a_n}{a_{n+1}} b_n - b_{n+1} \geq A \quad \text{------------------------------} \quad (A)$$

then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If there is a positive sequence $\{ b_n \}$ and a positive integer N such that for all integer $n \geq N$,

$$c_n = \frac{a_n}{a_{n+1}} b_n - b_{n+1} \leq 0 \quad \text{------------------------------} \quad (B)$$

then $\sum_{n=1}^{\infty} a_n$ is divergent if $\sum_{n=1}^{\infty} \frac{1}{b_n}$ is divergent.

Equivalently,
(i) $\sum_{n=1}^{\infty} a_n$ is convergent if $\liminf_{n\to\infty} c_n > 0$, and
(ii) $\sum_{n=1}^{\infty} a_n$ is divergent if $\sum_{n=1}^{\infty} \frac{1}{b_n}$ is divergent and $\limsup_{n\to\infty} c_n < 0$ or when $\limsup_{n\to\infty} c_n = 0$ and $c_n \leq 0$ for all integer $n \geq N$ for some positive integer N.

Proof.
Part (i). Suppose there is a positive sequence $\{ b_n \}$, a positive constant A, and a positive integer N such that for all integer $n \geq N$, inequality (A) is satisfied. Then for any integer $n \geq N$,

$$a_n b_n - a_{n+1} b_{n+1} \geq A a_{n+1} \quad \text{--------------------------} \quad (1).$$

Let p be any integer greater than or equal to 1. Summing (1) from $n = N$ to $n = N + p-1$ gives

$$a_N b_N - a_{N+p} b_{N+p} \geq A \sum_{k=N}^{N+p-1} a_{k+1} \text{ ------------------------ (2)}$$

Now let $s_n = \sum_{k=1}^{n} a_k$ be the n-th partial sum of the series $\sum_{n=1}^{\infty} a_n$. Then

$$\sum_{k=N}^{N+p-1} a_{k+1} = \sum_{k=N+1}^{N+p} a_k = s_{N+p} - s_N \leq \frac{1}{A}(a_N b_N - a_{N+p} b_{N+p}) \quad \text{by (2)}$$

$$\leq \frac{1}{A} a_N b_N.$$

It follows then that for all integer $p \geq 1$,

$$s_{N+p} \leq s_N + \frac{1}{A} a_N b_N.$$

Hence the set $\{s_{N+p} : p \in \boldsymbol{P}\}$ is bounded above by $s_N + \frac{1}{A} a_N b_N$. Consequently the sequence (s_n) is bounded. Therefore, by Proposition 11 Chapter 6, $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) Suppose $c_n = \frac{a_n}{a_{n+1}} b_n - b_{n+1} \leq 0$ for all integer $n \geq N$. Then

$$a_n b_n \leq a_{n+1} b_{n+1} \text{ for all integer } n \geq N.$$

Hence, for all integer $n \geq N$, $a_n b_n \geq a_N b_N$. It follows that for all integer $n \geq N$,

$$a_n \geq a_N b_N \frac{1}{b_n}.$$

Thus, if $\sum_{n=1}^{\infty} \frac{1}{b_n}$ is divergent, by the Comparison Test (Proposition 12 Chapter 6), $\sum_{n=1}^{\infty} a_n$ is divergent.

Remark. To apply Kummer's Test, we need to look for a suitable sequence (b_n) that satisfies the condition in (i) or (ii) of Theorem 1. We may of course choose the sequence for (b_n) and obtain more specialized test.

Example 2.

(1) The series $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$ is divergent.

For each integer $n \geq 1$, $\frac{a_{n+1}}{a_n} = \frac{2n+1}{2n+2}$ and so $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2n+1}{2n+2} = 1$ and so the Ratio Test is inconclusive. Now for any integer $n \geq 1$,

$$\frac{a_n}{a_{n+1}} = \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1}$$

so that

$$\frac{a_n}{a_{n+1}} n - (n+1) = \frac{n}{2n+1} - 1 = -\frac{n+1}{2n+1} \leq 0.$$

Therefore, by Kummer's Test (Theorem 1 (ii)) with (b_n) = (n) in the notation of Theorem 1, since $\sum_{n=1}^{\infty} \frac{1}{b_n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, $\sum_{n=1}^{\infty} a_n$ is divergent.

(2) The series $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{1}{n}$ is convergent.

For each integer $n \geq 1$, $\frac{a_{n+1}}{a_n} = \frac{(2n+1)n}{(2n+2)(n+1)}$ and so

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+1)n}{(2n+2)(n+1)} = 1$$

and the Ratio Test is again inconclusive.

Now for any integer $n \geq 1$,

$$\frac{a_n}{a_{n+1}} = \left(\frac{2n+2}{2n+1}\right)\frac{n+1}{n} = \left(1 + \frac{1}{2n+1}\right)\left(\frac{n+1}{n}\right)$$

so that

$$\frac{a_n}{a_{n+1}} n - (n+1) = \frac{n+1}{2n+1} \geq \frac{1}{2} > 0.$$

Therefore, by Kummer's Test (Theorem 1 (i)), with (b_n) = (n) in the notation of Theorem 1, , $\sum_{n=1}^{\infty} a_n$ is convergent.

13.2 Raabe's Test

Kummer's Test requires a suitable sequence (b_n) to test with the series. We may specify the sequence (b_n) to be the sequence (n) and obtain a special test, the Raabe's Test.

Theorem 3 (Raabe's Test)
Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms. Suppose that the limit

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = r.$$

Then
(i) $\sum_{n=1}^{\infty} a_n$ converges if $r > 1$
(ii) $\sum_{n=1}^{\infty} a_n$ diverges if $r < 1$ and
(iii) if $r = 1$, the series $\sum_{n=1}^{\infty} a_n$ may converge or diverge.

Proof. $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = r$ implies that given any ε > 0, there exists an integer N such that for all integer $n \geq N$,

$$r - \varepsilon < n\left(\frac{a_n}{a_{n+1}} - 1\right) < r + \varepsilon.$$

Hence, for integer $n \geq N$,

$$n\frac{a_n}{a_{n+1}} - (n+1) < (r-1) + \varepsilon \qquad \text{------------------------------} \quad (1)$$

and

$$n\frac{a_n}{a_{n+1}} - (n+1) > (r-1) - \varepsilon \qquad \text{------------------------------} \quad (2)$$

(i) If $r > 1$, then take $\varepsilon = \frac{r-1}{2} > 0$ and it follows from (2) that for all integer $n \geq N$,

$$n\frac{a_n}{a_{n+1}} - (n+1) > (r-1) - \varepsilon = \frac{r-1}{2} > 0.$$

Therefore, by Kummer's Test (Theorem 1(i) with $b_n = n$ and $A = \frac{r-1}{2} > 0$), $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $r < 1$, then take $\varepsilon = \frac{1-r}{2} > 0$ and it follows from (1) that for all integer $n \geq N$,

$$n\frac{a_n}{a_{n+1}} - (n+1) < (r-1) - \varepsilon = \frac{r-1}{2} < 0.$$

Therefore, by Kummer's Test (Theorem 1(ii) with $b_n = n$ so that $\sum_{n=1}^{\infty} \frac{1}{b_n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent), $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) Consider the series $\sum_{n=1}^{\infty} a_n$, where .$a_n = \left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right)^2$.

Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{2n+1}{2n+2}\right)^2 = 1$ and application of the Ratio Test gives no conclusion. For any integer $n \geq 1$,

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{2n}{2n+1} + \frac{n}{(2n+1)^2}.$$

Therefore, $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty}\left(\frac{2n}{2n+1}+\frac{n}{(2n+1)^2}\right) = 1.$

But for any integer $n \geq 1$,

$$\frac{a_n}{a_{n+1}}n-(n+1) = \frac{-1}{2n+1}+\frac{n}{(2n+1)^2} = -\frac{n+1}{(2n+1)^2} \leq 0.$$

Thus, by Kummer's Test (Theorem 1 (ii)) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right)^2$ is divergent.

