Chapter One. The Real Number System

We shall give a quick introduction to the real number system. It is imperative that we know how the set of real numbers behaves in the way that its completeness and consequences of completeness can tell us. We shall give the development of the natural numbers to the real numbers in a heuristic manner. A detail approach would be far too long and I refer the reader to my snippet "Real Numbers?"

1.1 The Natural Numbers N.

We start with the natural numbers. It is the set of numbers: $\mathbf{N} = \{0, 1,2,3,\dots \}$. We may describe the natural numbers by the following: $\mathbf{N}$ is a set containing a distinguished element 0, called zero, together with a successor function $S: \mathbf{N} \to \mathbf{N}$ into itself, which satisfies the following axioms:

      (S1) S is injective
      (S2) $0 \notin S(\mathbf{N})$
      (S3) If a subset $K \subseteq \mathbf{N}$ contains zero and is mapped into itself by S, then $K = \mathbf{N}$.

We may define $n + 1 = S(n)$. A set A is said to be *inductive*, if $0 \in A$ and whenever $a \in A$, then $a +1 \in A$. Then (S3) says that $\mathbf{N}$ is the smallest inductive set. A model for $\mathbf{N}$ can be taken to be the von Neumann's system: 0 is defined to be the empty set ϕ, 1 is defined to be $\{\phi\}$, 2 is defined to be $\{\phi, \{\phi\}\}$ and, in general, $n +1$ is defined by $n \cup \{n\}$. The successor function is defined by $S(n) = n \cup \{n\}$.

S3 expresses the principle of mathematical induction. It asserts in the more familiar form: if K is a subset of $\mathbf{N}$ such that $0 \in K$ and satisfies that $a \in K$ implies $a +1 \in K$, then $K = \mathbf{N}$. The uniqueness of the natural numbers is a consequence of the *Dedekind Recursion Theorem.* For details, please refer to any text on set theory, for example, Paul Halmos' Naive Set Theory.

Often we need the notion of countability as well as the notion of an infinite set. A set A is said to be *infinite* if there is an injective function $\alpha : A \to A$ such that $\alpha(A) \neq A$ *otherwise* it is *finite*. Therefore, (S2) expresses that $\mathbf{N}$ is infinite. We say a set A is countably infinite if there is a bijective map of $\mathbf{N}$ onto A. A set A is *countable* if it is either finite or countably infinite. Hence $\mathbf{N}$ is countable.

Here is an observation:

There exists an infinite set if and only if there is a set K and a function $S: K \to K$ satisfying the axioms S1 to S3 above.

Proof. (if part) Since axiom S1 says that S is injective and axiom S2 says that $S(K) \neq K$, by definition K is an infinite set.
(only if part). Suppose there exists an infinite set H. Then there exists an injective function $\alpha : H \to H$ such that $\alpha(H) \neq H$. Then take an element a in H but not in $\alpha(H)$. Consider the collection $\mathcal{C}$ of all subsets M of H such that $a \in M$ and $\alpha(M) \subseteq M$. Note that $a \notin \alpha(M)$ since $a \notin \alpha(H)$. Let $K = \cap \{M : M \in \mathcal{C} \}$. Let S be the restriction of α

to *K*.   Then *S* is injective and $S(K) \subseteq M$ for each *M* in $\mathcal{C}$. Therefore, $S(K) \subseteq \cap \{M : M \in \mathcal{C} \} = K$. Plainly $a \notin S(K)$.   Therefore, *K* satisfies axioms S1 and S2. Suppose *G* is a subset of *K* such that $a \in G$ and $S(G) \subseteq G$.   Then $G \in \mathcal{C}$.   Therefore, $K \subseteq G$. Hence $G = K$. So axiom S3 is also satisfied by *K*.   This completes the proof.

