# Function of Bounded Variation on Arbitrary Subset and Johnson's Indicatrix

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In this article, we aim to prove that if a function of bounded variation maps a subset to a set of measure zero, then its total variation function will map the same set to a set of measure zero too. We use the same technique used as before in my article, *Functions of Bounded Variation and Johnson's Indicatrix*.

We use the notation and definitions of terms given in *Functions of Bounded Variation and de La Vallée Poussin's Theorem* and *Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem*.

Suppose $f : A \to \mathbb{R}$ is a function of bounded variation and *A* is a subset of $\mathbb{R}$.

We shall assume initially that *A* is a bounded subset and extend the definition to arbitrary unbounded subset *A*.

Take $a_1, a_2 \in A$ with $a_1 < a_2$ and consider the closed interval $I = [a_1, a_2]$. Let $\tilde{I} = [a_1, a_2] \cap A$. Let $v_f = v_{f,a} : A \to \mathbb{R}$ be the total variation function of *f* on *A*. Then there exists a sequence, $\{P_n\}$, of partitions of the closed interval, *I* by points in *A* such that $P_n \subseteq P_{n+1} \subseteq \cdots$ and

$$\lim_{n\to\infty} \sum_{P_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right| = \text{total variation of } f \text{ on } \tilde{I} = v_f(a_2) - v_f(a_1) = M ,$$

where $P_n : a_1 = x_{0,n} < x_{1,n} < \cdots < x_{k_n,n} = a_2$, $\sum_{P_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right| = \sum_{j=1}^{k_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right|$.

Observe that by definition of the total variation over $\tilde{I}$, given any positive integer, *n*, there exists a partition, $P_n$, such that

$$M - \frac{1}{n} < \sum_{P_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right| \le M$$

and there exists a partition, *Q*, such that

$$M-\frac{1}{n+1}<\sum_{Q}\left|f(x_{j,n})-f(x_{j-1,n})\right|\leq M .$$

We can then choose $P_{n+1}$ to be the refinement $P_n \cup Q$. Starting with $n$ =1, we can then construct such a sequence $\{P_n\}$ and plainly, $\lim_{n\to\infty}\sum_{P_n}\left|f(x_{j,n})-f(x_{j-1,n})\right|=M$ .

For each partition, $P_n$ , we can define an indicatrix as follows.

For $1\leq j\leq k_n$, let $S_{j,n}$ be the closed intervals with $f(x_{j,n})$ and $f(x_{j-1,n})$ as end points, i.e., $S_{j,n}=\left[f(x_{j,n}),\ f(x_{j-1,n})\right]$ or $\left[f(x_{j-1,n}),\ f(x_{j,n})\right]$. Let $\chi\left(S_{j,n}\right)$ be the characteristic function of $S_{j,n}$. Then, plainly, $\chi\left(S_{j,n}\right)$ is Lebesgue integrable and

$$\int_{-\infty}^{\infty}\chi(S_{j,n})=\left|f(x_{j,n})-f(x_{j-1,n})\right| \quad \text{for } 1\leq j\leq k_n$$

For the partition, $P_n$ , let $T_n=\sum_{j=1}^{k_n}\chi\left(S_{j,n}\right)$. Then $T_n$ is a measurable function. In particular,

$$\int_{-\infty}^{\infty}T_n(y)dy=\sum_{j=1}^{k_n}\int_{-\infty}^{\infty}\chi(S_{j,n})=\sum_{j=1}^{k_n}\left|f(x_{j,n})-f(x_{j-1,n})\right|=\sum_{P_n}\left|f(x_{j,n})-f(x_{j-1,n})\right|.$$

