Functions of Bounded Variation and de La Vallée Poussin's Theorem

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This is the second of a series of articles towards the proof of the Denjoy Saks Young Theorem. Part of the idea of the proof of Denjoy Saks Young Theorem, is to somehow filter the domain of the function by an increasing sequence of subsets, on which *f* is locally of bounded variation. This is an intermediate result towards the proof of the Denjoy Saks Young Theorem. We present the idea of a function of bounded variation on arbitrary subset of $\mathbb{R}$. If the domain is an interval, then for a function of bounded variation on an interval, we have the well-known *de La Vallée Poussin's Theorem*, which gives a null subset where at each point, the function is not differentiable finitely or infinitely and outside of which, the function is differentiable finitely or infinitely and the modulus of the derivative is equal to the derivative of the associated total variation function and furthermore, its images under the function and the total variation function are null sets. We prove here the generalised de La Vallée Poussin's Theorem for function of bounded variation on arbitrary subset of $\mathbb{R}$. This is a key result towards the proof of the Denjoy Saks Young Theorem. Most proof of this intermediate result will involve some geometrical results concerning lines of tangents of the graph of the function as in Saks' proof.

Definition 1. Let *A* be an arbitrary subset of $\mathbb{R}$. Suppose $f : A \to \mathbb{R}$ is a finite valued function. Suppose $\Gamma = \{[x_i, y_i] : x_i, y_i \in A\}$ is any finite set of non-overlapping intervals. If there exists a real number $M > 0$ such that

$$\sum_i |f(y_i) - f(x_i)| < M$$

for every such finite set, Γ, of non-overlapping intervals, then the function *f* is said to be of *bounded variation* with bound *M*.

Note that if *f* is of bounded variation, then for any arbitrary collection of non-overlapping intervals, $\mathscr{C} = \{[x_i, y_i] : x_i, y_i \in A,\ i \text{ belongs to some index set}\}$,

$$\sum_i |f(y_i) - f(x_i)| \leq M ,$$

since $\mathscr{C}$ is at most denumerable. Thus, we could have stated the definition of bounded variation in terms of arbitrary collection of non-overlapping intervals $\mathscr{C}$. Plainly, these two conditions are equivalent.

We shall examine the continuity property of a function of bounded variation.

Since we know a function is continuous at a point if, and only if, the left and right limits of the function at the point are the same as the value of the function at the point, we examine the difference of the values of the function around the point. This leads to the notion of the saltus of the function at the point.

Saltus Function, Limits and Continuity

Definition 2. Let $f : A \to \mathbb{R}$ be a finite valued function and A an arbitrary subset of $\mathbb{R}$. Suppose $a \in A$ is a limit point of A. For each real number $\delta > 0$, let I_δ be an interval with length $\ell(I_\delta) < \delta$ and $a \in I_\delta$. Let

$$s(a,\delta) = \sup\left\{\left| f(y) - f(x)\right| : x, y \in A \cap I_\delta, \text{ for } I_\delta \text{ with } a \in I_\delta \text{ and } \ell(I_\delta) < \delta\right\}.$$

Then, evidently $s(a,\delta) \geq 0$ and $s(a,\delta)$ is a decreasing function of δ. The *saltus* of f at a is defined to be

$$s(a) = \lim_{\delta \to 0} s(a,\delta).$$

Note that f is always continuous at any isolated point of A. So, with this definition, it is only meaningful to talk about saltus at a limit point. By definition of continuity, it is easy to see that f is continuous at a limit point, a, if, and only if, the saltus of f at a is zero. We may thus extend the definition of saltus of f at an isolated point by defining it to be zero. Hence, f is continuous at a point in A, if, and only if, the saltus of f at the point is zero.

For each integer $n \geq 1$, let $D_n = \left\{ a \in A : s(a) > \frac{1}{n} \right\}$. Plainly, the set of discontinuities of f is equal to $\bigcup_{n=1}^{\infty} D_n$.

Theorem 3. If the finite valued function $f : A \to \mathbb{R}$ is of bounded variation on *A*, then D_n is finite for each integer $n \geq 1$.

Proof.

Note that D_n does not contain any isolated points of *A*.

Suppose on the contrary that for some $n \geq 1$, D_n is infinite. Then take an infinite subset, say $\{x_1, x_2, \cdots\}$ of D_n. Suppose *f* is of bounded variation with bound *M*. Let *k* be an integer such that $\frac{k}{n} > M$. Then for $x_1, x_2, \cdots, x_k$ we can find non-overlapping intervals $\left\{I_{\delta_i}(x_i), i = 1, \cdots, k\right\}$ with $I_{\delta_i}(x_i)$ containing x_i. Since each x_i is a limit point and $s(x_i) > \frac{1}{n}$, we can find $y_i, z_i \in I_{\delta_i}(x_i) \cap A$ such that

$$\left| f(y_i) - f(z_i) \right| > \frac{1}{n} .$$

We deduce this as follows:

Since $s(x_i) > \frac{1}{n}$, $\sup\left\{\left| f(y) - f(z) \right| : y, z \in A \cap I_{\delta_i}(x_i)\right\} \geq s(x_i) > \frac{1}{n}$. Therefore, by definition of supremum, there exists $y_i, z_i \in I_{\delta_i}(x_i) \cap A$ such that $\left| f(y_i) - f(z_i) \right| > \frac{1}{n}$.

Hence, we get $\sum_{i=1}^{k} \left| f(y_i) - f(z_i) \right| > \frac{k}{n} > M$, contradicting that *f* is of bounded variation with bound *M*. We conclude that each D_n is finite.

Theorem 4. Let $f : A \to \mathbb{R}$ be a finite valued function and *A* an arbitrary subset of $\mathbb{R}$. Suppose *f* is of bounded variation on *A*. Then the set of discontinuities of *f* is at most denumerable.

Proof.

Let *D* be the set of discontinuities of *f*. Then $D = \bigcup_{n=1}^{\infty} D_n$. Since each D_n is finite by Theorem 3, it follows that *D* is at most denumerable as it is the countable union of finite sets.

Theorem 5. Let $f : A \to \mathbb{R}$ be a finite-valued function of bounded variation on *A*. If *x* is a left limit point of *A*, then $f(x^-) = \lim_{y \to x^-, y \in A} f(y)$ exists. If *x* is a right limit point of *A*, then $f(x^+) = \lim_{y \to x^+, y \in A} f(y)$ exists.

Proof.

Suppose *x* is a limit point on the right of *A*. Since *f* is of bounded variation on *A*, *f* is bounded on *A*. Hence, $\limsup_{y \to x^+, y \in A} f(y)$ and $\liminf_{y \to x^+, y \in A} f(y)$ exist and are finite. Suppose $\alpha = \limsup_{y \to x^+, y \in A} f(y)$ and $\beta = \liminf_{y \to x^+, y \in A} f(y)$. We want to show that $\alpha = \beta$.

Suppose $\alpha - \beta > \lambda > 0$. For each integer $n \geq 1$, let $a_n = \sup\left\{ f(x) : x \in \left(x, x + \frac{1}{n} \right) \cap A \right\}$ and $b_n = \inf\left\{ f(x) : x \in \left(x, x + \frac{1}{n} \right) \cap A \right\}$. Then $a_n \geq \alpha$ and $b_n \leq \beta$. Hence, $a_n - b_n \geq \alpha - \beta > \lambda > 0$. Now $a_n - b_n = \sup\left\{ f(y) - f(z) : y, z \in \left(x, x + \frac{1}{n} \right) \cap A \right\} > \lambda > 0$. Therefore, by definition of supremum, there exist $y_n, z_n \in \left(x, x + \frac{1}{n} \right) \cap A$ such that

$$f(y_n) - f(z_n) > \lambda \ .$$

We shall find a sequence of points in *A*, $y_{n_1} > z_{n_1} > y_{n_2} > z_{n_2} > \cdots$ such that $\left| f(y_{n_i}) - f(z_{n_i}) \right| > \lambda$.

Starting with *n* =1, we can find $y_{n_1}, z_{n_1} \in (x, x+1) \cap A$ with $y_{n_1} > z_{n_1}$ such that $f(y_{n_1}) - f(z_{n_1}) = \left| f(y_{n_1}) - f(z_{n_1}) \right| > \lambda$. Let n_2 be such that $\frac{1}{n_2} < \min\left\{ y_{n_1} - x, z_{n_2} - x \right\} = z_{n_2} - x$. As $a_{n_1} - b_{n_2} > \lambda$, we can find $y_{n_2}, z_{n_2} \in \left(x, x + \frac{1}{n_2} \right) \cap A$ such that $f(y_{n_2}) - f(z_{n_2}) > \lambda$. In general, suppose $y_{n_k}, z_{n_k} \in \left(x, x + \frac{1}{n_k} \right) \cap A$ have been found with $y_{n_k} > z_{n_k}$ so that $f(y_{n_k}) - f(z_{n_k}) > \lambda$. Let n_{k+1} be such that $\frac{1}{n_{k+1}} < \min\left\{ y_{n_k} - x, z_{n_k} - x \right\}$. And as $a_{n_k} - b_{n_k} > \lambda$, we can find

$y_{n_{k+1}}, z_{n_{k+1}} \in \left(x, x+\frac{1}{n_{k+1}} \right) \cap A$ such that $f(y_{n_{k+1}}) - f(z_{n_{k+1}}) > \lambda$. We may assume that $y_{n_{k+1}} > z_{n_{k+1}}$, renaming if necessary and we always have $\left| f(y_{n_k}) - f(z_{n_k}) \right| > \lambda$.

Let *N* be an integer such that $N\lambda > M$, where *M* is the variation bound for *f* on *A*. Then, taking the sequence $y_{n_1} > z_{n_1} > y_{n_2} > z_{n_2} > \cdots > y_{n_N} > z_{n_N} \cdots$, we get

$$\sum_{k=1}^{N} \left| f(y_{n_k}) - f(z_{n_k}) \right| > N\lambda > M .$$

This contradicts that *f* is of bounded variation on *A* with bound *M*. Hence, $\alpha = \beta$ and this means $f(x^+) = \lim_{y \to x^+, y \in A} f(y)$ exists and is finite.

Similarly, we can prove that if *x* is a left limit point of *A*, then $f(x^-) = \lim_{y \to x^-, y \in A} f(y)$ exists and is finite.

Decomposition of Function of Bounded Variation

We shall show that any function of bounded variation is a difference of two increasing functions. Many results can be proved first on increasing function and then extend to the function of bounded variation via the total variation function of a function of bounded variation.