Addition and multiplication can be defined on **N** using the successor function *S*. For every fixed natural number *m* the addition can be defined starting from $m + 0 = m$, by the recursion formula $m + S(n) = S(m +n)$.   Similarly, the operation of multiplication by a fixed natural number *m* can be defined starting from $m \cdot 0 = 0$, recursively by $m \cdot (n + 1) = m \cdot n + m$. Of course, all the well known rules for addition and multiplication must now be proved.   We shall omit the details. The ordering $\leq$ can then be defined thus: $n \leq m$ if and only if there exists *t* in **N** such that $n + t = m$. Note that this is also equivalent to the ordering by set inclusion if **N** is taken to be the von Neumann's system.

Below we shall list some useful results concerning countable sets.

Theorem 1 Well ordering property.   Every non empty subset of **N** has a smallest element.

Proof.
We shall show that there are no non-empty subset of **N**, that has no least element.
Suppose $S \subseteq \mathbf{N}$ is a subset of **N** that has no least element. Then consider the complement $T = \mathbf{N} - S$.   Then $0 \notin S$, for if 0 were to be in S, then 0 would be the least element of *S*. Thus $0 \in T$.   We shall show that *T* is inductive. We shall start the argument by taking a subset of *T*, $U = \{n \in \mathbf{N} : \{0,1,2,...,n\} \subseteq T \}$. $U \neq \varnothing$ because $0 \in U$. Suppose $k \in U$, then $\{0,1,...,k\} \subseteq T$. If $k+1$ were in *S*, then $k+1$ would be the smallest element in *S*. Thus $k+1 \in T$ and so $\{0,1,...,k, k+1\} \subseteq T$. Therefore, $k+1 \in U$.   Hence *U* is inductive and so $U = \mathbf{N}$ and $T = \mathbf{N}$ and it follows that $S = \varnothing$.

We also have the following useful result.

Theorem 2. (1) Any subset of a countable set is either finite or countably infinite.
(2) Any infinite set has a countably infinite subset.
(3)   The union of a finite or countable family of countable sets is finite or countably infinite.
(4) $\mathbf{N} \times \mathbf{N}$ is countably infinite.
(5)   A finite product of countable sets is finite or countably infinite.

1.2 The Integers

To allow for the use of arithmetic, we would then introduce the notion of negative numbers.   View negative numbers as the solution to the linear equation,
$$x + n = 0$$

for each natural number *n*.

For this approach, of course the notion of zero would have to be introduced if it was not already introduced. It satisfies the following:

$m + 0 = m$ for all natural numbers m and $0 + 0 = 0$.

So we assume the existence of such a number zero. This is a very important number and we should indeed construct the natural numbers together with zero. Note that we have used 0 as the distinguished element in **N**. The counting numbers is defined to be the set **N** − {0}.

We can view the existence of the negative numbers as guaranteed by construction.

Thus the set of integers **Z** consists of the counting numbers, their negatives and zero.

Ordering comes naturally with the counting numbers together with zero. *Successor* comes after each number, starting from zero. The addition and multiplication can be extended to the set of integers and we can then define the order relation on **Z** by taking as positive cone the embedded set of counting numbers {1, 2, 3, …}. This ordering extends the usual ordering on **N**. For the construction and details see similar description below or Chapter 6 of "Real Numbers?".

1.3 The Rational Numbers

A ratio of $1 : n$ gives a fraction $\frac{1}{n}$ for each counting number n.

Then addition of m of this $\frac{1}{n}$ means $\frac{m}{n}$. This representation is not unique we have to allow for cancellation as $\frac{1}{2}$ means the same as $\frac{2}{4}$.

We say $\frac{a}{b}$ and $\frac{c}{d}$ are the same if and only if $a\,d = b\,c$. We write $\frac{a}{b} = \frac{c}{d}$ when this happens.

We can define negative fractions in the same way as we define integers as the solution $x + r = 0$ for each fraction r. The collection of the fractions, their negatives and zero constitute the rational numbers **Q**.

Positivity Axioms.