Since $P_{n+1}$ refines $P_n$, plainly, $T_{n+1}\geq T_n$. It follows that $\{T_n\}$ is an increasing sequence of non-negative measurable and integrable functions. We now define for this sequence of partitions, $\{P_n\}$ for $\tilde{I}$ ,

$$T_{\tilde{I}}=T_{[a_1,a_2]\cap A}=\lim_{n\to\infty}T_n .$$

By the Monotone Convergence Theorem, the function $T_{\tilde{I}}$ is Lebesgue integrable and

$$\int_{-\infty}^{\infty}T_{\tilde{I}}(y)dy=\lim_{n\to\infty}\int_{-\infty}^{\infty}T_n(y)dy=\lim_{n\to\infty}\sum_{P_n}\left|f(x_{j,n})-f(x_{j-1,n})\right|=\text{total variation of over } \tilde{I} ,$$

$$=v_f(a_2)-v_f(a_1) . \text{ -------------------------------(1)}$$

**Definition 1.** We define the *indicatrix* of $f_{\tilde{I}}$ , the restriction of $f$ to $\tilde{I}=[a_1,a_2]\cap A$ to be $T_{\tilde{I}}$. Note that the function $T_{\tilde{I}}$ depends on the sequence of partitions $\{P_n\}$ used to define it. Nevertheless, $T_{\tilde{I}}$ is unique up to a set of measure zero. That is to say, if $T\tilde{I}$ is defined using another sequence of partitions, $\{Q_n\}$, then $T\tilde{I} = T_{\tilde{I}}$ almost everywhere on $\tilde{I} = [a_1, a_2] \cap A$.

**Lemma 2.** With notation as above, $T_{\tilde{I}}$ is unique up to a set of measure zero.

**Proof.**

Denote $T_{\tilde{I},\{P_n\}}$ to be the indicatrix function defined by the sequence of partitions $\{P_n\}$ and $T_{\tilde{I},\{Q_n\}}$ to be the indicatrix function defined by the sequence of partitions $\{Q_n\}$. Let $\{R_n\}$ be the common refinement of $\{P_n\}$ and $\{Q_n\}$, with $R_n = P_n \cup Q_n$. Then $T_{\tilde{I},\{R_n\}} = \lim_{n\to\infty} T_{\tilde{I},R_n}$, where $T_{\tilde{I},R_n}$ is the function defined using the partition, $R_n$ and

$$\int_{-\infty}^{\infty} T_{\tilde{I},\{R_n\}}(y)dy = \lim_{n\to\infty}\int_{-\infty}^{\infty} T_{\tilde{I},R_n}(y)dy = \lim_{n\to\infty}\sum_{R_n}\left| f(x_{j,n}) - f(x_{j-1,n})\right| = v_f(a_2) - v_f(a_1).$$

Since $R_n$ is a refinement of both $P_n$ and $Q_n$, $T_{\tilde{I},R_n} \ge T_{\tilde{I},P_n}, T_{\tilde{I},Q_n}$. Passing to the limit we have then $T_{\tilde{I},\{R_n\}} \ge T_{\tilde{I},\{P_n\}}, T_{\tilde{I},\{Q_n\}}$. We claim that $T_{\tilde{I},\{R_n\}} = T_{\tilde{I},\{P_n\}}$ almost everywhere on $\tilde{I} = [a_1, a_2] \cap A$. Suppose there exists a subset $E$ of positive measure in $\tilde{I} = [a_1, a_2] \cap A$ such that $T_{\tilde{I},\{R_n\}} > T_{\tilde{I},\{P_n\}}$ then $\int_{-\infty}^{\infty} T_{\tilde{I},\{R_n\}}(y)dy > \int_{-\infty}^{\infty} T_{\tilde{I},\{P_n\}}(y)dy$. But $\int_{-\infty}^{\infty} T_{\tilde{I},\{R_n\}}(y)dy = \int_{-\infty}^{\infty} T_{\tilde{I},\{P_n\}}(y)dy = v_f(a_2) - v_f(a_1)$ give s a contradiction. Hence, we have that $T_{\tilde{I},\{R_n\}} = T_{\tilde{I},\{P_n\}}$ almost everywhere on $\tilde{I}$. Similarly, $T_{\tilde{I},\{R_n\}} = T_{\tilde{I},\{Q_n\}}$ almost everywhere on $\tilde{I}$. Therefore, $T_{\tilde{I},\{P_n\}} = T_{\tilde{I},\{Q_n\}}$ almost everywhere on $\tilde{I}$.

