Arbitrary Function, Limit Superior, Dini Derivative

and Lebesgue Density Theorem

by

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Suppose A is an arbitrary subset of the real numbers and $f: A \to \mathbb{R}$ is a finite-valued function. For the consideration of the limit of the function or the associated function involving f, only the limit points of A are relevant. We note that the set of points in A, that are not limit points are at most denumerable. We shall be considering the one-sided limits of the function f and its associated difference quotients, the Dini derivates. For deliberating some limit results for the function f, or any associated function, it is sometimes useful to restrict to a more useful and relevant subset in terms of measure theoretic considerations, i.e., that the procedure or process can be carried out outside a subset of measure zero. We use the Lebesgue density theorem to do this.

This article is the first of a series of four articles about general function on arbitrary subset of $\mathbb{R}$, its differential properties, the Denjoy Saks Young Theorem and consequences. The proof of Denjoy Saks Young Theorem is difficult, often presented in a very succinct and often times arduous manner. We aim to give a proof that includes all the necessary details, intermediate results, often of interest by themselves.

In this article, we set out the ideas and definitions towards a good understanding of the statement of the Denjoy Saks Young Theorem. Our first result is about the difference of the limit superior of a function and its one-sided limit superior. It turns out that up to a denumerable set, the limit superior is always less than or equal to the right limit superior. Similar deduction can be made of the limit inferior of the function. That is, up to a denumerable set, the limit inferior is greater or equal to the right limit inferior of the function. Similar result can be deduced for the left limit superior or the left limit inferior of the function.

Theorem 1. Suppose $f: A \to \mathbb{R}$ is a finite valued function and A is an arbitrary subset of $\mathbb{R}$. Let E be the set of points in A that are not limit points of A at least on one side. Then E is at most a denumerable set. Let $D = A - E$. Then every x in D is a two-sided limit point of A.

Then $\widetilde{D} = \left\{ x \in D : \limsup_{t \to x} f(t) > \limsup_{t \to x^+} f(t) \ \ or \liminf_{t \to x} f(t) < \liminf_{t \to x^+} f(t) \right\}$ is at most denumerable.

Proof.

Observe that if x in A is not a 2-sided limit point of A, then x belongs to at most a denumerable set. This is because x is either not a limit point or a one-sided only limit point and so $x \in \partial\left(\mathbb{R} - \overline{A}\right)$, which is countable. It follows that for every x in D, one-sided limit superior, one-sided limit inferior, limit superior and limit inferior exist at x.

For any rational number $\frac{p}{q}$ in its lowest term, let

$$A_{p,q} = \left\{ x \in D : \limsup_{t \to x} f(t) > \frac{p}{q} > \limsup_{t \to x^+} f(t) \right\}.$$

Now, $\limsup_{t \to x^+} f(t) < \frac{p}{q}$ implies that there exists $\delta_1 > 0$ such that

$$\sup\{ f(t) : t \in (x, x+h) \cap A \} < \frac{p}{q} \text{ for all } 0 < h < \delta_1 .$$

Hence,

$$f(t) < \frac{p}{q} \text{ for all } t \in \left(x, x+\delta_1\right) \cap A. \quad \text{----------------} \quad (1)$$

We also have that

$\limsup_{t \to x} f(t) > \frac{p}{q}$ implies that for all δ>0,

$$\sup\{ f(t) : t \in (x-\delta, x+\delta) \cap A - \{x\} \} > \frac{p}{q}. \quad \text{------------------} \quad (2)$$

It follows that for all $0 < \delta < \delta_1$, there exists a t_δ such that

$$t_\delta \in A-\{x\}, x-\delta < t_\delta < x+\delta \text{ and } f(t_\delta) > \frac{p}{q}.$$

But by (1), if $t_\delta \in (x, x+\delta_1) \cap A$, then $f(t_\delta) < \frac{p}{q}$ and so $t_\delta \in (x-\delta_1, x) \cap A$.