Ordering does not come easily this time with the rational numbers. The set of "positive" rational numbers has the following properties. It should of course contain the counting numbers. For any two "positive" rational numbers p and q,

      A. $p + q$ is again a "positive" rational number *and*

B. *p q* is again a "positive" rational number.

Notice that the counting numbers satisfy these two properties.

This meaning of "positive" is artificial and unnatural. It does apply to the counting numbers. Property **B** involves multiplication. It is easily seen that multiplication of a counting number by -1 gives a negative integer and so multiplication of a fraction by -1 gives us a negative rational number.

We now additionally insist that this set of "positive" rational numbers, together with its negative, that is the set consisting of the result of multiplying each "positive" rational numbers by -1, and zero form the entire set of the rational numbers.

We have a candidate for this set of "positive" rational numbers and it is unique. Our construction of the rational numbers involves the following ingredients, the fractions, their negatives and zero. Plainly the negative fractions are the result of multiplication of the fractions by -1. Obviously the fractions satisfy Properties **A** and **B**.

This definition of "positive" would capture the essence of the meaning of positive. Note that 1 is "positive", a notion we would accept readily. But with the new meaning, it would require some thought. We would use a contradiction argument to show this.

If 1 is not "positive", then it's negative -1 would be "positive" and by Property **B**, $(-1)(-1) = 1$ would be "positive", contradicting our assumption that 1 is not "positive".

Then by Property **A**, all the counting numbers, being defined successively by adding 1, are "positive".

Also note that for any counting number n, $\frac{1}{n}$ is "positive". This can be verified as follows. If $\frac{1}{n}$ is not "positive", then $\frac{1}{n}$ is the multiplication of a "positive" number by -1 because it is not 0. Thus $-\frac{1}{n}$ is "positive" and so since n is "positive", $-1 = -\frac{1}{n} \cdot n$ would be "positive" by Property **B** contradicting -1 is not "positive". Therefore, we conclude that $\frac{1}{n}$ is "positive".

It then follows from Property **B** that any fraction $\frac{m}{n}$ is "positive" for any counting numbers m and n, since $\frac{m}{n} = m \cdot \frac{1}{n}$.

Thus our fractions are "positive". Then the "positive" rational numbers are precisely the fractions.

This is because if there is a "positive" number p not a fraction, then since $p \neq 0$, $-p$ is a fraction and so $-p$ is "positive" (since any fraction is "positive") and so p is not "positive" contradicting p is "positive".

We call this subset of "positive" rational numbers, a *positive cone*. It is precisely the set of fractions.

The positive cone or the fractions would serve as a kind of reference for the ordering. It is a natural division of the rational numbers into two parts, a special part that decides the "direction" of an ordering and another. There is always a division of the rational

numbers at any point into the 'left' and 'right'; what we needed is a reference point, zero, and a translation operation to give meaning to 'left' and 'right'.

For any two rational numbers a and b, we say a is greater than b ($a > b$) if and only if $a - b$ is "positive", i.e., $a - b$ belongs to this special part.

This ordering is consistent with the ordering on the natural numbers. This is seen as follows. For any natural number n, $n + 1 > n$ because $(n + 1) - n = 1$ is "positive". Since $n + 1$ is the successor of n, this ordering is consistent with the previous ordering determined by the sequence of successor followed by successor. In particular for any fraction r, $r > 0$ because $r - 0 = r$ is "positive". If we now define any rational number r to be positive when $r > 0$, then positive would mean the same as "positive".

1.4 The Real Numbers.

The set of real numbers consists of the rational numbers and the irrational numbers. Morris Kline had this to say: "The irrational number logically defined is an intellectual monster." So we would not attempt to make a definition here. It is sufficient to know its workings and its properties.

If we can view the real numbers as the extension of the rational numbers, then we would want the properties that the rational numbers possessed that are so useful, to carry over to this extension. We shall describe in abstract terms these properties.