## Johnson's Indicatrix

**Definition 3.**

Suppose $f : A \to \mathbb{R}$ is a function of bounded variation and $A$ is a bounded subset of $\mathbb{R}$. Let $a = \inf A$ and $b = \sup A$. Then $A \subseteq [\inf A, \sup A] = [a, b]$.

Suppose inf *A* and sup *A* belongs to *A*. Then the indicatrix function $T_A$ is given by Definition 1.

Suppose $a \notin A$ and $b \in A$. Take a sequence $(a_n)$ in *A* such that $a_n \searrow a$. Define the indicatrix function on *A*

$$T_A = \lim_{n\to\infty} T_{[a_n,b]\cap A}.$$

Note that $\{T_{[a_n,b]\cap A}\}$ is an increasing sequence of functions, which are Lebesgue integrable. Moreover,

$$\int_{-\infty}^{\infty} T_A(y)dy = \lim_{n\to\infty}\int_{-\infty}^{\infty} T_{[a_n,b]\cap A}(y)dy = \lim_{n\to\infty} v_f(b) - v_f(a_n) = \text{the total variation of } f \text{ on } A.$$

Note that $v_f(b) - v_f(a_n)$ does not depend on the anchor point used to define the variation function. Note that $T_A$ does not depend on the sequence $(a_n)$. If $(a_n)$ is another sequence in *A* such that $a_n \searrow a$, then we can find a subsequence $(a_{n_k})$ such that $a_{n_k} < a_n$. Observe that since $(a_{n_k})$ is a subsequence of $(a_n)$,

$$\lim_{n\to\infty} T_{[a_n,b]\cap A} = \lim_{k\to\infty} T_{[a_{n_k},b]\cap A} \text{ almost everywhere on } A.$$

Therefore, $\lim_{n\to\infty} T_{[a_n,b]\cap A} = \lim_{k\to\infty} T_{[a_{n_k},b]\cap A} \geq \lim_{k\to\infty} T_{[a_k,b]\cap A}$ almost everywhere on *A*. By the same reasoning we can show that $\lim_{n\to\infty} T_{[a_n,b]\cap A} \geq \lim_{k\to\infty} T_{[a_n,b]\cap A}$ almost everywhere on *A*. Hence, $\lim_{n\to\infty} T_{[a_n,b]\cap A} = \lim_{n\to\infty} T_{[a_n,b]\cap A}$ almost everywhere on *A*. Thus, $T_A = \lim_{n\to\infty} T_{[a_n,b]\cap A}$ is defined.

Suppose $a \in A$ and $b \notin A$. Take a sequence $(b_n)$ in *A* such that $b_n \nearrow b$. Define the indicatrix function on *A*

$$T_A = \lim_{n\to\infty} T_{[a,b_n]\cap A}.$$

Note that

$$\int_{-\infty}^{\infty} T_A(y)dy = \lim_{n\to\infty}\int_{-\infty}^{\infty} T_{[a,b_n]\cap A}(y)dy = \lim_{n\to\infty} v_f(b_n) - v_f(a) = \text{the total variation of } f \text{ on } A.$$

Suppose $a \notin A$ and $b \notin A$. Take a sequence $(a_n)$ in *A* such that $a_n \searrow a$ and a sequence $(b_n)$ in *A* such that $b_n \nearrow a$. Consider $(a,b_n]\cap A$. We have just defined

$$T_{(a,b_n]\cap A} = \lim_{k\to\infty} T_{[a_k,b_n]\cap A}$$ and that

$\int_{-\infty}^{\infty} T_{(a,b_n]\cap A}(y)dy = \lim_{k\to\infty}\int_{-\infty}^{\infty} T_{[a_k,b_n]\cap A}(y)dy = \lim_{k\to\infty} v_f(b_n) - v_f(a_k)$ = total variation of *f* on $(a,b_n]\cap A$. Define $T_{(a,b)\cap A} = \lim_{n\to\infty} T_{(a,b_n]\cap A}$. Then

$$\int_{-\infty}^{\infty} T_A(y)dy = \int_{-\infty}^{\infty} T_{(a,b)\cap A}(y)dy = \lim_{n\to\infty}\int_{-\infty}^{\infty} T_{(a,b_n]\cap A}(y)dy = \lim_{n\to\infty}\left(v_f(b_n) - \lim_{k\to\infty} v_f(a_k)\right)$$