Hence we have that $\sum_{n=1}^{\infty} a_n$ is divergent and $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = 1.$

Next consider the series $\sum_{n=2}^{\infty} a_n$, where $a_n = \frac{1}{n(\ln(n))^2}$. Then for integer $n \geq 2$,

$$\frac{a_n}{a_{n+1}} = \frac{n+1}{n}\left(\frac{\ln(n+1)}{\ln(n)}\right)^2 = \left(1+\frac{1}{n}\right)\left(\frac{\ln(n+1)-\ln(n)+\ln(n)}{\ln(n)}\right)^2$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\ln(\frac{n+1}{n})}{\ln(n)}\right)^2 \quad \text{-------------------------------} \quad (3)$$

Therefore, $\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{\ln(\frac{n+1}{n})}{\ln(n)}\right)^2 = (1+0)(1+0)^2 = 1$ and consequently, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{\lim_{n\to\infty}\frac{a_n}{a_{n+1}}} = 1.$ It follows that applying the Ratio Test gives no conclusion. From (3) for integer $n \geq 2$,

$$n\frac{a_n}{a_{n+1}} = (n+1)\left(1+\frac{2\ln(1+\frac{1}{n})}{\ln(n)}+\left(\frac{\ln(\frac{n+1}{n})}{\ln(n)}\right)^2\right).$$

Thus $\quad n\left(\frac{a_n}{a_{n+1}}-1\right) = 1+(n+1)\left(\frac{2\ln(1+\frac{1}{n})}{\ln(n)}+\left(\frac{\ln(1+\frac{1}{n})}{\ln(n)}\right)^2\right).$

Now $\lim_{n\to\infty}(n+1)\ln(1+\frac{1}{n}) = \lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})}{\frac{1}{n+1}} = \lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n}}(-\frac{1}{n^2})}{-\frac{1}{(n+1)^2}} = 1$, $\lim_{n\to\infty}\ln(1+\frac{1}{n}) = 0$ and $\lim_{n\to\infty}\frac{1}{\ln(n)} = 0.$ Therefore,

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = 1+2\lim_{n\to\infty}((n+1)\ln(1+\tfrac{1}{n}))\lim_{n\to\infty}\frac{1}{\ln(n)}$$

$$+\lim_{n\to\infty}((n+1)\ln(1+\tfrac{1}{n}))\left(\lim_{n\to\infty}\frac{1}{\ln(n)}\right)^2\lim_{n\to\infty}\ln(1+\tfrac{1}{n})$$

$$= 1+2\cdot 1\cdot 0+1\cdot 0 = 1.$$

Let $f(x) = \frac{1}{x(\ln(x))^2}$. Then f is continuous and non-negative on $(1, \infty)$. Its derivative $f'(x) = -\frac{\ln(x)(2+\ln(x))}{(x(\ln(x))^2)^2} < 0$ for $x > 1$. Therefore, f is monotone decreasing on $[2, \infty)$.

We note that for $x \geq 2$, $\int_2^x \frac{1}{t(\ln(t))^2}dt = \left[-\frac{1}{\ln(t)}\right]_2^x = \frac{1}{\ln(2)}-\frac{1}{\ln(x)}$. Therefore,

$$\lim_{x\to\infty}\int_2^x \frac{1}{t(\ln(t))^2}dt = \lim_{x\to\infty}\left(\frac{1}{\ln(2)}-\frac{1}{\ln(x)}\right) = \frac{1}{\ln(2)}-0 = \frac{1}{\ln(2)}.$$

Since for integer $n \geq 2$, $a_n = f(n)$, by the Integral Test (Theorem 25 Chapter 6), $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^2}$ is convergent. But $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = 1.$

We have thus shown that, when $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = 1$, the series may converge or diverge. This completes the proof.

Remark.

1. We have similar conclusions to Theorem 3 when $r = \pm\infty$. That is, when $\sum_{n=1}^{\infty} a_n$ is a series of positive terms, (i) if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = +\infty$, then $\sum_{n=1}^{\infty} a_n$ is convergent and (ii) if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = -\infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent. The proof is similar to that of Theorem 3 part (i) and (ii) but with suitable modification.
2. The usefulness of Theorem 3 is that the test is carried out in terms of the limit $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right)$, which may be readily computed. The only disadvantage is when this limit turns out to be 1 and other tests may have to be used to determine convergence or divergence.

Example 4.

(1) The series $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{3n+1}{n(n+1)(n+2)}$ is convergent.

For integer $n \geq 1$, $\frac{a_n}{a_{n+1}} = \frac{(3n+1)(n+3)}{(3n+4)n} = \left(1 - \frac{3}{3n+4}\right)\left(\frac{n+3}{n}\right)$ and so

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \left(1 - \frac{3}{3n+4}\right)(n+3) - n = 3 - \frac{3(n+3)}{3n+4} \to 3 - 1 = 2 > 1.$$

Therefore, by Raabe's Test (Theorem 3 (i)), $\sum_{n=1}^{\infty} \frac{3n+1}{n(n+1)(n+2)}$ is convergent. One may also use the much simpler Comparison Test since

$$a_n = \frac{3n+1}{n(n+1)(n+2)} \leq \frac{3}{n(n+2)} \leq \frac{3}{n^2}$$

and , $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(2) The series $\sum_{n=1}^{\infty} a_n$ in Example 2 (1), where $a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$ is shown to be divergent by Kummer's test. We shall use Raabe's Test as follows.

For each integer $n \geq 1$, $n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{n}{2n+1} \to \frac{1}{2} < 1$ and so by Raabe's Test (Theorem 3 (ii)), $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$ is divergent.

The following theorem is a variation of Raabe's Test.

Theorem 5.

Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms. Suppose for integer $n \geq 1$,

$$\frac{a_n}{a_{n+1}} = 1 + \frac{r}{n} + v_n,$$

where v_n is of order $1/n$, i.e., $n\, v_n \to 0$.

Then (i) $\sum_{n=1}^{\infty} a_n$ converges if $r > 1$,

(ii) $\sum_{n=1}^{\infty} a_n$ diverges if $r < 1$,

(iii) if $r = 1$, the series $\sum_{n=1}^{\infty} a_n$ may converge or diverge.

Proof. Just observe that

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = r + nv_n \to r$$

Theorem 5 then follows from Theorem 3.

The next result is a slightly stronger version of Raabe's Test.

Theorem 6. Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.

(i) If there exists a fixed constant $A > 1$ and an integer $N \geq 1$ such that for any $n \geq N$,

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{A}{n},$$

then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If there exists an integer $N \geq 1$, such that for any $n \geq N$,

$$\frac{a_{n+1}}{a_n} \geq 1 - \frac{1}{n},$$

then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof.

(i) Since $\frac{a_{n+1}}{a_n} \leq 1 - \frac{A}{n}$ for $n \geq N$, $1 - \frac{a_{n+1}}{a_n} \geq \frac{A}{n}$. Thus,

$$\frac{a_n - a_{n+1}}{a_n} = \frac{a_n - a_{n+1}}{a_{n+1}} \cdot \frac{a_{n+1}}{a_n} \geq \frac{A}{n}$$

and so

$$\frac{a_n}{a_{n+1}} - 1 \geq \frac{a_n}{a_{n+1}} \cdot \frac{A}{n} \geq \frac{1}{1 - \frac{A}{n}} \cdot \frac{A}{n}.$$

Consequently, for integer $n \geq N$,

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq \frac{A}{1 - \frac{A}{n}} \qquad \text{-------------------------------} \quad (1)$$

It follows from (1) that for integer $n \geq N$,

$$\frac{a_n}{a_{n+1}} n - (n+1) \geq \frac{A - 1 + \frac{A}{n}}{1 - \frac{A}{n}} > A - 1 > 0.$$

Therefore, by Kummer's Test (Theorem 1 (i)) that $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) We may assume that $N > 1$. Then for any integer $n \geq N$,

$$\frac{a_{n+1}}{a_n} \geq 1 - \frac{1}{n} \Rightarrow \frac{a_n}{a_{n+1}} \leq \frac{n}{n-1} = 1 + \frac{1}{n-1}$$

$$\Rightarrow \frac{a_n}{a_{n+1}} - 1 \leq \frac{1}{n-1} \Rightarrow \frac{a_n}{a_{n+1}} n - n \leq \frac{n}{n-1} = 1 + \frac{1}{n-1}.$$

It then follows that for any integer $n \geq N$,

$$\frac{a_n}{a_{n+1}} n - (n+1) \leq \frac{1}{n-1} \qquad \text{--------------------------------} \quad (2)$$

Multiply (2) by ln(n), we obtain for any integer $n \geq N$,

$$\frac{a_n}{a_{n+1}} n \ln(n) - (n+1)\ln(n) \leq \frac{\ln(n)}{n-1}.$$

Consequently, for any integer $n \geq N$,

$$\frac{a_n}{a_{n+1}} n \ln(n) - (n+1)\ln(n+1)) \leq (n+1)\ln(\frac{n}{n+1}) + \frac{\ln(n)}{n-1}.$$

$$< -1 + \frac{\ln(n)}{n-1} \text{ ------------------ } (3)$$

because $(n+1)\ln(\frac{n+1}{n}) > 1$ for integer $n \geq 1$.

Since $\lim_{n\to\infty} \frac{\ln(n)}{n-1} = 0,$ there exists a positive integer N_0 such that for any integer n,

$$n \geq N_0 \Rightarrow \frac{\ln(n)}{n-1} < \frac{1}{2}. \quad \text{--------------------------------} \ (4)$$

Let $M = \max(N, N_0)$. Then it follows from (3) and (4) that for any integer $n \geq M$,

$$\frac{a_n}{a_{n+1}} n \ln(n) - (n+1)\ln(n+1)) \leq -1 + \frac{1}{2} = -\frac{1}{2} < 0.$$

Therefore, with $b_n = n \ln(n)$ (in the notation of Theorem 1 (ii)) by Kummer's test, $\sum_{n=1}^{\infty} a_n$ is divergent, since $\sum_{n=2}^{\infty} \frac{1}{b_n} = \sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ is divergent (by Integral Test, Theorem 25 Chapter 6).

Remark
For $\sum_{n=1}^{\infty} a_n$ a series of positive terms we normally apply the Ratio Test first. The expression for $\frac{a_{n+1}}{a_n}$ will be scrutinized and if the Ratio Test fails to make any conclusion, the expression for $\frac{a_{n+1}}{a_n}$ may be studied further to test for the condition of Theorem 6.

The next variation of Raabe's Test will make use of absolute convergence of some known series. It is stated below.