Field Axioms

The rational number system is an example of a mathematical object ---- a *field*.
It is a set F that comes with two binary operations called addition (+) and multiplication ($\times$), two unique elements called respectively 0 and 1, two unary operations, one on F denoted by $-: F \rightarrow F$ and the other on $F - \{0\}$, denoted by $*: F-\{0\} \rightarrow F-\{0\}$ satisfying the following 9 properties.
For all a in F,
       1. $a + 0 = 0 + a = a$;
       2. $a + (-a) = (-a) + a = 0.$
   For all a, b and c in F,
       3. $a + (b + c) = (a + b) + c;$ (Associativity)

4.	$a + b = b + a$;	(Commutativity)
5.	$a(b + c) = ab + ac$.	(Distributivity)

For all a in $F - \{0\}$,

6. $1 \times a = a \times 1 = a$
7. $a \times (*a) = (*a) \times a = 1$.

For all a, b and c in $F - \{0\}$,

8.	$a \times (b \times c) = (a \times b) \times c$;	(Associativity)
9.	$a \times b = b \times a$.	(Commutativity)

The unary operation $*$ for the rational numbers **Q** corresponds to taking reciprocals on non zero rational numbers. The other operations are suggestive of the symbols.

Positivity Axioms.

A *totally ordered field* F is a field F together with a positive cone P such that

0 does not belong to P, the union of P, its reflection $-P = \{- a : a$ belongs to $P\}$ and $\{0\}$ is equal to F and P satisfies the following two properties that for all a and b in P,

   (A) $a + b$ belongs to P and

   (B) $a \times b$ belongs to P.

The ordering, '>' , called a total ordering on F , is defined by $b > a$ if and only if $b - a$ belongs to P. Thus for any x in F, $x > 0$ if and only if x belongs to P.

The rational numbers **Q** satisfy all the 9 properties with the usual operations of addition, multiplication, taking negatives and reciprocals and has the total ordering described earlier with the (positive) fractions as the positive cone.

To see the desirable property the real number system should possess, we may have to reinvent the whole system of representing numbers.

Take for instance $\sqrt{2}$ the square root of 2. Is this a number? Geometric intuition says it is. One thing is sure - we can find fractions as close to $\sqrt{2}$ as we like 'before' and 'after' $\sqrt{2}$. We approximate" $\sqrt{2}$ by fractions. We cannot pin down $\sqrt{2}$ as a rational number. It is not a symbol readily understood as $- 2$ or $\frac{1}{2}$. But what we can say is this. If there is such a number, then its square would give us the integer 2.

We can be bold. We can extend, in any sense as we would, our rational numbers to some system containing the solution of the equation $x^2 = 2$. But then we would just open up a Pandora's box. What about $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$,..., $\sqrt{p}$, p a prime and so on? It is well-known none of these can be rational numbers. What about cube root of 2? These are solutions to polynomial equation of the form

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$$x^n = p \text{ where } p \text{ and } n \text{ are counting numbers.}$$

What about solution to all polynomial equations? It then becomes an impossible task to describe all these "numbers". In particular there are "numbers" that are not the solution of a polynomial equation such as the Euler constant e and π. So extending our rational numbers this way would not include these numbers. But what is plausible is that no matter what their origins may be, there are rational numbers as close to these numbers as we like 'before' and 'after' these numbers. This forms the basic concept of the *cut* of Dedekind. We have to think of numbers differently as if there is a hierarchy of numbers. We may not know what $\sqrt{2}$ is but we know there are rational numbers as close as we like on the 'left' of it or less than it if we can give an ordering on our set of real numbers. This gives enough information about $\sqrt{2}$ for all practical purposes. Indeed, it is a collection of rational numbers that can give us all the information we required about $\sqrt{2}$. Hence we need a collection of rational numbers to describe a *real number*.

The Archimedean Property

We say the totally ordered field F has the *Archimedean Property* if for all $x > 0$ in F, and for all $y > 0$ in F, there exists a counting number n such that

$$(n1) \times x > y.$$

We can rewrite the last inequality as $\dfrac{1}{n1} \times y < x$ or $(*(n1)) \times y < x$. What this says is that given any x and $y > 0$ in F, no matter how small x is, we can find a counting number n such that $\dfrac{1}{n} \times y < x$ if we identify $\dfrac{1}{n}$ with $\dfrac{1}{n1}$. Obviously the rational numbers $\mathbf{Q}$ has the Archimedean Property. This is a property that we would wish the real numbers to have.