= total variation of *f* on $(a,b)\cap A = A$.

Suppose $f : A \to \mathbb{R}$ is a function of bounded variation and *A* is an unbounded subset of $\mathbb{R}$. If it is not bounded above but bounded below, then we can take a sequence $(b_n)$ in *A* such that $b_n \nearrow \infty$ and define the indicatrix function $T_A$ as in the above procedure and $\int_{-\infty}^{\infty} T_A(y)dy$ = total variation of *f* on *A*. Similarly, we can define $T_A$ when *A* is not bounded below but bounded above. Finally, we can define $T_A$ when *A* is not bounded above and below in a similar fashion. Moreover, $\int_{-\infty}^{\infty} T_A(y)dy$ = total variation of *f* on *A*.

The nest result is an immediate consequence of the definition of the indicatrix function.

**Lemma 4.** Suppose $f : A \to \mathbb{R}$ is a function of bounded variation and *A* is a subset of $\mathbb{R}$. Suppose $a, b \in A$ and $a < b$. Let $\tilde{I} = [a,b]\cap A$. Suppose

$$y \notin \left[\inf\left\{f(x) : x \in \tilde{I}\right\}, \sup\left\{f(x) : x \in \tilde{I}\right\}\right].$$

Then $T_{\tilde{I}}(y) = 0$.

**Lemma 5.** Suppose $f : A \to \mathbb{R}$ is a function of bounded variation. Suppose $\{I_j\}$ is a sequence of pairwise disjoint closed intervals with end points in *A*. Then $T_A(y) \geq \sum_j T_{\tilde{I}_j}(y)$ almost everywhere on *A*.

**Proof.**

We prove the inequality for a finite number of the sequence $\{I_j\}$. Note that these are pairwise disjoint subsets. Let $\tilde{I}_j = I_j \cap A$. Take *k* of these sets, $\tilde{I}_1, \tilde{I}_2, \cdots, \tilde{I}_k$. Suppose $I_j = [a_j, b_j]$.

Let $a = \min\{a_1, a_2, \cdots, a_k\}$ and $b = \max\{b_1, b_2, \cdots, b_k\}$ and $I = [a,b]$. Take typical sequences of partitions for $\tilde{I} = [a,b] \cap A$, and $\tilde{I}_1, \tilde{I}_2, \cdots, \tilde{I}_k$ for the definition of the indicatrix functions. Refine the sequence of partitions for $\tilde{I} = [a,b] \cap A$ to include all the partitions for $\tilde{I}_1, \tilde{I}_2, \cdots, \tilde{I}_k$. Denote the new sequence of partitions for $\tilde{I}$ by $\{R_n\}$ and the sequences of partitions for $\tilde{I}_1, \tilde{I}_2, \cdots, \tilde{I}_k$ by $\{P_{1,n}\}, \{P_{2,n}\}, \cdots, \{P_{k,n}\}$. Observe that the members of the collection of sequences are each collection of disjoint sets, i.e., for each integer *n*, $\{P_{1,n}, P_{2,n}, \cdots, P_{k,n}\}$ is a collection of disjoint sets. Hence, $T_{\tilde{I},R_n}(y) \geq \sum_{j=1}^{k} T_{I_j,P_{j,n}}(y)$. Then passing to the limit we have then $T_{\tilde{I},\{R_n\}}(y) \geq \sum_{j=1}^{k} T_{I_j,\{P_{j,n}\}}(y)$ almost everywhere on $\tilde{I}$. Therefore, by Lemma 4, $T_A(y) \geq T_{\tilde{I},\{R_n\}}(y) \geq \sum_{j=1}^{k} T_{I_j,\{P_{j,n}\}}(y)$ almost everywhere on *A*. It follows that

