Chapter 12 Arithmetic of Power Series

Introduction. In Chapter 7, we have dealt with the representability of a function by a power series. A natural question arises as to the representability of the sums, products and quotients of functions by power series in terms of the power series of the functions and whether we can formally add, multiply or divide the power series representing the functions to give a power series for the sum, product and quotient.

12.1 Sums of Power Series

The situation of representing sum of functions by power series is quite simple.

Theorem 1. Suppose the function f is represented by $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence r_1 and g is represented by $\sum_{n=0}^{\infty} b_n x^n$ with radius of convergence r_2. Then the sum $f + g$ is represented by the power series $\sum_{n=0}^{\infty} (a_n + b_n) x^n$ with radius of convergence $r \geq \min(r_1, r_2)$.

Proof. If $\min(r_1, r_2) = 0$, then we have nothing to prove. Suppose now $0 < \min(r_1, r_2)$. Let c be a real number such that $0 < c < \min(r_1, r_2)$. Then the n-th partial sum $s_n(x) = \sum_{k=0}^{n} a_k x^k$ converges absolutely and uniformly to f on $[-c, c]$ by Theorem 4 and Remark 16 of Chapter 7 since $c < r_1$. Similarly we deduce that the n-th partial sum $t_n(x) = \sum_{k=0}^{n} b_k x^k$ converges absolutely and uniformly to g on $[-c, c]$, since $c < r_2$. This means given any $\varepsilon > 0$, there exists a positive integer N_1 such that for all integer $n \geq N_1$ and for all x in $[-c, c]$,
$$|s_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \text{-------------------------------- (1)}$$
and there exists a positive integer N_2 such that for all integer $n \geq N_2$ and for all x in $[-c, c]$,
$$|t_n(x) - g(x)| < \frac{\varepsilon}{2} \quad \text{-------------------------------- (2).}$$
Let $N = \max(N_1, N_2)$. It follows by (1) and (2) that for all integer $n \geq N$ and for all x in $[-c, c]$,
$$|s_n(x) + t_n(x) - (f(x) + g(x))| \leq |s_n(x) - f(x)| + |t_n(x) - gf(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
This means that $s_n(x) + t_n(x) = \sum_{k=0}^{n} (a_k + b_k) x^k$ converges uniformly to $f + g$ on $[-c, c]$.

Take a real number c' such that $0 < c < c' < \min(r_1, r_2)$. Then by Theorem 4 of Chapter 7, $s_n(c') + t_n(c') = \sum_{k=0}^{n} (a_k + b_k)(c')^k$ is convergent. It follows by Proposition 5 Chapter 7 that $s_n(x) + t_n(x) = \sum_{k=0}^{n} (a_k + b_k) x^k$ converges absolutely to $f + g$ on $[-c, c]$. (We may also deduce this directly by noting that for $|x| < c$, $\sum_{n=0}^{\infty} |a_n x^n|$ and $\sum_{n=0}^{\infty} |b_n x^n|$ are convergent and since for each integer $n \geq 0$, $|(a_n + b_n) x^n| \leq |a_n x^n| + |b_n x^n|$, by the Comparison Test, $\sum_{n=0}^{\infty} |(a_n + b_n) x^n|$ is convergent for $|x| \leq c$.) Therefore, $\sum_{n=0}^{\infty} (a_n + b_n) x^n$ converges for any x such that $|x| \leq c$ and for any c such that $0 < c < \min(r_1, r_2)$.

Therefore, if r is the radius of convergence of the series $\sum_{n=0}^{\infty}(a_n + b_n)x^n$, $r > c$ for any $0 < c < \min(r_1, r_2)$. Hence, $r \geq \min(r_1, r_2)$.

Remark. Theorem 1 is valid when any one of r_1 or r_2 is infinite and when both r_1 and r_2 are infinite and in this case r is infinity.

Example 2. We know from Example 21 Chapter 8 that
$$\ln(1 + x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \quad \text{for } -1 < x \leq 1.$$
and that by Theorem 19 Chapter 9,
$$\frac{1}{\sqrt{1-x}} = 1 + \sum_{n=1}^{\infty}\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}x^n \quad \text{for } -1 < x < 1.$$
Therefore, the power series $1 + \sum_{n=1}^{\infty}\left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} + \frac{(-1)^{n+1}}{n}\right)x^n$ converges to $\frac{1}{\sqrt{1-x}} + \ln(1+x)$ for $|x| < 1$. Now since $1 + \sum_{n=1}^{\infty}\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$ is divergent (by Comparison Test) and $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is convergent (by the Leibnitz's Alternating Series Test), the sum $1 + \sum_{n=1}^{\infty}\left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} + \frac{(-1)^{n+1}}{n}\right)$ is divergent. Similarly, because $1 + \sum_{n=1}^{\infty}\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}(-1)^n$ is convergent and $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n}$ is divergent, the sum $1 + \sum_{n=1}^{\infty}\left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} + \frac{(-1)^{n+1}}{n}\right)(-1)^n$ is divergent. Therefore, the radius of convergence of the series $1 + \sum_{n=1}^{\infty}\left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} + \frac{(-1)^{n+1}}{n}\right)x^n$ is less than or equal to 1. But by Theorem 1, its radius of convergence is exactly 1. In particular, it converges for all x such that $-1 < x < 1$ and diverges elsewhere.

Given two series, we can form two kinds of products, the first is the termwise product and the second is the Cauchy product which is a natural product for consideration of product of two functions represented by power series.

12.2 Termwise Product

Definition 3. If $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$, then the *termwise product* of these two series is defined to be $\sum_{n=0}^{\infty}a_n b_n$.

Theorem 4. If $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$ are absolutely convergent, then $\sum_{n=0}^{\infty}a_n b_n$ is also absolutely convergent.

Proof. Since $\sum_{n=0}^{\infty}b_n$ is convergent, $|b_n| \to 0$ as $n \to \infty$. Therefore, there exists a positive integer N such that $n \geq N \Rightarrow |b_n| < 1$. Therefore, for all integer n such that $n \geq N$, we have.

$$|a_n \, b_n \,| = |a_n| \, |b_n| \le | \, a_n \, | \qquad\qquad \text{------------------------ (1)}$$

Since $\sum_{n=0}^{\infty} |a_n|$ is convergent, $\sum_{n=0}^{\infty} |a_{N+n}|$ is also convergent. Therefore, by inequality (1) and the Comparison Test,

$$\sum_{n=0}^{\infty} |a_{N+n} b_{N+n}|$$

is convergent. Therefore, $\sum_{n=0}^{\infty} |a_n b_n|$ is convergent and so $\sum_{n=0}^{\infty} a_n b_n$ is absolutely convergent.