The (order) Completeness Axiom

We would need to add a new property that would tell us that the set of real numbers does exist in a different sense. We know there are rational numbers arbitrarily bigger than $\sqrt{2}$. We can think of all the rational numbers below or less than $\sqrt{2}$. In a way $\sqrt{2}$ would be the largest such number if it exists, bigger than all the rational numbers below $\sqrt{2}$. The existence of such a number would have guaranteed the meaning of $\sqrt{2}$. But of course we would have to think of rational numbers in a different way. To describe more precisely what we mean, we make the following definition.

Consider a subset A of F. A is said to be *bounded above* if there exists x in F such that for all a in A, $a \leq x$. We say A is *bounded below* if there exists y in F such that for all a in A, $y \leq a$.

The number x is called an *upper bound* for A and y a *lower bound* for A. We say A is *bounded* if it is both bounded above and bounded below.

If A is bounded above, then it has an upper bound. It is natural to ask if it has the smallest such upper bound. That means, if M is the smallest such upper bound, then of course M is in F and for any x in F with $x < M$, x cannot be an upper bound for A and so consequently, there exists an element a_0 in A such that $x < a_0$.

Definition 3. M is the *least upper bound* or *supremum (sup)* of a subset A of F if for all a in A, $a \leq M$ and for any $x < M$, there exists b in A such that $x < b$.

A more descriptive way of describing M is this: For any number x less than M, we can always find an element b in A such that $x < b \leq M$.

Similarly we can define the greatest lower bound or infimum of A.

Definition 4. m is the *greatest lower bound* or *infimum* (*inf*) of a subset A of F if for all a in A, $m \leq a$ and for any $x > m$, there exists b in A such that $b < x$.

We can thus characterize m by saying that for any x > m, we can always find an element b in A such that $x > b \geq m$

The notion of supremum or infimum would be in vain if they do not exist. We would like them to be included in our consideration. A totally ordered field in which every bounded subset has an infimum and a supremum is special in that *the boundaries of the bounded subsets are elements in the field*. This prompts the next important definition.

Definition 5.   A totally ordered field F is *complete* if every non-empty bounded above subset of F has a supremum (in F).

The significance of this definition is that the supremum is a member of F. That means any bounded above subset has its 'upper' boundary residing in F and there is no room for a gap to exist in F.

The term 'complete' has several meanings. The present meaning is sometimes referred to as *order complete*.

This property is new. It is desirable for $\sqrt{2}$ to have a meaning, to exist. The rational numbers **Q** is not complete. Take for example the subset

$$A = \{x \text{ in } \mathbf{Q}: x > 0 \text{ and } x^2 < 2 \} \text{ of } \mathbf{Q.}$$

It does not have a supremum (in **Q**). Note that we cannot as yet write $\sqrt{2}$ as its existence has not been established. We can only talk about it hypothetically. That $\sqrt{2}$ is not a rational number can easily be shown.   Thus $\sqrt{2}$ will belong to a different scheme of things. A is plainly bounded above for we see that for any a in A, $a < 2$. We shall now show that A has no supremum in **Q**.

Suppose A has a supremum M.

Then $M \geq a$ for all a in A and that if $k < M$, then there exists b in A such that $k < b$.

By definition of A, $a^2 < 2$ for all a in A. Also $a^2 \leq M^2$. We can compare M^2 and 2 to produce a contradiction.

Suppose $M^2 > 2$. Then $\dfrac{M^2 - 2}{M + 2} > 0$ since plainly $M + 2 > 0$.

Let $k = M - \dfrac{M^2 - 2}{M + 2} = \dfrac{2M + 2}{M + 2}$. Then $k < M$.

