Evaluation of $\int_0^\infty \frac{\ln(1+x^4)}{1+x^2}dx$

By Ng Tze Beng

We are going to evaluate this integral in two ways. The first is to employ the technique of differentiation under the integral sign and the second is to use Fubini's Theorem. We include an arduous way of evaluating this integral.

Using Differentiation under the integral sign

(1) Using $f(x,t)=\frac{\ln(1+t^4x^4)}{1+x^2}$

Define the function $f(x,t)=\frac{\ln(1+t^4x^4)}{1+x^2}$ for (x, t) in $[0,\infty)\times[0,K]$, where $K\geq 1$. Then f is a continuous function in both variables.

The partial derivative with respect to t exists for all (x, t) in $[0,\infty)\times[0,K]$ and is given by

$$\frac{\partial f}{\partial t}(x,t)=\frac{4t^3x^4}{(1+t^4x^4)(1+x^2)} .$$

Note that $f(x,0)=0$, $\frac{\partial f}{\partial t}(x,0)=0$ for all x in $[0,\infty)$.

We shall consider the case for $f(x,t)=\frac{\ln(1+t^4x^4)}{1+x^2}$ for (x, t) in $[0,\infty)\times[k,K]$, for $0 < k < K$ with $K > 1$. Note that, for $0\leq t\leq K$ and $x\geq 0$,

$$|f_t(x)|=\frac{\ln(1+t^4x^4)}{1+x^2}\leq\frac{\ln(1+K^4x^4)}{1+x^2}\leq 2\frac{\ln(1+K^2x^2)}{1+x^2}\leq 2\frac{\ln(K^2+K^2x^2)}{1+x^2}=2\frac{2\ln(K)+\ln(1+x^2)}{1+x^2}$$

and since $\int_0^\infty \frac{1}{1+x^2}dx=\frac{\pi}{2}$ and $\int_0^\infty \frac{\ln(1+x^2)}{1+x^2}dx=\ln(2)\pi$ are convergent, $\int_0^\infty f(x,t)dx$ is absolutely convergent for $0\leq t\leq K$

Let $G(t)=\int_0^\infty \frac{\partial f}{\partial t}(x,t)dx=\int_0^\infty \frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}dx$. Note that this is well defined since

$\int_0^\infty \frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}dx\leq\int_0^\infty \frac{4t^4x^4/t}{(1+t^4x^4)(1+x^2)}dx\leq\int_0^\infty \frac{4/k}{1+x^2}dx$ for all t in $[k, K]$ and $\int_0^\infty \frac{4/k}{1+x^2}dx$ is finite. Since $\int_0^\infty \frac{4/k}{1+x^2}dx$ is not dependent on t and the integral $\int_0^\infty \frac{4/k}{1+x^2}dx$ is convergent, this shows that $\int_0^\infty \frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}dx$ converges uniformly to $G(t)$ with respect to t in $[k, K]$.

Define $F(t)=\int_0^\infty f(x,t)dx=\int_0^\infty \frac{\ln(1+t^4x^4)}{1+x^2}dx$. Then by Theorem 60 of *Chapter 14 Improper integral and Lebesgue integral*, $F'(t)=\int_0^\infty \frac{\partial f}{\partial t}(x,t)dx$ for t in [k, K]. Since this is true for any $0 < k < K$, it follows hat $F'(t)=\int_0^\infty \frac{\partial f}{\partial t}(x,t)dx$ for t in (0, K]. Since K is arbitrary, we conclude that $F'(t)=\int_0^\infty \frac{\partial f}{\partial t}(x,t)dx$ for $t > 0$.

Observe that $f(x,t)\to 0$ as $t\to 0^+$ and as $\left|\frac{\ln(1+t^4x^4)}{1+x^2}\right|\le \frac{2\ln(1+t^2x^2)}{1+x^2}\le \frac{2\ln(1+x^2)}{1+x^2}$ for $t\le 1$, $\lim_{t\to 0^+}F(t)=0=F(0)$. It follows that F is continuous at 0.

