Change of Variable for Riemann Stieltjes integral and Riemann Stieltjes integrability

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Suppose the function $g:[a,b]\to\mathbb{R}$ is a continuous function but with unbounded variation. Let $[c,d]=g([a,b])$, Suppose $f:[c,d]\to\mathbb{R}$ is a bounded real valued function. Then we can define in the usual manner the Riemann Stieltjes integral of the composition $f\circ g$ with *g* as the integrator.

The question is: When is the following change of variable,

$$\int_a^b f\circ g(x)dg(x)=\int_{g(a)}^{g(b)} fdx \ , \quad \text{--------------------------- (1)}$$

where the right-hand integral is assumed to be a Riemann integral, for the Riemann Stieltjes integral holds?

The existence of the integral on the left-hand side of (1) does not necessary imply the existence of the right hand integral, nor does the existence of the right hand integral of (1) necessary imply the existence of the lefthand integral of (1).

However, if it is given that both integrals in (1) exist, then they are equal. This is due to Michael Bensimhoun, who proved that if the HK stieltjes integral of the left-hand side exists, then the right-hand side also exists as a HK integral. If $f\circ g$ is Riemann Stieltjes integrable with respect to *g*, then it is HK integrable with respect to *g*, this implies that the right-hand side exists as a HK integral and they are equal.

When *f* is continuous and *g* is continuous of bounded variation, then (1) holds. (See Theorem 20 of "*Limit of the Lebesgue Stieltjes Integral and Change of Variable*".)

When *g* is increasing and continuous and *f* is Borel, then (1) holds with the right hand integral of the function *f* being Lebesgue integrable. (See Theorem 46 or Corollary 61 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*".)

Example 1. There is a continuous function $g:[a,b]\to\mathbb{R}$ of unbounded variation and a Riemann integrable function *f* defined on the range of *g* such that $f\circ g$ is not Riemann Stieltjes integrable with respect to *g*.

Define g to be the function on [0, 1] by $g(x) = \begin{cases} \sqrt{x}\sin\left(\dfrac{\pi}{2x}\right), & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$.

Let $f(x) = x$. Then f is Riemann integrable on [0, 1].

Take any partition $P: x_0 = 0 < x_1 < x_2 < \cdots < x_M = 1$ with $\|P\| < \delta$. Then $x_1 < \delta$. We may assume that $x_1 = \dfrac{1}{2N+1} < \delta$ for some positive integer N. Take any $K > 0$. Then since $\sum_{i=0}^{\infty} \dfrac{1}{2i+1}$ is divergent, there exists an integer $N1 > N$ such that $\sum_{i=N}^{N1} \dfrac{1}{2i+1} > K$. Partition the subinterval $[0, x_1] = \left[0, \dfrac{1}{2N+1}\right]$ by division points $\left\{\dfrac{1}{2N1+1}, \dfrac{1}{2N1}, \dfrac{1}{2N1-1}, \ldots, \dfrac{1}{2N+2}, \dfrac{1}{2N+1}\right\}$. Let $y_i = \dfrac{1}{2i+1}$ and $z_i = \dfrac{1}{2i}$ for $i > 0$ and $z_{N1+1} = 0$. Then $Q: 0 < y_{N1} < z_{N1} < y_{N1-1} < z_{N1-1} \cdots, z_{N+1} < y_N = x_1$ is a subdivision of $[0, x_1]$. Take a Riemann Stieltjes sum corresponding to this partition,

$$SL = \sum_{i=N}^{N1} g(\xi_i)(g(y_i) - g(z_{i+1})) + \sum_{i=N+1}^{N1} g(\beta_i)(g(z_i) - g(y_i)) + \sum_{i=2}^{M} g(\eta_i)(g(x_i) - g(x_{i-1})) .$$

Let $L = \sum_{i=2}^{M} g(\eta_i)(g(x_i) - g(x_{i-1}))$. Then for any $K > 2|L|$. Take $\xi_i = y_i$ for $N \le i \le N1$ and $\beta_i = z_i$ for $N+1 \le i \le N1$. Then the Riemann sum

$$SL = \sum_{i=N}^{N1} \frac{1}{2i+1} + \sum_{i=2}^{M} g(\eta_i)(g(x_i) - g(x_{i-1})) > K .$$

This shows that the Riemann sums is unbounded. Hence, the Rieman Stieltjes integral $\int_a^b f \circ g(x)dg(x) = \int_a^b g(x)dg(x)$ does not exists but f is Riemann integrable on $[g(a), g(b)] = [0,1]$.

For a continuous function $g:[a,b] \to \mathbb{R}$ of bounded variation there does not exist a non-Riemann integrable bounded function f defined on the range of g such that $f \circ g$ is Riemann Stieltjes integrable with respect to g.

This is a consequence of the following theorem.

Theorem 1. Suppose $g:[a,b] \to \mathbb{R}$ is a continuous function of bounded variation. Let $J = g([a,b]) = [c,d]$ be the range of g. Assume that g is not a constant function. Suppose $f:[c,d] \to \mathbb{R}$ is a bounded Borel function. If the

Riemann Stieltjes integral $\int_a^b f \circ g(x)dg(x)$ exist, then *f* is Riemann integrable on [*c*, *d*].

Proof.

Since *f* is bounded, $f \circ g$ is Riemann Stieltjes integrable with respect to *g* on [*a*, *b*] implies that $f \circ g$ is Riemann Stieltjes integrable with respect to v_g, the total variation function of *g*. (See Chapter 14, Theorem 14.14, *Real Analysis* by N. L. Carothers.)

Let $D_{f\circ g} = \{x \in [a,b] : f \circ g \text{ is discontinuous at } x\}$ be the set of discontinuities of the composite function $f \circ g$.

Therefore, $\mu_{v_g}(D_{f\circ g}) = 0$. As v_g is continuous, $\mu_{v_g}(D_{f\circ g}) = m^*\left(v_g(D_{f\circ g})\right) = 0$, where *m** is the Lebesgue outer measure. (See Theorem 6 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*".) Therefore, $m^*\left(g(D_{f\circ g})\right) = 0$. (See Theorem 16 in "*Functions of Bounded Variation and Johnson's Indicatrix*".) Suppose *f* is non-Riemann integrable on the image of *g*. Then, there is a set *E* of positive measure such that $E \subseteq [c,d] - g(D_{f\circ g})$, where $[c,d]$ is the range of *g*, and *f* is discontinuous at every point in *E*. Let $F \subseteq [c,d]$ be the values of local constants of *g*. Then *F* is countable and is of zero measure. We may assume that $E \cap F = \varnothing$.

Let $G = g^{-1}(E)$. Then $f \circ g$ is discontinuous at every point in *G*. Therefore, $G \subseteq D_{f\circ g}$. This is impossible since this would imply $g(G) = E \subseteq g(D_{f\circ g})$ and $E \subseteq [c,d] - g(D_{f\circ g})$. Hence, *f* is Riemann integrable on [*c*, *d*].