Note that $k^2 - 2 = \dfrac{4(M + 1)^2 - 2(M + 2)^2}{(M + 2)^2} = \dfrac{2(M^2 - 2)}{(M + 2)^2} > 0$ since $M^2 > 2$.

Thus, $k^2 > 2$. But since $k < M$, there exists b in A such that $k < b$. Therefore, $k^2 < b^2 < 2$. This contradicts $k^2 > 2$. Therefore, $M^2 \leq 2$.

Since $M^2 \neq 2$ because M is a rational number, $M^2 < 2$.

We shall now derive another contradiction.

So we have $2 - M^2 > 0$. Let now $k = M + \dfrac{2 - M^2}{M + 2} = \dfrac{2M + 2}{M + 2}$. Then $k > M$

Also $k^2 - 2 = \dfrac{4(M + 1)^2 - 2(M + 2)^2}{(M + 2)^2} = \dfrac{2(M^2 - 2)}{(M + 2)^2} < 0$ since $M^2 < 2$. Therefore, $k^2 < 2$. Hence k belongs to A and so $k \leq M$. This contradicts $k > M$. Consequently, these two contradictions imply that $M = \sup A$ does not exist.

The following is a variation or equivalent definition for completeness. First, note that for any non-empty bounded below subset A of F, $-A = \{-a: a$ belongs to $A\}$ is bounded above. In particular, $\inf A = - \sup (-A)$. Thus if the supremum exists for any non empty bounded above subset of F, then the infimum too exists for any non-empty bounded below subset of F. The converse is similarly shown to be true using the fact that for any bounded above subset A of F, $-A = \{-a: a$ belongs to $A\}$ is bounded below and $\sup A = - \inf (-A)$. We can thus make the following equivalent definition of completeness.

Definition 6. A totally ordered field F is *complete* if every non-empty bounded below subset of F has an infimum.

If we assume we know how to construct the real numbers, then the following tells us just what it is.

Theorem 7. The real numbers **R** is a complete totally ordered field.

There is essentially one such complete ordered field. This is not to say that there is exactly one such complete totally ordered field but that any two are *isomorphic*. We interpret this to mean that for all intent and purposes they are the same although they may be constructs of a different nature.

Proposition 8. The real numbers **R** has the Archimedean Property.

That is to say for any $x, y > 0$ in **R**, there is a counting number n such that

$$n\,x > y. \qquad \text{------------------------------------- \quad (*)}$$

(Here we are using the notation inherited from the rational numbers.)

When a totally ordered field has the Archimedan Property, we say it is *archimedean*. Hence, **R** is archimedean.

Proof of Proposition 8.   We shall prove Proposition 8 by contradiction.

Suppose **R** is not archimedean. Then by negating the statement (*), we get

there exists $x, y > 0$ such that for all counting number n, $n\,x \le y$.   ------- (**)

Take the set $K = \{\, n\,x : n \text{ a counting number}\}$.

Then by (**) K is bounded above by y and is non-empty. Because **R** is complete, the supremum M of K exists. Hence for any counting number n, $n\,x \le M$ since $n\,x$ belongs to K.

Now $(n + 1)x$ belongs to K too. Therefore, $(n + 1)x \le M$.   That means

$$n\,x \le M - x.$$

Thus for any counting number n,

$$n\,x \le M - x < M. \qquad\qquad \text{------------------} \quad (***)$$

Thus $M - x$ is an upper bound for K.

Because $M - x < M$ and that M is the supremum of K, there is an element $n_0\,x$ in K, for some counting number n_0, such that

$$M - x < n_0\,x.$$

But by (***) $n_0\,x \le M - x$. This contradicts $M - x < n_0\,x$. Therefore, **R** is archimedean.

Real numbers are hard to think of conceptually. When we say take a small number $\varepsilon > 0$, we like to think of ε as a rational number, since we are more comfortable with the rational numbers. For all practical purposes, this is what we need to think of ε. We may indeed just say take a small rational number $\varepsilon > 0$ instead. The following justifies this.

