Anti-derivative of ln(1-e^x)

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Let $F(x)=\ln(1-e^x)$. Observe that $F(x)$ is only defined when e^x is less than 1, that is, when $x < 0$. Thus, the domain of $F(x)$ is the interval $(-\infty,0)$. Thus, an appropriate antiderivative of $F(x)$ would have $(-\infty,0)$ as its domain.

Note that $F(x)$ is just a composite of ln(1–x) and e^x. We shall begin by examining the function $\ln(1-x)$ and its power series expansion. It is easy to show that

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}+\cdots=-\sum_{n=0}^{\infty}\frac{1}{1+n}x^{n+1} \text{ for } |x|<1$$

and that the power series on the right-hand side converges uniformly on any closed interval in the interval (–1, 1) and so it converges uniformly on $[0,R]$ for $0 < R < 1$. Consider the function,

$$H(x)=\begin{cases}\dfrac{\ln(1-x)}{x}, \text{ for } 0<x<1,\\ -1, \quad \text{for } x=0\end{cases}$$

Since $\lim_{x\to 0^+}\dfrac{\ln(1-x)}{x}=-1$, $H(x)$ is continuous at x =0. Moreover, for $0\le x<1$,

$$H(x)=\begin{cases}-\sum_{n=0}^{\infty}\dfrac{1}{1+n}x^n, \text{ for } 0<x<1,\\ -1, \quad \text{for } x=0\end{cases}=\begin{cases}-\sum_{n=0}^{\infty}\dfrac{1}{1+n}x^n, \text{ for } 0<x<1,\\ -1, \quad \text{for } x=0\end{cases}=-\sum_{n=0}^{\infty}\frac{1}{1+n}x^n .$$

Note that this power series converges uniformly on $[0,R]$ for $0 < R < 1$. Therefore, we can integrate $H(x)$ term by term from 0 to R for $0 < R < 1$.

Thus, for $0\le x<1$,

$$G(x)=\int_0^x H(t)dt=-\sum_{n=0}^{\infty}\frac{1}{(1+n)^2}x^{n+1}=-\sum_{n=1}^{\infty}\frac{1}{n^2}x^n .$$

$G'(x)=H(x)$ for $0\le x<1$

Since $-\sum_{n=1}^{\infty}\frac{1}{n^2}x^n$ converges at x =1, the power series $-\sum_{n=1}^{\infty}\frac{1}{n^2}x^n$ converges uniformly on [0, 1]. $G(1)=\lim_{x\to 1^-}\int_0^x H(t)dt=-\lim_{x\to 1^-}\sum_{n=0}^{\infty}\frac{1}{(1+n)^2}x^{n+1}=-\sum_{n=1}^{\infty}\frac{1}{n^2}=-\frac{\pi^2}{6}$.

Now, let $L(x) = G(e^x) = -\sum_{n=1}^{\infty} \frac{1}{n^2} e^{nx}$, for $x \le 0$.

$$L'(x) = G'(e^x)e^x = H(e^x)e^x = \frac{\ln(1-e^x)}{e^x} e^x = \ln(1-e^x)$$ for $x < 0$. Thus,

$L'(x) = \ln(1-e^x) = F(x)$ for $x < 0$ and so $L(x)$ is the anti-derivative of $F(x)$.

Now,

$$\int_{-\infty}^{x} \ln(1-e^t)dt = \left[G(e^t)\right]_{-\infty}^{x} = G(e^x) - \lim_{t\to-\infty} G(e^t) = G(e^x) - G(0) = G(e^x)$$ for $x < 0$.

$$\int_{-\infty}^{0} \ln(1-e^t)dt = \lim_{x\to 0^-} \int_{-\infty}^{x} \ln(1-e^t)dt = \lim_{x\to 0^-} G(e^x) = G(1) = -\frac{\pi^2}{6} .$$

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