Theorem 2. Suppose $g:[a,b] \to \mathbb{R}$ is a continuous function of bounded variation. Assume that *g* is not a constant function. Let $J = g([a,b]) = [c,d]$ be the range of *g*. Suppose $f:[c,d] \to \mathbb{R}$ is a bounded Borel function.

Suppose *g* is absolutely continuous or *f* is the pointwise limit of a uniformly bounded sequence of continuous functions or *f* is a continuous function.

If the Riemann Stieltjes integral $\int_a^b f \circ g(x)dg(x)$ exist, then *f* is Riemann integrable on [*c*, *d*] and $\int_a^b f \circ g(x)dg(x) = \int_{g(a)}^{g(b)} f(x)dx$

Proof.

Suppose the Riemann Stieltjes integral $\int_a^b f\circ g(x)dg(x)$ exist. Then the Lebesgue Stieltjes integral $\int_a^b f\circ g(x)d\lambda_g$ exists. By Theorem 1, *f* is Riemann integrable on [*c*, *d*].

If *g* is absolutely continuous, then

$$\int_a^b f\circ g(x)d\lambda_g = \int_a^b f\circ g(x)g'(x)dx,$$

where the right-hand side is a Lebesgue integral. By Theorem 8 of "*Change of Variables Theorems*", since *f* is a bounded Lebesgue integrable function,

$$\int_a^b f\circ g(x)g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx,$$

as Lebesgue integrals. Since *f* is Riemann integrable, by Theorem 1 Part (2) of "*Change of Variable Theorem for Riemann Integral*", $f\circ g(x)g'(x)$ is Riemann integrable and $\int_a^b f\circ g(x)g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx$ as Riemann integrals. Hence, $\int_a^b f\circ g(x)dg(x) = \int_{g(a)}^{g(b)} f(x)dx$ as Riemann integrals.

If *f* is continuous, then the change of variable as Riemann integrals holds. (See Theorem 20 of "*Limit of the Lebesgue Stieltjes Integral and Change of Variable*".)

Suppose there exists a sequence of continuous function (f_n) such that f_n tends pointwise to *f* boundedly. Since f_n is continuous. $\int_a^b f_n\circ g(x)dg(x) = \int_{g(a)}^{g(b)} f_n(x)dx$ and so $\int_a^b f\circ g(x)dg(x) = \lim_{n\to\infty}\int_a^b f_n\circ g(x)dg(x) = \lim_{n\to\infty}\int_{g(a)}^{g(b)} f_n(x)dx = \int_{g(a)}^{g(b)} f(x)dx$ by the Bounded Lebesgue Convergence Theorem.

This completes the proof.

For unbounded function *f*, which is not Riemann integrable, you may have equality as Lebesgue Stieltjes and Lebesgue integrals but not as Riemann Stieltjes integral and Riemann integrals as in the following example.

Example 2. There is a continuous function $g:[a,b]\to\mathbb{R}$ of bounded variation and a non-Riemann integrable function *f* defined on the range of *g* such that $f\circ g$ is Lebesgue Stieltjes integrable with respect to *g*.

Let $g:[0,1]\to\mathbb{R}$ be defined by $g(x)=\begin{cases} x^4\sin^3\left(\dfrac{\pi}{2x}\right), & x>0 \\ 0, & x=0 \end{cases}$ and let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases} \dfrac{1}{x^{1/3}}, & x>0 \\ 0, & x=0 \end{cases}$. Then *f* is not bounded on [0, 1]. The function *f* is Lebesgue integrable and the integral is given by the improper Riemann integral $\int_0^1 f(x)dt=\lim\limits_{t\to 0^+}\int_t^1 f(x)dt=\lim\limits_{t\to 0^+}F(t)$, where $F:(0,1]\to\mathbb{R}$ is defined by $F(t)=\int_t^1 f(x)dx=\frac{3}{2}\left(1-t^{2/3}\right)$. Observe that

$$f(g(x))=\begin{cases} \dfrac{1}{x^{4/3}\sin\left(\frac{\pi}{2x}\right)}, & x>0 \text{ and } x\neq\dfrac{1}{2k}\text{ , integer } k\geq 1 \\ 0, & x=0 \text{ or } x=\dfrac{1}{2k}\text{ , integer } k\geq 1 \end{cases}$$

Note that $f\circ g$ is not bounded on [0, 1], hence it is not Riemann integrable on [0, 1]. Note that for any partition $P: x_0=0<x_1<x_2<\cdots<x_n=1$, for any Riemann Stieltjes sum $L=\sum_{i=1}^{n} f\circ g(\eta_i)(\phi\circ g(x_i)-\phi\circ g(x_{i-1}))$, since $\lim\limits_{x\to 0^+}|f\circ g(x)|=\infty$, by a suitable choice of η_1, we can make the Riemann Stieltjes sum *L* arbitrarily large. Hence, $f\circ g$ is also not Riemann Stieltjes integrable with respect to *g*.
The function *g* is differentiable on [0, 1] and

$$g'(x)=\begin{cases} 4x^3\sin^3\left(\dfrac{\pi}{2x}\right)-\pi x^2\dfrac{3}{2}\sin^2\left(\dfrac{\pi}{2x}\right)\cos\left(\dfrac{\pi}{2x}\right), & x>0 \\ 0, & x=0 \end{cases}$$

Note that g' is bounded and continuous on [0, 1]. Hence, it is Riemann integrable on [0, 1].
Therefore, *g* is absolutely continuous on [0, 1] and
$\int_a^b f\circ g(x)d\lambda_g=\int_a^b f\left(g(x)\right)g'(x)d(x)$ as Lebesgue Stieltjes integrals.

$$f(g(x))g'(x)=\begin{cases} 4x^{5/3}x\sin^2\left(\dfrac{\pi}{2x}\right)-\pi x^{2/3}\sin\left(\dfrac{\pi}{2x}\right)\cos\left(\dfrac{\pi}{2x}\right), x>0 \text{ and } x\neq\dfrac{1}{2k}\text{ , integer } k\geq 1 \\ 0, x=0 \text{ or } x=\dfrac{1}{2k}\text{, integer } k\geq 1 \end{cases}$$

$f(g(x))g'(x)$ is continuous [0, 1] and so it is Riemann integrable on [0, 1].
By Theorem 9 of "*Change of Variables Theorems*",
$\int_0^1 f(g(x))\cdot g'(x)dx=\int_{g(0)}^{g(1)} f(x)dx$ as Lebesgue integrals and $\int_0^1 f(g(x))\cdot g'(x)dx=\frac{2}{3}$.