Corollary 9. For any $\varepsilon > 0$, there is a counting number n such that $\frac{1}{n} < \varepsilon$.

Proof. By the Archimedean Property of **R**, there exists a counting number n such that $n\,\varepsilon > 1$. Therefore, $\frac{1}{n} < \varepsilon$.

Note that $\frac{1}{n}$ is a rational number. So the Corollary says that for any $\varepsilon > 0$, no matter how small ε is, we can find a rational number $\frac{1}{n}$ such that $0 < \frac{1}{n} < \varepsilon$. So for all pratical purposes, in place of ε, we can use $\frac{1}{n}$ instead.

Now that we have recognized a class of numbers which consists of numbers that are not rational numbers and that there is an ordering that applies to the whole of **R**, we might ask ourselves the question, "How often can we find a rational number or for that matter,

irrational number ?" Very often is the answer. In mathematical terms we mean the rational numbers or the irrational numbers are *dense*. The following Corollary gives meaning to the term 'density of the rational numbers'.

Corollary 10. For any x and y in **R** and $x < y$, there exists an integer n and a counting number m such that $x < \frac{n}{m} < y$.

A descriptive way of stating Corollary 4 will be "between any two real numbers there is a rational number".

Proof of Corollary 10. The proof goes like this. Take two real numbers x and y such that $x < y$. Then $y - x > 0$.

It follows by Corollary 3 that there is a counting number m such that $\frac{1}{m} < y - x$.

The rest of the proof will be divided into 3 cases.

The easiest case will be when $x = 0$.
Then we have $0 < \frac{1}{m} < y$ and so the required rational number is $\frac{1}{m}$ for this case.

The second case is when $x > 0$. We now invoke the Archimedean Property of **R**. By this property there is a counting number n, such that $n\frac{1}{m} > x$.

Having established the existence of such an integer n, we can then by successively taking one away from this number n to obtain the least integer n such that $n\frac{1}{m} > x$.

That means $(n-1)\frac{1}{m} \le x$.

Therefore,

$$y = (y - x) + x > \frac{1}{m} + (n-1)\frac{1}{m} = \frac{n}{m} > x.$$

For this case, the required rational number is $\frac{n}{m}$.

The remaining case is for $x < 0$. That is $-x > 0$.

Then by the Archimedean property there is a counting number n such that $n\frac{1}{m} > -x$.

As before we choose the least integer n such that $n\frac{1}{m} \ge -x$.

Then we have $(n-1)\frac{1}{m} < -x$.

Therefore,

$$y = (y - x) + x > \frac{1}{m} - \frac{n}{m} = (1 - n)\frac{1}{m} > x.$$

For this case the required rational number is $\frac{1-n}{m}$.

Having established the density of the rational numbers, we expect that the irrational numbers are also dense in **R**. More is true here. The irrational numbers are more numerous than the rational numbers. This statement will make sense only when we have some means of "measuring" subsets of the real numbers. Indeed, the "measure" of the set of rational numbers is zero but not so for the set of irrational numbers. We would need a

theory of measure to establish this. The fact that **R** is uncountable and the rational numbers **Q** is countable gives us some idea of the difference in "size" of the set of irrational numbers and the set of rational numbers. A set is said to be countable if we can match its elements one to one with elements of the counting numbers. It can be shown that **Q** is countable though not finite. But it is much harder to show that **R** is not countable. One can do this by showing that the real numbers between 0 and 1 is not countable. This can be shown by way of contradiction. First, by assuming that we have a matching function from the counting numbers to the real numbers between 0 and 1 and thus we can write them as a sequence. Then by assuming that each term of this sequence can be written as an infinite decimal and using this sequence of infinite decimals to produce a number different from any term of this sequence and thus showing that we can never have a matching function. This approach will need some criterion to distinguish infinite decimals converging to different limits. For now we are content with the following.