We claim that in the interval $(x, x+\delta_1)$, there does not exist a point $y \in A$ such that

$$\limsup_{t \to y^+} f(t) < \frac{p}{q} < \limsup_{t \to y} f(t).$$

If $y \in (x, x+\delta_1) \cap A$ and $\limsup_{t \to y^+} f(t) < \frac{p}{q} < \limsup_{t \to y} f(t)$, then by (1), for any $\gamma > 0$ such that $(y, y+\gamma) \subseteq (x, x+\delta_1)$, we have that for all $t \in (y, y+\gamma) \cap A$,

$$\sup\{f(t) : t \in (y, y+\gamma) \cap A\} \le \sup\{f(t) : t \in (x, x+\delta_1) \cap A\} \le \frac{p}{q}$$

and $\limsup_{t \to y} f(t) > \frac{p}{q}$ implies that for all $\beta > 0$,

$$\sup\{f(t) : t \in (y-\beta, y+\beta) \cap A - \{y\}\} > \frac{p}{q}.$$

But if we take β such that $y+\beta < x+\delta_1$ and $y-\beta > x$, then for all $t \in (y-\beta, y+\beta) \cap A - \{y\}$, $t \in (x, x+\delta_1)$ so that by (1) $f(t) < \frac{p}{q}$ and so

$$\sup\{f(t) : t \in (y-\beta, y+\beta) \cap A - \{y\}\} \le \frac{p}{q},$$

contradicting $\sup\{f(t) : t \in (y-\beta, y+\beta) \cap A - \{y\}\} > \frac{p}{q}$.

Thus, for every $x \in A_{p,q}$, there exists a $\delta_x > 0$ such that $[x, x+\delta_x) \cap A_{p,q} = \{x\}$. In particular the collection of half open intervals $\{[x, x+\delta_x) : x \in A_{p,q}\}$ are pairwise disjoint. Hence the collection $\{(x, x+\delta_x) : x \in A_{p,q}\}$ is a collection of disjoint open intervals. Since $\mathbb{R}$ is of second countable, this collection is at most countable. Therefore, $A_{p,q}$ is at most denumerable.

Since $\left\{ x \in D : \limsup_{t \to x} f(t) > \limsup_{t \to x^+} f(t) \right\} = \bigcup_{p,q} A_{p,q}$, it follows that

$\left\{ x \in D : \limsup_{t \to x} f(t) > \limsup_{t \to x^+} f(t) \right\}$ is at most denumerable.

The set $\left\{ x \in D : \liminf_{t \to x} f(t) < \liminf_{t \to x^+} f(t) \right\} = \left\{ x \in D : \limsup_{t \to x} \left(-f(t)\right) > \limsup_{t \to x^+} \left(-f(t)\right) \right\}$

and so by what we have just proved, $\left\{ x \in D : \liminf_{t \to x} f(t) < \liminf_{t \to x^+} f(t) \right\}$ is at most denumerable. It follows that $\widetilde{D}$ is at most denumerable.

The next result is about points assuming strict extreme value of a function.

Theorem 2. Suppose $f : A \to \mathbb{R}$ is a finite valued function. Then the set of points consisting of strict maximizer for *f* or strict minimizer for *f* is at most denumerable.

Proof.

Recall that *x* in *A* is a strict maximizer for *f*, if there exist a positive integer, *n*, such that for all $t \in \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap A - \{x\}$, $f(t) < f(x)$. That is to say, $f(x)$ is a strict maximum in $\left\{ f(t) : t \in \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap A \right\}$. Similarly, *x* in *A* is a strict minimizer for *f*, if there exist a positive integer, *n*, such that for all $t \in \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap A - \{x\}$, $f(t) > f(x)$.

Let $E \subseteq A$ be the set of strict maximizer for *f*. Then *E* does not contain any isolated points of *A for* if *x* is an isolated point of *A*, then there exists a positive integer *n* so that $\left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap A - \{x\} = \varnothing$.