An easy consequence of Theorem 4 is the following:

If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=0}^{\infty} a_n^2$ is absolutely convergent.

The converse of this statement is evidently false as $\sum_{n=0}^{\infty} \dfrac{1}{n^2}$ is convergent but $\sum_{n=0}^{\infty} \dfrac{1}{n}$ is divergent. It is of course not true that if $\sum_{n=0}^{\infty} a_n$ is convergent, then $\sum_{n=0}^{\infty} a_n^2$ is absolutely convergent. For instance $\sum_{n=0}^{\infty} (-1)^n \dfrac{1}{\sqrt{n}}$ is convergent but $\sum_{n=0}^{\infty} \left((-1)^n \dfrac{1}{\sqrt{n}} \right)^2$ is divergent.

Definition 5. If $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are power series then the *termwise product* of these two power series is defined to be $\sum_{n=0}^{\infty} a_n b_n x^n$.

Theorem 6. If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R_1 > 0$ and $\sum_{n=0}^{\infty} b_n x^n$ has radius of convergence $R_2 > 0$, then the termwise product $\sum_{n=0}^{\infty} a_n b_n x^n$ has radius of convergence $\ge R_1 R_2$.

Proof. Take any real number c such that $0 < c < R_1 R_2$. We shall show that $\sum_{n=0}^{\infty} a_n b_n c^n$ is convergent. Now $c < R_1 R_2 \Rightarrow \dfrac{c}{R_1} < R_2$ and $\dfrac{c}{R_2} < R_1$. Therefore, $x_1 = \frac{1}{2}(R_1 + \frac{c}{R_2}) < R_1$ and $x_2 = \frac{1}{2}(R_2 + \frac{c}{R1}) < R_2$. Hence $\sum_{n=0}^{\infty} a_n x_1^n$ and $\sum_{n=0}^{\infty} b_n x_2^n$ are absolutely convergent. Therefore, by Theorem 4, the termwise product $\sum_{n=0}^{\infty} a_n x_1^n b_n x_2^n = \sum_{n=0}^{\infty} a_n b_n (x_1 x_2)^n$ is absolutely convergent. Observe that

$$x_1 x_2 = \tfrac{1}{4}(R_1 R_2 + 2c + \tfrac{c^2}{R_1 R_2}) = \tfrac{c}{2} + \tfrac{1}{4}(R_1 R_2 + \tfrac{c^2}{R_1 R_2}) > \tfrac{c}{2} + \tfrac{1}{2}(\sqrt{R_1 R_2} \cdot \sqrt{\tfrac{c^2}{R_1 R_2}}) = c$$

Hence by Proposition 5 Chapter 7, $\sum_{n=0}^{\infty} a_n b_n c^n$ is absolutely convergent. Thus, by Proposition 5 Chapter 7 $\sum_{n=0}^{\infty} a_n b_n x^n$ is absolutely convergent for all x such that $|x| \le c$. Since c is any real number such that $0 < c < R_1 R_2$, this means that $\sum_{n=0}^{\infty} a_n b_n x^n$ is

absolutely convergent for all x such that $|x| < R_1R_2$. Therefore the radius of convergence of $\sum\limits_{n=0}^{\infty} a_n b_n x^n$ is greater than or equal to R_1R_2.

Example 7. If $\sum\limits_{n=0}^{\infty} a_n x^n$ has radius of convergence $R_1 > 0$ and $\sum\limits_{n=0}^{\infty} b_n x^n$ has radius of convergence $R_2 > 0$, then the termwise product $\sum\limits_{n=0}^{\infty} a_n b_n x^n$ may have radius of convergence exactly equal to R_1R_2. Take $\sum\limits_{n=0}^{\infty} a_n x^n = \sum\limits_{n=0}^{\infty} \frac{1}{2^n} x^n$ and $\sum\limits_{n=0}^{\infty} b_n x^n = \sum\limits_{n=0}^{\infty} \frac{1}{3^n} x^n$. Then $R_1 = 2$ and $R_2 = 3$. The termwise product $\sum\limits_{n=0}^{\infty} a_n b_n x^n = \frac{1}{6^n} x^n$. By the Ratio Test (Theorem 18 Chapter 7), since $\lim\limits_{n\to\infty} \frac{a_{n+1}b_{n+1}}{a_n b_n} = \frac{1}{6}$, the radius of convergence is 6, which is precisely equal to R_1R_2.

Example 8. If $\sum\limits_{n=0}^{\infty} a_n x^n$ has radius of convergence $R_1 > 0$ and $\sum\limits_{n=0}^{\infty} b_n x^n$ has radius of convergence $R_2 > 0$, a termwise product may have its radius of convergence strictly greater than R_1R_2.

Let $\sum\limits_{n=0}^{\infty} a_n x^n = \sum\limits_{n=0}^{\infty}\left(\left(\frac{x}{3}\right)^{2n+1} + \left(\frac{x}{5}\right)^{2n} \right)$ and $\sum\limits_{n=0}^{\infty} b_n x^n = \sum\limits_{n=0}^{\infty}\left(\left(\frac{x}{5}\right)^{2n+1} + \left(\frac{x}{3}\right)^{2n} \right)$. That is,

$$a_n = \begin{cases} \left(\frac{1}{5}\right)^n & \text{if } n \text{ is even and } n \geq 0 \\ \left(\frac{1}{3}\right)^n & \text{if } n \text{ is odd positive integer} \end{cases}$$

and $b_n = \begin{cases} \left(\frac{1}{3}\right)^n & \text{if } n \text{ is even and } n \geq 0 \\ \left(\frac{1}{5}\right)^n & \text{if } n \text{ is odd positive integer} \end{cases}$.