$$T_A(y) \geq \lim_{k\to\infty} \sum_{j=1}^{k} T_{I_j,\{P_{j,n}\}}(y) = \sum_{j=1}^{\infty} T_{I_j,\{P_{j,n}\}}(y) \text{ almost everywhere on } A.$$

Dropping the reference to the partitions used to defined the indicatrix functions, we obtain, $T_A(y) \geq \sum_{j=1}^{\infty} T_{I_j}(y)$ almost everywhere on *A*.

The next result gives a bound to the image, under the total variation function, of the points of *A* in a closed interval, with end points in *A*, by the integral of the indicatrix function. This is a crucial inequality used to limit the bound of the image of the total variation function.

**Lemma 6.** Suppose $f : A \to \mathbb{R}$ is a function of bounded variation and *A* is a subset of $\mathbb{R}$. Suppose $a, b \in A$ and $a < b$. Let $\tilde{I} = [a,b] \cap A$. Then

$$m^*(v_f(\tilde{I})) = m^*(v_f([a,b] \cap A)) \leq \int_{-\infty}^{\infty} T_{\tilde{I}}(y)\,dy,$$

where $m^*$ is the Lebesgue outer measure.

**Proof.**

$$m^*(\nu_f(\tilde{I})) = m^*(\nu_f([a,b]\cap A)) \le \nu_f(b) - \nu_f(a) = \int_{-\infty}^{\infty} T_{\tilde{I}}(y)dy .$$

**Lemma 7.** Suppose $f : A \to \mathbb{R}$ is a function of bounded variation and $A$ is a subset of $\mathbb{R}$. Suppose $\{I_j\}$ is a sequence of pairwise disjoint closed intervals with end points in $A$. Let $I_j = [a_j, b_j]$, $a_j < b_j$, $a_j, b_j \in A$. Let $S = \bigcup_j I_j$, $\tilde{I}_j = [a_j, b_j] \cap A$ and $S = \left(\bigcup_j I_j\right) \cap A = \bigcup_j \tilde{I}_j$. Suppose $E$ is a measurable subset of $\mathbb{R}$ such that $\left[\inf\left\{f(x) : x \in I_j \cap A\right\}, \sup\left\{f(x) : x \in I_j \cap A\right\}\right] = \left[\inf f(\tilde{I}_j), \sup f(\tilde{I}_j)\right] \subseteq E$ for each integer $j$. Then

$$m^*\left(\nu_f(S)\right) = m^*\left(\nu_f(S\cap A)\right) \le \int_E \sum_j T_{\tilde{I}_j}(y)dy \le \int_E T_A(y)dy .$$

**Proof.**

$m^*\left(\nu_f(S)\right) \le \sum_j m^*\left(\nu_f(\tilde{I}_j)\right) \le \sum_j \int_{-\infty}^{\infty} T_{\tilde{I}_j}(y)dy$, by Lemma6,

$\le \sum_j \int_E T_{\tilde{I}_j}(y)dy$, by Lemma 4,

since $\left[\inf\left\{f(x) : x \in I_j \cap A\right\}, \sup\left\{f(x) : x \in I_j \cap A\right\}\right] = \left[\inf f(\tilde{I}_j), \sup f(\tilde{I}_j)\right] \subseteq E$,

$\le \int_E T_A(y)dy$, by Lemma 5.

The next result is useful for the approach to using finite union of subsets before passing to infinite union of subsets.

**Lemma 8**. Suppose $\{A_j\}$ is a sequence of subsets of $\mathbb{R}$, uniformly bounded. Then there exists an integer $k$ such that

$$m^*\left(\bigcup_{n=1}^{k} A_n\right) \ge \frac{1}{2} m^*\left(\bigcup_{n=1}^{\infty} A_n\right) ,$$

where $m^*$ is the Lebesgue outer measure.