Theorem 6. Let $f : A \to \mathbb{R}$ be a finite-valued function of bounded variation on *A*. Pick a point *a* in *A* as the anchor point. Then there is a decomposition $f(x) - f(a) = \varphi_1(x) - \varphi_2(x)$, where φ_1 and φ_2 are respectively the positive and negative variation of *f* satisfying $\varphi_1(a) = \varphi_2(a) = 0$ and are increasing functions on *A*. Moreover, $\varphi_1(x)$ and $\varphi_2(x)$ are optimal in the sense that if we have another decomposition of $f(x) - f(a)$, $f(x) - f(a) = g(x) - h(x)$, where *g* and *h* are increasing functions with $g(a) = h(a) = 0$, then for $x \geq a$, $0 \leq \varphi_1(x) \leq g(x)$ and $0 \leq \varphi_2(x) \leq h(x)$ and for $x < a$ $0 \geq \varphi_1(x) \geq g(x)$ and $0 \geq \varphi_2(x) \geq h(x)$. The function *f* is the difference of two increasing function, $f(x) = \psi(x) - \varphi_2(x)$, where $\psi(x) = \varphi_1(x) + f(a)$.

Proof.

Fix a point *a* in *A*. We shall use this point as the anchor point for the variation function of *f*.

Suppose x is in A and $x > a$. Let $Q: a = x_0 < x_1 < x_2 < \cdots < x_n = x$ be any subdivision of (a, x) with points in A. Let

$$P(x,Q) = \sum_i \max\{0, f(x_i) - f(x_{i-1})\}$$

and $$N(x,Q) = \sum_i \min\{0, f(x_i) - f(x_{i-1})\}.$$

Then, $f(x) - f(a) = P(x,Q) + N(x,Q)$.

Let $p(x) = \sup\{P(x,Q) : Q \text{ a subdivision of } (a,x) \text{ by points in } A\}$ and

$n(x) = \inf\{N(x,Q) : Q \text{ a subdivision of } (a,x) \text{ by points in } A\}$.

We claim that $f(x) - f(a) = p(x) + n(x)$. We deduce this as follows.

Take any $\varepsilon > 0$. Then there exists a subdivision $\tilde{Q}$ of (a, x) such that

$$p(x) \ge P(x,\tilde{Q}) > p(x) - \varepsilon.$$

It follows that $f(x) - f(a) = P(x,\tilde{Q}) + N(x,\tilde{Q}) > p(x) + n(x) - \varepsilon$. Since ε is arbitrary, we conclude that $f(x) - f(a) \ge p(x) + n(x)$.

By definition of infimum, there exists a subdivision Q' of (a, x) such that

$$n(x) \le N(x,Q') < n(x) + \varepsilon.$$

Hence, $f(x) - f(a) = P(x,Q') + N(x,Q') < p(x) + n(x) + \varepsilon$. As ε is arbitrary, we conclude that $f(x) - f(a) \le p(x) + n(x)$. It follows that $f(x) - f(a) = p(x) + n(x)$.

Plainly, $p(x)$ is an increasing function on $[a,\infty) \cap A$ and $n(x)$ is a decreasing function on $[a,\infty) \cap A$. It follows that

$$f(x) - f(a) = p(x) - (-n(x))$$

is the difference of two increasing functions on $[a,\infty) \cap A$.

We now consider points x in A with $x < a$, i.e., $(-\infty, a] \cap A$. Take a subdivision

$Q: x = x_0 < x_1 < x_2 < \cdots < x_n = a$.

Let $P(x,Q) = \sum_i \max\{0, f(x_i) - f(x_{i-1})\}$ and $N(x,Q) = \sum_i \min\{0, f(x_i) - f(x_{i-1})\}$.

This time we have $f(a)-f(x)=P(x,Q)+N(x,Q)$.

As before, define for $x<a$,

$$p(x)=\sup\{P(x,Q):Q \text{ a subdivision of } (x,a) \text{ by points in } A\}$$

and $$n(x)=\inf\{N(x,Q):Q \text{ a subdivision of } (x,a) \text{ by points in } A\}.$$

And we deduce as before that for $x<a$, $f(a)-f(x)=p(x)+n(x)$ so that

$$f(x)-f(a)=-p(x)-n(x).$$

Note that for all x in A, $p(x)\geq 0$ and $n(x)\leq 0$.

Note that if $y<x<a$, then $p(y)>p(x)$, $n(y)<n(x)$ and we have that

$-p(y)<-p(x)$ and $-n(y)>-n(x)$.

We define for x in A,

$$\varphi_1(x)=\begin{cases} p(x), x>a \\ 0, x=a, \\ -p(x), x<a \end{cases} \quad \text{and} \quad \varphi_2(x)=\begin{cases} -n(x), x>a \\ 0, x=a, \\ n(x), x<a \end{cases}.$$

Plainly, φ_1 and φ_2 are increasing functions. Moreover,

$$\varphi_1(x)-\varphi_2(x)=\begin{cases} p(x)+n(x), x>a \\ 0, x=a, \\ -p(x)-n(x), x<a \end{cases} = f(x)-f(a).$$

Therefore, $f(x)=f(a)+\varphi_1(x)-\varphi_2(x)$. If we let $\psi(x)=\varphi_1(x)+f(a)$, then $\psi(x)$ is an increasing function on A and so $f(x)=\psi(x)-\varphi_2(x)$ is a difference of two increasing function.

Take any x in A. Assume $x>a$. Let $Q: a=x_0<x_1<\cdots<x_n=x$ be a partition for $[a,x]$, with points in A. Then

$$f(x_i)-f(x_{i-1})=g(x_i)-g(x_{i-1})-[h(x_i)-h(x_{i-1})]\leq g(x_i)-g(x_{i-1}).$$

Therefore, as $P(x,Q)$ is the sum over the terms for which $f(x_i)-f(x_{i-1})\geq 0$,

$$P(x,Q)\leq\sum_{i=1}^{n}\left(g(x_i)-g(x_{i-1})\right)=g(x)-g(a)=g(x) .$$

It follows that, $p(x)\leq g(x)$. Hence, $\varphi_1(x)\leq g(x)$.

Suppose now $x < a$. Let $Q: x = x_0 < x_1 < \cdots < x_n = a$ be a partition of $[x,a]$ by points in A. As before, we have

$$f(x_i) - f(x_{i-1}) = g(x_i) - g(x_{i-1}) - \left[h(x_i) - h(x_{i-1})\right] \le g(x_i) - g(x_{i-1}).$$

Now, for $x < a$, $P(x,Q) = \sum_i \max\{0, f(x_i) - f(x_{i-1})\} \le -g(x)$. Hence, $p(x) \le -g(x)$ and so $\varphi_1(x) = -p(x) \ge g(x)$.

Now, for $x > a$, $g(x) - h(x) = f(x) - f(a) = p(x) + n(x) \le g(x) + n(x)$. Therefore, $n(x) \ge -h(x)$. Hence, $\varphi_2(x) = -n(x) \le h(x)$.

Suppose now $x < a$. $g(x) - h(x) = f(x) - f(a) = -p(x) - n(x) \ge g(x) - n(x)$. Therefore, $n(x) \ge h(x)$. It follows that $\varphi_2(x) = n(x) \ge h(x)$.

Upper and Lower Derivates

Next, we examine the derivative property of a function of bounded variation.

Firstly, we recall the definition of the upper and lower derivates of a function.

Definition 7. Let $f: A \to \mathbb{R}$ be a finite valued function. Let x be a limit point of A. The *upper derivate* of f at x is defined as ${}_A\overline{D}f(x) = \limsup_{t \to x} \left\{ \frac{f(t) - f(x)}{t - x} : t \in A \right\}$ and the *lower derivate* of f at x is ${}_A\underline{D}f(x) = \liminf_{t \to x} \left\{ \frac{f(t) - f(x)}{t - x} : t \in A \right\}$.

Let $a_\delta = \sup\left\{ \frac{f(x+h) - f(x)}{h} : h \ne 0, x + h \in (x - \delta, x + \delta) \cap A \right\}$. Then we have,

$$\begin{aligned} {}_A\overline{D}f(x) &= \limsup_{t \to x} \left\{ \frac{f(t) - f(x)}{t - x} : t \in A \right\} = \limsup_{\delta \to 0^+} \left\{ \frac{f(x+h) - f(x)}{h} : h \ne 0, x + h \in (x - \delta, x + \delta) \cap A \right\} \\ &= \lim_{\delta \to 0^+} a_\delta . \end{aligned}$$

Likewise, let $b_\delta = \inf\left\{ \frac{f(x+h) - f(x)}{h} : h \ne 0, x + h \in (x - \delta, x + \delta) \cap A \right\}$ and we get

$$\begin{aligned} {}_A\underline{D}f(x) &= \liminf_{t \to x} \left\{ \frac{f(t) - f(x)}{t - x} : t \in A \right\} = \liminf_{\delta \to 0^+} \left\{ \frac{f(x+h) - f(x)}{h} : h \ne 0, x + h \in (x - \delta, x + \delta) \cap A \right\} \\ &= \lim_{\delta \to 0^+} b_\delta . \end{aligned}$$

Note that a_δ is a decreasing function of δ and b_δ is an increasing function of δ.

Theorem 8. Suppose $f : A \to \mathbb{R}$ is a finite valued function of bounded variation on *A*. Then, for almost all *x* in *A*, the upper and lower derivates of *f* at *x* exist, are finite and equal. The common value ${}_A Df(x)$ is called the *derivative* of *f* over *A* at *x*.

Remark. If *A* is an interval, then ${}_A Df(x)$ is just the usual definition of the derivative of f at *x*. Note that ${}_A\overline{D}f(x) = {}_A\underline{D}f(x)$ if, and only if, ${}_A D^+ f(x) = {}_A D_+ f(x) = {}_A D^- f(x) = {}_A D_- f(x)$. (See *Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem*.)

Theorem 9. Suppose $f : A \to \mathbb{R}$ is a finite-valued function of bounded variation on *A*. Assuming that *A* is either bounded or of finite outer measure. Then, except for a set of measure zero in *A*, ${}_A\overline{D}f(x)$ and ${}_A\underline{D}f(x)$ exist and are finite.

Proof.

Observe that if $\limsup_{\delta\to 0^+}\left\{\frac{f(x+h)-f(x)}{h} : h \neq 0, x+h \in (x-\delta, x+\delta)\cap A\right\} = \infty$, then

$\limsup_{\delta\to 0^+}\left\{\left|\frac{f(x+h)-f(x)}{h}\right| : h \neq 0, x+h \in (x-\delta, x+\delta)\cap A\right\} = \infty$ and if

$\liminf_{\delta\to 0^+}\left\{\frac{f(x+h)-f(x)}{h} : h \neq 0, x+h \in (x-\delta, x+\delta)\cap A\right\} = -\infty$, then

$\limsup_{\delta\to 0^+}\left\{\left|\frac{f(x+h)-f(x)}{h}\right| : h \neq 0 x+h \in (x-\delta, x+\delta)\cap A\right\} = \infty$.