Then $\limsup\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{3}$ and $\limsup\limits_{n\to\infty} |b_n|^{\frac{1}{n}} = \frac{1}{3}$. Therefore, by the Cauchy-Hadamard Formula (Theorem 19 Chapter 7), the radius of convergence R_1 of $\sum\limits_{n=0}^{\infty} a_n x^n$ and the radius of convergence R_2 of $\sum\limits_{n=0}^{\infty} b_n x^n$ are the same and equals 3. But the termwise product $\sum\limits_{n=0}^{\infty} a_n b_n x^n = \frac{1}{(15)^n} x^n$ whose radius of convergence by the Cauchy Hadamard Formula is 15 and is strictly greater than $R_1R_2 = 9$.

Remark. Note that if $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum\limits_{n=0}^{\infty} b_n x^n$, then the termwise product $\sum\limits_{n=0}^{\infty} a_n b_n x^n$ is not equal to $f(x)\, g(x)$ in general. Take the two series in Example 7. $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n = \sum\limits_{n=0}^{\infty} \frac{1}{2^n} x^n$ and $g(x) = \sum\limits_{n=0}^{\infty} b_n x^n = \sum\limits_{n=0}^{\infty} \frac{1}{3^n} x^n$. Then $f(1) = 2$ and $g(1) = \frac{3}{2}$. Therefore, $f(1)\, g(1) = 3$. The termwise product of the two power series is $\sum\limits_{n=0}^{\infty} a_n b_n x^n = \frac{1}{6^n} x^n$ and converges to the value $\frac{6}{5} \neq f(1)g(1)$ when $x = 1$.

© Ng Tze Beng

12. 3 Cauchy Product

Suppose $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum\limits_{n=0}^{\infty} b_n x^n$. We now come to a product of power series of $\sum\limits_{n=0}^{\infty} a_n x^n$ and $\sum\limits_{n=0}^{\infty} b_n x^n$ which when it converges gives the value of the product $f(x)\, g(x)$.

First we give the definition for series of constant terms.

Definition 9. Suppose $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ are two series.   Then we define the *Cauchy product* of these two series to be the series $\sum\limits_{n=0}^{\infty} c_n$, where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Theorem 10. If $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ are absolutely convergent, then the Cauchy product $\sum\limits_{n=0}^{\infty} c_n$ is also absolutely convergent. Moreover, if $\sum\limits_{n=0}^{\infty} a_n = A$, $\sum\limits_{n=0}^{\infty} b_n = B$ and $\sum\limits_{n=0}^{\infty} c_n = C$, then $C = A\,B$.

Proof.   We shall prove Theorem 10 when both a_n and b_n are non-negative for all integer $n \geq 0$. Suppose $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ are convergent. Then the product,

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(a_n \sum_{k=0}^{\infty} b_k\right) = \lim_{n\to\infty}\sum_{k=0}^{n}\left(a_k \lim_{n\to\infty}\sum_{j=0}^{n} b_j\right) = \left(\lim_{n\to\infty}\sum_{k=0}^{n} a_k\right)\left(\lim_{n\to\infty}\sum_{j=0}^{n} b_j\right) \text{ -- (1)}$$

Now for each integer $n \geq 0$, $a_n \sum\limits_{k=0}^{\infty} b_k$ is convergent means $\sum\limits_{k=0}^{\infty} a_n b_k$ is convergent for each integer $n \geq 0$, or equivalently that $a_n \lim\limits_{n\to\infty}\sum\limits_{j=0}^{n} b_j$ exists. Since $\sum\limits_{n=0}^{\infty} a_n$ is convergent $\sum\limits_{k=0}^{\infty}\left(a_k \lim\limits_{n\to\infty}\sum\limits_{j=0}^{n} b_j\right)$ is convergent

and

$$\sum_{k=0}^{\infty}\left(a_k \lim_{n\to\infty}\sum_{j=0}^{n} b_j\right) = \lim_{n\to\infty}\sum_{k=0}^{n}\left(a_k \lim_{n\to\infty}\sum_{j=0}^{n} b_j\right) = \left(\lim_{n\to\infty}\sum_{k=0}^{n} a_k\right)\left(\lim_{n\to\infty}\sum_{j=0}^{n} b_j\right).$$

This explains the identity (1) above.   For each integer $n \geq 0$, let $l_n = \left(\sum\limits_{k=0}^{n} a_k\right)\left(\sum\limits_{k=0}^{n} b_k\right)$.

Thus by (1), l_n is convergent and converges to $\left(\sum\limits_{n=0}^{\infty} a_n\right)\left(\sum\limits_{n=0}^{\infty} b_n\right)$.

Now
$$l_{n+1} = \left(\sum_{k=0}^{n+1} a_k\right)\left(\sum_{k=0}^{n+1} b_k\right) = \left(\sum_{k=0}^{n} a_k\right)\left(\sum_{k=0}^{n} b_k\right) + \left(\sum_{k=0}^{n} a_k\right)b_{n+1} + a_{n+1}\left(\sum_{k=0}^{n+1} b_k\right)$$
$$= l_n + a_{n+1}\left(\sum_{k=0}^{n} b_k\right) + b_{n+1}\left(\sum_{k=0}^{n} a_k\right) \qquad \text{------------------------- (2)}$$

Consider the array

$$a_0b_0| \quad a_0b_1| \quad a_0b_2| \quad a_0b_3| \quad a_0b_4 \cdots\cdots \quad a_0b_k \quad a_0b_{k+1} \cdots$$

$$a_1b_0 \quad a_1b_1| \quad a_1b_2| \quad a_1b_3| \quad a_1b_4 \cdots\cdots \quad a_1b_k \quad a_1b_{k+1} \cdots$$

$$a_2b_0 \quad a_2b_1 \quad a_2b_2| \quad a_2b_3| \quad a_2b_4 \cdots\cdots \quad a_2b_k \quad a_2b_{k+1} \cdots$$

$$a_3b_0 \quad a_3b_1 \quad a_3b_2 \quad a_3b_3| \quad a_3b_4 \cdots\cdots \quad a_3b_k \quad a_3b_{k+1} \cdots$$

$$a_4b_0 \quad a_4b_1 \quad a_4b_2 \quad a_4b_3 \quad a_4b_4 \cdots\cdots \quad a_4b_k \quad a_4b_{k+1} \cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_kb_0 \quad a_kb_1 \quad a_kb_2 \quad a_kb_3 \quad a_kb_4 \cdots\cdots \quad a_kb_k \quad a_kb_{k+1} \cdots$$

$$a_{k+1}b_0 \quad a_{k+1}b_1 \quad a_{k+1}b_2 \quad a_{k+1}b_3 \quad a_{k+1}b_4 \cdots\cdots a_{k+1}b_k \quad a_{k+1}b_{k+1} \cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