However, $f \circ g$ is Riemann Stieltjes integrable with respect to *g* on $[k,1]$ for any *k* such that 0 < *k* <1 and $\int_k^1 f \circ g(x)dg = \int_k^1 f(x)dx$. Therefore, if we define the improper Riemann Stieltjes integral to be $\lim_{k \to 0^+} \int_k^1 f \circ g(x)dg$, then it is equal to the improper Riemann integral, $\lim_{k \to 0^+} \int_k^1 f(x)dx$.

There is no non-Riemann integrable bounded function *f* such that for continuous increasing function *g* on [*a*, *b*], $f \circ g$ is Riemann Stieltjes integrable with respect to *g*.

Theorem 3. Suppose $g:[a,b] \to \mathbb{R}$ is an increasing and continuous function. Let $J = g([a,b]) = [c,d]$ be the range of *g*. Suppose $f:[c,d] \to \mathbb{R}$ is a bounded Borel function. Then $f \circ g$ is Riemann Stieltjes integrable with respect to *g* on [*a*, *b*] implies that *f* is Riemann integrable on [*c*, *d*] and $\int_a^b f \circ g(x)dg(x) = \int_{g(a)}^{g(b)} f(x)dx$ as Riemann integrals.

Proof.

Since *f* is bounded, $f \circ g$ is Riemann Stieltjes integrable with respect to *g*, if, and only if, $\mu_g(D_{f \circ g}) = 0$, where $D_{f \circ g}$ is the set of discontinuities of $f \circ g$. If $g(a) = g(b)$,then *g* is a constant function and we have nothing to prove. Assume now $g(a) < g(b)$. Suppose *g* is continuous, then by Theorem 6 of "Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals", $\mu_g(D_{f \circ g}) = m^*\left(g(D_{f \circ g})\right)$. Hence, $m^*\left(g(D_{f \circ g})\right) = 0$. Let D_f be the set of discontinuities of *f* on [*c*, d]. Let *F* be the values of *g* where *g* is locally constant. Then *F* is countable and so is a set of zero Lebesgue measure. Let $G = D_f - F$. Since *g* is continuous, $g^{-1}(G) \subseteq D_{f \circ g}$. Therefore $m^*(G) = m^*\left(g\left(g^{-1}(G)\right)\right) = 0$. It follows that $m^*(D_f) = 0$. Therefore, *f* is Riemann integrable on [*c*, *d*].

If $f \circ g$ is Riemann Stieltjes integrable with respect to *g* on [*a, b*], then it is Lebesgue Stieltjes integrable with respect to *g*. Then by Theorem 46 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*", $\int_a^b f \circ g(x)dg(x) = \int_{g(a)}^{g(b)} f(x)dx$ as Lebesgue integrals. Hence, they are equal as Riemann integrals.

Theorem 4. Suppose $g:[a,b]\to\mathbb{R}$ is a continuous function of bounded variation. Let $J=g([a,b])=[c,d]$, with $c < d$, be the range of g. Suppose $f:[c,d]\to\mathbb{R}$ is a bounded Borel function. If the Riemann Stieltjes integral $\int_a^b f\circ g(x)dg(x)$ exists, then f is Riemann integrable on [c, d] and $\int_a^b f\circ g(x)dg(x)=\int_{g(a)}^{g(b)} f(x)dx$.

Proof.

By Theorem 1, the function f must be Riemann integrable on [c, d].

If f is continuous, then $f\circ g$ is continuous and by Theorem 20 of "*Limit of the Lebesgue Stieltjes Integral and Change of Variable*", $\int_a^b f\circ g(x)d\lambda_g=\int_{g(a)}^{g(b)} f(x)dx$ as Lebesgue integrals. Moreover, the Riemann Stieltjes integral $\int_a^b f\circ g(x)dg(x)$ is equal to the Lebesgue Stieltjes integral $\int_a^b f\circ g(x)d\lambda_g$. Hence, $\int_a^b f\circ g(x)dg(x)=\int_{g(a)}^{g(b)} f(x)dx$ as Riemann integrals and the change of variable formula holds.

Now we suppose that f is not necessarily continuous.

We shall approximate f by a sequence of continuous functions differing from f by a set of Lebesgue measure tending to 0. Let E be the set in [c, d], where f is continuous at every point in E and the measure of the complement of E is zero. By the inner regularity of the Lebesgue measure, there exists a sequence of compact sets K_n such that $K_n\subseteq E$, $K_n\subseteq K_{n+1}$ and the Lebesgue measure $m(E-K_n)<\frac{1}{n}$, where we denote the Labesgue measure by m. By the Tietze Extension Theorem, for each positive integer n, we can extend the restriction of f to K_n, to a continuous function f_n on [c, d] such that $\sup\{|f_n(x)|:x\in[c,d]\}\le\sup\{|f(x)|:x\in[c,d]\}$. Then we have, since f_n is continuous,

$$\int_a^b f_n\circ g(x)dg(x)=\int_{g(a)}^{g(b)} f_n(x)dx=\int_{g(a)}^{g(b)} f_n(x)dx$$

By the Lebesgue Dominated Convergence Theorem, $\int_{g(a)}^{g(b)} f_n(x)dx\to\int_{g(a)}^{g(b)} f(x)dx$, since f_n converges boundedly almost everywhere to f and f is integrable.

Note that for $K=\bigcup_{n=1}^{\infty}K_n$, $m(E-K)=0$. Let $h_n = f - f_n$. Then $h_n(x)=0$ for *x* in K_n, $m(x: h_n(x)\neq 0)\leq \frac{1}{n}$ and $|h_n(x)|\leq 2C$, where $C=\sup\{|f(x)| : x\in[c,d]\}$. Note that $m([c,d]-K_n)\leq \frac{1}{n}$ and $m([c,d]-K)=0$. Let $H_n = g^{-1}(K_n)$. We have $H_n \subseteq H_{n+1}$ and $H=\bigcup_{n=1}^{\infty}H_n = g^{-1}(K)$. $[a,b]-H=\bigcap_{n=1}^{\infty}([a,b]-H_n)=g^{-1}([c,d]-K)$ and

$$\int_a^b (f-f_n)\circ g(x)dg(x)=\int_a^b h_n\circ g(x)dg(x)=\int_{H_n\cup[a,b]-H_n} h_n\circ g(x)d\lambda_g$$

$$=\int_{H_{nn}} h_n\circ g(x)d\lambda_g+\int_{[a,b]-H_n} h_n\circ g(x)d\lambda_g = 0+\int_{[a,b]-H_n} h_n\circ g(x)d\lambda_g$$

$$=\int_{[a,b]-H_n} h_n\circ g(x)d\lambda_g = \int_{g^{-1}([c,d]-K_n)} h_n\circ g(x)d\lambda_g .$$