Note that if $\limsup_{\delta\to 0^+}\left\{\frac{f(x+h)-f(x)}{h} : h \neq 0, x+h \in (x-\delta, x+\delta)\cap A\right\} = -\infty$, then

$\liminf_{\delta\to 0^+}\left\{\frac{f(x+h)-f(x)}{h} : h \neq 0, x+h \in (x-\delta, x+\delta)\cap A\right\} = -\infty$ and if

$$\liminf_{\delta\to 0^+}\left\{\frac{f(x+h)-f(x)}{h}:h\neq 0, x+h\in(x-\delta,x+\delta)\cap A\right\}=+\infty$$, then

$$\limsup_{\delta\to 0^+}\left\{\frac{f(x+h)-f(x)}{h}:h\neq 0, x+h\in(x-\delta,x+\delta)\cap A\right\}=\infty.$$

Let $E_1=\left\{x\in A: {}_A\overline{D}f(x)=+\infty\right\}$ and $E_2=\left\{x\in A: {}_A\underline{D}f(x)=-\infty\right\}$. Observe that $\left\{x\in A: {}_A\overline{D}f(x)=\pm\infty \text{ or } {}_A\underline{D}f(x)=\pm\infty\right\}=E_1\cup E_2$ and

$$E_1\cup E_2\subseteq E=\left\{x\in A:\limsup_{\delta\to 0^+}\left\{\left|\frac{f(x+h)-f(x)}{h}\right|:h\neq 0, x+h\in(x-\delta,x+\delta)\cap A\right\}=\infty\right\}.$$

We shall show that E is of zero measure.

Let $a_\delta=\sup\left\{\left|\frac{f(x+h)-f(x)}{h}\right|:h\neq 0, x+h\in(x-\delta,x+\delta)\cap A\right\}$. If $x\in E$, then $\lim_{\delta\to 0^+}a_\delta=\infty$.

Hence, for an arbitrary positive number K, there exists a $\tilde{\delta}>0$ such that for all $0<\delta<\tilde{\delta}$, $a_\delta>K$.

Choose an integer N such that $\frac{1}{N}<\tilde{\delta}$.

Thus, for any integer $n\geq 1$, we can find $x_n\in\left(x-\frac{1}{N+n},x+\frac{1}{N+n}\right)\cap A$ such that $\left|\frac{f(x_n)-f(x)}{x_n-x}\right|>K$ and $0<\left|x_n-x\right|<\frac{1}{N+n}$. In this way we obtain a sequence (x_n) such that $x_n\to x$. Let $v_n(x)$ be the closed interval determined by the points x and x_n, i.e., $v_n(x)=[x,x_n]$ if $x_n>x$ and $v_n(x)=[x_n,x]$ if $x_n<x$. Let $\Lambda_x=\{v_i(x)\}$ be the family of closed intervals. Then $\Omega=\bigcup\{\Lambda_x:x\in E\}$ constitutes a covering of E in the Vitali sense. Since A is of finite outer measure, E is also of finite outer measure. Therefore, by the Vitali Covering Theorem, for any $\varepsilon>0$, there exists a finite disjoint set J of closed intervals in Ω such that

$$m^*\left(\bigcup_{I_i\in J}I_i\right)>m^*(E)-\varepsilon\ .$$

Suppose the number of members in J is L. For $I_i\in J$, there exists $y_i\in E$ such that $I_i=v_{n_i}(y_i)=\left[y_i,x_{n_i}\right]$ or $\left[x_{n_i},y_i\right]$, where $x_{n_i}\to y_i$. Thus, we have

$$\sum_{i=1}^{L}\left|f\left(x_{n_i}\right)-f\left(y_i\right)\right|>K\sum_{i=1}^{L}\left|x_{n_i}-y_i\right| .$$

Now $m^*\left(\bigcup_{I_i\in J} I_i\right)=\sum_{i=1}^{L}\left|x_{n_i}-y_i\right|$ and so $\sum_{i=1}^{L}\left|f\left(x_{n_i}\right)-f\left(y_i\right)\right|>K\left(m^*(E)-\varepsilon\right)$.

Suppose the variation bound of *f* is *M*. Suppose $m^*(E) > 0$. We can take the number *K* such that $Km^*(E)>M+1$. By taking $\varepsilon > 0$ sufficiently small so that $K\varepsilon<1$. We then obtain

$$\sum_{i=1}^{L}\left|f\left(x_{n_i}\right)-f\left(y_i\right)\right|>K\left(m^*(E)-\varepsilon\right)>M ,$$

contradicting that *f* is of bounded variation on *A* with bound *M*. Therefore, $m^*(E) = 0$. Hence, E_1 and E_2 are of measure zero. It follows that ${}_A\overline{D}f(x)$ and ${}_A\underline{D}f(x)$ exist and are finite for almost all *x* in *A*.

We may remove the assumption imposed on *A* in Theorem 9.

Corollary 10. Suppose $f:A\to\mathbb{R}$ is a finite valued function of bounded variation on *A*. Then, except for a set of measure zero in *A*, ${}_A\overline{D}f(x)$ and ${}_A\underline{D}f(x)$ exist and are finite.

Proof. If *A* is bounded, then the conclusion is given by Theorem 9. Suppose A is unbounded. Subdivide *A* into countable non-overlapping bounded pieces. We can do this by taking each piece as $A_n=A\cap[n,n+1]$. Since the conclusion is valid for each A_n , it follows that it also holds for the whole space *A*.

Proof of Theorem 8.

Suppose $f:A\to\mathbb{R}$ is a function of bounded variation on *A*. By Theorem 6, *f* is a difference of two increasing and bounded function. It suffices to prove the theorem for *f* an increasing and bounded function. We shall assume that *A* is either bounded or of finite outer measure. By Theorem 9 or Corollary 10, outside of a set of measure zero, ${}_A\overline{D}f(x)$ and ${}_A\underline{D}f(x)$ exist and are finite. Let $E=\left\{x\in A: {}_A\overline{D}f(x) \text{ and } {}_A\underline{D}f(x) \text{ exist and are finite}\right\}$. Since *f* is increasing, ${}_A\underline{D}f(x)\geq 0$.

For any rational number h, k such that $0 < h < k$, let

$$E_{h,k} = \left\{ x \in E : \text{ and } {}_A\underline{D}f(x) < h < k < {}_A\overline{D}f(x) \right\}.$$

Let $x \in E$. By definition of ${}_A\underline{D}f(x)$, there exists a sequence (x_i) in A such that $x_i \to x$, $x_i \neq x$ and

$$\frac{f(x_i) - f(x)}{x_i - x} = \left| \frac{f(x_i) - f(x)}{x_i - x} \right| < h. \text{ -------------------------- } (1)$$

Thus, we have arbitrary small closed intervals $[x_i, x]$ or $[x, x_i]$ satisfying (1). These intervals for each of the points x in $E_{h,k}$ form a Vitali covering for $E_{h,k}$. Hence by the Vitali Covering Theorem, there exists a finite set of disjoint closed intervals, $v_1, v_2, \cdots, v_n$ such that

$$m^*(E_{h,k}) - \varepsilon \leq \sum_{i=1}^{n} m^*(E_{h,k} \cap v_i) \leq \sum_{i=1}^{n} m^*(v_i) < m^*(E_{h,k}) + \varepsilon .$$

If x_i is the point $x \in E_{h,k}$ associated with the closed interval, v_i , then

$v_i = [x_i, x_{n_i}]$ or $[x_{n_i}, x_i]$, where $x_{n_i} \neq x_i$ and $\left| \dfrac{f(x_{n_i}) - f(x_i)}{x_{n_i} - x_i} \right| < h$. We then have

$$\sum_{i=1}^{n} \left| f(x_{n_i}) - f(x_i) \right| < h \sum_{i=1}^{n} m^*(v_i) < h\left(m^*(E_{h,k}) + \varepsilon \right). \text{ ---------------------- } (2)$$

Consider the set $\left(\bigcup_{i=1}^{n} v_i \right) \cap E_{h,k} \subseteq E_{h,k}$. For each integer $1 \leq i \leq n$, let $\operatorname{int}(v_i)$ be the interior of v_i. For each x in $\left(\bigcup_{i=1}^{n} \operatorname{int}(v_i) \right) \cap E_{h,k}$, we have that ${}_A\overline{D}f(x) > k$.

Therefore, by definition of ${}_A\overline{D}f(x)$, for each x in $\left(\bigcup_{i=1}^{n} \operatorname{int}(v_i) \right) \cap E_{h,k}$, there exists a sequence (y_i) such that $y_i \neq x$, $y_i \to x$ and

$$\left| f(y_i) - f(x) \right| > k \left| y_i - x \right|. \text{ -------------------------- } (3)$$

Thus, we have arbitrarily small closed intervals $[y_i, x]$ or $[x, y_i]$ satisfying (3). Furthermore, we may restrict these small intervals to be in $\bigcup_{i=1}^{n} \operatorname{int}(v_i)$, since

$x \in \bigcup_{i=1}^{n} \mathrm{int}(v_i)$. These collection of arbitrarily small closed intervals for each *x* in $\left(\bigcup_{i=1}^{n} \mathrm{int}(v_i)\right) \cap E_{h,k}$ forms a Vitali covering for $\left(\bigcup_{i=1}^{n} \mathrm{int}(v_i)\right) \cap E_{h,k}$ Hence, by the Vitali Covering Theorem, there is a finite disjoint sets of closed intervals, $v_1', v_2', \cdots, v_p'$ in $\bigcup_{i=1}^{n} \mathrm{int}(v_i)$ such that

$$\sum_{i=1}^{p} m^*(v_i') > m^*(E_{h,k} \cap \bigcup_{i=1}^{n} v_i) - \varepsilon > m^*(E_{h,k}) - 2\varepsilon \ .$$

Thus, if y_i and y_i' are end points of v_i' , then by (3) we have

$$\sum_{i=1}^{p} \left| f(y_i) - f(y_i') \right| > k \sum_{i=1}^{p} m^*(v_i') > k\left(m^*(E_{h,k}) - 2\varepsilon\right). \text{ --------------- } (4)$$

Since $v_i' \subseteq \bigcup_{i=1}^{n} v_i$ and *f* is increasing, $\sum_{i=1}^{p} \left| f(y_i) - f(y_i') \right| \leq \sum_{i=1}^{n} \left| f(x_{n_i}) - f(x_i) \right|$. It follows then from (2) that

$$\sum_{i=1}^{p} \left| f(y_i) - f(y_i') \right| \leq \sum_{i=1}^{n} \left| f(x_{n_i}) - f(x_i) \right| < h\left(m^*(E_{h,k}) + \varepsilon\right). \text{ ------------ } (5)$$

Therefore, from (4) and (5) we get, $k\left(m^*(E_{h,k}) - 2\varepsilon\right) < h\left(m^*(E_{h,k}) + \varepsilon\right)$

and so $(k-h)\left(m^*(E_{h,k}) - 2\varepsilon\right) < \varepsilon(2h+h)$. Since ε is arbitrary, we conclude that

$$(k-h) m^*(E_{h,k}) \leq 0 .$$

This is only possible if $m^*(E_{h,k}) = 0$ as $k - h > 0$.

Let $\widetilde{E} = \left\{ x \in E : {}_A\underline{D}f(x) < {}_A\overline{D}f(x) \right\}$. Then $\widetilde{E} = \bigcup_{h,k \text{ rational}, h<k} E_{h,k}$. Since each $E_{h,k}$ is of measure zero by we have just shown, $\widetilde{E}$ is of measure zero. This means except for a set of measure zero, ${}_A\overline{D}f(x)$ and ${}_A\underline{D}f(x)$ exist, are finite and equal.