**Proof.** Note that $\bigcup_{n=1}^{\infty} A_n$ is bounded and so $m^*\left(\bigcup_{n=1}^{\infty} A_n\right)$ is finite.

If $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$, then we have nothing to prove since both sides of the inequality are zero.

Suppose now $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) > 0$. Then by the continuity from below property of Lebesgue outer measure,

$$\lim_{j\to\infty} m^*\left(\bigcup_{n=1}^{j} A_n\right) = m^*\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Therefore, there exists an integer $k > 0$, such that for all $j \geq k$, we have that

$$\left| m^*\left(\bigcup_{n=1}^{j} A_n\right) - m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \right| < \frac{1}{2} m^*\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Hence, $m^*\left(\bigcup_{n=1}^{k} A_n\right) > m^*\left(\bigcup_{n=1}^{\infty} A_n\right) - \frac{1}{2} m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \frac{1}{2} m^*\left(\bigcup_{n=1}^{\infty} A_n\right)$.

It is easier to prove the result we stated at the outset on set where the function is continuous. We formulate the special case in the next theorem.

**Theorem 9.** Suppose $f : A \to \mathbb{R}$ is a function of bounded variation and $A$ is a subset of $\mathbb{R}$. Suppose $E$ is a subset of $A$ such that $f$ is continuous at every point of $E$ and that the measure of its image under $f$, $m(f(E))$, is zero. Then $m(v_f(E)) = 0$.

**Proof.**

We may assume that every point of $E$ is a two-sided limit point of $A$ because isolated points and one-sided only limit points constitute at most a denumerable set.

Since $m(f(E)) = 0$ and $f(E)$ is bounded, for any positive integer, $n$, there exists a bounded open set $U_n$ such that $f(E) \subseteq U_n$ and $m(U_n) \leq \frac{1}{n}$ .

Take $e \in E$. Then $f(e) \in U_n$ and so there exists $\varepsilon_e > 0$ such that $\left( f(e) - \varepsilon_e, f(e) + \varepsilon_e \right) \subseteq U_n$. As *f* is continuous at *e*, there exists $\delta_e > 0$ such that $f\left( (e-\delta_e, e-\delta_e) \cap A \right) \subseteq \left( f(e) - \frac{\varepsilon_e}{2}, f(e) + \frac{\varepsilon_e}{2} \right)$. Since *e* is a two-sided limit point of *A*, there exists $a_e \in (e - \frac{\delta_e}{2}, e) \cap A$ and $b_e \in (e, e + \frac{\delta_e}{2}) \cap A$. Let $I_e = [a_e, b_e]$. Then

$f\left( I_e \cap A \right) \subseteq \left( f(e) - \frac{\varepsilon_e}{2}, f(e) + \frac{\varepsilon_e}{2} \right) \subseteq U_n$. Therefore,

$\left[ \inf f\left( I_e \cap A \right), \sup f\left( I_e \cap A \right) \right] \subseteq \left[ f(e) - \frac{\varepsilon_e}{2}, f(e) + \frac{\varepsilon_e}{2} \right] \subseteq \left( f(e) - \varepsilon_e, f(e) + \varepsilon_e \right) \subseteq U_n$.

The collection $\Gamma = \{(a_e, b_e) : e \in E\}$ is an open cover for *E*. Therefore, by Lindelöf Theorem, Γ has a countable subcover, $\mathscr{C} = \left\{ \operatorname{int} I_{e_i} ; i = 1, 2, \cdots \right\}$.