We shall filter the collection of strict maximizer for *f*. For each positive integer *n*, let E_n be the collection of strict maximizer *x* for *f* such that *x* is the centre of the interval, $\left(x - \frac{1}{n}, x + \frac{1}{n} \right)$ on which $f(x)$ is the strict maximum, i.e., $f(x)$ is a strict maximum in $\left\{f(t): t \in\left(x-\frac{1}{n}, x+\frac{1}{n}\right) \cap A\right\}$. Then the interval $\left(x-\frac{1}{n}, x+\frac{1}{n}\right)$ cannot contain another maximizer for f in E_n. If $y \in\left(x-\frac{1}{n}, x+\frac{1}{n}\right) \cap A-\{x\}$ is a strict maximizer for f in E_n, then $f(y)<f(x)$. Note that $x \in\left(y-\frac{1}{n}, y+\frac{1}{n}\right) \cap A-\{y\}$. It follows that $f(x)<f(y)$ giving a contradiction. Thus, for all $y \in E_n$, $y \neq x \Rightarrow |y-x| \geq \frac{1}{n}$. Hence, each point of E_n is an isolated point of E_n. We can now conclude using the fact that isolated points of any set in $\mathbb{R}$ is at most countable and so E_n is at most denumerable.

We may also deduce this fact as follows. Note that the collection $\mathscr{C}$ = $\left\{\left(x-\frac{1}{2n}, x+\frac{1}{2n}\right): x \in E_n\right\}$ is a collection of disjoint open intervals covering E_n such that each interval $\left(x-\frac{1}{2n}, x+\frac{1}{2n}\right)$ contains exactly one point x in E_n. If $x \in E_n$, then $\left(x-\frac{1}{2n}, x+\frac{1}{2n}\right) \cap E_n=\{x\}$. Thus, if $x, y \in E_n$ and $x \neq y$, then $y \notin\left(x-\frac{1}{2n}, x+\frac{1}{2n}\right)$ and $\left(x-\frac{1}{2n}, x+\frac{1}{2n}\right) \cap\left(y-\frac{1}{2n}, y+\frac{1}{2n}\right)=\varnothing$ since $|y-x| \geq \frac{1}{n}$. This shows that $\mathscr{C}$ is a collection of disjoint open intervals. Hence, E_n is at most denumerable since $\mathscr{C}$ is at most countable as $\mathbb{R}$ is second countable as a metric topological space with the usual metric. Now $E=\bigcup_{n=1}^{\infty} E_n$ and so E is at most denumerable since each E_n is at most denumerable.

Similarly, we can show that the set $F = \{x \in A: x \text{ is a strict minimizer for } f\}$ is at most denumerable. It follows that the set of points at which f is a strict extremum is at most denumerable.

Definition 3.

Suppose $f: A \rightarrow \mathbb{R}$ is a finite valued function. Then we can define the four *Dini derivates* for f at x in A as follows.

The *upper right derivate* of f at x, ${}_A D^{+} f(x)=\limsup_{t \rightarrow x^{+}}\left\{\frac{f(t)-f(x)}{t-x}: t \in A\right\}$, the *upper left derivate* of f at x, ${}_A D^{-} f(x) = \limsup_{t \to x^{-}} \left\{ \frac{f(t) - f(x)}{t - x} : t \in A \right\}$,

the *lower right derivate* of f at x, ${}_A D_{+} f(x) = \liminf_{t \to x^{+}} \left\{ \frac{f(t) - f(x)}{t - x} : t \in A \right\}$,

the *lower left derivate* of f at x, ${}_A D_{-} f(x) = \liminf_{t \to x^{-}} \left\{ \frac{f(t) - f(x)}{t - x} : t \in A \right\}$.

The *upper derivate* of f at x is defined as ${}_A \overline{D} f(x) = \limsup_{t \to x} \left\{ \frac{f(t) - f(x)}{t - x} : t \in A \right\}$ and the *lower derivate* of f at x is ${}_A \underline{D} f(x) = \liminf_{t \to x} \left\{ \frac{f(t) - f(x)}{t - x} : t \in A \right\}$.

We refer to ${}_A D^{+} f(x), {}_A D_{-} f(x)$ and ${}_A D^{-} f(x), {}_A D_{+} f(x)$ as pairs of *opposite derivates*.