We note from (2) that l_{n+1} is obtained from l_n by adding the term,

$$a_{n+1}\left(\sum_{k=0}^{n+1} b_k\right) + b_{n+1}\left(\sum_{k=0}^{n} a_k\right).$$

Note also that l_n is obtained by adding the terms in the $n \times n$ square in the above array. Hence (l_n) is convergent means the series obtained by summing up in succession the terms in the $k \times k$ square by adding the next $(2k+1)$ terms $\left(a_k\left(\sum_{i=0}^{k} b_i\right) + b_k\left(\sum_{i=0}^{k-1} a_i\right)\right)$ is convergent. That is the series, starting with $a_0 b_0$, then add $(a_1b_0 + a_1b_1 + a_0b_1)$, then add $(a_2b_0 + a_2b_1 + a_2b_2 + b_2a_0 + b_2a_1)$ and so on, is convergent. Since all the terms are non-negative, the series obtained from the above by adding only one term at a time is convergent and converges to the same limit. We define this series as below by defining its terms. We let $d_1 = a_0 b_0$, $d_2 = a_1b_0, d_3 = a_1b_1, d_4 = a_0b_1,$
$d_5 = a_2b_0, d_6 = a_2b_1, d_7 = a_2b_2, d_8 = b_2a_0, d_9 = b_2a_1$
and in general,

$$d_{k^2+i} = a_kb_{i-1}, \text{for } i = 1, \cdots, k+1, \quad d_{k^2+i} = b_ka_{i-(k+1)}, \text{for } i = k+1, \cdots, 2k.$$

Hence, for integer $n \geq 1$,

$$l_n = \left(\sum_{k=0}^{n} a_k\right)\left(\sum_{k=0}^{n} b_k\right) = \sum_{k=1}^{(n+1)^2} d_k \quad \text{-------------------------- (3)}$$

Since (l_n) is convergent and since it is non-negative and monotone increasing, it is bounded above. The n-th partial sum for each integer $n \geq 1$

$$s_n = \sum_{k=1}^{n} d_k$$

is also bounded above. This can be deduced as follows.

$$s_n = \sum_{k=1}^{n} d_k \leq \sum_{k=1}^{n^2} d_k = l_{n-1} \leq \lim_{n \to \infty} l_n$$

Since (s_n) is monotone increasing and bounded above, by the Monotone Convergence Theorem (Theorem 15 Chapter 2), (s_n) is convergent. Then any subsequence of (s_n) is convergent and converges to the same limit (see Proposition 19 Chapter 3). By (3), for each integer $n \geq 1$, $s_{n^2} = l_{n-1}$. Therefore,

$$\lim_{n\to\infty} s_{n^2} = \lim_{n\to\infty} l_n = \lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k\right)\left(\sum_{k=0}^{n} b_k\right) = AB$$

It follows that $\lim_{n\to\infty} s_n = AB$. This means $\sum_{n=1}^{\infty} d_n = AB$. Now we examine the series

$\sum_{n=0}^{\infty} c_n$. Recall that $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$. Let $t_n = \sum_{k=0}^{n} c_k$ be the n-th partial sum for

each integer $n \geq 0$.

Then, $t_0 = d_0 = a_0 b_0$ and

$$t_{n+1} = \sum_{k=0}^{n} c_k + c_{n+1} = \sum_{k=0}^{n} c_k + \sum_{k=0}^{n+1} a_k b_{n+1-k} \quad \text{-----------------} (4)$$

Let $\sum_{n=0}^{\infty} g_n$ be the series obtained from $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$ without the bracketing.

Since each $a_i b_j$ corresponds uniquely to some d_k and conversely each d_k corresponds

uniquely to some $a_i b_j$, by (4), $\sum_{n=0}^{\infty} g_n$ is a rearrangement of $\sum_{n=1}^{\infty} d_n$. Therefore, since

$\sum_{n=1}^{\infty} d_n$ is (absolutely) convergent , $\sum_{n=0}^{\infty} g_n = \sum_{n=1}^{\infty} d_n = AB$. (see Tut 10 Question 1 and

solution). Now $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$ is obtained from $\sum_{n=0}^{\infty} g_n$ by group the terms in

brackets without altering the order of the terms. Therefore,

$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) = \sum_{n=0}^{\infty} g_n = AB$. (See Tut 10 Question 2 and solution.) This

proves the theorem for the case when a_n, b_n are non-negative for all integer $n \geq 0$.

In general, if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent, then $\sum_{n=0}^{\infty}|a_n|$ and $\sum_{n=0}^{\infty}|b_n|$ are

convergent. It follows that $\sum_{n=1}^{\infty} d_n$, where d_n is defined as before, is absolutely

convergent, because by the above argument $\sum_{n=1}^{\infty}|d_n|$ is convergent and converges to

$\left(\sum_{n=0}^{\infty}|a_n|\right)\left(\sum_{n=0}^{\infty}|b_n|\right)$. Let $\sum_{n=0}^{\infty} g_n$ be the series obtained from $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$

without the bracketing. Then as noted and observed before, $\sum_{n=0}^{\infty} g_n$ is a rearrangement

of $\sum_{n=1}^{\infty} d_n$. Since $\sum_{n=1}^{\infty} d_n$ is absolutely convergent,

$$\sum_{n=0}^{\infty} g_n = \sum_{n=1}^{\infty} d_n.$$

We have observed above that $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} g_n$. Hence $\sum_{n=0}^{\infty} c_n = \sum_{n=1}^{\infty} d_n$. Now from (3)

$$s_{(n+1)^2} = \sum_{k=1}^{(n+1)^2} d_k = l_n = \left(\sum_{k=0}^{n} a_k\right)\left(\sum_{k=0}^{n} b_k\right).$$

Therefore, $\lim_{n\to\infty} s_{(n+1)^2} = \lim_{n\to\infty} l_n = \lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k\right)\left(\sum_{k=0}^{n} b_k\right) = AB$. Since we already knew

(s_n) is convergent, $\sum_{n=1}^{\infty} d_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{(n+1)^2} = AB$. Hence, $\sum_{n=0}^{\infty} c_n = \sum_{n=1}^{\infty} d_n = AB$.