Now, $\left|\int_{g^{-1}([c,d]-K_n)} h_n\circ g(x)d\lambda_g\right|\leq 2C\int_{g^{-1}([c,d]-K_n)} d\mu_{v_g}$, where v_g is the total variation function of *g*. Since $g^{-1}([c,d]-K_n)=[a,b]-H_n \supseteq [a,b]-H_{n+1}$, $g^{-1}([c,d]-K_n)$ tends to $g^{-1}([c,d]-K)$. Therefore, $\int_{g^{-1}([c,d]-K_n)} d\mu_{v_g}$ tends to $\int_{g^{-1}([c,d]-K)} d\mu_{v_g}$. Since v_g is continuous and increasing, $\int_{g^{-1}([c,d]-K)} d\mu_{v_g}=m\left(v_g\left(g^{-1}([c,d]-K)\right)\right)$. Now the Lebesgue measure of $g(g^{-1}([c,d]-K))=[c,d]-K$ is zero and so $m\left(v_g\left(g^{-1}([c,d]-K)\right)\right)=0$. Thus, we have $\int_a^b (f-f_n)\circ g(x)dg(x)\to 0$ as *n* trends to infinity. On the other hand,

$$\int_a^b (f-f_n)\circ g(x)dg(x)=\int_a^b f\circ g(x)dg(x)-\int_a^b f_n\circ g(x)dg(x)$$

tends to $\int_a^b f\circ g(x)dg(x)-\int_{g(a)}^{g(b)} f(x)d(x)$. Therefore, $\int_a^b f\circ g(x)dg(x)=\int_{g(a)}^{g(b)} f(x)d(x)$.

Theorem 5. Suppose $g:[a,b]\to\mathbb{R}$ is an increasing continuous function. Let $J=g([a,b])=[c,d]$ be the range of *g*. Suppose $f:[c,d]\to\mathbb{R}$ is a bounded Borel function.

(i) Suppose $\phi:[c,d]\to\mathbb{R}$ is an increasing continuous function. If the Riemann Stieltjes integral $\int_a^b f\circ g(x)d(\phi\circ g)$ exist, then *f* is Riemann Stieltjes integrable with respect to ϕ on [*c*, *d*] and $\int_a^b f\circ g(x)d(\phi\circ g)=\int_{g(a)}^{g(b)} f(x)d\phi$.

(ii) Suppose $\phi:[c,d]\to\mathbb{R}$ is a function of bounded variation. Suppose *f* is Riemann Stieltjes integrable with respect to ϕ on [*c*, *d*]. Then the Riemann Stieltjes integral $\int_a^b f\circ g(x)d(\phi\circ g)$ exist and $\int_a^b f\circ g(x)d(\phi\circ g)=\int_{g(a)}^{g(b)} f(x)d\phi$.

If *g* is strictly increasing and continuous and if $f\circ g$ is Riemann Stieltjes integrable with respect to $\phi\circ g$ on [*a*, *b*], then *f* is Riemann Stieltjes integrable with respect to ϕ on [*c*, *d*] and $\int_a^b f\circ g(x)d(\phi\circ g)=\int_{g(a)}^{g(b)} f(x)d\phi$.

(iii) Suppose $\phi:[c,d]\to\mathbb{R}$ is a continuous function of bounded variation. $\int_a^b f\circ g(x)d(\phi\circ g)$ exists if, and only if, $\int_c^d f(x)d\phi$ exists.

If $f\circ g$ is Riemann Stieltjes integrable with respect to $\phi\circ g$ on [*a*, *b*] or the function *f* is Riemann Stieltjes integrable with respect to ϕ on [*c*, *d*], then $\int_a^b f\circ g(x)d(\phi\circ g)=\int_{g(a)}^{g(b)} f(x)d\phi$.

Remark.

Contrast this with the Lebesgue Stieltjes integral versions. (See Theorem 55, Corollary 61, Corollary 62 of *Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals.*).

Proof.

We assume that $g(a)<g(b)$ for if $g(a)=g(b)$, then we have nothing to prove.

(i) If the Riemann Stieltjes integral $\int_a^b f\circ g(x)d(\phi\circ g)$ exists, then it is equal to the Lebesgue Stieltjes integral $\int_a^b f\circ g(x)d\mu_{\phi\circ g}$ and by Corollary 61 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*", $\int_a^b f\circ g(x)d\mu_{\phi\circ g}=\int_{g(a)}^{g(b)} fd\mu_\phi$ as Lebesgue Stieltjes integral. Since the Riemann Stieltjes integral $\int_a^b f\circ g(x)d(\phi\circ g)$ exist, $\mu_{\phi\circ g}(D_{f\circ g})=0$, where $D_{f\circ g}$ is the set of points where $f\circ g$ is discontinuous. Since $\phi\circ g$ is continuous, $m^*(\phi\circ g(D_{f\circ g}))=\mu_{\phi\circ g}(D_{f\circ g})=0$. Let D_f be the set of points at which *f* is discontinuous. We may assume that D_f does not meet the set of local constant values of *g*, which is countable and of μ_ϕ measure zero. Since *g* is continuous, $g^{-1}(D_f)\subseteq D_{f\circ g}$. Therefore, $m^*(\phi\circ g(g^{-1}(D_f)))=0$. This means $m^*(\phi(D_f))=0$. Since ϕ is continuous,

$\mu_\phi(D_f) = m^*(\phi(D_f)) = 0$. Hence, f is Riemann Stieltjes integrable with respect to ϕ on [c, d] and hence is also Lebesgue Stieltjes integrable with respect to ϕ. Thus, $\int_a^b f \circ g(x) d(\phi \circ g) = \int_{g(a)}^{g(b)} f(x) d\phi$ as Riemann Stieltjes integrals.

(ii).

Let $\int_c^d f(x) d\phi = \int_{g(a)}^{g(b)} f(x) d\phi = L$. Then given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition $P: y_0 = c < y_1 < y_2 < \cdots < y_n = d$ for [c, d] with $\|P\| < \delta$, any Riemann Stieltjes sum, with any $\xi_i \in [y_{i-1}, y_i]$,

$$\left| \sum_{i=1}^{n} f(\xi_i)(\phi(y_i) - \phi(y_{i-1})) - L \right| < \varepsilon .$$

Since g is continuous on [a, b], there exists $\delta_1 > 0$ such that $|g(x) - g(y)| < \delta$ whenever $|x - y| < \delta_1$. Let $Q: x_0 = a < x_1 < x_2 < \cdots < x_n = b$ be any partition for [a, b] with $\|Q\| < \delta_1$.

Suppose now *g* is strictly increasing and continuous.