Theorem 7. Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.
Suppose there exists an integer $N \geq 1$ such that for any $n \geq N$,

$$\frac{a_{n+1}}{a_n} = 1 - \frac{A}{n} + v_n , \qquad \text{--------------------------} \quad (C)$$

where $\sum_{n=1}^{\infty} v_n$ is absolutely convergent.
Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $A > 1$.

We shall need an estimate of $\ln(1-x)$ for $|x| < 1$ for the proof of Theorem 7. We state the result as follows.

Lemma 8. $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $|x| < 1$. In particular,

$$-x - x^2 \frac{1}{1-|x|} \leq \ln(1-x) \leq -x + x^2 \frac{1}{1-|x|} \text{ ----------------------} \quad (D)$$

and for $|x| < \frac{1}{2}$,

$$-x - 2x^2 \leq \ln(1-x) \leq -x + 2x^2 \text{ --------------------------------} \ (E)$$

Proof. The expansion for $\ln(1-x)$ for $|x| < 1$ is as given in Example 21 of Chapter 8. Thus

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - x^2 \sum_{n=2}^{\infty} \frac{x^{n-2}}{n} \text{ for } |x| < 1.$$

Note that $\left| x^2 \sum_{n=2}^{\infty} \frac{x^{n-2}}{n} \right| \leq x^2 \frac{1}{1-|x|}$ for $|x| < 1$ and so we have

$$-x - x^2 \frac{1}{1-|x|} \leq \ln(1-x) \leq -x + x^2 \frac{1}{1-|x|},$$

which is (D).
(E) follows from (D).

Proof of Theorem 7.
Since $\sum_{n=1}^{\infty} v_n$ is absolutely convergent, $v_n \to 0$. It follows that, $\frac{A}{n} - v_n \to 0$. Therefore, there exists a positive integer N_1 such that for any integer n,

$$n \geq N_1 \Rightarrow \left| \frac{A}{n} - v_n \right| < \frac{1}{2} \text{ --} \ (1)$$

Let $M = \max(N, N_1)$.
Now apply the logarithmic function on both sides of (C), we obtain for $n \geq M$,

$$\ln(a_{n+1}) - \ln(a_n) = \ln(1 - (\frac{A}{n} - v_n)),$$

Then summing from M onwards we obtain for $n \geq M$,

$$\sum_{k=M}^{n}(\ln(a_{k+1})-\ln(a_k)) = \sum_{k=M}^{n}\ln(1-(\frac{A}{k}-v_k))$$

Thus, $$\ln(a_{n+1})-\ln(a_M) = \sum_{k=M}^{n}\ln(1-(\frac{A}{k}-v_k))$$

or for integer $n \geq M$,

$$\ln(a_{n+1}) = \ln(a_M) + \sum_{k=M}^{n}\ln(1-(\frac{A}{k}-v_k)) \quad \text{-----------------------------} \quad (2)$$

It then follows from (1) and (E) that for integer $n \geq M$,

$$\ln(a_{n+1}) \leq \ln(a_M) - \sum_{k=M}^{n}\frac{A}{k} + \sum_{k=M}^{n} v_k + 2\sum_{k=M}^{n}(\frac{A}{k}-v_k)^2$$

$$\leq \ln(a_M) - \sum_{k=1}^{n}\frac{A}{k} + \sum_{k=1}^{M-1}\frac{A}{k} + \sum_{k=1}^{\infty}|v_k| + 2\sum_{k=M}^{n}(\frac{A}{k}-v_k)^2$$

since we know $\sum_{k=1}^{\infty}|v_k|$ is convergent.

Now note that $\sum_{k=1}^{\infty}(\frac{A}{k}-v_k)^2$ is convergent. This is because

$$(\frac{A}{k}-v_k)^2 = \frac{A^2}{k^2} + v_k^2 - 2A\frac{1}{k}v_k,$$

and $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is convergent, $\sum_{k=1}^{\infty}v_k^2$ is convergent since $\sum_{k=1}^{\infty}v_k$ is absolutely convergent and $\sum_{k=1}^{\infty}\frac{1}{k}v_k$ is absolutely convergent by a simple Comparison Test. Therefore, for $A > 1$ and $n \geq M$,

$$\ln(a_{n+1}) \leq \ln(a_M) - A\ln(n+1) + \sum_{k=1}^{M-1}\frac{A}{k} + \sum_{k=1}^{\infty}|v_k| + 2\sum_{k=1}^{\infty}(\frac{A}{k}-v_k)^2$$

since $\sum_{k=1}^{n}\frac{1}{k} > \ln(n+1)$.

Letting $C = \ln(a_M) + \sum_{k=1}^{M-1}\frac{A}{k} + \sum_{k=1}^{\infty}|v_k| + 2\sum_{k=1}^{\infty}(\frac{A}{k}-v_k)^2$, we then have for $A > 1$ and $n \geq M$,

$$\ln(a_{n+1}) \leq C + \ln(\frac{1}{(n+1)^A}) . \quad \text{-------------------------------} \quad (3)$$

Therefore, applying exponential function to (3), we obtain for $A > 1$ and $n \geq M$,

$$a_{n+1} \leq e^C\frac{1}{(n+1)^A}.$$

Since $A > 1$, $\sum_{n=1}^{\infty}\frac{1}{(n+1)^A}$ is convergent and so by the Comparison test, $\sum_{n=M}^{\infty}a_{n+1}$ is convergent and consequently $\sum_{n=1}^{\infty}a_n$ is convergent.

Suppose now $A \leq 1$. Then it follows from (2), (1) and (E) that for integer $n \geq M$,

$$\ln(a_{n+1}) \geq \ln(a_M) - \sum_{k=M}^{n}\frac{A}{k} + \sum_{k=M}^{n}v_k - 2\sum_{k=M}^{n}(\frac{A}{k}-v_k)^2 \quad \text{------------------} \quad (4)$$

Thus, if $0 \leq A \leq 1$, for integer $n \geq M$,

$$\ln(a_{n+1}) \geq \ln(a_M) - \sum_{k=2}^{n}\frac{A}{k} + \sum_{k=2}^{M-1}\frac{A}{k} - \sum_{k=1}^{\infty}|v_k| - 2\sum_{k=1}^{\infty}(\frac{A}{k}-v_k)^2$$

since we know $\sum_{k=1}^{\infty}|v_k|$ and $\sum_{k=1}^{\infty}(\frac{A}{k}-v_k)^2$ are convergent,

$$\geq \ln(a_M) - A\ln(n) + \sum_{k=2}^{M-1}\frac{A}{k} - \sum_{k=1}^{\infty}|v_k| - 2\sum_{k=1}^{\infty}(\frac{A}{k}-v_k)^2$$

since $\sum_{k=2}^{n}\frac{1}{k} < \ln(n)$.

Hence letting $K = \ln(a_M) + \sum_{k=2}^{M-1} \frac{A}{k} - \sum_{k=1}^{\infty}|v_k| - 2\sum_{k=1}^{\infty}(\frac{A}{k} - v_k)^2$, for $0 \le A \le 1$ and integer $n \ge M$,

$$\ln(a_{n+1}) \ge K - A\ln(n) = K + \ln(\frac{1}{n^A}).$$

Applying the exponential function we obtain, for $0 \le A \le 1$ and integer $n \ge M$,

$$a_{n+1} \ge e^K \frac{1}{n^A}.$$

Therefore, by the Comparison Test $\sum_{n=1}^{\infty} a_n$ is divergent because for $0 \le A \le 1$ $\sum_{n=1}^{\infty} \frac{1}{n^A}$ is divergent.

If $A < 0$, it follows from (4) that for integer $n \ge M$,

$$\ln(a_{n+1}) \ge \ln(a_M) - \sum_{k=1}^{n} \frac{A}{k} + \sum_{k=1}^{M-1} \frac{A}{k} - \sum_{k=1}^{\infty}|v_k| - 2\sum_{k=1}^{\infty}(\frac{A}{k} - v_k)^2$$

$$\ge \ln(a_M) - A\ln(n+1) + \sum_{k=1}^{M-1} \frac{A}{k} - \sum_{k=1}^{\infty}|v_k| - 2\sum_{k=1}^{\infty}(\frac{A}{k} - v_k)^2.$$

Let $K = \ln(a_M) + \sum_{k=1}^{M-1} \frac{A}{k} - \sum_{k=1}^{\infty}|v_k| - 2\sum_{k=1}^{\infty}(\frac{A}{k} - v_k)^2$. We then have for $A < 0$ and integer $n \ge M$,

$$\ln(a_{n+1}) \ge K + \ln((n+1)^{-A}).$$

Therefore, for $A < 0$ and integer $n \ge M$,

$$a_{n+1} \ge e^K (n+1)^{-A}.$$

Plainly, by the Comparison Test, $\sum_{n=1}^{\infty} a_n$ is divergent. This completes the proof of Theorem 7.

Remark. Usually the following specialization of Theorem 7 is used:

Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.

Suppose there exists an integer $N \ge 1$ such that for any $n \ge N$,

$$\frac{a_{n+1}}{a_n} = 1 - \frac{A}{n} + v_n$$

and $v_n = O\left(\frac{1}{n^{1+k}}\right)$ for $k > 0$, that is $|v_n| < C\frac{1}{n^{1+k}}$ for some positive constant C.

Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $A > 1$.

This follows from Theorem 7 since $v_n = O\left(\frac{1}{n^{1+k}}\right)$ implies that $\sum_{n=1}^{\infty} v_n$ is absolutely convergent.