Note that for each integer $n \geq 0$, $|c_n| \leq \sum_{k=0}^{n}|a_k||b_{n-k}|$. We have already shown that

$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}|a_k||b_{n-k}|\right)$ is convergent. It follows then by the Comparison test that $\sum_{n=0}^{\infty}|c_n|$ is

convergent and so $\sum_{n=0}^{\infty} c_n$ is absolutely convergent.

If $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ are conditionally convergent, it need not follow that the Cauchy product $\sum\limits_{n=0}^{\infty} c_n$ is convergent. See the following example:

Example 11. Let $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ be two series where

$a_0 = a_1 = b_0 = b_1 = 0$ and $a_n = b_n = (-1)^n \dfrac{1}{\ln(n)}$ for integer $n \geq 2$.

By the Alternating Series Test, $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ are convergent. Since $\dfrac{1}{\ln(n)} > \dfrac{1}{n}$ for

integer $n \geq 2$ and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is divergent, by the Comparison Test $\sum\limits_{n=2}^{\infty} \dfrac{1}{\ln(n)}$ is divergent.

Hence $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ are conditionally convergent.

$c_0 = c_1 = c_2 = c_3 = 0$ and for integer $n > 3$,

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = (-1)^n \left\{ \frac{1}{\ln(2)\ln(n-2)} + \frac{1}{\ln(3)\ln(n-3)} + \cdots + \frac{1}{\ln(n-2)\ln(2)} \right\}.$$

Thus, for n even and > 3, $c_n \geq \dfrac{n-3}{(\ln(n-2))^2}$. By application of L'Hôpital's Rule (see

remark after Example 39 Chapter 4), $\dfrac{n-3}{(\ln(n-2))^2} \to \infty$, Therefore, by the

Comparison test, $c_{2n} \to \infty$.

Similarly for n odd and > 3, $c_n \leq -\dfrac{n-3}{(\ln(n-2))^2} \to -\infty$. Hence the series $\sum\limits_{n=0}^{\infty} c_n$ cannot converge.

12.4 Multiplication of Power Series

Theorem 12. Suppose $f_1(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ has radius of convergence R_1 and

$f_2(x) = \sum\limits_{n=0}^{\infty} b_n x^n$ has radius of convergence R_2. Then the product $f(x) = f_1(x) f_2(x)$ is represented by a power series

$$\sum_{n=0}^{\infty} c_n x^n, \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k},$$

in the interval $(-R, R)$, where $R = \min(R_1, R_2)$

Proof. Let x be in the interval $(-R, R)$. Then $x \in (-R_1, R_1)$ and $x \in (-R_2, R_2)$. Therefore, $\sum\limits_{n=0}^{\infty} a_n x^n$ and $\sum\limits_{n=0}^{\infty} b_n x^n$ are absolutely convergent (see Theorem 4 Chapter 7). Then by Theorem 10, the Cauchy product of these two series,

$$\sum_{n=0}^{\infty} d_n, \text{ where } d_n = \sum_{k=0}^{n} (a_k x^k)(b_{n-k} x^{n-k}) = \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n,$$

is absolutely convergent. That is, if $c_n = \sum\limits_{k=0}^{n} a_k b_{n-k}$ for each integer ≥ 0, then $\sum\limits_{n=0}^{\infty} c_n x^n$ is absolutely convergent. Moreover, the Cauchy product,

$$\sum_{n=0}^{\infty} c_n x^n = \left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) = f_1(x) f_2(x) \quad .$$

Hence $\sum\limits_{n=0}^{\infty} c_n x^n$ is absolutely convergent for $|x| < R$.

Example 13. We can express $\dfrac{\ln(1+x)}{1+x}$ as a power series for $|x| < 1$.

First we note that $\dfrac{1}{1+x} = \sum\limits_{n=0}^{\infty}(-1)^n x^n$ and $\ln(1+x) = -\sum\limits_{n=1}^{\infty}(-1)^n \dfrac{x^n}{n}$ for $|x| < 1$. That is, the radius of convergence for both series is 1. Therefore, by Theorem 12, for $|x| < 1$,

$$\frac{\ln(1+x)}{1+x} = -\sum_{n=1}^{\infty} x^n \left(\sum_{k=1}^{n}(-1)^{n-k}\frac{(-1)^k}{k} \right) = -\sum_{n=1}^{\infty}(-1)^n x^n \left(\sum_{k=1}^{n}\frac{1}{k} \right) = \sum_{n=1}^{\infty}(-1)^{n+1} H_n x^n,$$

where $H_n = \sum\limits_{k=1}^{n}\dfrac{1}{k}$.

12. 5. Quotient of Power Series.

We now consider division of power series. Plainly it is sufficient to study the quotient of a power series $1/\sum\limits_{n=0}^{\infty} a_n x^n$ since

$$\frac{\sum\limits_{n=0}^{\infty} b_n x^n}{\sum\limits_{n=0}^{\infty} a_n x^n} = \left(\sum_{n=0}^{\infty} b_n x^n \right) \cdot \left(\frac{1}{\sum\limits_{n=0}^{\infty} a_n x^n} \right),$$

for if we can represent $\dfrac{1}{\sum\limits_{n=0}^{\infty} a_n x^n}$ as a power series we can then obtain the Cauchy

product with $\sum\limits_{n=0}^{\infty} b_n x^n$ by Theorem 12.