We claim that

$$m^*\left( \nu_f \left( \bigcup_{i=1}^{\infty} I_{e_i} \cap A \right) \right) = m^*\left( \nu_f \left( \bigcup_{i=1}^{\infty} \tilde{I}_{e_i} \right) \right) \le 2\int_{U_n} T_A(y)dy \text{, where } \tilde{I}_{e_i} = I_{e_i} \cap A \text{. ------ (*)}$$

By Lemma 8, $\frac{1}{2} m^*\left( \nu_f \left( \bigcup_{i=1}^{\infty} \tilde{I}_{e_i} \right) \right) \le m^*\left( \nu_f \left( \bigcup_{i=1}^{k} \tilde{I}_{e_i} \right) \right)$ for some positive integer *k*.

Thus,

$$m^*\left( \nu_f \left( \bigcup_{i=1}^{\infty} \tilde{I}_{e_i} \right) \right) \le 2m^*\left( \nu_f \left( \bigcup_{i=1}^{k} \tilde{I}_{e_i} \right) \right). \quad \text{---------------- (1)}$$

Note that $\bigcup_{i=1}^{k} I_{e_i}$ is a finite collection of closed intervals. Hence $\bigcup_{i=1}^{k} I_{e_i}$ is a finite disjoint collection of closed intervals, say, $C_1, C_2, \cdots, C_J$. Each $C_j$ is a union of a finite number of closed intervals in $\left\{ I_{e_i} : i = 1, 2, \cdots, k \right\}$, say , $I_1, I_2, \cdots, I_{n_j}$, where the union $\bigcup \left\{ I_1, I_2, \cdots, I_{n_j} \right\}$ cannot be partitioned into two disjoint collections. It follows that the corresponding collections

$$\left\{ \left[ \inf f(I_i \cap A), \sup f(I_i \cap A) \right], i = 1, 2, \cdots, n_j \right\},$$

also have the property that their union cannot be partitioned into two disjoint collections. We deduce this as follows. Suppose

$$\left[\inf f(I_1 \cap A), \sup f(I_1 \cap A)\right] \cap \left[\inf f(I_2 \cap A), \sup f(I_2 \cap A)\right] = \varnothing .$$

Then, $(I_1 \cap A) \cap (I_2 \cap A) = \varnothing$, for if $(I_1 \cap A) \cap (I_2 \cap A)$ were to be non- empty then there exists $a \in (I_1 \cap A) \cap (I_2 \cap A)$ and

$f(a) \in \left[\inf f(I_1 \cap A), \sup f(I_1 \cap A)\right]$ and $f(a) \in \left[\inf f(I_2 \cap A), \sup f(I_2 \cap A)\right]$,

contradicting that $\left[\inf f(I_1 \cap A), \sup f(I_1 \cap A)\right] \cap \left[\inf f(I_2 \cap A), \sup f(I_2 \cap A)\right] = \varnothing$.

Because each $\left[\inf f(I_j \cap A), \sup f(I_j \cap A)\right] \subseteq U_n$, it follows that

$$\left[\min_{1 \le i \le n_j} \left\{\inf f(\tilde{I}_i)\right\}, \max_{1 \le i \le n_j} \left\{\sup f(\tilde{I}_i)\right\}\right] \subseteq \left[\inf f(C_j \cap A), \sup f(C_j \cap A)\right] \subseteq U_n .$$

Hence, by Lemma 7,

$$m^*\left(\nu_f\left(\bigcup_{i=1}^{k} \tilde{I}_{e_i}\right)\right) = m^*\left(\nu_f\left(\bigcup_{i=1}^{J} C_i\right)\right) \le \int_{U_n} T_A(y)dy .$$

Therefore, it follows from inequality (1) that

$$m^*\left(\nu_f\left(\bigcup_{i=1}^{\infty} \tilde{I}_{e_i}\right)\right) \le 2m^*\left(\nu_f\left(\bigcup_{i=1}^{k} \tilde{I}_{e_i}\right)\right) \le 2\int_{U_n} T_A(y)dy .$$

This proves the claim.

Since $E \subseteq \left(\bigcup_{i=1}^{\infty} \tilde{I}_{e_i}\right)$, $m^*\left(\nu_f(E)\right) \le m^*\left(\nu_f\left(\bigcup_{i=1}^{\infty} \tilde{I}_{e_i}\right)\right) \le 2\int_{U_n} T_A(y)dy$ .