If ${}_A D^{+} f(x) = {}_A D_{+} f(x)$, then the *right derivative* of f at x is defined to be this common value, likewise if ${}_A D^{-} f(x) = {}_A D_{-} f(x)$, then the *left derivative* of f at x is defined to be this common value. If ${}_A \overline{D} f(x) = {}_A \underline{D} f(x)$, or equivalently ${}_A D^{+} f(x) = {}_A D_{+} f(x) = {}_A D^{-} f(x) = {}_A D_{-} f(x)$, then we say f is differentiable at x and the *derivative*, ${}_A Df(x)$, is defined to be the common value.

Theorem 4. Let $f : A \to \mathbb{R}$ be a finite valued function. Then the set of points x at which ${}_A D^{+} f(x) < {}_A D_{-} f(x)$ or ${}_A D^{-} f(x) < {}_A D_{+} f(x)$ is at most denumerable.

Proof.

We may assume that A has no isolated points for the Dini derivates of isolated points are not defined. We may further assume that every point of A is a two-sided limit point of A, since points which are not a limit points at least on one side constitute a denumerable set. Thus, we may assume that Dini derivates are all defined (finite or infinite) for all x in A.

Let $C = \left\{ x \in A : {}_A D^+ f(x) < {}_A D_- f(x) \right\}$. For each integer $q > 0$, and p let

$$C_{p,q} = \left\{ x \in A : {}_A D^+ f(x) < \frac{p}{q} < {}_A D_- f(x) \right\} .$$

Take x in $C_{p,q}$. Let $f_{p,q}(x) = f(x) - p\frac{x}{q}$ for x in A. Then

$$\frac{f_{p,q}(x+h) - f_{p,q}(x)}{h} \text{ for } x+h \in A \text{ and } h \neq 0,$$

$$= \frac{f(x+h) - f(x)}{h} - \frac{p}{q} .$$

Then ${}_A D^+ f_{p,q}(x) = \limsup_{h \to 0^+, x+h \in A} \frac{f(x+h) - f(x)}{h} - \frac{p}{q} = {}_A D^+ f(x) - \frac{p}{q} < 0$ for x in $C_{p,q}$. Note that this holds even if ${}_A D^+ f(x) = -\infty$.

Similarly, ${}_A D_- f_{p,q}(x) = \liminf_{h \to 0^-, x+h \in A} \frac{f(x+h) - f(x)}{h} - \frac{p}{q} = {}_A D_- f(x) - \frac{p}{q} > 0$. We remark that this holds too even if ${}_A D_- f(x) = \infty$. Hence, for all x in $C_{p,q}$,

$${}_A D^+ f_{p,q}(x) < 0 < {}_A D_- f_{p,q}(x) .$$

This means that there exists $\delta_1 > 0$ such that

$$\sup_{0 < h < \delta_1, x+h \in A} \left\{ \frac{f_{p,q}(x+h) - f(x)}{h} \right\} < 0 .$$

Hence, for all $t \in (x, x+\delta_1) \cap A$,

$$f_{p,q}(t) < f_{p,q}(x) . \quad \text{---------------------------------} \ (1)$$

Similarly, ${}_A D_- f_{p,q}(x) > 0$ implies there exists $\delta_2 > 0$ such that

$$\inf_{-\delta_2 < h < 0, x+h \in A} \left\{ \frac{f_{p,q}(x+h) - f(x)}{h} \right\} > 0$$

Thus, for all $t \in (x - \delta_2, x) \cap A$,

$$f_{p,q}(t) < f_{p,q}(x). \quad \text{--------------------------------- (2)}$$

Thus, letting $\delta = \min\{\delta_1, \delta_2\}$, we have that for all $t \in (x-\delta, x+\delta) \cap A$,

$$f_{p,q}(t) < f_{p,q}(x).$$

Thus, every *x* in $C_{p,q}$ is a strict maximizer of $f_{p,q}$. It follows by Theorem 2, that $C_{p,q}$ is at most denumerable. Now $C = \bigcup_{p,q\in\mathbb{N},q>0} C_{p,q}$ is a countable union of at most denumerable sets and so is at most denumerable.

We show similarly that the set of points *x* in *A* such that ${}_AD^- f(x) < {}_AD_+ f(x)$ is at most denumerable.