Then $P: g(x_0) = g(a) = c < g(x_1) < g(x_2) < \cdots < g(x_n) = g(b) = d$ is a partition for [c, d] with $\|P\| < \delta$. Therefore, for any $\eta_i \in [x_{i-1}, x_i]$,

$$\left| \sum_{i=1}^{n} f \circ g(\eta_i)(\phi \circ g(x_i) - \phi \circ g(x_{i-1})) - L \right| = \left| \sum_{i=1}^{n} f(g(\eta_i))(\phi(y_i) - \phi(y_{i-1})) - L \right| < \varepsilon .$$

This implies that the Riemann Stieltjes integral $\int_a^b f \circ g(x) d(\phi \circ g)$ exists and $\int_a^b f \circ g(x) d(\phi \circ g) = \int_{g(a)}^{g(b)} f(x) d\phi$.

If *g* is just increasing and continuous, the proof is a little more delicate. Let $\int_c^d f(x) d\phi = \int_{g(a)}^{g(b)} f(x) d\phi = L$. Then given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition $P: y_0 = c < y_1 < y_2 < \cdots < y_n = d$ for [c, d] with $\|P\| < \delta$, any Riemann Stieltjes sum, with any $\xi_i \in [y_{i-1}, y_i]$,

$$\left| \sum_{i=1}^{n} f(\xi_i)(\phi(y_i) - \phi(y_{i-1})) - L \right| < \varepsilon .$$

Since g is continuous on [a, b], there exists $\delta_1 > 0$ such that $|g(x) - g(y)| < \delta$ whenever $|x - y| < \delta_1$. Let $Q: x_0 = a < x_1 < x_2 < \cdots < x_n = b$ be any partition for [a, b] with $\|Q\| < \delta_1$.

Then $g(x_0)=g(a)=c\le g(x_1)\le g(x_2)\le\cdots\le g(x_n)=g(b)=d$. This gives rise to a partition $P_1: g(z_0)=g(a)=c\le g(z_1)<g(z_2)<\cdots<g(z_m)=g(b)=d$, with $m\le n$ and $\|P_1\|<\delta$. Consider the Riemann sum $\sum_{i=1}^{m} f\circ g(\eta_i')(\phi\circ g(z_i)-\phi\circ g(z_{i-1}))$ with $\eta_i'\in[z_{i-1},z_i]$.

We shall show that this is a Riemann Stieltjes sum for the integral $\int_c^d f(x)d\phi$.

Note that $\{z_1,z_2,\cdots,z_m\}\subseteq\{x_1,x_2,\cdots,x_m\}$. Suppose $z_{i-1}<x_k<x_{k+1}<\cdots<x_j<z_i$.

Then if $g(z_{i-1})<g(x_k)$, $g(x_k)=g(x_{k+1})=\cdots g(x_j)=g(z_i)$, $g(z_i)=g(x_j)<g(x_{j+1})$, $z_i=x_j$ and $z_{i-1}=x_{k-1}$. Moreover, $|g(z_i)-g(z_{i-1})|=|g(x_k)-g(x_{k-1})|<\delta$.

$$f\circ g(\eta_i')(\phi\circ g(z_i)-\phi\circ g(z_{i-1}))=f(g(\eta_i'))(\phi(g(z_i))-\phi(g(z_{i-1})))$$

$$=f(g(\eta_i'))(\phi(g(x_j))-\phi(g(x_{k-1})))=f(g(\eta_i'))(\phi(g(x_k))-\phi(g(x_{k-1})))\,,$$

Note that

$$f(g(\eta_i'))(\phi(g(z_i))-\phi(g(z_{i-1})))=f(g(\eta_i'))(\phi(g(x_j))-\phi(g(x_{j-1}))+\phi(g(x_{j-2}))-\phi(g(x_{j-2}))+\cdots$$

$$+\phi(g(x_{k+1}))-\phi(g(x_{k+1}))+\phi(g(x_k))-\phi(g(x_k))-\phi(g(x_{k-1})))$$

$\eta_i'\in[x_{k-1},x_j]$. Since $g(x_k)=g(x_{k+1})=\cdots g(x_j)=g(z_i)$, $g(\eta_i')$ lies between

$g(x_{k-1})$ and $g(x_k)$, $g(\eta_i')=g(\eta_k)$ for some $\eta_k\in[x_{k-1},x_k]$ and

$$f\circ g(\eta_i')(\phi\circ g(z_i)-\phi\circ g(z_{i-1}))=f(g(\eta_k))(\phi(g(x_k))-\phi(g(x_{k-1})))\,.$$

If $g(z_{i-1})=g(x_k)$,i.e., $z_{i-1}=x_k$, then $g(x_k)=g(x_{k+1})=\cdots g(x_j)<g(z_i)$, $g(x_k)=g(x_{k+1})=\cdots<g(z_i)=g(x_{j+1})$, $z_i=x_{j+1}$ and $|g(z_i))-g(z_{i-1})|=|g(x_{j+1}))-g(x_j)|<\delta$.

$$f\circ g(\eta_i')(\phi\circ g(z_i))-\phi\circ g(z_{i-1}))=f(g(\eta_i'))(\phi(g(z_i))-\phi(g(z_{i-1})))$$

$$=f(g(\eta_i'))(\phi(g(x_{j+1}))-\phi(g(x_k)))=f(g(\eta_i'))(\phi(g(x_{j+1}))-\phi(g(x_j)))\,,$$ where

$\eta_i'\in[z_{i-1},z_i]=[x_k,x_{j+1}]$.

Since $g(x_k)=g(x_{k+1})=\cdots g(x_j)<g(z_i)=g(x_{j+1})$, $g(\eta_i')$ lies between

$g(x_j)$ and $g(x_{j+1})$, $g(\eta_i')=g(\eta_{j+1})$ for some $\eta_{j+1}\in[x_j,x_{j+1}]$.

Hence, $f\circ g(\eta_i')(\phi\circ g(z_i)-\phi\circ g(z_{i-1}))=f(g(\eta_{j+1}))(\phi(g(x_{j+1}))-\phi(g(x_j)))$.

Therefore, $\sum_{i=1}^{m} f(g(\eta_i'))(\phi(g(z_i))-\phi(g(z_{i-1})))$ is a Riemann Stieltjes sum for the partition P_1 with $\|P_1\|<\delta$.

Thus, $\sum_{i=1}^{n} f\circ g(\eta_i)(\phi\circ g(x_i)-\phi\circ g(x_{i-1}))=\sum_{i=1}^{m} f(g(\eta_i'))(\phi(g(z_i))-\phi(g(z_{i-1})))$ is a Riemann Stieltjes sum for the partition P_1 with $\|P_1\|<\delta$.

Therefore,

$$\left|\sum_{i=1}^{n} f\circ g(\eta_i)(\phi\circ g(x_i)-\phi\circ g(x_{i-1}))-L\right|=\left|\sum_{i=1}^{m} f(g(\eta_i'))(\phi(g(z_i))-\phi(g(z_{i-1})))-L\right|<\varepsilon .$$

This shows that the Riemann Stieltjes integral $\int_a^b f\circ g(x)d(\phi\circ g)$ exists and $\int_a^b f\circ g(x)d(\phi\circ g)=\int_{g(a)}^{g(b)} f(x)d\phi$.