13.3 Bertrand's Test

The next test that we shall present may be useful when Theorem 3 is inconclusive, that is when $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = 1$.

Theorem 9 (Bertrand's Test). Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.

(i) Suppose there exists an integer $N \ge 2$ such that for any $n \ge N$,

$$\frac{a_n}{a_{n+1}} \ge 1 + \frac{1}{n} + \frac{A}{n\ln(n)} . \qquad \text{---------------------------} \quad \text{(F)}$$

If $A > 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) Suppose there exists an integer $N \ge 2$ such that for any $n \ge N$,

$$\frac{a_n}{a_{n+1}} \le 1 + \frac{1}{n} + \frac{1}{n\ln(n)} . \qquad \text{---------------------------} \quad (G)$$

Then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof.

(i). From (F) we have, for any integer $n \ge N$,

$$\frac{a_n}{a_{n+1}} n\ln(n) \ge (n+1)\ln(n) + A$$

and so

$$\frac{a_n}{a_{n+1}} n\ln(n) - (n+1)\ln(n+1) \ge (n+1)\ln(\frac{n}{n+1}) + A \ge -1 - \frac{1}{n} + A$$

since $(n+1)\ln(\frac{n}{n+1}) > -1 - \frac{1}{n}$.

If $A > 1$, then $A - 1 > 0$. Then there exists a positive integer N_1 such that for any integer n,

$$n \ge N_1 \Rightarrow \frac{1}{n} < \frac{1}{2}(A-1).$$

Therefore, if $A > 1$, then for any integer $n \ge \max(N, N_1)$,

$$\frac{a_n}{a_{n+1}} n\ln(n) - (n+1)\ln(n+1) > -1 - \frac{1}{n} + A > \frac{1}{2}(A-1) > 0.$$

Hence, by Kummer's Test (Theorem 1 (i)), $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) Similarly it follows from (G) that for any integer $n \ge N$,

$$\frac{a_n}{a_{n+1}} n\ln(n) \le (n+1)\ln(n) + 1$$

and so

$$\frac{a_n}{a_{n+1}} n\ln(n) - (n+1)\ln(n+1) \le (n+1)\ln(\frac{n}{n+1}) + 1 \le -1 + 1 \le 0$$

since $(n+1)\ln(\frac{n}{n+1}) < -1$.

Therefore, by Kummer's Test (Theorem 1(ii)), since $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ is divergent, $\sum_{n=1}^{\infty} a_n$ is divergent.

Next we have the following variation of Bertrand's Test in terms of $\frac{a_{n+1}}{a_n}$.

Theorem 10. Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.

(i) Suppose there exists an integer $N \ge 2$ such that for any n integer $\ge N$,

$$\frac{a_{n+1}}{a_n} \le 1 - \frac{1}{n} - \frac{A}{n\ln(n)} . \qquad \text{---------------------------} \quad (H)$$

If $A > 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) Suppose there exists an integer $N \ge 2$ such that for any integer $n \ge N$,

$$\frac{a_{n+1}}{a_n} \ge 1 - \frac{1}{n} - \frac{1}{n\ln(n)} . \qquad \text{---------------------------} \quad (I)$$

Then $\sum_{n=1}^{\infty} a_n$ is divergent.

We can prove Theorem 10 in the same way Theorem 9 is proved but using a version of Kummer's Test in terms of $\frac{a_{n+1}}{a_n}$:

Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.

(i) If there is a positive sequence $\{ b_n \}$, a positive constant A, and a positive integer N such that for all integer $n \ge N$,

$$b_n - b_{n+1}\frac{a_{n+1}}{a_n} \ge A,$$

then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If there is a positive sequence { b_n } and a positive integer N such that for all integer $n \geq N$, $\quad b_n - b_{n+1}\dfrac{a_{n+1}}{a_n} \leq 0,$

hen $\sum_{n=1}^{\infty} a_n$ is divergent if $\sum_{n=1}^{\infty} \dfrac{1}{b_n}$ is divergent.

We shall give a different proof using Theorem 9.

Proof of Theorem 10.

(i) From (H), we have for any integer $n \geq N$, $\dfrac{a_{n+1}}{a_n} \leq 1 - \dfrac{1}{n} - \dfrac{A}{n\ln(n)}$.

If $A > 1$, since $\dfrac{1}{n} + \dfrac{A}{n\ln(n)} \to 0$, there exists a positive integer N_1 such that for any integer n,

$$n \geq N_1 \Rightarrow 0 < \frac{1}{n} + \frac{A}{n\ln(n)} < 1.$$

Therefore, for any integer $n \geq max(N, N_1)$,

$$\frac{a_n}{a_{n+1}} \geq \frac{1}{1 - \frac{1}{n} - \frac{A}{n\ln(n)}} \geq 1 + \frac{1}{n} + \frac{A}{n\ln(n)} .$$

It follows by Theorem 9 (i) that $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) We shall prove part (ii) along the lines of the proof of Theorem 7.

Applying the logarithmic function to both sides of (I) we obtain for any integer $n \geq N$,

$$\ln(a_{n+1}) - \ln(a_n) \geq \ln\left(1 - \frac{1}{n} - \frac{1}{n\ln(n)}\right) \quad \text{--------------------} \quad (1)$$

We may assume that $n \geq N \Rightarrow 0 < \dfrac{1}{n} + \dfrac{1}{n\ln(n)} < \dfrac{1}{2}$. We shall assume that $N > 4$.

It then follows from (1) and Lemma 8 or (E) that,

$$\ln(a_{n+1}) - \ln(a_n) \geq -\frac{1}{n} - \frac{1}{n\ln(n)} - 2\left(\frac{1}{n} + \frac{1}{n\ln(n)}\right)^2,$$

Thus, summing from N onwards we obtain for integer $n \geq N$,

$$\ln(a_{n+1}) - \ln(a_N) \geq -\sum_{k=N}^{n} \frac{1}{k} - \sum_{k=N}^{n} \frac{1}{k\ln(k)} - 2\sum_{k=N}^{n}\left(\frac{1}{k} + \frac{1}{k\ln(k)}\right)^2$$

$$\geq -\sum_{k=2}^{n} \frac{1}{k} + \sum_{k=2}^{N-1} \frac{1}{k} - \sum_{k=4}^{n} \frac{1}{k\ln(k)} + \sum_{k=4}^{N-1} \frac{1}{k\ln(k)} - 2\sum_{k=N}^{n}\left(\frac{1}{k} + \frac{1}{k\ln(k)}\right)^2$$

assuming without loss of generality that $N > 4$,

$$\geq -\ln(n) + \sum_{k=2}^{N-1} \frac{1}{k} - \ln(\ln(n)) + \sum_{k=4}^{N-1} \frac{1}{k\ln(k)} - 2\sum_{k=N}^{n}\left(\frac{1}{k} + \frac{1}{k\ln(k)}\right)^2$$

since $\sum_{k=2}^{n} \dfrac{1}{k} < \ln(n)$ and $\sum_{k=4}^{n} \dfrac{1}{k\ln(k)} < \ln(\ln(n)) - \ln(\ln(3))$,

$$\geq -\ln(n\ln(n)) + \sum_{k=2}^{N-1} \frac{1}{k} + \sum_{k=4}^{N-1} \frac{1}{k\ln(k)} - 2\sum_{k=2}^{\infty}\left(\frac{1}{k} + \frac{1}{k\ln(k)}\right)^2$$

since plainly $\sum_{k=2}^{\infty}\left(\frac{1}{k}+\frac{1}{k\ln(k)}\right)^2$ is convergent by a simple Comparison Test with the convergent $\sum_{k=2}^{\infty}\frac{1}{k^2}$.

Therefore, for integer $n \geq N$,

$$\ln(a_{n+1}) \geq \ln(a_N) + \sum_{k=2}^{N-1}\frac{1}{k} + \sum_{k=4}^{N-1}\frac{1}{k\ln(k)} - 2\sum_{k=2}^{\infty}\left(\frac{1}{k}+\frac{1}{k\ln(k)}\right)^2 - \ln(n\ln(n)).$$

Letting $C = \ln(a_N) + \sum_{k=2}^{N-1}\frac{1}{k} + \sum_{k=4}^{N-1}\frac{1}{k\ln(k)} - 2\sum_{k=2}^{\infty}\left(\frac{1}{k}+\frac{1}{k\ln(k)}\right)^2$, we then get for integer $n \geq N$,

$$\ln(a_{n+1}) \geq C - \ln(n\ln(n)) \text{------------------------------------ (2)}.$$

Therefore, applying the exponential function to (2), we obtain for any integer $n \geq N$,

$$a_{n+1} \geq e^C \frac{1}{n\ln(n)}.$$

Therefore, by the Comparison Test, since $\sum_{k=2}^{\infty}\frac{1}{n\ln(n)}$ is divergent, $\sum_{n=1}^{\infty} a_n$ is divergent. This completes the proof of Theorem 10.

13.4 Gauss Test

We now introduce Gauss test. The first result is in the form that is usually applied.

Theorem 11 (Gauss Test). Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.
Suppose there exists an integer $N \geq 1$ such that for any $n \geq N$,

$$\frac{a_n}{a_{n+1}} = 1 + \frac{A}{n} + \frac{A_n}{n^2},$$

where (A_n) is a bounded sequence.
Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $A > 1$.