An immediate deduction is the following:

Corollary 5. Let $f : A \to \mathbb{R}$ be a finite-valued function. Then except for a set of at most denumerable points in *A*, ${}_AD^+ f(x) \geq {}_AD_- f(x)$ and ${}_AD^- f(x) \geq {}_AD_+ f(x)$.

Corollary 6. Suppose $f : A \to \mathbb{R}$ is a finite-valued function. Then the set of points, where the right and left derivatives exist and are not equal is at most denumerable.

Proof.

If the right and left derivatives exist at *x*, then ${}_AD^+ f(x) = {}_AD_+ f(x)$ and ${}_AD^- f(x) = {}_AD_- f(x)$. Thus, if ${}_AD^+ f(x) = {}_AD_+ f(x) \neq {}_AD^- f(x) = {}_AD_- f(x)$,then either ${}_AD^+ f(x) < {}_AD_- f(x)$ or ${}_AD^- f(x) < {}_AD_+ f(x)$ and it follows by Theorem 4, that *x* belongs to a set, which is at most denumerable.

Definition 7.

Suppose *A* is an arbitrary subset of $\mathbb{R}$. Let *x* be a point in $\mathbb{R}$. The *upper density* of *A* at *x* is defined by

$$\limsup_{m(I)\to 0} \left\{ \frac{m^*(A \cap I)}{m(I)} : I \text{ an interval containing } x \right\},$$

where *m** is the Lebegsue outer measure and *m* the Lebesgue measure. Note that for an interval *I*, *m*(*I*) is just the length of the interval *I*.

Similarly, the *lower density* of A at x is defined by

$$\liminf_{m(I)\to 0}\left\{\frac{m^*(A\cap I)}{m(I)} : I \text{ an interval containing } x\right\}.$$

If the upper density of A at x is equal to the lower density of A at x, then the common valued is defined to be the *density* of A at x.

Lebesgue Density Theorem.

We shall state the more restrictive form of the Lebesgue density theorem.

Theorem 8. Let A be any bounded set in $\mathbb{R}$. Then for almost all points x in A, the density of A at x exists and is unity.

Proof.

Plainly, for any x in A and any interval I containing x, $\dfrac{m^*(A\cap I)}{m(I)}\leq 1$. Thus, the upper density of A at x is not greater than 1.

Suppose that there exists a set $B\subseteq A$ with $m^*(B)>0$ and such that at each x in B,

$$\liminf_{m(I)\to 0}\left\{\frac{m^*(A\cap I)}{m(I)} : x\in I\right\}<\lambda<1.$$

If $x\in B$, we can associate with x, a sequence of closed intervals, $\{v_i\}$ with $x\in v_i$ for each positive integer i, such that $m(v_i)\to 0$ and with

$$\frac{m^*(A\cap v_i)}{m(v_i)}<\lambda.$$

We deduce this as follows. Let $\liminf_{m(I)\to 0}\left\{\dfrac{m^*(A\cap I)}{m(I)} : x\in I\right\}=\beta<\lambda$. If we let $\delta_k=\inf_{m(I)\leq\frac{1}{k}}\left\{\dfrac{m^*(A\cap I)}{m(I)} : x\in I\right\}$ for each positive integer k, then $\delta_k\nearrow\beta$. Then for each positive integer n, there exists a $k_n>n$ such that $\beta-\dfrac{1}{n}<\delta_{k_n}\leq\beta$.

Consequently, there exists an interval I_n containing x such that

$\delta_{k_n} \le \dfrac{m^*(A \cap I_n)}{m(I_n)} < \dfrac{\beta+\lambda}{2} < \lambda$ and $m(I_n) \le \dfrac{1}{k_n} < \dfrac{1}{n}$. We may choose the intervals I_n to be closed intervals. For each positive integer *i*, let $v_i = I_i$.