Let $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$. Suppose $\sum\limits_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. We consider first the case that $f(x) \neq 0$ at least in a neighbourhood containing 0. Hence the leading coefficient $a_0 \neq 0$. We may assume that $a_0 = 1$. This is seen as follows. We can write

$$\sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{a_n}{a_0} x^n.$$

The series $\sum\limits_{n=0}^{\infty} a_n x^n$ and $\sum\limits_{n=0}^{\infty} \dfrac{a_n}{a_0} x^n$ have the same radius of convergence R while the leading coefficient of $\sum\limits_{n=0}^{\infty} \dfrac{a_n}{a_0} x^n$ is 1. Then

$$\frac{1}{\sum\limits_{n=0}^{\infty} a_n x^n} = \frac{1}{a_0} \cdot \frac{1}{\sum\limits_{n=0}^{\infty} \frac{a_n}{a_0} x^n}.$$

We seek a power series $\sum\limits_{n=0}^{\infty} b_n x^n$ such that $\dfrac{1}{\sum\limits_{n=0}^{\infty} a_n x^n} = \sum\limits_{n=0}^{\infty} b_n x^n$. Hence for such a

solution, we have

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = 1 .$$

Hence by the definition of Cauchy product,

$$\sum_{n=0}^{\infty} c_n x^n = 1$$

9

and so we have the equations
$$c_0 = 1$$
and $$c_n = 0 \text{ for all integer } n \geq 1.$$
That means,
$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1 \end{cases},$$

That is,

$$\left.\begin{array}{l} b_0 = 1 \\ b_1 = -a_1 b_0 \\ b_2 = -(a_1 b_1 + a_2 b_0) \\ \quad\vdots \\ b_n = -\displaystyle\sum_{k=1}^{n} a_k b_{n-k} \end{array}\right\} \text{----------------------------------- (A)}$$

In this way we define a formal inverse $\sum_{n=0}^{\infty} b_n x^n$ to $\sum_{n=0}^{\infty} a_n x^n$.

Theorem 14. Suppose $\sum_{n=0}^{\infty} a_n x^n$ is a power series with $a_0 = 1$ and with a non-zero radius of convergence R. Then the formal inverse of $f(x) = \sum_{n=0}^{\infty} a_n x^n$ as given by (A) is a power series expansion for $\dfrac{1}{f(x)}$ with a non-zero radius of convergence. That is,
$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} b_n x^n,$$
where the b_n's are determined recursively by (A).

Proof. It is sufficient to show that $\sum_{n=0}^{\infty} b_n x^n$ has a non-zero radius of convergence. Note that by assumption, $f(0) = a_0 = 1 \neq 0$. Since the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is positive, by continuity $f(x) \neq 0$ in a neighbourhood of 0. Thus $\dfrac{1}{f(x)}$ is at least defined in a neighbourhood of 0.
We shall give some estimate of b_k's to use the Cauchy Hadamard formula to show that the formal inverse $\sum_{n=0}^{\infty} b_n x^n$ has a positive radius of convergence.

Let r be any real number such that $0 < r < R$. Then $\sum_{n=0}^{\infty} a_n r^n$ is convergent. Therefore, $a_n r^n \to 0$. (See Proposition 10 Chapter 6). Therefore, the sequence ($a_n r^n$) is bounded and so there exists a constant $M \geq 1$ such that
$$| a_n r^n | \leq M \text{ for integer } n \geq 0.$$
Hence, for integer $n \geq 0$,
$$|a_n| \leq \frac{M}{r^n} \text{ ----------------------------------- (1)}$$
Then using (1), we get,
$$|b_0| = 1 \leq M,$$
$$|b_1| = |a_1 b_0| = |a_1| \leq M/r ,$$
$$|b_2| = |(a_1 b_1 + a_2 b_0)| \leq |a_1 b_1| + |a_2 b_0| \leq \frac{M}{r}\frac{M}{r} + \frac{M}{r^2}M = 2\frac{M^2}{r^2}$$
$$|b_3| \leq |a_1 b_2| + |a_2 b_1| + |a_3 b_0| \leq \frac{M}{r} \cdot 2\frac{M^2}{r^2} + \frac{M}{r^2} \cdot \frac{M}{r} + \frac{M}{r^3} \cdot M \leq 2^2\frac{M^3}{r^3}$$

10

We claim that $|b_n| \leq 2^n \dfrac{M^n}{r^n}$ for any integer $n \geq 0$. Evidently from the above it is true for $n = 0,1,2$ and 3. Assuming that the claim is true for $k = 0, 1,2, \ldots, n\text{-}1$, we shall show that it is true for $k = n$.

By (A),

$$|b_n| \leq \sum_{k=1}^{n} |a_k||b_{n-k}| \leq \sum_{k=1}^{n} \frac{M}{r^k} \cdot \frac{2^{n-k}M^{n-k}}{r^{n-k}} \quad \text{by (1) and the induction hypothesis}$$

$$\leq \frac{1}{r^n} \sum_{k=1}^{n} 2^{n-k}M^{n+1-k} \leq \frac{1}{r^n} \sum_{k=1}^{n} 2^{n-k}M^n$$

$$\leq \frac{M^n}{r^n} \sum_{k=1}^{n} 2^{n-k} \leq 2^n \frac{M^n}{r^n}$$

Hence, by mathematical induction $|b_n| \leq 2^n \dfrac{M^n}{r^n}$ for any integer $n \geq 0$.

Therefore, for all integer $n \geq 1$, $|b_n|^{\frac{1}{n}} \leq 2\dfrac{M}{r}$. It follows that $\sup\{|b_n|^{\frac{1}{n}} : n \geq k\} \leq 2\dfrac{M}{r}$ for integer $k \geq 1$. Therefore, $\limsup\limits_{n \to \infty} b_n|^{\frac{1}{n}} \leq 2\dfrac{M}{r}$. By Theorem 19 Chapter 7 (Cauchy Hadamard Formula), the radius of convergence of $\sum\limits_{n=0}^{\infty} b_n x^n$ is given by

$$\frac{1}{\limsup\limits_{n \to \infty} b_n|^{\frac{1}{n}}} \geq \frac{r}{2M} > 0.$$

In practice we can obtain the coefficients b_n's by long division and for region of convergence estimate of M for a particular r need to be sought.

We now come to the case when the leading coefficient $a_0 = 0$ so that $f(0) = 0$. We can find the reciprocal $1/f(x)$ as a power series in a deleted neighbourhood of 0.