Suppose now g is strictly increasing and continuous. Then g has a continuous inverse $\eta:[c,d]\to[a,b]$. Suppose $f\circ g$ is Riemann Stieltjes integrable with respect to $\phi\circ g$ on [a, b]. Let $\int_a^b f\circ g(x)d(\phi\circ g)=M$. $f\circ g$ is Riemann Stieltjes integrable with respect to $\phi\circ g$ implies that for any $\varepsilon>0$, there exists a $\delta>0$ such that for any partition, $Q: x_0=a<x_1<x_2<\cdots<x_n=b$, with $\|Q\|<\delta$, for any Riemann Stieltjes sum $\sum_{i=1}^{n} f\circ g(\eta_i)(\phi\circ g(x_i)-\phi\circ g(x_{i-1}))$ for any $\eta_i\in[x_{i-1},x_i]$,

$$\left|\sum_{i=1}^{n} f\circ g(\eta_i)(\phi\circ g(x_i)-\phi\circ g(x_{i-1}))-M\right|<\varepsilon .$$

Since η is uniformly continuous, there exists $\delta_1>0$ such that $|\alpha-\beta|<\delta_1\Rightarrow|\eta(\alpha)-\eta(\beta)|<\delta$.

Take $P: y_0=c<y_1<y_2<\cdots<y_m=d$ a partition for [c, d] with $\|P\|<\delta_1$. Then $Q1: x_0=a<x_1<x_2<\cdots<x_m=b$, where $x_i=\eta(y_i)$, is a partition for [a, b], with $\|Q1\|<\delta$. Note that $g(x_i)=y_i$. Take any Riemann Stieltjes sum with respect to the parttition P, $\sum_{i=1}^{n} f(\xi_i)(\phi(y_i)-\phi(y_{i-1}))$, where $\xi_i\in[y_{i-1},y_i]$. Then there exists $\eta_i\in[x_{i-1}.x_i]$ such that $\xi_i=g(\eta_i)$. Therefore,

$$\sum_{i=1}^{n} f(\xi_i)(\phi(y_i)-\phi(y_{i-1}))=\sum_{i=1}^{n} f(g(\eta_i)(\phi(g(x_i)-\phi(g(x_{i-1})) .$$

It follows that $\left|\sum_{i=1}^{n} f(\xi_i)(\phi(y_i)-\phi(y_{i-1}))-M\right| = \left|\sum_{i=1}^{n} f(g(\eta_i)(\phi(g(x_i)-\phi(g(x_{i-1}))-M\right| < \varepsilon$.

This shows that *f* is Riemann Stieltjes integrable with respect to ϕ and

$\int_{g(a)}^{g(b)} f(x)d\phi = M = \int_a^b f\circ g(x)d(\phi\circ g)$.

(iii)

Write $\phi = \phi(a)+P-N$, where *P* and *N* are the positive and negative variation functions of ϕ. Then *P* and *N* are continuous. If $\int_a^b f\circ g(x)d(\phi\circ g)$ exists then $\int_a^b f\circ g(x)d(P\circ g)$ and $\int_a^b f\circ g(x)d(N\circ g)$ exist. By Part (i),
$\int_a^b f\circ g(x)d(P\circ g) = \int_{g(a)}^{g(b)} f(x)dP$ and $\int_a^b f\circ g(x)d(N\circ g) = \int_{g(a)}^{g(b)} f(x)dN$.

$$\int_{g(a)}^{g(b)} f(x)d\phi = \int_{g(a)}^{g(b)} f(x)dP - \int_{g(a)}^{g(b)} f(x)dN$$

$$= \int_a^b f\circ g(x)d(P\circ g) - \int_a^b f\circ g(x)d(N\circ g) = \int_a^b f\circ g(x)d(\phi\circ g) .$$

If *f* is Riemann Stieltjes integrable with respect to ϕ on [*c*, *d*], then by part (ii) $\int_a^b f\circ g(x)d(\phi\circ g)$ exists and $\int_a^b f\circ g(x)d(\phi\circ g) = \int_{g(a)}^{g(b)} f(x)d\phi$.

The next result dispenses with the condition of being strictly increasing for the function *g* in Theorem 5 part (ii).

Theorem 6. Suppose $g:[a,b]\to\mathbb{R}$ is an increasing continuous function. Let $J = g([a,b]) = [c,d]$ be the range of *g*. Suppose $\phi:[c,d]\to\mathbb{R}$ is a function of bounded variation. Suppose $f:[c,d]\to\mathbb{R}$ is a bounded Borel function. Then

$$\int_a^b f\circ g(x)d(\phi\circ g) = \int_{g(a)}^{g(b)} f(x)d\phi ,$$

whenever $f\circ g$ is Riemann Stieltjes integrable with respect to $\phi\circ g$ on [*a*, *b*] or f is Riemann Stieltjes integrable with respect to ϕ on [*c*, *d*].

Proof.

Note that $\phi\circ g$ is a function of bounded variation on [*a*, *b*].

By Corollary 62 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\int_a^b f\circ g(x)d\lambda_{\phi\circ g} = \int_{g(a)}^{g(b)} f(x)d\lambda_\phi$$

as Lebesgue Stieltjes integrals.

If f is Riemnn Stieltjes integrable with respect to ϕ on [c, d], then by Theorem 5 Part (ii), $f\circ g$ is Riemann Stieltjes integrable with respect to $\phi\circ g$ on [a, b]. Hence, $f\circ g$ is Lebesgue Stieltjes integrable with respect to $\phi\circ g$ on [a, b]. Note that f is Lebesgue Stieltjes integrable with respect to ϕ. Hence,
$\int_a^b f\circ g(x)d(\phi\circ g) = \int_{g(a)}^{g(b)} f(x)d\phi$

Suppose $f\circ g$ is Riemann Stieltjes integrable with respect to $\phi\circ g$ on [a, b]. Then $f\circ g$ is Riemann Stieltjes integrable with respect to $v_{\phi\circ g}$ on [a, b]. Therefore, $f\circ g$ is Lebesgue Stieltjes integrable with respect to $\phi\circ g$ and also with respect to $v_{\phi\circ g}$ on [a, b]. Let $\phi = \phi(c) + P - N$, where P and N are the positive and negative variations functions of ϕ. Let $D_{f\circ g}$ be the set of discontinuities of $f\circ g$. Then $\mu_{v_{f\circ g}}(D_{f\circ g}) = \mu_P(D_{f\circ g}) = \mu_N(D_{f\circ g}) = \lambda_{f\circ g}(D_{f\circ g}) = 0$.