Since $B \subseteq A$, it follows that for $x \in B$,

$$\frac{m^*(B \cap v_i)}{m(v_i)} < \lambda \text{ for all positive integer } i. \quad \text{-------------} \ (1)$$

Thus, these collections of intervals associated with each point *x* in *B*, form a Vitaly family of closed sets covering the set *B*. Therefore, by the Vitaly Covering Theorem, since *B* is bounded, for any $\varepsilon > 0$, there exists a finite mutually exclusive set of these closed intervals, say, $v_1, v_2, \cdots, v_N$ for which

$$m^*\left(\bigcup_{i=1}^{N} B \cap v_i\right) = \sum_{i=1}^{N} m^*\left(B \cap v_i\right) > m^*(B) - \varepsilon$$

and
$$m^*\left(\bigcup_{i=1}^{N} v_i\right) = \sum_{i=1}^{N} m(v_i) < m^*(B) + \varepsilon. \qquad \text{-----------------------} \ (2)$$

We have thus,

$$m^*(B) - \varepsilon < \sum_{i=1}^{N} m^*\left(B \cap v_i\right) < \lambda \sum_{i=1}^{N} m(v_i) < \lambda\left(m^*(B) + \varepsilon\right).$$

Since $\lambda < 1$, we have then

$$m^*(B) < \lambda m^*(B) + 2\varepsilon.$$

This leads to a contradiction if $2\varepsilon < (1-\lambda) m^*(B)$. Thus, we conclude that $m^*(B) = 0$.

Denote the set *B* associated with $0 < \lambda < 1$ by B_λ. Let (λ_n) be a sequence of values such that $\lambda_n \nearrow 1$. Denote the associated set B_{λ_n} by B_n. Then for each positive integer *n*, $m^*(B_n) = 0$.

Let $\mathscr{C} = \{x \in A : \text{lower density of } A \text{ at } x < 1\} = \bigcup_{n=1}^{\infty} B_n$. Then by the continuity from below property of outer measure, $m^*(\mathscr{C}) = 0$. It follows that for almost all *x* in *A*, the lower density of *A* at *x* is 1. Consequently, for almost all *x* in *A*, the density of *A* at *x* is 1.

Corollary 9. Let *A* be any arbitrary set in $\mathbb{R}$. Then for almost all points *x* in *A*, the density of *A* at *x* exists and is unity.

Proof.

If *A* is bounded, then the conclusion follows from Theorem 8. If *A* is unbounded, we can take a countable partition of $\mathbb{R}$ by non-overlapping non-trivial bounded intervals, $\{C_n\}$. Then each $C_n \cap A$ is bounded. Moreover, for *x* in $C_n \cap A - \partial C_n$, for sufficiently small interval *I* containing *x*,

$$\frac{m^*(A \cap I)}{m(I)} = \frac{m^*(A \cap C_n \cap I)}{m(I)}.$$

Hence, the lower density and upper density of *A* at *x* are the same as the lower and upper density of $A \cap C_n$. Therefore, by Theorem 8, for almost all *x* in $A \cap C_n$ the density of *A* at *x* exists and is unity. Since this is true for each integer *n* and the set of boundary points of C_n are countable, it follows that for almost all *x* in *A*, the density of *A* at *x* exists and is unity.

We next present a result concerning the effect of bounds on the Dini derivative on the outer measure of the image of sets.

Theorem 10. Let $f : A \to \mathbb{R}$ be a finite valued function. Suppose *D* is a subset of *A* such that ${}_A D^+ f(x) \le M$ and ${}_A D_- f(x) \ge -M$, where *M* is a finite non-negative number for every *x* in *D*. Then $m^*(f(D)) \le Mm^*(D)$, where m^* is the Lebesgue outer measure.

Proof.

If $m^*(D) = \infty$ and $M > 0$, we have nothing to prove. So, we are left with the case $m^*(D) < \infty$ or $M = 0$.

We now assume that $m^*(D) < \infty$.

Let $\widetilde{D} = \left\{ x \in D : {}_A D^+ f(x) < {}_A D_- f(x) \right\}$. By Theorem 4, $\widetilde{D}$ is at most denumerable. Hence, $m^*(f(\widetilde{D})) = 0$. We may thus assume that for all *x* in *D*, ${}_A D^+ f(x) \ge {}_A D_- f(x)$.

The hypothesis of the theorem implies that every x in D is a two-sided limit point of A.