We shall prove this by proving the following slightly more generalized version.

Theorem 12 (Gauss Test). Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.
Suppose there exists an integer $N \geq 1$ such that for any $n \geq N$,

$$\frac{a_n}{a_{n+1}} = 1 + \frac{A}{n} + v_n, \qquad \text{-------------------------------- (J)}$$

where v_n is of order $\frac{1}{n^{1+k}}, k > 0$, i.e., there is a positive integer M such that $|v_n| \leq C\frac{1}{n^{1+k}}$, for some positive constant C and for all integer $n \geq M$.

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $A > 1$.

Proof. By supposition, for any integer $n \geq N$,

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = A + nv_n \quad \text{--------------------------------} \quad (1)$$

Since $|v_n| \leq C\frac{1}{n^{1+k}}$ integer $n \geq M$, we have then that for $n \geq M$, $|nv_n| \leq C\frac{1}{n^{1+k}}$. Therefore, because $\lim_{n\to\infty} \frac{1}{n^{1+k}} = 0$ for $k > 0$, by the Comparison Test (Proposition 8 Chapter 2), $\lim_{n\to\infty} nv_n = 0$.

It follows then from (1) that

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = A + \lim_{n\to\infty} nv_n = A + 0 = A.$$

Therefore, by Raabe's Test (Theorem 3), $\sum_{n=1}^{\infty} a_n$ converges if $A > 1$ and diverges if $A < 1$.

If $A = 1$, by (J) we have that for integer $n \geq N_1 = \max(N, M)$,

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{n\ln(n)v_n}{n\ln(n)}.$$

Note that for integer $n \geq N_1$, $|n\ln(n)v_n| \leq C\frac{\ln(n)}{n^k}$. Since $\lim_{n\to\infty} C\frac{\ln(n)}{n^k} = 0$ for $k > 0$, there exists a positive integer N_2 such that $n \geq N_2$ implies that $|n\ln(n)v_n| \leq 1$. Therefore, for integer $n \geq \max(N_1, N_2)$,

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n} + \frac{1}{n\ln(n)}.$$

It then follows by Bertrand's Test (Theorem 9 (ii)), $\sum_{n=1}^{\infty} a_n$ is divergent. This completes the proof of Theorem 12.

Theorem 11 follows from Theorem 12 since $\frac{A_n}{n^2}$ is of order $\frac{1}{n^2}$ if the sequence (A_n) is bounded.

A further generalization may be called Gauss Test too. Indeed Theorem 7 may be called a "Gauss-like" test too.

Theorem 13. Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.

Suppose there exists an integer $N \geq 1$ such that for any $n \geq N$,

$$\frac{a_n}{a_{n+1}} = 1 + \frac{A}{n} + v_n \quad \text{--------------------------} \quad (K)$$

where $\sum_{n=1}^{\infty} v_n$ is absolutely convergent.

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $A > 1$.

Proof. The proof is similar to that of Theorem 7. We reproduce here for convenience.

Since $\sum_{n=1}^{\infty} v_n$ is absolutely convergent, $v_n \to 0$. It follows that, $\frac{A}{n} + v_n \to 0$. Therefore, there exists a positive integer N_1 such that for any integer n,

$$n \geq N_1 \Rightarrow \left|\frac{A}{n} + v_n\right| < \frac{1}{2} \quad \text{--} \quad (1)$$

Let $M = \max(N, N_1)$.

Now applying the logarithmic function on both sides of (K), we obtain for $n \geq M$,

$$\ln(a_{n+1}) - \ln(a_n) = -\ln\left(1 + \left(\frac{A}{n} + v_n\right)\right),$$

Then summing from M onwards we obtain for $n \geq M$,

$$\sum_{k=M}^{n} (\ln(a_{k+1}) - \ln(a_k)) = -\sum_{k=M}^{n} \ln\left(1 + \left(\frac{A}{k} + v_k\right)\right)$$

Thus, $\ln(a_{n+1}) - \ln(a_M) = -\sum_{k=M}^{n} \ln(1 + (\frac{A}{k} + v_k))$

or for integer $n \geq M$,

$$\ln(a_{n+1}) = \ln(a_M) - \sum_{k=M}^{n} \ln(1 + (\frac{A}{k} + v_k)) \quad \text{-----------------------------} \quad (2)$$

It then follows from (1) and (E) that for integer $n \geq M$,

$$\ln(a_{n+1}) \leq \ln(a_M) - \sum_{k=M}^{n} \frac{A}{k} - \sum_{k=M}^{n} v_k + 2\sum_{k=M}^{n} (\frac{A}{k} + v_k)^2$$

$$\leq \ln(a_M) - \sum_{k=1}^{n} \frac{A}{k} + \sum_{k=1}^{M-1} \frac{A}{k} + \sum_{k=1}^{\infty} |v_k| + 2\sum_{k=M}^{n} (\frac{A}{k} + v_k)^2$$

since we know $\sum_{k=1}^{\infty} |v_k|$ is convergent.

Now note that $\sum_{k=1}^{\infty} (\frac{A}{k} + v_k)^2$ is convergent. This is because

$$(\frac{A}{k} + v_k)^2 = \frac{A^2}{k^2} + v_k^2 + 2A\frac{1}{k}v_k,$$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent, $\sum_{k=1}^{\infty} v_k^2$ is convergent since $\sum_{k=1}^{\infty} v_k$ is absolutely convergent and $\sum_{k=1}^{\infty} \frac{1}{k} v_k$ is absolutely convergent by a simple Comparison Test (compare with $\sum_{k=1}^{\infty} v_k$, ref Proposition 12 Chapter 6). Therefore, for $A > 1$ and $n \geq M$,

$$\ln(a_{n+1}) \leq \ln(a_M) - A\ln(n+1) + \sum_{k=1}^{M-1} \frac{A}{k} + \sum_{k=1}^{\infty} |v_k| + 2\sum_{k=1}^{\infty} (\frac{A}{k} + v_k)^2$$

since $\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1)$.

Letting $C = \ln(a_M) + \sum_{k=1}^{M-1} \frac{A}{k} + \sum_{k=1}^{\infty} |v_k| + 2\sum_{k=1}^{\infty} (\frac{A}{k} + v_k)^2$, we then have for $A > 1$ and $n \geq M$,

$$\ln(a_{n+1}) \leq C + \ln(\frac{1}{(n+1)^A}) \ . \quad \text{------------------------------} \ (3)$$

Therefore, applying exponential function to (3), we obtain for $A > 1$ and $n \geq M$,

$$a_{n+1} \leq e^C \frac{1}{(n+1)^A}.$$

Since $A > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^A}$ is convergent and so by the Comparison test, $\sum_{n=M}^{\infty} a_{n+1}$ is convergent and consequently $\sum_{n=1}^{\infty} a_n$ is convergent.

Suppose now $A \leq 1$. Then it follows from (2), (1) and (E) that for integer $n \geq M$,

$$\ln(a_{n+1}) \geq \ln(a_M) - \sum_{k=M}^{n} \frac{A}{k} - \sum_{k=M}^{n} v_k - 2\sum_{k=M}^{n} (\frac{A}{k} + v_k)^2 \ \text{------------------} \ (4)$$

Thus, if $0 \leq A \leq 1$, for integer $n \geq M$,

$$\ln(a_{n+1}) \geq \ln(a_M) - \sum_{k=2}^{n} \frac{A}{k} + \sum_{k=2}^{M-1} \frac{A}{k} - \sum_{k=1}^{\infty} |v_k| - 2\sum_{k=1}^{\infty} (\frac{A}{k} + v_k)^2$$

since we know $\sum_{k=1}^{\infty} |v_k|$ and $\sum_{k=1}^{\infty} (\frac{A}{k} - v_k)^2$are convergent,

$$\geq \ln(a_M) - A\ln(n) + \sum_{k=2}^{M-1} \frac{A}{k} - \sum_{k=1}^{\infty} |v_k| - 2\sum_{k=1}^{\infty} (\frac{A}{k} + v_k)^2$$

since $\sum_{k=2}^{n} \frac{1}{k} < \ln(n)$.

Hence letting $K = \ln(a_M) + \sum_{k=2}^{M-1} \frac{A}{k} - \sum_{k=1}^{\infty} |v_k| - 2\sum_{k=1}^{\infty} (\frac{A}{k} + v_k)^2$, for $0 \leq A \leq 1$ and integer $n \geq M$,

$$\ln(a_{n+1}) \geq K - A\ln(n) = K + \ln(\frac{1}{n^A}).$$

Applying the exponential function we obtain, for $0 \leq A \leq 1$ and integer $n \geq M$,

$$a_{n+1} \geq e^K \frac{1}{n^A}.$$

Therefore, by the Comparison Test, $\sum_{n=1}^{\infty} a_n$ is divergent because for $0 \leq A \leq 1$, $\sum_{n=1}^{\infty} \frac{1}{n^A}$ is divergent.