Since $m(U_n) \to 0$, $\lim_{n \to \infty} \int_{U_n} T_A(y)dy = 0$. It follows that $m^*\left(\nu_f(E)\right) = 0$ .

This completes the proof of Theorem 9.

Finally, we state our main theorem as follows.

**Theorem 10.** Suppose $f : A \to \mathbb{R}$ is a function of bounded variation and *A* is a subset of $\mathbb{R}$. Suppose *E* is a subset of *A* such that $m(f(E))$ is zero. Then $m(\nu_f(E)) = 0$.

**Proof.** By Theorem 4 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, the set *D* of discontinuities of *f* is at most denumerable. It follows that $m\left(f(D)\right)=m\left(\nu_f(D)\right)=0$. Since $m(f(E))=0$, $m(f(E-D))=0$. Note that *f* is continuous at every point of $E-D$. Therefore, by Theorem 9, $m(\nu_f(E-D))=0$. Hence, $m^*(\nu_f(E))\le m^*(\nu_f(E-D))+m^*\left(\nu_f(E\cap D)\right)=0+0=0$. It follows that $m(\nu_f(E))=0$.

**Corollary 11.** Suppose $f:A\to\mathbb{R}$ is a function of bounded variation and *A* is a subset of $\mathbb{R}$. Suppose *E* is a subset of *A*. Then $m(f(E))=0$ if, and only if, $m(\nu_f(E))=0$.

**Proof.** If $m(f(E))=0$, then by Theorem 11 $m(\nu_f(E))=0$. If $m(\nu_f(E))=0$, then by Theorem 16 of *Functions of Bounded Variation and Johnson's Indicatrix*, $m(f(E))=0$. Note that this theorem applies to arbitrary function of bounded variation as the same proof is valid for the general function of bounded variation.

**Theorem 12.** Suppose $f:A\to\mathbb{R}$ is a function of bounded variation and *A* is a subset of $\mathbb{R}$. Then *f* is a Lusin function if, and only if, its total variation function, $\nu_f$, is a Lusin function.

**Proof.** Suppose *E* is a subset of *A* of zero measure. Then by Corollary 11, $m(f(E))=0$ if, and only if, $m(\nu_f(E))=0$. Thus, *f* maps a null set to a null set if, and only if, $\nu_f$ does the same. Hence, Theorem 12 follows.

**Theorem 13.** Suppose *A* is a measurable closed and bounded subset of $\mathbb{R}$ or an interval and $f:A\to\mathbb{R}$ is a finite valued function of bounded variation on *A*. Then *f* is absolutely continuous, if and only if, $\nu_f:A\to\mathbb{R}$ is absolutely continuous on *A*.

**Proof.** By Theorem 13 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, *f* is continuous if, and only if, $\nu_f$ is continuous. So, we assume that *f* is a continuous function of bounded variation. Since $|f(y)-f(x)|\le|\nu_f(y)-\nu_f(x)|$ for any $x,y\in A$, it follows that if $\nu_f$ is absolutely continuous, then *f* is absolutely continuous.

Note that the total variation function, $\nu_f$, of *f* is a bounded increasing function and so is of bounded variation.

Suppose now *f* is absolutely continuous. By Lemma 3 of *Absolutely Continuous Functions on Arbitrary Domain and Function of Bounded variation*, *f* is a Lusin function. By Theorem 10, since *f* is of bounded variation, $v_f$ is also a Lusin Function.

If *A* is closed and bounded, by Theorem 4 of *Absolutely Continuous Functions on Arbitrary Domain and Function of Bounded variation*, $v_f$ is absolutely continuous

Suppose *A* is an interval. Since the total variation function, $v_f$ , is of bounded variation, continuous and a Lusin function, by Theorem 15 of *Absolutely Continuous Functions on Arbitrary Domain and Function of Bounded variation*, $v_f$ is absolutely continuous.