For x in D,

$$ {}_{A}D^{+} f(x) = \limsup_{t\to x^{+}} \left\{ \frac{f(t)-f(x)}{t-x} : t \in A \right\} = \lim_{\delta\to 0^{+}} a_{\delta}, $$

where $a_{\delta} = \sup \left\{ \frac{f(t)-f(x)}{t-x} : t \in (x, x+\delta) \cap A \right\}$.

Let ${}_{A}D^{+} f(x) = k$. Then $k \le M$. Given any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that for all $0 < \delta < \delta_1$, $k \le a_{\delta} < k + \varepsilon$.

Hence, for all $t \in (x, x+\delta) \cap A$ and $0 < \delta < \delta_1$, $\frac{f(t)-f(x)}{t-x} \le a_{\delta} < k + \varepsilon$. Therefore, for all $t \in (x, x+\delta) \cap A$ and $0 < \delta < \delta_1$,

$$ f(t) - f(x) < (t-x)(k+\varepsilon) \le (M+\varepsilon)(t-x). \quad \text{---------------------------- (1)} $$

Next, ${}_{A}D_{-} f(x) = \liminf_{t\to x^{-}} \left\{ \frac{f(t)-f(x)}{t-x} : t \in A \right\} = \lim_{\delta\to 0^{+}} b_{\delta}$, where

$$ b_{\delta} = \inf \left\{ \frac{f(t)-f(x)}{t-x} : t \in (x-\delta, x) \cap A \right\}. $$

Let ${}_{A}D_{-} f(x) = \beta$. Hence, there exists $\delta_2 > 0$ such that for all $0 < \delta < \delta_2$, $\beta - \varepsilon < b_{\delta} \le \beta$. It follows that for all $t \in (x-\delta, x) \cap A$ and $0 < \delta < \delta_2$,

$$ \frac{f(t)-f(x)}{t-x} \ge b_{\delta} > \beta - \varepsilon \ge -M - \varepsilon. $$

Thus, for all $t \in (x-\delta, x) \cap A$ and $0 < \delta < \delta_2$,

$$ f(t) - f(x) < (t-x)(-M-\varepsilon) = (M+\varepsilon)|t-x|. \quad \text{--------------------- (2)} $$

Let $0 < \delta_x < \min(\delta_1, \delta_2)$. Then, for all $t \in (x-\delta_x, x+\delta_x) \cap A$,

$$ f(t) - f(x) \le (M+\varepsilon)|t-x|. \quad \text{----------------------------------- (3)} $$

Thus, for each x in D, (3) holds for some $\delta_x > 0$.

For each integer $n \ge 1$, let

$$D_n = \left\{ x \in D : f(t) - f(x) \leq (M+\varepsilon)|t-x|, |t-x| < \frac{1}{n}, t \in A \right\}$$

$$= \left\{ x \in D : f(t) - f(x) \leq (M+\varepsilon)|t-x|, t \in \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap A \right\}.$$

Evidently, $D_n \subseteq D_{n+1}$.

Now, for each *x* in *D*, there exists $\delta_x > 0$ such that for all $t \in (x - \delta_x, x + \delta_x) \cap A$,

$$f(t) - f(x) \leq (M+\varepsilon)|t-x|.$$

Take any integer *n* such that $0 < \frac{1}{n} < \delta_x$, $f(t) - f(x) \leq (M+\varepsilon)|t-x|$ for all $t \in \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap A$. Hence, $x \in D_n$. It follows that $D = \bigcup_{n=1}^{\infty} D_n$. Since $m^*(D) < \infty$, by the continuity from below property of Lebesgue outer measure,

$$m^*(D) = \lim_{n\to\infty} m^*(D_n) < \infty \ .$$

Thus, $m^*(D_n) < \infty$. Therefore, we can find an open set *U* containing D_n such that $m(U) \leq m^*(D_n) + \varepsilon$. Now, *U* is a countable union of disjoint open intervals. To each of these open intervals, we can further partition it into at most countable number of non-overlapping intervals, each with length less than $\frac{1}{n}$. Now, we collect all these intervals with non-empty intersection with D_n. These then form a countable covering of D_n. Let $\{I_k\}$ denote this countable covering.