Note that if *f* is of bounded variation on *A*, then by Theorem 6, $f = \varphi_1 - \varphi_2$, where φ_1 and φ_2 are increasing functions on *A*. Since, except for a set of measure zero in *A*, ${}_A\overline{D}\varphi_1(x)$ and ${}_A\underline{D}\varphi_1(x)$ exist, are finite and equal and also ${}_A\overline{D}\varphi_2(x)$ and ${}_A\underline{D}\varphi_2(x)$ exist, are finite and equal, it follows that except for a set of measure zero in A, ${}_A\overline{D}f(x)$ and ${}_A\underline{D}f(x)$ exist, are finite and equal. This means f is differentiable almost everywhere on A.

If A is unbounded or of infinite outer measure, then we can subdivide A into countable union of bounded subsets. Since the conclusion is true for each bounded piece, it is true also for the countable union as countable union of set of measure zero is of measure zero.

Now we examine the measurable property of the derivative of measurable function. If f is measurable, then its derivative is also measurable.

Theorem 11. Suppose $f: A \to \mathbb{R}$ is a measurable finite valued function and A is measurable. Suppose ${}_A Df(x)$ exists and is finite for every x in A. Then ${}_A Df(x)$ is a measurable function on A.

We shall need the well-known Lusin Theorem (see *Royden and Fitzpatrick, Real Analysis*) for approximation of measurable function by continuous function. We shall state the theorem without proof.

Theorem 12. Suppose $f: E \to \mathbb{R}$ is a measurable finite valued function and E is measurable. Then, for any $\varepsilon > 0$, there exists a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which $f = g$ on F and $m(E - F) < \varepsilon$.

Proof of Theorem 11.

We shall prove this theorem in stages.

Firstly, suppose A is an interval, say, (a,b) and f is defined on $\mathbb{R}$. Extend f trivially to a measurable function on $\mathbb{R}$. Then since f is differentiable on A, for any sequence (h_n) with $h_n \neq 0$ and $h_n \to 0$, the function

$$g_n(x) = \frac{f(x+h_n) - f(x)}{h_n}$$

is measurable. Since ${}_A Df(x)$ exists for x in A, $g_n(x) \to {}_A Df(x)$ on A. This implies that ${}_A Df(x)$ is measurable on A.

If f is only defined on A and A is not an interval, then it is not necessary that for x in A, that $x + h_n$ is in A. Thus, we need to find a function defined on $\mathbb{R}$ and closed to f on A or some subset of A. We use Lusin Theorem to furnish this function.

By Lusin Theorem, for each integer n, there exists a closed set $C_n \subseteq A$ and a continuous function $\varphi_n : \mathbb{R} \to \mathbb{R}$ such that $\varphi_n(x) = f(x)$ for all $x \in C_n$ and $m(A - C_n) < \frac{1}{n}$.

Now since C_n is closed the complement of C_n is at most a countable union of disjoint open intervals. Therefore, the boundary of C_n is of measure zero. Now $\varphi_n = f$ on the interior of C_n, $\operatorname{int} C_n$. Note that ${}_A Df(x)$ is defined for all x in $\operatorname{int} C_n$.

Now for each $x \in \operatorname{int} C_n$, there exists an open interval $(x-\varepsilon, x+\varepsilon) \subseteq \operatorname{int} C_n \subseteq A$. It follows that $D\varphi_n(x) = {}_A Df(x)$ on $\operatorname{int} C_n$. Thus, $D\varphi_n(x) = {}_A Df(x)$ almost everywhere on C_n. Now $\operatorname{int} C_n$ is a countable disjoint union of open intervals. Therefore, by what we have just shown, $D\varphi_n(x)$ is measurable on each of the open intervals and so it is measurable on $\operatorname{int} C_n$ and hence on C_n. It follows that ${}_A Df(x)$ is measurable on C_n. Note that as $m(A - C_n) < \frac{1}{n}$, $m(A - C_n) \to 0$ as $n \to \infty$.

For each integer $n \geq 1$, let $K_n = C_1 \cup C_2 \cup \cdots \cup C_n$. Let

$$\chi_n(x) = \begin{cases} {}_A Df(x), x \in K_n \\ 0, x \in K_n{}^c \end{cases} .$$

Then χ_n is measurable. Note that $K_n \subseteq A$ and $m(A - K_n) \to 0$. Take $\widetilde{K} = \bigcup_{n=1}^{\infty} K_n \subseteq A$. As $m(K_n) = m(A) - m(A - K_n)$, $m(K_n) \to m(A)$. By the continuity from below property of measure, $m\left(\widetilde{K}\right) = m(A)$ and so $m\left(A - \widetilde{K}\right) = 0$

Hence, $\chi_n(x) \to {}_A Df(x)$ for almost all x in A. Since each χ_n is measurable, ${}_A Df(x)$ is measurable.

Total Variation

Suppose $f : A \to \mathbb{R}$ is a finite valued function of bounded variation on A. Take an anchor point a in A. Then we have that $f = f(a) + \varphi_1 - \varphi_2$, where

$$\varphi_1(x) = \begin{cases} p(x), x > a \\ 0, x = a, \\ -p(x), x < a \end{cases} \quad \text{and} \quad \varphi_2(x) = \begin{cases} -n(x), x > a \\ 0, x = a, \\ n(x), x < a \end{cases}$$

and $p(x)$ and $n(x)$ are defined in the proof of Theorem 6.

We define the *total variation* function of f, $v_f : A \to \mathbb{R}$ by

$$v_f(x) = \varphi_1(x) + \varphi_2(x) = \begin{cases} p(x) - n(x), x > a \\ 0, x = a, \\ n(x) - p(x), x < a \end{cases}.$$

Note that v_f is an increasing function. Moreover, $v_f(x) \geq 0$ for all $x \geq a$ in A and $v_f(x) \leq 0$ for all x in A with $x \leq a$. Observe that for x and y in A,

$$|f(y) - f(x)| \leq v_f(y) - v_f(x) \text{ for } y \geq x.$$

Theorem 13. Suppose $f : A \to \mathbb{R}$ is a finite valued function of bounded variation on A. Then $v_f : A \to \mathbb{R}$ is continuous at x in A if, and only if, f is continuous at x.

Proof.

Note that we choose a point a in A as the anchor point for the definition of the positive and negative variations of f. Observe that $\varphi_1(x), \varphi_2(x) \geq 0$ for $x \geq a$ and $\varphi_1(x), \varphi_2(x) \leq 0$ for $x \leq a$. It is to be noted that for all x in A, $p(x) \geq 0$ and $n(x) \leq 0$.

We shall first consider the isolated points of A. f is of course always continuous at any isolated point of A. Suppose c is an isolated point of A. Since it is an isolated point, there exists $\delta > 0$ such that $(c - \delta, c + \delta) \cap A = \varnothing$. Suppose $c \neq a$. Let $c' = \sup\{y \in A, y < c\}$ and $d' = \inf\{y \in A, y > c\}$. Then $c' \leq c - \delta < c$ and

$d' \geq c+\delta > c$. Then φ_1 and φ_2 are not defined at any point in $(c', d') - \{c\}$. Hence, trivially $v_f(x) = \varphi_1(x) + \varphi_2(x)$ is continuous at *c*.

We may assume that the anchor point, *a*, is not an isolated point of *A*.

Take *c* a limit point of *A*. Suppose that *f* is continuous at *c*. Then for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|x-c| < \delta_1 \text{ and } x \in A \Rightarrow |f(x) - f(c)| < \frac{\varepsilon}{2}. \text{ ------------------- } (1)$$

Suppose *c* is a right limit point of *A*. We assume that $c \geq a$. Then for any $\delta > 0$, $(c, c+\delta) \cap A \neq \varnothing$. Take any $b \in (c, c+\delta_1) \cap A$. Now, we take the variation function of *f* anchor at the point *c*, $v_{f,c}: A \to \mathbb{R}$.

Then by definition of the total variation of *f*,

$$v_{f,c}(b) = \sup\left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| : c = x_0 < \cdots < x_n = b, \text{ a subdivision of } [c,b] \text{ by points in } A \right\}.$$

Hence, given $\varepsilon > 0$, there exists a partition of [*c*, *b*] by points in *A*,

$$c = x_0 < x_1 < \cdots < x_n = b \ , \ x_i \in A$$

such that

$$v_{f,c}(b) - \frac{\varepsilon}{2} < \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq v_{f,c}(b).$$

Let $\delta = (x_1 - x_0) = (x_1 - c)$. Then for $x \in (c, c+\delta) \cap A$,

$$v_{f,c}(b) - \frac{\varepsilon}{2} < |f(x_1) - f(c)| + \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})|$$

$$\leq |f(x_1) - f(x)| + |f(x) - f(c)| + \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})|$$

$$= |f(x) - f(c)| + \left(|f(x_1) - f(x)| + \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})| \right)$$

$$\leq |f(x) - f(c)| + v_{f,x}(b)$$

$$\leq \frac{\varepsilon}{2} + v_{f,x}(b) \quad \text{by (1).}$$

Hence,

$$v_{f,c}(b) - v_{f,x}(b) < \varepsilon . \quad \text{---------------------} \quad (2)$$

Next we claim that

$$v_{f,a}(x) - v_{f,a}(c) \leq v_{f,c}(b) - v_{f,x}(b) .$$

Now

$$v_{f,a}(x) = \sup\left\{ \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| : a = x_0 < \cdots < x_n = x, \text{ a subdivision of } [a,x] \text{ by points in } A \right\}$$

$$\leq \sup\left\{ \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| : a = x_0 < \cdots < x_n = c, \text{ a subdivision of } [a,c] \text{ by points in } A \right\}$$

$$+ \sup\left\{ \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| : c = y_0 < \cdots < y_m = x, \text{ a subdivision of } [c,x] \text{ by points in } A \right\}.$$

Thus, we have

$$v_{f,a}(x) \leq v_{f,a}(c) + v_{f,c}(x) \leq v_{f,a}(x) . \quad \text{------------------------} \quad (3)$$

Hence, $v_{f,a}(x) - v_{f,a}(c) = v_{f,c}(x)$.

Similarly, we can derive that

$$v_{f,c}(b) \leq v_{f,c}(x) + v_{f,x}(b) \leq v_{f,c}(b)$$

and so

$$v_{f,c}(b) - v_{f,x}(b) = v_{f,c}(x) . \quad \text{---------------------------} \quad (4)$$

Therefore, $v_{f,a}(x) - v_{f,a}(c) = v_{f,c}(x) = v_{f,c}(b) - v_{f,x}(b) < \varepsilon$.

This shows that $v_{f,a}(x)$ is continuous from the right at *c*.