Suppose $f(x) = x^p \sum\limits_{n=0}^{\infty} a_n x^n$ and p is an integer ≥ 1 and $a_0 \neq 0$. Suppose $f(x)$ has radius of convergence $R > 0$. Then for $x \neq 0$ and x in $(-R, R)$,

$$\frac{f(x)}{x^p} = \sum_{n=0}^{\infty} a_n x^n.$$

Note that $\sum\limits_{n=0}^{\infty} a_n x^n$ has the same radius of convergence $R > 0$. By Theorem 14, the formal quotient of $\sum\limits_{n=0}^{\infty} a_n x^n$,

$$\frac{1}{\sum\limits_{n=0}^{\infty} a_n x^n} = \sum_{n=0}^{\infty} b_n x^n$$

has radius of convergence $R' > 0$. Therefore, by Theorem 12,

$$\frac{f(x)}{x^p} \sum_{n=0}^{\infty} b_n x^n = 1$$

for $x \neq 0$ and x in $(-R_2, R_2)$, where $R_2 = \min(R, R')$. Then in $(-R_2, R_2) - \{0\}$,

$$\frac{1}{f(x)} = \frac{1}{x^p} \sum_{n=0}^{\infty} b_n x^n = \frac{b_0}{x^p} + \frac{b_1}{x^{p-1}} + \cdots + b_p + \sum_{k=1}^{\infty} b_{p+k} x^k$$

In view of Theorem 14, division of power series $\dfrac{f(x)}{g(x)}$ can be performed as forming the product of $f(x)$ and $1/g(x)$ and is equivalent to formal power series division.

Example 15. Find the first few terms of $\dfrac{\sin(x)}{\cos(x)}$ by power series division. Recall from Chapter 11,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^n}{(2n)!} + \cdots.$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

We perform long division as follows:

$$x + \frac{1}{3}x^2 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \;\Big|\; x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots$$

$$x - \frac{1}{2}x^3 + \frac{1}{4!}x^5 - \frac{1}{6!}x^7 + \frac{1}{8!}x^9 + \cdots$$

$$\frac{1}{3}x^3 - \frac{4}{5!}x^5 + \frac{6}{7!}x^7 - \frac{8}{7!}x^9 + \cdots$$

$$\frac{1}{3}x^3 - \frac{1}{6}x^5 + \frac{6}{4!3}x^7 - \frac{8}{6!3}x^9 + \cdots$$

$$\frac{2}{15}x^5 - \frac{4}{315}x^7 + \frac{1}{2268}x^9 + \cdots$$

$$\frac{2}{15}x^5 - \frac{1}{15}x^7 + \frac{1}{180}x^9 + \cdots$$

$$\frac{17}{315}x^7 - \frac{29}{5670}x^9 + \cdots$$

$$\frac{17}{315}x^7 - \frac{17}{630}x^9 + \cdots$$

$$\frac{62}{2835}x^9 + \cdots$$

Therefore, $\dfrac{\sin(x)}{\cos(x)} = x + \frac{1}{3}x^2 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots$

Remark.

Is the composition of two power series expandable as a power series function? The answer is "yes". But the proof will require analytic function theory. If the power series function $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ satisfies that $f'(x_0) \neq 0,$ then by continuity $f'(x$ $) \neq 0$ for some neighbourhood of x_0 and so by Darboux Theorem (Theorem 47 Chapter 4) in that neighbourhood either $f'(x) > 0$ or $f'(x) < 0$. Therefore, in a small neighbourhood of x_0, $f(x)$ is either strictly increasing or strictly decreasing. Hence $f(x)$ is injective in some interval containing x_0. Therefore, its inverse f^{-1} exists in that

interval. The question we can ask is whether the inverse function is representable as powers of ($y - f(x_0)$). The answer is 'yes' but the proof would require an invariance of domain result.

12.6 Analytic Function

We have seen that not all infinitely differentiable functions can be expandable as a power series, (See Example 13 Chapter 8). For manipulation of function by power series, we need to ascertain if the function to be manipulated has a power series expansion, i.e., if it is analytic. The next theorem gives a necessary and sufficient condition for a function to be analytic at the origin. This gives another criterion for analyticity in addition to Theorem 15 of Chapter 8.

Theorem 16. Suppose the function f is defined in a neighbourhood of the origin. Then f is analytic at the origin if and only if f is infinitely differentiable and that there exists a real number $r > 0$ and a real number $K > 0$ such that

$$|f^{(n)}(x)| \leq \frac{r\,K\,n!}{(r - |x|)^{n+1}}$$

for $-r < x < r$.

Proof. Suppose f is analytic at the origin. Then f has a power series expansion in some interval say $(-R, R)$ containing the origin. That is to say,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

for all $|x| < R$.

Then by Theorem 11 of Chapter 8, we can differentiate $f(x)$ termwise any number of times in the interval $(-R, R)$ and

$$f^{(n)}(x) = \frac{d^n}{dx^n} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k \frac{d^n}{dx^n} x^k$$

$$= \sum_{k=0}^{\infty} a_k \frac{d^n}{dx^n} x^k$$

$$= \sum_{k=n}^{\infty} a_k k(k-1)\cdots(k-(n-1)) x^{k-n}$$

$$= \sum_{k=n}^{\infty} a_k \frac{k!}{(k-n)!} x^{k-n} \qquad \text{-------------------------------- (1)}$$

for all $|x| < R$. Let constant $r > 0$ be such that $0 < r < R$. Then $f(r) = \sum_{n=0}^{\infty} a_n r^n$ is absolutely convergent. Therefore, $\sum_{n=0}^{\infty} |a_n r^n|$ is convergent and so for each integer $n \geq 0$,

$$|a_n r^n| \leq \sum_{k=0}^{\infty} |a_k r^k|. \qquad \text{-------------------------------- (2)}$$

Let $K = \sum_{k=0}^{\infty} |a_k r^k|$. Then from (1) we obtain by the triangle inequality,

$$|f^{(n)}(x)| \leq \sum_{k=n}^{\infty} |a_k x^{k-n}| \frac{k!}{(k-n)!}$$

$$\leq \sum_{k=n}^{\infty} |a_k| \left|\frac{x^{k-n}}{r^{k-n}}\right| r^{k-n} \frac{k!}{(k-n)!}$$

$$\leq \sum_{k=n}^{\infty} |a_k r^k| \frac{k!}{(k-n)!} \cdot \frac{1}{r^n} \left(\frac{|x|}{r}\right)^{k-n}.$$