Thus, we have $\sum_k m^*(I_k) \leq m^*(D_n) + \varepsilon$.

For each non empty $I_k \cap D_n$, if $x, y \in I_k \cap D_n$,

$$f(y) - f(x) \leq (M+\varepsilon)|y-x| \text{ and } f(x) - f(y) \leq (M+\varepsilon)|x-y|$$

so that we get $|f(y) - f(x)| \leq (M+\varepsilon)|y-x|$. It follows that the diameter of $f(I_k \cap D_n)$ is less than or equal to $(M+\varepsilon)m^*(I_k)$.

Hence,

$$m^*(f(D_n)) \le m^*\left(f\left(\bigcup_k I_k \cap D_n \right)\right) \le \sum_k m^*\big(f(I_k \cap D_n)\big)$$

$$\le \sum_k (M+\varepsilon) m^*\big(I_k\big) = (M+\varepsilon)\sum_k m^*(I_k) \le (M+\varepsilon)\big(m^*(D_n)+\varepsilon\big)$$

Thus, taking limit as $n \to \infty$,

$$m^*(f(D) = \lim_{n\to\infty} m^*(f(D_n)) \le \lim_{n\to\infty}(M+\varepsilon)\big(m^*(D_n)+\varepsilon\big) = (M+\varepsilon)\big(m^*(D)+\varepsilon\big). \text{ ------ (4)}$$

Since ε is arbitrarily small, $m^*(f(D)) \le Mm^*(D)$.

In particular, if $m^*(D)=0$, from (4), $m^*(f(D)) \le (M+\varepsilon)(\varepsilon)$, for arbitrary ε >0. This implies that $m^*(f(D))=0$ and obviously the inequality holds..

Note that if M =0 and $m^*(D) < \infty$, then except for a denumerable subset in D, ${}_AD^+ f(x) = {}_AD_- f(x) = M = 0$ for x in D. It follows from (4) that $m^*(f(D) \le \varepsilon\big(m^*(D)+\varepsilon\big)$. As ε is arbitrarily small, we conclude that $m^*(f(D)) \le 0$. Therefore, $m^*(f(D))=0$ and we have nothing to prove.

If M = 0 and $m^*(D)=\infty$. Partition D into countable pieces by setting $E_n = D \cap [n, n+1]$. Then D is a countable union of $\{E_n\}$. Since each E_n has finite measure, it follows that $m^*(f(E_n))=0$ and so $m^*(f(D))=0$ and the inequality is trivially true if we set the multiplication rule $0*\infty=0$.

The next result is an application of Theorem 10, often used, for instance in the proof of the change of variable theorem for Lebesgue integration.

Theorem 11. Let $f : A \to \mathbb{R}$ be a finite valued function. Suppose D is a subset of A such that at every point x of D, ${}_ADf(x)$ exists and is zero. Then $m^*(f(D))=0$, where m^* is the Lebesgue outer measure.

Proof. We may assume that every point of D is a two-sided limit point of A, since non-limit points or only one-sided limit points constitute a denumerable set. For x in D, ${}_ADf(x) = {}_AD^+ f(x) = {}_AD_- f(x) = 0$ and so

$$-\frac{1}{n} \le {}_AD_- f(x) \text{ and } {}_AD^+ f(x) \le \frac{1}{n} \text{ for any positive integer } n.$$

Suppose $m^*(D) < \infty$. Then by Theorem 10, $m^*(f(D)) \leq \frac{1}{n} m^*(D)$ for any positive integer n and so as $\frac{1}{n} \to 0$, we conclude that $m^*(f(D)) = 0$.

Suppose $m^*(D) = \infty$. Let $D_n = D \cap [-n, n]$ and so $m^*(D_n) < \infty$. We have then $m^*\left(f(D)\right) = m^*\left(\bigcup_{n=1}^{\infty} f(D_n)\right) = \lim_{n\to\infty} m^* f(D_n)$ by the continuity from below property of Lebesgue outer measure. By what we have just shown, $m^*(f(D_n)) = 0$ and so we can conclude that $m^*\left(f(D)\right) = 0$.