Suppose now *c* is a left limit point of *A*. Take any $\delta_1 > 0$ such that $\delta_1 < c - a$ and

$$|x - c| < \delta_1 \text{ and } x \in A \Rightarrow \left| f(x) - f(c) \right| < \frac{\varepsilon}{2} .$$

Take a point *b* in $(c-\delta_1, c)\cap A$. Then we can find a partition of [*b*, *c*] by points in *A*, $x_0 = b < x_1 < \cdots < x_n = c$, such that

$$v_{f,b}(c) - \frac{\varepsilon}{2} < \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| \le v_{f,b}(c).$$

Let $\delta = c - x_{n-1}$. Then for *x* in $(c-\delta, c)\cap A$,

$$v_{f,b}(c) < \frac{\varepsilon}{2} + \sum_{i=1}^{n-1} \left| f(x_i) - f(x_{i-1}) \right| + \left| f(c) - f(x_{n-1}) \right|$$

$$\le \frac{\varepsilon}{2} + \sum_{i=1}^{n-1} \left| f(x_i) - f(x_{i-1}) \right| + \left| f(x) - f(x_{n-1}) \right| + \left| f(c) - f(x) \right|$$

$$< \frac{\varepsilon}{2} + v_{f,b}(x) + \frac{\varepsilon}{2} = v_{f,b}(x) + \varepsilon.$$

Hence, $v_{f,b}(c) - v_{f,b}(x) < \varepsilon$ and so

$$v_{f,a}(c) - v_{f,a}(x) = v_{f,x}(c) = v_{f,b}(c) - v_{f,x}(x) < \varepsilon.$$

This shows that $v_{f,a}(x)$ is continuous from the left at *c*.

We have thus shown that if $c \ge a$ and *c* is either a left or right limit point or both, then *f* is continuous at *c* implies that $v_{f,a}(x)$ is continuous at *c*.

Similarly, we can show that if *f* is continuous at *c* and $c < a$, then $v_{f,a}(x)$ is continuous at *c*.

Suppose now *c* is not an isolated point of *A* and $v_{f,a}(x)$ is continuous at *c*. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| x - c \right| < \delta \text{ and } x \in A \Rightarrow \left| v_{f,a}(x) - v_{f,a}(c) \right| < \varepsilon.$$

It follows that $\left| f(x) - f(c) \right| \le \left| v_{f,a}(x) - v_{f,a}(c) \right| < \varepsilon$. Hence, *f* is continuous at *c*. This completes the proof of Theorem 13.

Derived Numbers

Suppose $f : A \to \mathbb{R}$ is a finite-valued function defined on an arbitrary subset of $\mathbb{R}$. Let x be in A. Suppose there exists a sequence (a_n) in A such that $a_n \to x$ and $\dfrac{f(a_n) - f(x)}{a_n - x}$ tends to a limit, finite or infinite. Then this limit is called a *derived number* of *f*. If ${}_A Df(x)$ exist (finitely or infinitely), then *f* can have only one derived number at *x*.

The next theorem is a key technical result to a *de La Vallée Poussin's Theorem* for a function of bounded variation.

Theorem 14. Suppose $f : A \to \mathbb{R}$ is a finite-valued function of bounded variation on *A*. Let *h* an *k* be positive numbers such that *h* < *k*. Take $a \in A$ be the anchor point for a total variation function of the function *f* as defined above. We assume *a* is a limit point of *A* and denote the total variation function by v_f. Let

E = {$x \in A$: there is a derived number of v_f at *x* greater than *k* and a derived number of *f* at *x*, whose absolute value is less than *h*.} and

S = {$x \in A$: there is a positive derived number and a negative derived number of *f* at *x*.}.

Then $m^*(v_f(E \cup S)) = m^*(f(E \cup S)) = m^*(E \cup S) = 0$.

Proof.

By Theorem 4, the set of discontinuity of *f* is at most denumerable. Since denumerable subset of *A* and its image under *f* are null sets, we may thus assume that *f* is continuous on *A*. We may assume that *A* has no isolated points, that is, every point of *A* is a limit point of *A*.

If $E \cup S$ is denumerable, then trivially the conclusion of the theorem holds.

We now assume that $E \cup S$ is non-denumerable.

We assume that *A* is bounded.

Let $\ell = \inf A$ and $L = \sup A$. Since *A* is bounded, ℓ and L are finite.

We begin by considering the case when $\ell, L \in A$.

Thus $A \subseteq [\ell, L]$.

Suppose f is a function of bounded variation with bound M. Then, given any $\varepsilon > 0$, there exists a partition of $[\ell, L]$ by points in A,

$$P : x_0 = \ell < x_1 < \cdots < x_n = L,$$

such that

$$M - \varepsilon < \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| \le M . \quad \text{--------------------------} \quad (1)$$

Note that we may take $M = v_{f,\ell}(L)$.

Observe that the derived number of v_f is independent of the anchor point for v_f since $v_{f,a}(x) - v_{f,a}(y) = v_{f,b}(x) - v_{f,b}(y)$ for a, b, x, y in A.

Thus,

$$\sum_{i=1}^{t} \left(v_{f,\ell}(z_i) - v_{f,\ell}(z_{i-1}) \right) = v_{f,\ell}(L) \le M < \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| + \varepsilon$$

$$\le \sum_{i=1}^{t} \left| f(z_i) - f(z_{i-1}) \right| + \varepsilon , \quad \text{--------------------------------------} \quad (2)$$

for any partition, $Q : z_0 = \ell < z_1 < \cdots < z_t = L$, containing all the points of the partition P. We shall denote this special partition P by

$$P : u_0 = \ell < u_1 < \cdots < u_n = L .$$

Now f and $v_{f,\ell}$ are bounded functions. Thus, $v_{f,\ell}(E)$ has finite outer measure. Thus, there exists an open set U containing $v_{f,\ell}(E)$ such that

$$m(U) < m^*\left(v_{f,\ell}(E) \right) + \varepsilon . \quad \text{----------------------------------} \quad (3)$$

Since f is continuous on A, by Theorem 13, $v_{f,\ell}$ is continuous on A and hence is continuous on E. Since U is open, for each $e \in E$, there exists $\zeta > 0$ such that $\left(v_{f,\ell}(e) - \zeta, v_{f,\ell}(e) + \zeta \right) \subseteq U$. By continuity of $v_{f,\ell}$ at $e \in E$, there exists $\delta > 0$ such that

$$x \in (e-\delta, e+\delta) \cap A \Rightarrow \nu_{f,\ell}(x) \in \left(\nu_{f,\ell}(e)-\zeta, \nu_{f,\ell}(e)+\zeta\right) \subseteq U .$$

Thus, we can find arbitrary small closed intervals $[x, y]$ with $x \le e \le y$, $x < y$, $x,y \in A$ such that

$$\nu_{f,\ell}(e) \in \left[\nu_{f,\ell}(x), \nu_{f,\ell}(y)\right] \subseteq \left(\nu_{f,\ell}(e)-\zeta, \nu_{f,\ell}(e)+\varsigma\right).$$

Note that e is a limit point of A, it may be a left limit point only or a right limit point only or both. Since $\nu_{f,\ell}$ has a positive derived number $> k$ at e, we can find such arbitrary small intervals $[x, y]$ such that

$$\frac{\nu_{f,\ell}(y)-\nu_{f,\ell}(x)}{y-x} > k . \quad \text{----------------------} \quad (4)$$

We deduce this claim as follows.

Since there is a derived number at e greater than k , there is a sequence (x_n) in A such that $x_n \neq e$ and $\lim\limits_{n\to\infty} \dfrac{\nu_{f,\ell}(x_n)-\nu_{f,\ell}(e)}{x_n - e} > k$.

Note that for any $\delta > 0$,

$$\sup\left\{ \frac{\nu_{f,\ell}(x)-\nu_{f,\ell}(y)}{x-y} : x > e \ge y \text{ or } x < e \le y, |x-y| < \delta, x, y \in A \right\} \ge \lim_{n\to\infty} \frac{\nu_{f,\ell}(x_n)-\nu_{f,\ell}(e)}{x_n - e} > k \ .$$

Hence, there exist $x, y \in A$ with $|x-y| < \delta$ such that

$$\frac{\nu_{f,\ell}(x)-\nu_{f,\ell}(y)}{x-y} > k .$$

Thus, we can cover $\nu_{f,\ell}(E)$ by arbitrary such small closed intervals $\left[\nu_{f,\ell}(x), \nu_{f,\ell}(y)\right]$ for each $\nu_{f,\ell}(e)$ corresponding to arbitrary small choice of ζ . Note that x may be equal to e or y may be equal to e. Therefore, by the Vitali Covering Theorem, we can cover $\nu_{f,\ell}(E)$ almost everywhere by countable mutually disjoint closed intervals $\left\{\left[\nu_{f,\ell}(a_i), \nu_{f,\ell}(b_i)\right], a_i < b_i, a_i, b_i \in A\right\}$. It follows that the collection of intervals $\left\{[a_i, b_i], a_i < b_i, a_i, b_i \in A\right\}$ are also mutually disjoint.

Note that for each i, $\left[\nu_{f,\ell}(a_i), \nu_{f,\ell}(b_i)\right] \subseteq U$ and

$$\nu_{f,\ell}(b_i)-\nu_{f,\ell}(a_i)>k\left(b_i-a_i\right) \text{ ----------------- (5)}$$

by inequality (4).

Hence, $m^*\left(\nu_{f,\ell}(E)\right)+\varepsilon>m(U)\geq\sum_i\left(\nu_{f,\ell}(b_i)-\nu_{f,\ell}(a_i)\right)>k\sum_i\left(b_i-a_i\right)$ by inequality (5).

It follows that

$$\sum_i\left(b_i-a_i\right)<\frac{1}{k}\left(m^*\left(\nu_{f,\ell}(E)\right)+\varepsilon\right). \text{ ----------------- (6)}$$

Without loss of generality we may assume that the set of points of the partition $P:u_0=\ell<u_1<\cdots<u_n=L$ does not contain any point of *E*. If *P* contains a point in *E*, we may just remove this point from *E* without affecting the conclusion of the theorem. We may thus remove all the points in *P* that are in *E* from *E*. Observe that the partition *P* depends on *ε*. By taking $\varepsilon=\frac{1}{N}$ and letting *N* tends to infinity, we would remove at most a denumerable number of points from *E*. Since denumerable set of points has zero measure, the removal of these points does not affect the conclusion of the theorem.

Next, we observe that if the anchor point chosen is $a\in(\ell,L)$, then $\nu_{f,a}(x)=\nu_{f,\ell}(x)-\nu_{f,\ell}(a)$. Thus, the definition of *E* and *S* does not depend on the anchor point.

We now assume that for any of the partition *P*, $P\cap E=\varnothing$. Now we shall refine our covering. Firstly, we examine the end points of the intervals $\{[a_i,b_i],a_i<b_i,a_i,b_i\in A\}$. We shall examine the points in $E-\{a_i,b_i\}$. Note that the collection $\{a_i,b_i\}$ is at most denumerable and so by not considering these points will not affect the conclusion of the theorem. So, we now remove the end points $\{a_i,b_i\}$ from *E*.