If $A < 0$, it follows from (4) that for integer $n \geq M$,

$$\ln(a_{n+1}) \geq \ln(a_M) - \sum_{k=1}^{n} \frac{A}{k} + \sum_{k=1}^{M-1} \frac{A}{k} - \sum_{k=1}^{\infty}|v_k| - 2\sum_{k=1}^{\infty}(\frac{A}{k} + v_k)^2$$

$$\geq \ln(a_M) - A\ln(n+1) + \sum_{k=1}^{M-1} \frac{A}{k} - \sum_{k=1}^{\infty}|v_k| - 2\sum_{k=1}^{\infty}(\frac{A}{k} + v_k)^2.$$

Let $K = \ln(a_M) + \sum_{k=1}^{M-1} \frac{A}{k} - \sum_{k=1}^{\infty}|v_k| - 2\sum_{k=1}^{\infty}(\frac{A}{k} + v_k)^2$. We then have for $A < 0$ and integer $n \geq M$,

$$\ln(a_{n+1}) \geq K + \ln((n+1)^{-A}).$$

Therefore, for $A < 0$ and integer $n \geq M$,

$$a_{n+1} \geq e^K (n+1)^{-A}.$$

Plainly, by the Comparison Test, $\sum_{n=1}^{\infty} a_n$ is divergent. This completes the proof of Theorem 13.

Note that in the proof of Theorem 13 we use only inequalities. Thus we may formulate Theorem 13 as follows:

Theorem 14. Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.

(i) Suppose there exists an integer $N \geq 1$ such that for any $n \geq N$,

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{A}{n} + v_n ,$$

where $\sum_{n=1}^{\infty} v_n$ is absolutely convergent. If $A > 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) Suppose there exists an integer $N \geq 1$ such that for any $n \geq N$,

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n} + v_n ,$$

where $\sum_{n=1}^{\infty} v_n$ is absolutely convergent. Then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. The proof is embedded in the proof of Theorem 13.

For part (i) note that for any $n \geq N$, assuming that $1 + (\frac{A}{n} + v_n) > 0$,

$$\ln(a_{n+1}) - \ln(a_n) \leq -\ln(1 + (\frac{A}{n} + v_n)).$$

With this inequality, the rest of the proof is exactly the same as in the proof of Theorem 13.

Likewise for part (ii), $\ln(a_{n+1}) - \ln(a_n) \geq -\ln(1 + (\frac{1}{n} + v_n))$ and the proof proceeds in exactly the same manner as in the proof of Theorem 13.

Thus specializing Theorem 14 we obtain:

Theorem 15. Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.

(i) Suppose there exists an integer $N \geq 1$ such that for any $n \geq N$,

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{A}{n} + \frac{A_n}{n^2} ,$$

where (A_n) is a bounded sequence. If $A > 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) Suppose there exists an integer $N \geq 1$ such that for any $n \geq N$,

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n} + \frac{A_n}{n^2} ,$$

where (A_n) is a bounded sequence. Then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 16.

For a, $b > 0$ the series $\sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)}{b(b+1)(b+2)\cdots(b+n-1)}$ converges if and only if $b-a > 1$.

Proof. Let $a_n = \frac{a(a+1)(a+2)\cdots(a+n-1)}{b(b+1)(b+2)\cdots(b+n-1)}$. Then for integer $n \geq 1$,

$$\frac{a_n}{a_{n+1}} = \frac{b+n}{a+n} = 1 + \frac{b-a}{n} - \frac{a(b-a)}{(1+\frac{a}{n})} \Big/ n^2$$

and so since the sequence $\left(-\frac{a(b-a)}{(1+\frac{a}{n})}\right)$ is bounded, by Gauss Test (Theorem 11) the series $\sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)}{b(b+1)(b+2)\cdots(b+n-1)}$ is convergent if and only if $(b-a) > 1$.

13.5 Cauchy Condensation Test

There is one interesting test, the Cauchy condensation test, particularly useful for certain logarithmic series. This test applies only to series with monotone decreasing terms.

Theorem 17.

Suppose (a_n) is a monotone decreasing sequence of non-negative terms. Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ is convergent.

Proof. Let $s_n = \sum_{j=1}^{n} a_j$ be the n-th partial sum of the series for each integer $n \geq 1$. Since each a_n is nonnegative, the sequence (s_n) is an increasing sequence. Therefore, by the Monotone Convergence Theorem (Theorem 15 Chapter 2), $\sum_{n=1}^{\infty} a_n$ is convergent if and only if (s_n) is convergent if and only if the sequence (s_n) is bounded above.

Let Let $v_n = 2^n a_{2^n}$ for each integer $n \geq 0$. Let Let $t_n = \sum_{j=0}^{n-1} v_j$. Similarly we deduce that $\sum_{n=0}^{\infty} v_n$ is convergent if and only if the sequence (t_n) is bounded above. Now we make some simple observation.

Since (a_n) is a monotone decreasing, for each integer $n \geq 0$ and any $j \geq 0$, $a_{2^n} \geq a_{2^n+j}$.

Hence for each integer $n \geq 0$,

$$2^n a_{2^n} \geq \sum_{j=0}^{2^n-1} a_{2^n+j} = \sum_{j=2^n}^{2^{n+1}-1} a_j.$$

We also have that for each integer $n \geq 0$ and any $0 \leq j \leq 2^n - 1$, $a_{2^{n+1}-1} \leq a_{2^n+j}$. Hence,

$$2^n a_{2^{n+1}-1} \leq \sum_{j=0}^{2^n-1} a_{2^n+j} = \sum_{j=2^n}^{2^{n+1}-1} a_j.$$

Therefore, for each integer $n \geq 0$, $2^n a_{2^{n+1}} \leq 2^n a_{2^{n+1}-1} \leq \sum_{j=2^n}^{2^{n+1}-1} a_j$. It then follows that for each integer $n \geq 0$,

$$\left.\begin{array}{r} \frac{1}{2}2^{n+1} a_{2^{n+1}} \leq \sum_{j=2^n}^{2^{n+1}-1} a_j \leq 2^n a_{2^n} \\ \text{or } \frac{1}{2} v_{n+1} \leq \sum_{j=2^n}^{2^{n+1}-1} a_j \leq v_n \end{array}\right\}. \quad \text{-----------------------------} \quad (1)$$

Suppose that the series $\sum_{n=0}^{\infty} v_n$ is convergent, i.e., the sequence (t_n) is bounded above. We shall next show that the sequence (s_n) is also bounded above.
Take any integer $n \geq 1$. Then for some $k \geq 1$, $n \leq 2^k - 1$. We have

$$\begin{aligned} s_n &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + a_n \\ &\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^{k-1}} + a_{2^{k-1}+1} + \cdots + a_{2^k-1}) \\ &\leq v_0 + v_1 + v_2 + \cdots + v_{k-1} = t_k \end{aligned} \quad \text{--------------------------------------} \quad (2)$$

by (1).
Since (t_n) is bounded above, there exists a positive constant C such that $t_n \leq C$. Therefore, by (2), $s_n \leq t_k \leq C$. It follows that $s_n \leq C$ for all integer $n \geq 1$. This means (s_n) is also bounded above. Hence $\sum_{n=1}^{\infty} a_n$ is convergent.
Conversely suppose $\sum_{n=1}^{\infty} a_n$ is convergent. This means (s_n) is bounded above and so there exists a positive number K such that $s_n \leq K$ for all integer $n \geq 1$. We shall show that (t_n) is also bounded above. Take any integer $n \geq 2$. Then

$$\begin{aligned} t_n &= \sum_{j=0}^{n-1} v_j = v_0 + v_1 + \cdots + v_{n-1}. \\ &\leq a_1 + 2a_1 + 2(a_2 + a_3) + 2(a_4 + a_5 + a_6 + a_7) \cdots + 2(a_{2^{n-2}} + a_{2^{n-2}+1} + \cdots + a_{2^{n-1}-1}) \quad \text{by (1)} \\ &\leq a_1 + 2s_{2^{n-1}-1} \leq a_1 + 2K = D. \end{aligned}$$

Plainly $t_1 \leq D$. Therefore, the sequence (t_n) is bounded above by D. This means $\sum_{n=1}^{\infty} v_n$ is convergent.

Example 18. $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ is divergent.

Note that $\left(\frac{1}{n \ln(n)} \right)$ is a monotone decreasing sequence of positive terms. Therefore, by the Cauchy Condensation Test, $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ is convergent if and only if

$\sum_{n=1}^{\infty} \frac{2^n}{2^n \ln(2^n)}$ is convergent. But $\sum_{n=1}^{\infty} \frac{1}{n\ln(2)}$ is divergent by Comparison Test since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Hence, $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ is divergent.

13. 6 Examples of the use of the tests.

1. The series $\sum_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdot \cdots \cdot(3n-2)}{3\cdot 6\cdot 9\cdot \cdots \cdot(3n)}$ is divergent.

   Let $a_n = \frac{1\cdot 4\cdot 7\cdot \cdots \cdot(3n-2)}{3\cdot 6\cdot 9\cdot \cdots \cdot(3n)}$ for integer $n \geq 1$.

   Then for integer $n \geq 1$,
$$\frac{a_n}{a_{n+1}} = \frac{3n+3}{3n+1} = 1+\frac{2}{3n+1} = 1+\frac{2}{3n}\cdot\frac{3n}{3n+1} = 1+\frac{2}{3n}\cdot\left(1-\frac{1}{3n+1}\right)$$
$$= 1+\frac{2/3}{n}-\frac{2}{3n(3n+1)} = 1+\frac{2/3}{n}-\frac{1}{n^2}\frac{2n}{(3n+1)}.$$
   Since $\left|-\frac{2n}{(3n+1)}\right| \leq \frac{2}{3}$, the sequence $\left(A_n = -\frac{2n}{(3n+1)}\right)$is bounded and so because in the notation of Theorem 11, $A=\frac{2}{3}<1$, by Gauss Test (Theorem 11), $\sum_{n=1}^{\infty} a_n$ is divergent.