We now proceed with the special case that ϕ is an increasing function so that we can apply the result to the functions P and N and give the desire conclusion.

If g is an increasing function and ϕ an increasing function, by the proof of Theorem 48 in "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*", for any Borel set B in [a, b], $\mu_{\phi\circ g}(B) = \mu_\phi(v^{-1}(B))$, where v is the generalized inverse of g. Note that v is an increasing left continuous function.

Let D_f be the set of discontinuities of f. We may assume that D_f does not meet the set of local constant values of g for this set is countable and is μ_{v_ϕ} null. As g is continuous, $g^{-1}(D_f) \subseteq D_{f\circ g}$.

Since g is increasing and continuous, then
$\mu_{\phi\circ g}(g^{-1}(D_f)) = \mu_\phi(v^{-1}(g^{-1}(D_f))) = \mu_\phi(D_f)$. The last equality is the consequence of the fact that g is the left inverse of v because g is also continuous.

Thus, if g is increasing and continuous and $f\circ g$ is Riemann Stieltjes integrable with respect to $\phi\circ g$ on [a, b], then $\mu_{\phi\circ g}(g^{-1}(D_f)) = 0$ and so $\mu_\phi(D_f) = 0$. Thus, we can conclude that if g is increasing and continuous and ϕ is increasing and if $f\circ g$ is Riemann Stieltjes integrable with respect to $\phi\circ g$ on [a, b], then f is

Riemann Stieltjes integrable with respect to ϕ on [*c*, *d*] and so
$\int_a^b f \circ g(x) d(\phi \circ g) = \int_{g(a)}^{g(b)} f(x) d\phi$.

Now if ϕ is of bounded variation, then $\phi = (\phi(c) + P) - N$, where *P* and *N* are the positive and negative variation functions of ϕ and they are increasing functions. By what we have just shown, *f* is Riemann Stieltjes integrable with respect to *P* and *N* on [*c*, *d*]. Hence, *f* is Riemann Stieltjes integrable with respect to ϕ on $[c, d]$.

Therefore, $\int_a^b f \circ g(x) d(P \circ g) = \int_{g(a)}^{g(b)} f(x) dP$, $\int_a^b f \circ g(x) d(N \circ g) = \int_{g(a)}^{g(b)} f(x) dN$ and

$$\int_a^b f \circ g(x) d(\phi \circ g) = \int_a^b f \circ g(x) d(P \circ g) + \int_a^b f \circ g(x) d(N \circ g)$$

$$= \int_{g(a)}^{g(b)} f(x) dP + \int_{g(a)}^{g(b)} f(x) dN = \int_{g(a)}^{g(b)} f(x) d\phi$$

Therefore, it follows that if *g* is increasing and continuous, ϕ is of bounded variation and if $f \circ g$ is Riemann Stieltjes integrable with respect to $\phi \circ g$ on [*a*, *b*], then *f* is Riemann Stieltjes integrable with respect to ϕ on [*c*, *d*] and
$\int_a^b f \circ g(x) d(\phi \circ g) = \int_{g(a)}^{g(b)} f(x) d\phi$.

Theorem 7. Suppose $g : [a, b] \to \mathbb{R}$ is a continuous function. Let $J = g([a, b]) = [c, d]$ be the range of *g*. Assume that *c* < *d*. Suppose $\phi : [c, d] \to \mathbb{R}$ is a continuous function of bounded variation, $\phi \circ g$ is a function of bounded variation and $f : [c, d] \to \mathbb{R}$ is a bounded Borel function.

If $f \circ g$ is Riemann Stieltjes integrable with respect to $\phi \circ g$ on [*a, b*], then *f* is Riemann Stieltjes integrable with respect to ϕ on [*c*, *d*].

Proof.

If $f \circ g$ is Riemann Stieltjes integrable with respect to $\phi \circ g$ on [*a*, *b*], then it is Riemann Stieltjes integrable with respect to the total variation function of $\phi \circ g$, $v_{\phi \circ g}$, on [*a*, *b*]. Therefore, $\mu_{v_{\phi \circ g}}(D_{f \circ g}) = 0$, where $D_{f \circ g}$ is the set of discontinuities of $f \circ g$ on [*a*, *b*]. Since $\phi \circ g$ is continuous, $v_{\phi \circ g}$ is also continuous. Therefore, $m^*(v_{\phi \circ g}(D_{f \circ g})) = \mu_{v_{f \circ g}}(D_{f \circ g}) = 0$. Hence, $m^*(\phi \circ g(D_{f \circ g})) = 0$. (See Theorem 16 of "*Functions of Bounded Variation and Johnson's Indicatrix*.) Let D_f be the set of discontinuities of the function *f*. We may assume that D_f does not meet the set of local constant values of *g*. As *g* is continuous, $g^{-1}(D_f) \subseteq D_{f\circ g}$. Therefore, $m^*(\phi\circ g(g^{-1}(D_f)))=0$. That is, $m^*(\phi(D_f))=0$. As ϕ is a function of bounded variation, by Theorem 1 of "*Functions of Bounded Variation and Johnson's Indicatrix*", $m^*(v_\phi(D_f))=0$. As v_ϕ is continuous, $\mu_{v_\phi}(D_f)=m^*(v_\phi(D_f))=0$. Therefore, $m^*(P(D_f))=m^*(N(D_f))=0$, where *P* and *N* are the positive and negative variation functions of ϕ. Note that $\mu_P(D_f)=\mu_N(D_f)=0$. Hence, *f* is Riemann Stieltjes integrable with respect to *P* and *N* on [*c*, *d*] and consequently, *f* is Riemann Stieltjes integrable with respect to ϕ on [*c*, *d*].

Theorem 8. Suppose $g:[a,b]\to\mathbb{R}$ is a continuous function, the range of *g*, $g([a,b])=J=[c,d]$ and $c<d$. Suppose $\phi:J\to\mathbb{R}$ is a continuous function of bounded variation, $\phi\circ g$ is of bounded variation and $f:J\to\mathbb{R}$ is a bounded Borel function. Suppose $f\circ g$ is Riemann Stieltjes integrable with respect to $\phi\circ g$ on [*a*, *b*]. Then the function *f* is Riemann Stieltjes integrable with respect to ϕ on [*c*, *d*] and $\int_a^b f\circ g(x)d(\phi\circ g)=\int_{g(a)}^{g(b)} f(x)d\phi$ as Riemann Stieltjes integrals.

Proof.

By Theorem 7, *f* is Riemann Stieltjes integrable with respect to ϕ on [*c*, *d*].