For each $e\in E-\{a_i,b_i\}$, corresponding to any $\zeta>0$ with $\left(\nu_{f,\ell}(e)-\zeta,\nu_{f,\ell}(e)+\zeta\right)\subseteq U$, as $P\cap E=\varnothing$, there exists a $\delta>0$ such that

$$(e-\delta,e+\delta)\cap P=\varnothing,\ (e-\delta,e+\delta)\subseteq\bigcup_i(a_i,b_i)$$

and

$$x \in (e-\delta, e+\delta) \cap A \Rightarrow \nu_{f,\ell}(x) \in \left(\nu_{f,\ell}(e)-\zeta, \nu_{f,\ell}(e)+\zeta\right) \subseteq U .$$

Thus, we can find for *e* in *E*, arbitrary small interval [*x*, *y*] with $x \leq e \leq y$, $x \neq y, x, y \in A$ such that $x, y \in A - \{a_i, b_i\} - P$,

$$\nu_{f,\ell}(e) \in \left[\nu_{f,\ell}(x), \nu_{f,\ell}(e)\right] \subseteq \left(\nu_{f,\ell}(e)-\zeta, \nu_{f,\ell}(e)+\zeta\right).$$

Now we consider the other property that the points in *E* have, namely that at each point of *E*, there is a derived number whose absolute value is less than *h*.

So, at each point *e* of $E - \{a_i, b_i\}$ there is a derived number of *f* whose absolute value is less than *h*.

We claim that we can find arbitrary small interval, [*c*, *d*], with $c, d \in A$, such that *e* is either one of the end points of the interval such that

$$\frac{|f(d)-f(c)|}{d-c} < h,$$

$[c,d] \subseteq \bigcup_i [a_i, b_i]$ and that $P \cap [c,d] = \varnothing$.

We elaborate and prove this claim as follows.

We may assume that every point of *A* is a two-sided limit point of *A*. As one-sided only limit points or non-limit points constitute at most a denumerable set, this assumption would not affect the conclusion of the theorem. If need be, we shall remove these points from *A*.

Let $e \in E - \{a_i, b_i\}$. Suppose β is a derived number of *f* at the point *e* such that $\alpha = |\beta| < h$. Then there exists a sequence, (x_n), such that $x_n \neq e$, $x_n \to e$ and

$$\left|\frac{f(x_n)-f(e)}{x_n - e}\right| \to \alpha < h .$$

Thus, there exists *N* such that

$$n \geq N \Rightarrow 0 \leq \left|\frac{f(x_n)-f(e)}{x_n - e}\right| < h .$$

As $x_n \to e$, we may choose N such that $n \geq N \Rightarrow 0 < |x_n - e| < \delta$. Hence, we can choose $N_0 > N$ such that $\left|\dfrac{f(x_{N_0}) - f(e)}{x_{N_0} - e}\right| < h$ and $0 < |x_{N_0} - e| < \delta$. In particular we may choose $\delta > 0$ sufficiently small so that $(e-\delta, e+\delta) \cap \left(P \cup \bigcup_i \{a_i, b_i\}\right) = \varnothing$.

Take $c = x_{N_0}$. Then $c > e$ or $c < e$. If $c < e$, let $d = e$. If $c > e$, rename c as d and let $c = e$. Hence, we have $P \cap [c,d] = \varnothing$ and $[c,d] \subseteq \bigcup_i (a_i, b_i)$.

If $\alpha > 0$, then we can choose c such that $0 < \left|\dfrac{f(d) - f(c)}{d - c}\right| < h$. This is because by definition of limit, by taking $\varepsilon = \min\left\{\dfrac{h-\alpha}{2}, \dfrac{\alpha}{2}\right\}$, there exists integer $\widetilde{N}$ such that

$$n \geq N \Rightarrow 0 < \frac{\alpha}{2} < \left|\frac{f(x_n) - f(e)}{x_n - e}\right| < \alpha + \varepsilon < h.$$

and then we can pick c arbitrarily closed to e as before. It follows after renaming c and d so that $c < d$, $|f(d) - f(c)| > 0$ and $v_{f,\ell}(d) - v_{f,\ell}(c) > 0$.

If $\alpha = 0$, then for any $0 < \varepsilon < h$, there exists an integer N such that

$$n \geq N \Rightarrow 0 \leq \left|\frac{f(x_n) - f(e)}{x_n - e}\right| < \varepsilon < h.$$

Since f is continuous at e and $x_n \to e$, we have that $f(x_n) \to f(e)$. We may assume that $x_n \neq e$ for all integer $n \geq 1$.

Suppose the sequence $\{f(x_n)\}$ is not eventually constant, i.e., for any integer N, there exists $n_N > N$ such that $f(x_{n_N}) \neq f(e)$. We may then replace the sequence (x_n) by the subsequence $\left(x_{n_N}\right)$. We shall then obtain the closed interval $[c, d]$ with $v_{f,\ell}(d) - v_{f,\ell}(c) > 0$.

Suppose the sequence $\{f(x_n)\}$ is eventually constant, i.e., there exists an integer N, and $n \geq N \Rightarrow f(x_n) = f(e)$. If there exists an integer $\tilde{n}$ such that $v_{f,\ell}(x_{\tilde{n}}) = v_{f,\ell}(e) > 0$, then, as $x_{\tilde{n}} \neq e$, $v_{f,\ell}$ is constant on $[x_{\tilde{n}}, e] \cap A$ or on $[e, x_{\tilde{n}}] \cap A$. It follows that $(x_{\tilde{n}}, e] \cap E = \{e\}$ or $[e, x_{\tilde{n}}) \cap E = \{e\}$ as every point x in $(x_{\tilde{n}}, e) \cap A$ or

$(e,x_{\tilde{n}})\cap A$ would have ${}_A D\nu_{f,\ell}(x)=0$ so that every derived number of $\nu_{f,\ell}$ at x in $(x_{\tilde{n}},e)\cap A$ or $(e,x_{\tilde{n}})\cap A$ is zero. Thus, $e\in\partial\left(\mathbb{R}-\overline{E}\right)$ and so e belongs to a set, which is at most denumerable. We may remove these points from E without affecting the conclusion of the theorem. We may thus assume that $\nu_{f,\ell}(x_n)\neq\nu_{f,\ell}(e)>0$ for all integer $n\geq 1$, that is, $\left|\nu_{f,\ell}(x_n)-\nu_{f,\ell}(e)\right|>0$ for all integer $n\geq 1$.

Hence, we can cover $\nu_{f,\ell}(E)$ by arbitrary small such closed interval $\left[\nu_{f,\ell}(c),\nu_{f,\ell}(d)\right]$ with $[c,d]\subseteq\bigcup_i[a_i,b_i]$ and that $P\cap[c,d]=\varnothing$. Therefore, by the Vitali Covering Theorem, we can cover $\nu_{f,\ell}(E)$ almost everywhere by countable disjoint closed intervals $\left\{\left[\nu_{f,\ell}(c_i),\nu_{f,\ell}(d_i)\right]\right\}$ with $P\cap[c_i,d_i]=\varnothing$, $[c_i,d_i]\subseteq\bigcup_i(a_i,b_i)$ and

$$\left|f(d_i)-f(c_i)\right|<h(d_i-c_i)\,. \text{ -------------- (7)}$$

Now, consider the partition $Q=P\cup\{c_i,d_i\}_{i=1}^N$.

$$\sum_{i=1}^N\left(\nu_{f,\ell}(d_i)-\nu_{f,\ell}(c_i)\right)+\sum_{c_i',d_i'\in Q,c_i'\neq c_j\text{ or }d_i'\neq d_j}\left|f(d_i')-f(c_i')\right|$$

$$\leq\sum_{i=1}^N\left(\nu_{f,\ell}(d_i)-\nu_{f,\ell}(c_i)\right)+\sum_{c_i',d_i'\in Q,c_i'\neq c_j\text{ or }d_i'\neq d_j}\left|\nu_{f,\ell}(d_i')-\nu_{f,\ell}(c_i')\right|=\nu_{f,\ell}(L)$$

$$<\sum_{i=1}^N\left|f(d_i)-f(c_i)\right|+\sum_{c_i',d_i'\in Q,c_i'\neq c_j\text{ or }d_i'\neq d_j}\left|f(d_i')-f(c_i')\right|+\varepsilon \text{ by inequality (2).}$$

Hence,

$$\sum_{i=1}^N\left(\nu_{f,\ell}(d_i)-\nu_{f,\ell}(c_i)\right)\leq\sum_{i=1}^N\left|f(d_i)-f(c_i)\right|+\varepsilon\,.$$

Now letting N tends to infinity we get,

$$\sum_i\left(\nu_{f,\ell}(d_i)-\nu_{f,\ell}(c_i)\right)\leq\sum_i\left|f(d_i)-f(c_i)\right|+\varepsilon\,. \text{ --------------- (8)}$$

Note that $m^*(\nu_{f,\ell}(E))\leq\sum_i\left(\nu_{f,\ell}(d_i)-\nu_{f,\ell}(c_i)\right)$ and so

$$m^*(\nu_{f,\ell}(E))\leq\sum_i\left|f(d_i)-f(c_i)\right|+\varepsilon\leq h\sum_i\left(d_i-c_i\right)+\varepsilon\text{, by inequality (7),}$$

$$\leq h\sum_i (b_i - a_i) + \varepsilon \text{, since } \bigcup_i [c_i, d_i] \subseteq \bigcup_i [a_i, b_i].$$

Hence,

$$\frac{m^*(\nu_{f,\ell}(E)) - \varepsilon}{h} \leq \sum_i (b_i - a_i)$$

$$< \frac{1}{k}\left(m^*\left(\nu_{f,\ell}(E)\right) + \varepsilon\right) \text{ by inequality (6).}$$

As $\varepsilon = \frac{1}{N}$, by passing N to infinity, we get

$$\frac{m^*(\nu_{f,\ell}(E))}{h} \leq \frac{1}{k} m^*\left(\nu_{f,\ell}(E)\right).$$

Since $h < k$, this is possible only if $m^*\left(\nu_{f,\ell}(E)\right) = 0$.

We shall now show that $m^*\left(\nu_{f,\ell}(S)\right) = 0$.

We proceed almost exactly as for the set E.

For $\varepsilon = \frac{1}{N}$, we pick a partition $P: u_0 = \ell < u_1 < \cdots < u_n = L$,

such that

$$M - \varepsilon < \sum_{i=1}^{n} \left| f(u_i) - f(u_{i-1}) \right| \leq M .$$

We may remove the points in P from S for $\varepsilon = \frac{1}{N}$ and for all positive integer N.

We assume that $S \cap P = \varnothing$.