Corollary 11. For any two rational numbers a and b with $a < b$, there is an irrational number α such that $a < \alpha < b$.

Proof. The proof of this Corollary is by actually producing the required irrational number α by making use of a known irrational number.

Take an irrational number $k > 0$. For instance we can take $k = \sqrt{2}$.

By the Archimedean Property of **R**, there is a counting number n such that $n(b-a) > k$. That is, $a + \frac{k}{n} < b$. Then $a < a + \frac{k}{n} < b$. Since k is irrational, $a + \frac{k}{n}$ is also irrational because a is rational and $\frac{k}{n}$ is irrational. Take $\alpha = a + \frac{k}{n}$ to be our required irrational number. This establishes the truth of this corollary.

Thus as a consequence of Corollary 10 and 11, for any x and y in **R** and $x < y$, there exists an irrational number α such that $x < \alpha < y$.

We can expect to find an integer between x and $x + 1$ for any real number x. This is stated more precisely as follows.

Corollary 12. For any real number x, there is an integer n_0 such that $x < n_0 \leq x + 1$.

Proof. If x is an integer, then we only have to take n_0 to be $x + 1$.

Now assume x is not an integer.

Suppose $x > 0$. Then by the Archimedean Property of **R**, there is a counting number n such that $n = n \times 1 > x$. Then take the least such integer N with $N > x$. Thus we have $N - 1 \leq x$. Therefore, we have $x < N \leq x + 1$. So take $n_0 = N$.

Observe that we actually have $x < N < x + 1$, since $x + 1$ is not an integer.

Now for the case x is not an integer and $x < 0$. Then as before there exists an integer M such that $-x < M < -x + 1$. Thus $x - 1 < -M < x$ and so $x < -M + 1 < x + 1$. For this case take $n_0 = 1 - M$.

Exercises 13.

1. Using the order axioms prove:
   (a) If $a > b$, then $a + c > b + c$ for any c.
   (b) If $a > b$ and $c > 0$, then $ac > bc$.
   (c) If $a > b$ and $b > c$, then $a > c$.
   (d) For any two elements a and b , one and only one of the following relations hold:
   $a > b, a = b, b > a$.
   Conversely, suppose that a field exists which has a relation $>$ satisfying 1(a)- (d), show that the subset $P = \{a : a > 0\}$ satisfies the order axioms.

2. Show that in a totally ordered field $a \neq 0$ implies $a^2 > 0$. Deduce $1 > 0$. Show that the complex numbers cannot be ordered. (Hint: $i^2 = -1$).

3. Suppose a , $b \in \mathbf{R}$ and $a > b - \varepsilon$ for all $\varepsilon > 0$, ε in $\mathbf{R}$, prove that $a \geq b$

4. Let k be an upper bound for $S \subseteq \mathbf{R}$. Show that $k \in S \Rightarrow k$ is the supremum of S.

5. Suppose A is a non-empty bounded subset of $\mathbf{R}$. Let B be a non-empty subset of $\mathbf{A}$. Prove that sup $A \geq$ sup B and that inf $A \leq$ inf B.

6. Find, if they exist, the supremum and infimum of each of the following subsets of the real numbers. Also decide which of these sets have the greatest element (maximum), least element (minimum).
   (i) $\{ 1/n : n \in \mathbf{Z}_+ \}$ (ii) $\{ x : 0 \leq x \leq \sqrt{2} , x \text{ rational} \}$
   (iii) $\{ x : x^2 + x + 1 \geq 0\}$ (iv) $\{x : x < 0 \text{ and } x^2 + x - 1 < 0 \}$
   (v) $\{ 1/n + (-1)^n : n \in \mathbf{Z}_+ \}$ (vi) $\{ x : x = n^3 \text{ for some positive integer } n \}$

7. Let A and B be non empty bounded above subsets of $\mathbf{R}$.
   Let $A + B = \{x + y : x \in A \text{ and } y \in B \}$ Prove that sup $(A + B) =$ sup $A +$ sup B.