Therefore, for $|x| < r$,

$$|f^{(n)}(x)| \leq \sum_{k=n}^{\infty} |a_k r^k| \frac{k!}{(k-n)!} \cdot \frac{1}{r^n} \left(\frac{|x|}{r} \right)^{k-n}$$

$$\leq \sum_{k=n}^{\infty} K \frac{k!}{(k-n)!} \cdot \frac{1}{r^n} \left(\frac{|x|}{r} \right)^{k-n} \qquad \text{by (2)}$$

$$\leq \frac{K}{r^n} \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} \left(\frac{|x|}{r} \right)^{k-n} = \frac{K\,n!}{r^n} \sum_{k=n}^{\infty} \frac{k!}{(k-n)!\,n!} \left(\frac{|x|}{r} \right)^{k-n}$$

$$\leq \frac{K\,n!}{r^n} \sum_{k=n}^{\infty} \binom{k}{n} \left(\frac{|x|}{r} \right)^{k-n}$$

$$= \frac{K\,n!}{r^n} \left(1 - \frac{|x|}{r} \right)^{-(n+1)} \qquad \text{----------------------- (3)}$$

since $\left| \frac{x}{r} \right| < 1$ so that $\sum_{k=n}^{\infty} \binom{k}{n} \left(\frac{|x|}{r} \right)^{k-n} = \left(1 - \frac{|x|}{r} \right)^{-(n+1)}$.   We deduce this series

expansion as a special case of the following:

First note that $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$.   Then by Theorem 11 Chapter 8, we can

differentiate the function termwise any number of times in the interval (-1, 1).
Therefore, for any integer $n \geq 0$ and for $|x| < 1$,

$$\frac{d^n}{dx^n} \frac{1}{1-x} = \frac{d^n}{dx^n} \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} \frac{d^n}{dx^n} x^k$$

$$= \sum_{k=n}^{\infty} k(k-1) \cdots (k-(n-1)) x^{k-n}$$

$$= \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} x^{k-n}$$

Hence, $\dfrac{n!}{(1-x)^{n+1}} = \sum_{k=n}^{\infty} \dfrac{k!}{(k-n)!} x^{k-n}.$

It follows that

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!\,n!} x^{k-n} = \sum_{k=n}^{\infty} \binom{k}{n} x^{k-n} \qquad \text{------------------ (4)}$$

for $|x| < 1$ and any integer $n \geq 0$.

It follows from (3) that

$$|f^{(n)}(x)| \leq \frac{K\,n!}{r^n} \left(1 - \frac{|x|}{r} \right)^{-(n+1)} = \frac{K\,n!}{r^n} \left(\frac{r}{r-|x|} \right)^{(n+1)} = \frac{r\,K\,n!}{(r-|x|)^{n+1}}$$

for $|x| < r$ and any integer $n \geq 0$.
Now for the proof of the converse statement. Suppose f is infinitely differentiable
and that there exists real numbers $r > 0$ and $K > 0$ such that for $|x| < r$ and integer $n \geq 0$,

$$|f^{(n)}(x)| \leq \frac{r\,K\,n!}{(r-|x|)^{n+1}}.$$

By Theorem 44 Chapter 4, the Lagrange form of the remainder for the Taylor
expansion for f around the origin is given by

$$R_n(x) = \tfrac{1}{(n+1)!} x^{n+1} f^{(n+1)}(\eta)$$

for some η between 0 and x. Therefore for $|x| < r/2$,

$$|R_n(x)| = \left| \frac{1}{(n+1)!} x^{n+1} f^{(n+1)}(\eta) \right| \text{for some } \eta \text{ between 0 and } x.$$

$$\leq \frac{r\,K\,(n+1)!}{(r-|\eta|)^{n+2}} \frac{1}{(n+1)!} |x^{n+1}| = \frac{r\,K}{(r-|\eta|)^{n+2}} |x^{n+1}|$$

$$\leq \frac{r\,K}{(r-|x|)^{n+2}} |x|^{n+1} \qquad \text{since } |\eta| \leq |x|$$

14

$$= \frac{r\,K}{(r-|x|)}\left(\frac{|x|}{(r-|x|)}\right)^{n+1} \quad \text{----------------------- (5)}$$

Now because $|x| < r/2$, $\frac{|x|}{(r-|x|)} < 1$ and so $\left(\frac{|x|}{(r-|x|)}\right)^{n+1} \to 0$ as $n \to \infty$. Hence it follows from (5) that $|R_n(x)| \to 0$. Therefore, f is analytic on the interval $(-\frac{r}{2}, \frac{r}{2})$. This completes the proof.

Exercises 17.

1. By squaring the left hand side of $\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$ and using the Cauchy product of the right hand side with itself show that $\frac{1}{(1-x)^2} = \sum\limits_{n=0}^{\infty} (n+1)x^n$ for $|x| < 1$.

2. By using power series multiplication (Cauchy product), compute the first few terms in the expansion about $x = 0$ of the following:
   (a) $\cos^2(x)$ (b) $e^x \ln(1+x)$ (c) $(\sin^{-1}(x))^2$ (d) $\sin(x)\sin^{-1}(x)$
   (e) $e^x \cos(x)$ (f) $\sin(x)\cosh(x)$

3. By using division of power series find the first few terms in the expansion about $x = 0$ of the following functions.
   (a) $\frac{\cos(x)}{1+\sin(x)}$ (b) $\frac{x}{e^x - 1}$ (c) $\frac{e^x}{1-x^2}$ (d) $\frac{1}{\cosh(x)}$ (e) $\frac{\cos(x)}{\cosh(x)}$.

4. The *Bernoulli numbers* B_n are defined by $\frac{x}{e^x - 1} = \sum\limits_{n=0}^{\infty} \frac{B_n}{n!}x^n$. Prove that
   $B_{2k+1} = 0$ for $k = 1,2,3, \dots$ and that $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$ and $B_8 = -1/30$.

5. Show that $\frac{1}{e^x + 1} = -\sum\limits_{n=0}^{\infty} \frac{B_n}{n!}(2^n - 1)x^{n-1}$, where the B_n 's are defined in question 4 above.

6. Show by using Cauchy product that
   $$\frac{1}{1-x}\ln(\frac{1}{1-x}) = \sum\limits_{n=1}^{\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)x^n$$
   and state the range for which it is valid.

7. Show that for $|x| < 1$,
   $$2\sum\limits_{n=0}^{\infty} x^n \cos(n\theta) \sum\limits_{n=0}^{\infty} x^n \sin(n\theta) = \sum\limits_{n=0}^{\infty} x^n(n+1)\sin(n\theta) .$$

15