Note that $\nu_{f,\ell}(S)$ has finite outer measure. Thus, there exists an open set V containing $\nu_{f,\ell}(S)$ such that

$$m(V) < m^*\left(\nu_{f,\ell}(S)\right) + \varepsilon. \text{ ------------------------- (9)}$$

Since $\nu_{f,\ell}$ is continuous for each e in S, there exists a $\zeta > 0$ such that $\left(\nu_{f,\ell}(e) - \zeta, \nu_{f,\ell}(e) + \zeta\right) \subseteq V$. By continuity of $\nu_{f,\ell}$ at $e \in S$, there exists $\delta > 0$ such that

$$x \in (e-\delta, e+\delta) \cap A \Rightarrow v_{f,\ell}(x) \in \left(v_{f,\ell}(e)-\zeta, v_{f,\ell}(e)+\zeta\right) \subseteq V .$$

Now, *f* has a derived number at *e* greater than 0. Thus, we may pick arbitrary small closed interval, [*r, s*], such that *e* is one of the end points of the interval, $[r,s] \cap P = \varnothing$, $[v_{f,\ell}(r), v_{f,\ell}(s)] \subseteq \left(v_{f,\ell}(e)-\zeta, v_{f,\ell}(e)+\zeta\right) \subseteq V$ and

$$\frac{f(s)-f(r)}{s-r} > 0 .$$

Since we can choose $\zeta > 0$ to be arbitrarily small, these arbitrary small intervals $[v_{f,\ell}(r), v_{f,\ell}(s)]$ for *e* and for each *e* in *S* form a Vitali covering for $v_{f,\ell}(S)$. Therefore, by the Vitali Covering Theorem, we may cover $v_{f,\ell}(S)$ almost everywhere by a countable mutually disjoint such closed intervals $\left\{[v_{f,\ell}(r_i), v_{f,\ell}(s_i)]\right\}$ such that $[r_i, s_i] \cap P = \varnothing$, $f(s_i)-f(r_i) > 0$ and

$$m^*\left(v_{f,\ell}(S)\right) \leq \sum_i \left(v_{f,\ell}(s_i)-v_{f,\ell}(r_i)\right). \text{ ---------------------- } (10)$$

We deduce similarly as for inequality (8) that

$$\sum_i \left(v_{f,\ell}(s_i)-v_{f,\ell}(r_i)\right) \leq \sum_i \left| f(s_i)-f(r_i)\right| + \varepsilon = \sum_i \left(f(s_i)-f(r_i)\right) + \varepsilon . \text{ --------- } (11)$$

Now, we remove the end points of the collection $\{r_i, s_i\}$ from *S*. Since $\{r_i, s_i\}$ is countable, this will not affect the conclusion of the theorem. We now proceed similarly as for the case of the set *E*. We now assume that $S \cap \{r_i, s_i\} = \varnothing$. Each point *e* in $S - \{r_i, s_i\}$ has a negative derived number. Following the case for the set *E*, we cover $v_{f,\ell}(S)$ almost everywhere by a countable mutually disjoint such closed intervals $\left\{[v_{f,\ell}(p_i), v_{f,\ell}(q_i)]\right\}$ such that $f(p_i) > f(q_i)$, $[p_i, q_i] \subseteq \bigcup_i (r_i, s_i)$, $[p_i, q_i] \cap P = \varnothing$, and

$$m^*\left(v_{f,\ell}(S)\right) \leq \sum_i \left(v_{f,\ell}(q_i)-v_{f,\ell}(p_i)\right)$$

$$\leq \sum_i \left| f(q_i)-f(p_i)\right| + \varepsilon = \sum_i \left(f(p_i)-f(q_i)\right) + \varepsilon . \text{ ------------ } (12)$$

We deduce the last inequality as for inequality (8).

Since $\bigcup_i [p_i, q_i] \subseteq \bigcup_i [r_i, s_i]$ and both collections are collections of disjoint intervals, as $f(q_i) - f(p_i) \le 0$ for each *i*,

$$\sum_i \left(f(q_i) - f(p_i) \right) \ge \sum_i n_{f,\ell}(s_i) - n_{f,\ell}(r_i),$$

where $n_{f,\ell}$ is the negative variation of *f* as defined in the proof of Theorem 6.

Note that by definition, as $\ell \in A$ and $\ell = \inf A$, $n_{f,\ell} \le 0$ and $n_{f,\ell}$ is decreasing.

As $f(s_i) - f(r_i) \ge 0$ for each *i*,

$$\sum_i \left(f(s_i) - f(r_i) \right) \le \sum_i p_{f,\ell}(s_i) - p_{f,\ell}(r_i),$$

where $p_{f,\ell}$ is the positive variation of *f* as defined in the proof of Theorem 6.

Note also that $p_{f,\ell} \ge 0$ and $p_{f,\ell}$ is increasing.

Therefore,

$$\sum_i \left(f(p_i) - f(q_i) \right) + \sum_i \left(f(s_i) - f(r_i) \right) \le \sum_i n_{f,\ell}(r_i) - n_{f,\ell}(s_i) + \sum_i p_{f,\ell}(s_i) - p_{f,\ell}(r_i)$$

$$\le \sum_i \left(p_{f,\ell}(s_i) - n_{f,\ell}(s_i) \right) - \sum_i \left(p_{f,\ell}(r_i) - n_{f,\ell}(r_i) \right) = \sum_i v_{f,\ell}(s_i) - \sum_i v_{f,\ell}(r_i).$$

Hence,

$$\sum_i \left(f(p_i) - f(q_i) \right) + \sum_i \left(f(s_i) - f(r_i) \right) \le \sum_i \left(v_{f,\ell}(s_i) - v_{f,\ell}(r_i) \right). \text{ ------------- } (13)$$

Thus, it follows from inequalities (11) and (12) that

$$\sum_i \left(v_{f,\ell}(s_i) - v_{f,\ell}(r_i) \right) - \varepsilon + \sum_i \left(v_{f,\ell}(q_i) - v_{f,\ell}(p_i) \right) - \varepsilon$$

$$\le \sum_i \left(f(s_i) - f(r_i) \right) + \sum_i \left(f(p_i) - f(q_i) \right) \le \sum_i \left(v_{f,\ell}(s_i) - v_{f,\ell}(r_i) \right).$$

Hence, $\sum_i \left(v_{f,\ell}(q_i) - v_{f,\ell}(p_i) \right) \le 2\varepsilon$. Therefore,

$$m^*\left(v_{f,\ell}(S) \right) \le \sum_i \left(v_{f,\ell}(q_i) - v_{f,\ell}(p_i) \right) \le 2\varepsilon.$$

As $\varepsilon = \dfrac{1}{N}$, and so as $N \to \infty$, $m^*\left(\nu_{f,\ell}(S)\right) \le 0$. We conclude that $m^*\left(\nu_{f,\ell}(S)\right) = 0$.

Next, we assert that $m^*(f(E \cup S)) = 0$. This is a consequence of the fact that for a function *f* of bounded variation, if *H* is a subset such that $m^*\left(\nu_f(H)\right) = 0$, then $m^*(f(H)) = 0$.

We give a proof here for the case when $\ell = \inf(A) \in A$. The proof for the general case is exactly the same.

Suppose *H* is a subset of *A* such that $m^*\left(\nu_f(H)\right) = 0$. Then given any $\varepsilon > 0$, there exists an open set *U* such that $\nu_{f,\ell}(H) \subseteq U$ and $m(U) < \varepsilon$. Since *U* is open, *U* is a disjoint union of at most countable number of open intervals, i.e., $U = \bigcup_n I_n$ and $m(U) = \sum_i m(I_i) < \varepsilon$. Moreover, $\nu_{f,\ell}{}^{-1}(U) \supseteq H$. Let $A_i = f\left(\nu_{f,\ell}{}^{-1}(I_i)\right)$. For any *x*, *y* in A_i, there exist $a, b \in \nu_{f,\ell}{}^{-1}(I_i)$ such that $x = f(a)$ and $y = f(b)$. Then

$$|x - y| = |f(a) - f(b)| \le |\nu_{f,\ell}(a) - \nu_{f,\ell}(b)| \le m^*(I_i).$$

It follows that the diameter of A_i is less than or equal to $m^*(I_i)$. Hence, $m^*(A_i) \le m^*(I_i)$.

Now, $f(H) \subseteq f\left(\nu_{f,\ell}{}^{-1}(U)\right) = f\left(\nu_{f,\ell}{}^{-1}\left(\bigcup_i I_i\right)\right) = f\left(\bigcup_i \nu_{f,\ell}{}^{-1}(I_i)\right) = \bigcup_i f\left(\nu_{f,\ell}{}^{-1}(I_i)\right)$.

Therefore,

$$m^*(f(H)) \le \sum_i m^*\left(f\left(\nu_{f,\ell}{}^{-1}(I_i)\right)\right) = \sum_i m^*(A_i) \le \sum_i m^*(I_i) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $m^*(f(H)) = 0$. This completes the proof of our assertion.

We shall now show that $m(E \cup S) = 0$.

Since *f* is of bounded variation, by Theorem 8, *f* is differentiable almost everywhere. Evidently, *f* is not differentiable at every point of *S* and so $m(S) = 0$.

At each point of $e \in E$, $\nu_{f,\ell}$ has a derived number greater than *k*. We may assume that *f* is continuous at every point of *E* as the discontinuities of *f* form at most a denumerable set.

Since we know that $m\left(\nu_{f,\ell}(E)\right)=0$,given $\varepsilon > 0$, there exists an open set *U* such that $\nu_{f,\ell}(E) \subseteq U$ and $m(U) < \varepsilon$. Since *U* is open, for each *e* in *E*, there exists $\zeta > 0$ such that $\left(\nu_{f,\ell}(e)-\zeta, \nu_{f,\ell}(e)+\zeta\right) \subseteq U$. By continuity of $\nu_{f,\ell}$ at $e \in E$, there exists $\delta > 0$ such that

$$x \in (e-\delta, e+\delta) \cap A \Rightarrow \nu_{f,\ell}(x) \in \left(\nu_{f,\ell}(e)-\zeta, \nu_{f,\ell}(e)+\zeta\right) \subseteq U .$$

Thus, as in the first part of the proof, at each point *e*, we can find arbitrary small closed intervals [*x*, *y*] with $x \le e \le y$, $x < y$, $x, y \in A$ such that

$$\nu_{f,\ell}(e) \in \left[\nu_{f,\ell}(x), \nu_{f,\ell}(y)\right] \subseteq \left(\nu_{f,\ell}(e)-\zeta, \nu_{f,\ell}(e)+\varsigma\right)$$

and $\dfrac{\nu_{f,\ell}(y)-\nu_{f,\ell}(x)}{y-x} > k$.

Thus, these arbitrary small intervals at each point of *E* and for all points in *E* form a Vitali covering for *E*. Therefore, there exists a countable sequence of pairwise disjoint closed intervals, $\{[a_i, b_i]\}$, covering *E* almost everywhere.