2. The series $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}{2^n(n+1)!}$ is convergent.

   For integer $n \geq 1$ let $a_n = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}{2^n(n+1)!}$. Then for $n \geq 1$,
$$\frac{a_n}{a_{n+1}} = \frac{2(n+2)}{(2n+1)} = 1+\frac{3}{2n+1}.$$
   Therefore, for integer $n \geq 1$,
$$\left(\frac{a_n}{a_{n+1}}-1\right)n = \frac{3n}{2n+1}.$$
   Thus $\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}}-1\right)n = \lim_{n\to\infty}\frac{3n}{2n+1} = \frac{3}{2} > 1.$

   Therefore, by Raabe's Test (Theorem 3 (i)), $\sum_{n=1}^{\infty} a_n$ is convergent.

3. The series $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot(2n)}\frac{4n+1}{2n+2}$ is divergent.

   Here $a_n = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot(2n)}\frac{4n+1}{2n+2}$ for integer $n \geq 1$. Then for $n \geq 1$,
$$\frac{a_n}{a_{n+1}} = \frac{(2n+4)(4n+1)}{(2n+1)(4n+5)} = \left(1+\frac{3}{2n+1}\right)\left(1-\frac{4}{4n+5}\right).$$
$$= 1+\frac{3}{2n+1}-\frac{4}{4n+5}-\frac{12}{(2n+1)(4n+5)}$$
$$= 1+\frac{3}{2n}\frac{2n}{2n+1}-\frac{1}{n}\frac{4n}{4n+5}-\frac{12}{(2n+1)(4n+5)}$$
$$= 1+\frac{3}{2n}\left(1-\frac{1}{2n+1}\right)-\frac{1}{n}\left(1-\frac{5}{4n+5}\right)-\frac{12}{(2n+1)(4n+5)}$$
$$= 1+\frac{1}{2n}-\frac{3}{2n(2n+1)}+\frac{5}{n(4n+5)}-\frac{12}{(2n+1)(4n+5)}$$

$$= 1 + \frac{1}{2n} + \frac{1}{n^2}\left\{-\frac{3n}{2(2n+1)} + \frac{5n}{4n+5} - \frac{12n^2}{(2n+1)(4n+5)}\right\}.$$

Now $|A_n| = \left|-\frac{3n}{2(2n+1)} + \frac{5n}{4n+5} - \frac{12n^2}{(2n+1)(4n+5)}\right| \le \frac{3}{4} + \frac{5}{4} + 2 = 4$ for all integer $n \ge 1$. Therefore, by Gauss Test (Theorem 11), ,$\sum_{n=1}^{\infty} a_n$ is divergent.

4. The series ,$\sum_{n=1}^{\infty} a_n$, where $a_n = \left(\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}\right)^k$, diverges for $k \le 2$, converges for $k > 2$.

   For integer $n \ge 1$, $\frac{a_n}{a_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^k = \left(1 + \frac{1}{2n+1}\right)^k$.

   Therefore,

$$\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}} - 1\right)n = \lim_{n\to\infty}\frac{\left(1+\frac{1}{2n+1}\right)^k - 1}{\frac{1}{n}} = \lim_{n\to\infty}\frac{k\left(1+\frac{1}{2n+1}\right)^{k-1}\left(-\frac{2}{(2n+1)^2}\right)}{-\frac{1}{n^2}}$$

by L' Hôpital's Rule,

$$= \lim_{n\to\infty} k\left(1 + \frac{1}{2n+1}\right)^{k-1}\frac{2n^2}{(2n+1)^2} = \frac{k}{2}.$$

   Therefore, by Raabe's Test (Theorem 3), $\sum_{n=1}^{\infty} a_n$ converges if $\frac{k}{2} > 1$, i.e., $k > 2$ and $\sum_{n=1}^{\infty} a_n$ diverges if $\frac{k}{2} < 1$, i.e., $k < 2$. If $k = 2$, for integer $n \ge 1$,

$$\frac{a_n}{a_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^k = \left(1 + \frac{1}{2n+1}\right)^2 = 1 + \frac{2}{2n+1} + \frac{1}{(2n+1)^2}$$
$$= 1 + \frac{1}{n}\left(1 - \frac{1}{2n+1}\right) + \frac{1}{(2n+1)^2}$$
$$= 1 + \frac{1}{n} - \frac{1}{n(2n+1)} + \frac{1}{(2n+1)^2}$$
$$= 1 + \frac{1}{n} + \frac{1}{n^2}\left\{-\frac{n}{(2n+1)} + \frac{n^2}{(2n+1)^2}\right\}.$$

   Since $|A_n| = \left|-\frac{n}{(2n+1)} + \frac{n^2}{(2n+1)^2}\right| \le \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, the sequence

$$\left(A_n = -\frac{n}{(2n+1)} + \frac{n^2}{(2n+1)^2}\right)$$

   is bounded and we have for integer $n \ge 1$,

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{A_n}{n^2}.$$

   Therefore, by the Gauss Test (Theorem 11), $\sum_{n=1}^{\infty} a_n$ is divergent.

5. The series ,$\sum_{n=1}^{\infty} a_n$, where $a_n = (1 + \frac{1}{2n})^2 - (1 + \frac{1}{2n+1})^2$ is convergent.

   Now $a_n = \left(1 + \frac{1}{2n} + 1 + \frac{1}{2n+1}\right)\left(\frac{1}{2n} - \frac{1}{2n+1}\right) = \frac{8n(n+1)+1}{4n^2(2n+1)^2}$ for integer $n \ge 1$.

   Therefore, for integer $n \ge 1$,

$$\frac{a_n}{a_{n+1}} = \left(\frac{8n(n+1)+1}{n^2(2n+1)^2}\right)\left(\frac{(n+1)^2(2n+3)^2}{8(n+1)(n+2)+1}\right)$$
$$= \left(\frac{8n(n+1)+1}{8(n+1)(n+2)+1}\right)\left(\frac{(n+1)(2n+3)}{n(2n+1)}\right)^2$$
$$= \left(1 - \frac{16(n+1)}{8(n+1)(n+2)+1}\right)\left(1 + \frac{4n+3}{n(2n+1)}\right)^2$$

$$= 1 + \frac{2(4n+3)}{n(2n+1)} + \left(\frac{4n+3}{n(2n+1)}\right)^2 - \frac{16(n+1)}{8(n+1)(n+2)+1}\left(1 + \frac{4n+3}{n(2n+1)}\right)^2.$$

It follows then that for integer $n \geq 1$,

$$\left(\frac{a_n}{a_{n+1}} - 1\right)n = \frac{2(4n+3)}{(2n+1)} + \frac{1}{n}\left(\frac{4n+3}{(2n+1)}\right)^2 - \frac{16(n+1)n}{8(n+1)(n+2)+1}\left(1 + \frac{4n+3}{n(2n+1)}\right)^2.$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}} - 1\right)n$$

$$= \lim_{n\to\infty}\frac{2(4n+3)}{(2n+1)} + \lim_{n\to\infty}\frac{1}{n}\left(\frac{4n+3}{(2n+1)}\right)^2 - \lim_{n\to\infty}\frac{16(n+1)n}{8(n+1)(n+2)+1}\left(1 + \frac{4n+3}{n(2n+1)}\right)^2$$

$= 4 + 0 - 2 = 2 > 1.$

Hence, by Raabe's Test (Theorem 3), $\sum_{n=1}^{\infty} a_n$ is convergent.

6. It is well known that the binomial series, $\sum_{n=0}^{\infty} a_n x^n$, where $a_n = \binom{\beta}{n}$, is absolutely convergent for $|x| < 1$ and any real number β. We shall establish its convergence at the end points $x = \pm 1$.

First of all note that for β = 0, the series $\sum_{n=0}^{\infty} a_n x^n$ is convergent for any x since $\binom{\beta}{n} = 0$ for $n > 0$.

(i) When β > 0, the series is absolutely convergent at $x = \pm 1$.
(ii) When β < 0 and $x = -1$, the series is divergent.
(iii) When β ≤ −1 and $x = 1$, the series is divergent.
(iv) When −1 < β < 0 and $x = 1$, the series is conditionally convergent.

Note that for integer $n > 0$,

$$\frac{a_n}{a_{n+1}} = \frac{\beta(\beta-1)\cdot\cdots\cdot(\beta-(n-1))}{1\cdot 2\cdot\cdots\cdot n}\frac{1\cdot 2\cdot\cdots\cdot(n+1)}{\beta(\beta-1)\cdot\cdots\cdot(\beta-n)}$$
$$= \frac{n+1}{\beta-n}$$

Therefore, $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{\beta-n}\right| = 1$. Hence the radius of convergence is 1.

For integer $n > \beta$,

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+1}{n-\beta} = 1 + \frac{1+\beta}{n-\beta} = 1 + \frac{1+\beta}{n}\left(1 + \frac{\beta}{n-\beta}\right)$$
$$= 1 + \frac{1+\beta}{n} + \frac{\beta(1+\beta)}{n(n-\beta)} = 1 + \frac{1+\beta}{n} + \frac{1}{n^2}\left(\frac{n\beta(1+\beta)}{(n-\beta)}\right)$$

In the notation of Theorem 11, $A = 1 + \beta$ and $A_n = \dfrac{n\beta(1+\beta)}{(n-\beta)}$ for integer $n > \beta$.