Let $E=[c,d]-D_f$. Then *f* is continuous at every point of *E* and $\mu_{v_\phi}([c,d]-E)=\mu_P([c,d]-E)=\mu_N([c,d]-E)=0$. As μ_{v_ϕ} is a positive Radon measure, by the inner regularity of μ_{v_ϕ}, there exists a sequence (K_n) of compact set in [*c*, *d*] such that $K_n\subseteq E$, $K_n\subseteq K_{n+1}$ and $\mu_{v_\phi}(E-K_n)<\frac{1}{n}$. In particular, $\mu_P(E-K_n)<\frac{1}{n}$, $\mu_N(E-K_n)<\frac{1}{n}$ and $\mu_{v_\phi}([c,d]-K_n)<\frac{1}{n}$.

By the Tietze Extension Theorem, for each positive integer *n*, we can extend the restriction of *f* to K_n, to a continuous function f_n on [*c*, *d*] such that $\sup\{|f_n(x)|:x\in[c,d]\}\le\sup\{|f(x)|:x\in[c,d]\}$. Since f_n is continuous, by Theorem 20 of "*Limit of the Lebesgue Stieltjes Integral and Change of Variable*",

$$\int_a^b f_n\circ g(x)d\lambda_{\phi\circ g}=\int_{g(a)}^{g(b)} f_n(x)d\lambda_\phi .$$

By the Dominated Convergence Theorem, as f_n converges boundedly to f almost everywhere with respect to μ_P and μ_N and f is Lebesgue Stieltjes integrable with respect to μ_P and μ_N, $\int_{g(a)}^{g(b)} f_n(x)d\mu_P \to \int_{g(a)}^{g(b)} f(x)d\mu_P$ and $\int_{g(a)}^{g(b)} f_n(x)d\mu_N \to \int_{g(a)}^{g(b)} f(x)d\mu_N$. Therefore, $\int_{g(a)}^{g(b)} f_n(x)d\lambda_\phi \to \int_{g(a)}^{g(b)} f(x)d\lambda_\phi$. Hence, $\int_a^b f_n \circ g(x)d\lambda_{\phi\circ g} \to \int_{g(a)}^{g(b)} f(x)d\lambda_\phi$.

Let $K = \bigcup_{n=1}^{\infty} K_n$. Then $\mu_{v_\phi}(E-K)=0$. Let $h_n = f - f_n$. Then $h_n(x)=0$ for x in K_n, $\mu_{v_\phi}(x : h_n(x)\neq 0) \le \frac{1}{n}$ and $|h_n(x)| \le 2C$, where $C = \sup\{|f(x)| : x\in[c,d]\}$. Note that $\mu_{v_\phi}([c,d]-K_n) \le \frac{1}{n}$ and $\mu_{v_\phi}([c,d]-K)=0$. Since ϕ is continuous, v_ϕ is also continuous and so $m^*(v_\phi([c,d]-K)) = \mu_{v_\phi}([c,d]-K)=0$. Hence, $m^*(P([c,d]-K)) = \mu_P([c,d]-K)=0$ and $m^*(N([c,d]-K)) = \mu_N([c,d]-K)=0$. Note that $m^*(\phi([c,d]-K))=0$. Let $H_n = g^{-1}(K_n)$. We have $H_n \subseteq H_{n+1}$ and $H = \bigcup_{n=1}^{\infty} H_n = g^{-1}(K)$. Then $[a,b]-H = \bigcap_{n=1}^{\infty}\left([a,b]-H_n\right) = g^{-1}([c,d]-K)$ and

$$\int_a^b \left(f-f_n\right)\circ g(x)d\lambda_{\phi\circ g} = \int_a^b h_n \circ g(x)d\lambda_{\phi\circ g} = \int_{H_n\cup([a,b]-H_n)} h_n \circ g(x)d\lambda_{\phi\circ g}$$

$$= \int_{H_n} h_n \circ g(x)d\lambda_{\phi\circ g} + \int_{[a,b]-H_n} h_n \circ g(x)d\lambda_{\phi\circ g} = 0 + \int_{[a,b]-H_n} h_n \circ g(x)d\lambda_{\phi\circ g}$$

$$= \int_{[a,b]-H_n} h_n \circ g(x)d\lambda_{\phi\circ g} = \int_{g^{-1}([c,d]-K_n)} h_n \circ g(x)d\lambda_{\phi\circ g}.$$

Now, $$\left|\int_{g^{-1}([c,d]-K_n)} h_n \circ g(x)d\lambda_{\phi\circ g}\right| \le 2C\int_{g^{-1}([c,d]-K_n)} d\mu_{v_{\phi\circ g}}. \quad \text{------------ (1)}$$

Since $g^{-1}\left([c,d]-K_n\right) = [a,b]-H_n \supseteq [a,b]-H_{n+1}$, $g^{-1}\left([c,d]-K_n\right)$ tends to $g^{-1}([c,d]-K)$. Therefore, $\int_{g^{-1}([c,d]-K_n)} d\mu_{v_{\phi\circ g}}$ tends to $\int_{g^{-1}([c,d]-K)} d\mu_{v_{\phi\circ g}}$. Since $v_{\phi\circ g}$ is continuous and increasing, $\int_{g^{-1}([c,d]-K)} d\mu_{v_{\phi\circ g}} = m^*\left(v_{\phi\circ g}\left(g^{-1}([c,d]-K)\right)\right)$.

Now, $m^*(\phi\circ g(g^{-1}([c,d]-K))) = m^*(\phi([c,d]-K)) = 0$. Since $\phi\circ g$ is a continuous function of bounded variation, $m^*\left(v_{\phi\circ g}\left(g^{-1}([c,d]-K)\right)\right)=0$. Hence, $\int_{g^{-1}([c,d]-K_n)} d\mu_{v_{\phi\circ g}}$ tends to 0. It follows from (1) that $\int_{g^{-1}([c,d]-K_n)} h_n \circ g(x)d\lambda_{\phi\circ g}$ tends to 0 and so $\int_a^b \left(f-f_n\right)\circ g(x)d\lambda_{\phi\circ g}$ tends to 0.

On the other hand,

$$\int_a^b (f - f_n) \circ g(x) d\lambda_{\phi\circ g} = \int_a^b f \circ g(x) d\lambda_{\phi\circ g} - \int_a^b f_n \circ g(x) d\lambda_{\phi\circ g}$$

tends to $\int_a^b f \circ g(x) d\lambda_{\phi\circ g} - \int_{g(a)}^{g(b)} f d\lambda_\phi$. Therefore, $\int_a^b f \circ g(x) d\lambda_{\phi\circ g} = \int_{g(a)}^{g(b)} f d\lambda_\phi$. It follows that $\int_a^b f \circ g(x) d(\phi \circ g) = \int_{g(a)}^{g(b)} f d\phi$ as Riemann Stieltjes integrals.

Remark. Taking g to be the identity function on $[a, b]$, we get Theorem 4 as a special case of Theorem 8.

Jan 2026