We claim that $\left\{\left[\nu_{f,\ell}(a_i), \nu_{f,\ell}(b_i)\right]\right\}$ is a family of non-overlapping intervals. If $[a_1, b_1] \cap [a_2, b_2] = \varnothing$, then either $b_1 < a_2$ or $b_2 < a_1$. Hence, $\nu_{f,\ell}(b_1) \le \nu_{f,\ell}(a_2)$ or $\nu_{f,\ell}(b_2) \le \nu_{f,\ell}(a_1)$, consequently, $\left[\nu_{f,\ell}(a_1), \nu_{f,\ell}(b_1)\right]$ and $\left[\nu_{f,\ell}(a_2), \nu_{f,\ell}(b_2)\right]$ are non-overlapping. Therefore,

$$m^*(E) \le \sum_i m^*([a_i, b_i]) = \sum_i (b_i - a_i) < \sum_i \frac{1}{k}\left(\nu_{f,\ell}(b_i) - \nu_{f,\ell}(a_i)\right)$$

$$= \sum_i \frac{1}{k} m^*\left(\left[\nu_{f,\ell}(a_i), \nu_{f,\ell}(b_i)\right]\right) = \frac{1}{k} m^*\left(\bigcup_i \left[\nu_{f,\ell}(a_i), \nu_{f,\ell}(b_i)\right]\right)$$

$$\le \frac{1}{k} m^*(U) \le \frac{1}{k}\varepsilon .$$

Since ε is arbitrary, we conclude that $m^*(E) = 0$.

If $\ell = \inf(A) \in A$ and $L = \sup(A) \notin A$, then take a sequence (b_n) in A such that $b_n \nearrow L$. A may not be bounded above and so L may be ∞.

Apply the argument to $A \cap [\ell, b_n]$ with the corresponding E and S in $A \cap [\ell, b_n]$ denoted by E_n and S_n. Plainly, $E = \bigcup_n E_n$ and $S = \bigcup_n S_n$. It then follows by the continuity from below property of Lebesgue outer measure that the conclusion of the theorem holds for E and S since it holds for each E_n and S_n.

Now if $\ell = \inf(A) \notin A$, take a sequence (a_n) in A such that $a_n \searrow \ell$. We note that if A is not bounded below, then $\ell = -\infty$. Then apply the theorem to $A \cap [a_n, \infty)$ with a_n as the anchor point for the variation function of f and the reference partition P. If A is not bounded above and below, we can apply the theorem to each bounded part $A \cap [n, n+1]$ and conclude that the conclusion of the theorem holds.

We are now ready to state and prove our main theorem.

Theorem 15. Suppose $f : A \to \mathbb{R}$ is a finite-valued function of bounded variation on A. Then there is a subset N of A such that

$$m\big(v_f(N)\big) = m\big(f(N)\big) = m(N) = 0,$$

where v_f is any total variation of f defined using a point a in A and for each $x \in A - N$, ${}_A Df(x)$ and ${}_A Dv_f(x)$ exist (finitely or infinitely) and that ${}_A Dv_f(x) = \left| {}_A Df(x) \right|$.

Proof.

We note that for different anchor points for the total variation of f, the image of one of them is a translation of the other. Thus, for different anchor points, say, a and b, $m\big(v_{f,a}(N)\big) = m\Big(\big(v_{f,b}(N)\big)\Big)$. Moreover, if the derivative, ${}_A Dv_f(x)$, exists, it is independent of the anchor point used for the definition of the total variation function v_f.

For any rational numbers, $0 < h < k$, let $E_{h,k}$ = {$x \in A$: there is a derived number of v_f at x greater than k and a derived number of f at x, whose absolute value is less than h.}

Let S = {$x \in A$: there is a positive derived number and a negative derived number of f at x.}.

Let $E = \bigcup\left\{E_{h,k} : 0 < h < k, h \text{ and } k \text{ are rational numbers.}\right\}$

Then, E = {$x \in A$: there is a derived number of ν_f greater than the absolute value of a derived number of f at x.}.

Let $N = E \cup S$. We have already shown in the proof of Theorem 14, that

$$m(S) = m\big(f(S)\big) = m\big(\nu_f(S)\big) = 0.$$

By Theorem 14, $m(E_{h,k}) = 0$, for each pair (h, k) of rational numbers with $h < k$. Therefore, $m(E) = m\left(\bigcup\left\{E_{h,k} : 0 < h < k, h \text{ and } k \text{ are rational numbers.}\right\}\right) = 0$. It follows that $m(N) = m(E \cup S) = 0$. Note that $m\big(f(E)\big) \le \sum_{0<h<k,\, h,k \text{ rational}} m\big(f(E_{h,k})\big) = 0$, since the set $f(E) = \bigcup\left\{f\left(E_{h,k}\right) : 0 < h < k, h \text{ and } k \text{ are rational numbers.}\right\}$ is a countable union of sets $f(E_{h,k})$ each of measure zero. Hence, $m\big(f(E)\big) = 0$. Similarly, we can show that $m\big(\nu_f(E)\big) = 0$. It follows that $m\big(\nu_f(N)\big) = 0$.

We now prove the remaining assertion of the theorem.

Take any x in $A - N$. Then $x \notin S$ and $x \notin E$. This means firstly, that f does not have both a negative and a positive derived number at x and secondly, that for any finite derived number, DV, of ν_f at x, $DV \le |Df|$ for any derived number, Df, of f at x.

Therefore, if ν_f has a finite derived number DV at x, then

$$DV \le \inf\left\{|Df| : Df \text{ is a derived number of } f \text{ at } x\right\}. \text{ ------------- (14)}$$

This statement is meaningful if $\inf\left\{|Df| : Df \text{ is a derived number of } f \text{ at } x\right\}$ exists.

Now, by definition of derived number of ν_f, there exists a sequence (x_n) in A such that $x_n \ne x$, $x_n \to x$ and $DV = \lim_{n\to\infty} \dfrac{\nu_f(x_n) - \nu_f(x)}{x_n - x}$. Hence, the sequence $\left\{\dfrac{\nu_f(x_n) - \nu_f(x)}{x_n - x}\right\}$ is bounded. Since for each n, $\left|\dfrac{f(x_n) - f(x)}{x_n - x}\right| \le \left|\dfrac{\nu_f(x_n) - \nu_f(x)}{x_n - x}\right|$, the sequence $\left\{\frac{f(x_n)-f(x)}{x_n-x}\right\}$ is also bounded. Therefore, by the Bolzano Weierstrass Theorem, it has a convergent subsequence $\left\{\frac{f(x_{n_i})-f(x)}{x_{n_i}-x}\right\}$. Let $Df_1=\lim_{i\to\infty}\frac{f(x_{n_i})-f(x)}{x_{n_i}-x}$, then this is a derived number of *f* at *x*. Hence statement (14) is meaningful. Note that the subsequence $\left\{\frac{v_f(x_{n_i})-v_f(x)}{x_{n_i}-x}\right\}$ must converge to the same value *DV*. Thus, we must have

$$|Df_1|\le DV .$$

By (14), since $x\in A-N$, $DV\le|Df_1|$ and so $DV=|Df_1|$. It follows that

$$DV=|Df_1|\le\inf\{|Df|: Df \text{ is a derived number of } f \text{ at } x\}$$

and so $|Df_1|=\inf\{|Df|: Df \text{ is a derived number of } f \text{ at } x\}$. Therefore, since infimum is unique, there can be only one derived number for v_f at *x*. We have thus shown that if v_f has a finite derived number at *x*, then v_f is differentiable at *x*. It then follows that for any derived number *Df* of *f* at *x*, $|Df|\le {}_ADv_f(x)$. But ${}_ADv_f(x)=\inf\{|Df|: Df \text{ is a derived number of } f \text{ at } x\}\le|Df|$. Hence, ${}_ADv_f(x)=|Df|$ for any derived number of *f* at *x*. Since $x\notin S$, *f* does not have positive and negative derived number at *x* and so there can only be one derived number for *f* at *x* and so *f* is differentiable at *x*. Therefore, ${}_ADv_f(x)=|{}_ADf(x)|$.

Since, v_f is an increasing function, either it has a finite derived number or it has only one derived number, equal to infinity. We have shown that if $x\in A-N$, and v_f has a finite derived number, then v_f and *f* are both differentiable at *x* and ${}_ADv_f(x)=|{}_ADf(x)|$.

Suppose now v_f has an infinite derived number at $x\in A-N$. Since $x\notin E$, any derived number *Df* of *f* at *x* must have $|Df|=\infty$. Since $x\notin S$, *f* cannot have derived number of opposite sign and so as any derived number of *f* at *x* must have its modulus equal to ∞, *f* can have only one derived number at *x*, either ∞

or $-\infty$. It follows that *f* is differentiable infinitely at *x*. Thus, v_f is differentiable infinitely at *x* and *f* is differentiable infinitely at *x* with ${}_A Dv_f(x) = |{}_A Df(x)| = \infty$.

This completes the proof of Theorem 15.

Definition 16.

Suppose $f : A \to \mathbb{R}$ is a finite-valued function and *A* is an arbitrary subset of $\mathbb{R}$.

The function *f* is said to satisfy a *Lusin condition* or is a *Lusin function*, if it maps sets of measure zero to sets of measure zero.

Theorem 17. Suppose $f : A \to \mathbb{R}$ is a finite valued function and *A* is an arbitrary subset of $\mathbb{R}$. Suppose *E* is a subset of *A*, where *f* is differentiable at every *x* in *E*. Then *f* is a Lusin function on *E*.

Proof.

Since *f* is differentiable at *x* in *E*, ${}_A Df(x) = {}_A D^+ f(x) = {}_A D_- f(x)$.

For each positive integer *n*, let $E_n = \{x \in E : |{}_A Df(x)| < n\}$. Then $E = \bigcup_{n=1}^{\infty} E_n$ and $E_n \subseteq E_{n+1} \subseteq \cdots$.

Suppose *B* is a subset of *E* of measure zero. Then $m(B \cap E_n) = 0$. Now for every point *x* in E_n, $-n < {}_A D_- f(x) = {}_A Df(x) = {}_A D^+ f(x) < n$. It follows from Theorem 10, *Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem* that $m^*(f(B \cap E_n)) \le nm^*(B \cap E_n) = 0$. Therefore, $m^*(f(B \cap E_n)) = 0$.

Now, $m^*\big(f(B)\big) = m^*\left(f\left(\bigcup_{n=1}^{\infty} B \cap E_n\right)\right) \le \sum_{n=1}^{\infty} m^*\big(f(B \cap E_n)\big)$ and it follows from this that $m^*\big(f(B)\big) = 0$. Thus, *f* is a Lusin function on *E*.

Theorem 18. Suppose $f : A \to \mathbb{R}$ is a finite-valued function of bounded variation on *A*. Let *N* be the subset of *A* given by Theorem 15 such that

$$m\left(v_f(N)\right)=m\left(f(N)\right)=m(N)=0,$$

where v_f is any total variation of *f* defined using a point *a* in *A* and for each $x\in A-N$, ${}_A Df(x)$ and ${}_A Dv_f(x)$ exist (finitely or infinitely) and that ${}_A Dv_f(x)=\left|{}_A Df(x)\right|$. Let $E\subseteq A-N$ be the subset $E=\left\{x\in A-N: {}_A Df(x)=\pm\infty\right\}$. Then *m*(*E*) =0 and *f* is a Lusin function on $A-(E\cup N)$.

Proof. By Theorem 15, *f* is differentiable and has finite derivative at every point in $A-(E\cup N)$. By Theorem 8, *m*(*E*) =0. By Theorem 17, *f* is a Lusin function on $A-(E\cup N)$.