Now the sequence (A_n) is bounded since for $n > \beta$,

$$|A_n| = \left|\frac{\beta(1+\beta)}{(1-\beta/n)}\right| \leq \begin{cases} \dfrac{|\beta(1+\beta)|}{(1-\beta/([\beta]+1))} & \text{if } \beta > 0 \\ |\beta(1+\beta)| & \text{if } \beta < 0 \end{cases}.$$

Therefore, by Gauss Test (Theorem 11), the series is absolutely convergent when $x = \pm 1$ and $A = 1 + \beta > 1$, i.e., β > 0. This proves part (i).

By Gauss Test (Theorem 11), the series $\sum_{n=0}^{\infty}|a_n|$ is divergent when 1+β < 1, 1+β < 1, i.e., β < 0.
When β < 0 and x = –1, $a_n x^n = a_n(-1)^n = |a_n|$ for integer $n \geq 0$. Therefore, the series

$$\sum_{n=0}^{\infty} a_n(-1)^n = \sum_{n=0}^{\infty}|a_n|$$

is divergent. This proves part (ii).
When β ≤ –1 and x = 1,

$$|a_n| = \left|\frac{\beta(\beta-1)\cdot\cdots\cdot(\beta-(n-1))}{1\cdot 2\cdot\cdots\cdot n}\right| = \left|\frac{|\beta|(|\beta|+1)\cdot\cdots\cdot(|\beta|+(n-1))}{1\cdot 2\cdot\cdots\cdot n}\right| \geq 1,$$

and so $a_n \nrightarrow 0$. Therefore, when β ≤ –1 and x = 1, the series $\sum_{n=0}^{\infty} a_n x^n$ diverges. This proves part (iii).
(iv) When –1 < β < 0, 1 > – β > 0 so that for any integer k > 0,

$$1 + k > -\beta + k > k \quad \text{-------------------------------------} \quad (1)$$

Now,

$$a_n = \frac{\beta(\beta-1)\cdot\cdots\cdot(\beta-(n-1))}{1\cdot 2\cdot\cdots\cdot n} = (-1)^n\frac{(-\beta)(-\beta+1)\cdot\cdots\cdot(-\beta+(n-1))}{1\cdot 2\cdot\cdots\cdot n}.$$

Let $b_n = \frac{(-\beta)(-\beta+1)\cdot\cdots\cdot(-\beta+(n-1))}{1\cdot 2\cdot\cdots\cdot n}$ for integer $n \geq 1$ and $b_0 = 1$. Then by (1), $0 < b_n < 1$. Thus by (1) for integer $n \geq 1$,

$$\frac{b_{n+1}}{b_n} = \frac{-\beta+n}{n+1} < 1.$$

Hence the sequence (b_n) is a positive decreasing sequence.
Now we write for integer n > 1,

$$b_n = \frac{b_n}{b_{n-1}}\frac{b_{n-1}}{b_{n-2}}\cdot\cdots\cdot\frac{b_2}{b_1}b_1$$

Applying the logarithmic function we obtain for integer $n \geq 1$,

$$\ln(b_n) = \sum_{k=1}^{n}\ln\left(\frac{b_k}{b_{k-1}}\right) \leq \sum_{k=1}^{n}\left(\frac{b_k}{b_{k-1}}-1\right) = \sum_{k=1}^{n}\left(\frac{-\beta+k-1}{k}-1\right) = (-\beta-1)\sum_{k=1}^{n}\frac{1}{k}.$$

(Here we have used the fact that for 0 < x < 1, ln(x) < x–1.)
Therefore, since (–β –1) < 0 and $\sum_{k=1}^{\infty}\frac{1}{k} = +\infty$, $\lim_{n\to\infty}\ln(b_n) = -\infty$ and consequently, $b_n \to 0$.
It now follows by the Leibnitz's Alternating Series Test (Theorem 20 Chapter 6), that when –1 < β < 0 and x =1, $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty}(-1)^n b_n$ is convergent. Since we already knew that $\sum_{n=0}^{\infty}|a_n|$ diverges for β < 0, the convergence is conditional.

Exercises 19.

1. Test $\sum a_n$ for convergence where a_n is given as follows.
(a) $\frac{1}{n^p+\alpha}$, $n^p + \alpha \neq 0$, (b) $\frac{1}{2^n+x}$, $2^n + x \neq 0$, (c) $e^{-n^2 x}$,
(d) $\frac{1}{\sqrt{n+\sqrt{n+1}}}$, (e) $\sqrt{\frac{n}{n^4+1}}$, (f) $\frac{\sqrt{n^2+n+1}-\sqrt{n^2-n+1}}{n}$,

(g) $\frac{1}{(\ln(n))^{\alpha}}, n \geq 2$, (h) $\frac{n^{\alpha}}{n!}$, (i) $\frac{(n!)^2}{(2n)!}x^{2n}$, (j) $\frac{1}{n^2+a^2}$,

(k) $\left(\frac{n}{2n+1}\right)^{n^3}$, (l) $\frac{1}{n(\ln(n))^2}$

2. Determine the convergence of the series.

(a) $\sum_{n=1}^{\infty} \frac{n!}{x(x+1)\ldots(x+n-1)}, x>0$, (b) $\sum_{n=1}^{\infty}\left\{\frac{(2n)!}{2^{2n}(n!)^2}\right\}^{\alpha}$,

(c) $\sum_{n=1}^{\infty}\sqrt{\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{\beta(\beta+1)\ldots(\beta+n-1)}}, \alpha, \beta > 0$, (d) $\sum_{n=1}^{\infty}\frac{x(x+1)\ldots(x+n-1)}{n^n}$,

(e) $\sum_{n=1}^{\infty}\left(n\ln\left(\frac{3n+2}{3n-2}\right)-1\right)$, (f) $\sum_{n=1}^{\infty}\left\{\sin\left(\frac{x}{n}\right)\right\}^{\alpha}, x>0$,

(g) $\sum_{n=2}^{\infty}\frac{1}{(\ln(n))^{3n}}$, (h) $\sum_{n=2}^{\infty}\frac{1}{(\ln(n))^{\ln(n)}}$,

(i) $\sum_{n=3}^{\infty}\frac{1}{(\ln(\ln(n)))^{\ln(n)}}$, (j) $\sum_{n=3}^{\infty}\frac{1}{(\ln(n))^{\ln(\ln(n))}}$,

(k) $\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\frac{1}{n^{\alpha}}$, (l) $\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^2}$,

(m) $\sum_{n=1}^{\infty}\frac{\sqrt{n}\,\ln(n)}{n^2+2n+3}$.

3. Test $\sum a_n$ for absolute convergence or conditional convergence where a_n is given as follows.

(a) $(-1)^n\frac{(n+1)}{n^3\ln(n)}, n\geq 2$, (b) $\frac{(-3)^{n+1}}{n(n!)}$, (c) $\frac{(-1)^n}{n^{1+1/n}}$, (d) $(-1)^n\ln(1+\frac{2}{n})$

(e) $\frac{(-1)^n}{\sqrt{n+\frac{1}{n}}}$, (f) $\frac{(-3)^{n+1}}{n(n!)}$ (g) $\sin\left((n+\frac{1}{n})\pi\right)$, (h) $(-1)^{n-1}(\ln(1+\frac{1}{n}))^n$.

4. Determine the values of x for which the following series is (i) absolutely convergent , (ii) convergent.

(a) $\sum_{n=1}^{\infty}\frac{n!}{(2n)!}x^n$, (b) $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}x^n$, (c) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\ln\left(\frac{2n+1}{n}\right)x^n$,

(d) $\sum_{n=1}^{\infty}\frac{x^n}{n^x}$, (e) $\sum_{n=1}^{\infty}(\ln(x))^n\ln\left(\frac{n+1}{n}\right)$, (f) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{(nx)^n}$.

5. Determine the region of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{3n+4}{n(n+1)(n+2)}x^n.$$

Find the sum when $x = 1$.

6. Determine the values of x for which the following series is convergent.

(a) $\sum_{n=1}^{\infty}\frac{\cos(nx)}{\sqrt{n+1}}$, (b) $\sum_{n=1}^{\infty}\frac{\sin(nx)}{\ln(n+1)}$, (c) $\sum_{n=1}^{\infty}\cos(nx)\sin\left(\frac{x}{\sqrt{n}}\right)$.

7. Prove that the series

$$1+\frac{1}{3^{\beta}}-\frac{1}{2^{\beta}}+\frac{1}{5^{\beta}}+\frac{1}{7^{\beta}}-\frac{1}{4^{\beta}}+\cdots+\frac{1}{(4n-3)^{\beta}}+\frac{1}{(4n-1)^{\beta}}-\frac{1}{(2n)^{\beta}}+\cdots$$

is properly divergent when $\beta < 1$ and convergent when $\beta \geq 1$. Show that when $\beta = 1$, the sum of the series is $\ln(2\sqrt{2})$.

8. Prove that the series

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$$

is divergent whereas the series

$$1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \cdots$$

is convergent and converges to $\ln(3)$.