We shall now find an anti-derivative of $\frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}$ with respect to x.

$$\frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}=\frac{4t^3x^4}{(1+\sqrt{2}tx+t^2x^2)(1-\sqrt{2}tx+t^2x^2)(1+x^2)}$$

$$=\frac{\sqrt{2}t^2(1+t^2)x^4}{(1+t^4)(1-\sqrt{2}tx+t^2x^2)}-\frac{2t^3}{(1+t^4)(1-\sqrt{2}tx+t^2x^2)}$$

$$-\frac{\sqrt{2}t^2(1+t^2)x^4}{(1+t^4)(1+\sqrt{2}tx+t^2x^2)}-\frac{2t^3}{(1+t^4)(1+\sqrt{2}tx+t^2x^2)}+\frac{4t^3}{(1+t^4)(1+x^2)}$$

Now,

$$\frac{\sqrt{2}t^2(1+t^2)x^4}{(1+t^4)(1-\sqrt{2}tx+t^2x^2)}=\frac{\sqrt{2}(1+t^2)}{1+t^4}\cdot\frac{\frac{2xt^2-\sqrt{2}t}{2}+\frac{\sqrt{2}t}{2}}{1-\sqrt{2}tx+t^2x^2}$$

$$=\frac{1}{2}\frac{\sqrt{2}(1+t^2)}{(1+t^4)}\cdot\frac{2xt^2-\sqrt{2}t}{1-\sqrt{2}tx+t^2x^2}+\frac{(1+t^2)t}{(1+t^4)}\cdot\frac{1}{1-\sqrt{2}tx+t^2x^2}$$

$$=\frac{\sqrt{2}}{2}\left(\frac{1+t^2}{1+t^4}\right)\cdot\frac{2xt^2-\sqrt{2}t}{1-\sqrt{2}tx+t^2x^2}+\left(\frac{1+t^2}{(1+t^4)t}\right)\cdot\frac{1}{\left(x-\frac{1}{\sqrt{2}t}\right)^2+\frac{1}{2t^2}},$$

$$\frac{\sqrt{2}t^2(1+t^2)x^4}{(1+t^4)(1+\sqrt{2}tx+t^2x^2)}=\frac{\sqrt{2}(1+t^2)}{1+t^4}\cdot\frac{\frac{2xt^2+\sqrt{2}t}{2}-\frac{\sqrt{2}t}{2}}{1+\sqrt{2}tx+t^2x^2}$$

$$=\frac{1}{2}\frac{\sqrt{2}(1+t^2)}{(1+t^4)}\cdot\frac{2xt^2+\sqrt{2}t}{1+\sqrt{2}tx+t^2x^2}-\frac{(1+t^2)t}{(1+t^4)}\cdot\frac{1}{1+\sqrt{2}tx+t^2x^2}$$

$$=\frac{\sqrt{2}}{2}\left(\frac{1+t^2}{1+t^4}\right)\cdot\frac{2xt^2+\sqrt{2}t}{1+\sqrt{2}tx+t^2x^2}-\left(\frac{1+t^2}{(1+t^4)t}\right)\cdot\frac{1}{\left(x+\frac{1}{\sqrt{2}t}\right)^2+\frac{1}{2t^2}},$$

Therefore,

$$\left(\int\frac{\sqrt{2}t^2(1+t^2)x^4}{(1+t^4)(1-\sqrt{2}tx+t^2x^2)}-\frac{\sqrt{2}t^2(1+t^2)x^4}{(1+t^4)(1+\sqrt{2}tx+t^2x^2)}\right)dx$$

$$=\frac{1}{\sqrt{2}}\left(\frac{1+t^2}{1+t^4}\right)\ln\left(\frac{1-\sqrt{2}tx+t^2x^2}{1+\sqrt{2}tx+t^2x^2}\right)+\sqrt{2}\left(\frac{1+t^2}{1+t^4}\right)\left(\tan^{-1}\left(\sqrt{2}tx-1\right)+\tan^{-1}\left(\sqrt{2}tx+1\right)\right)+C.$$

$$\int\frac{2t^3}{(1+t^4)(1-\sqrt{2}tx+t^2x^2)}dx=\int\frac{2t}{1+t^4}\frac{1}{\left(x-\frac{1}{\sqrt{2}t}\right)^2+\frac{1}{2t^2}}dx=\frac{2\sqrt{2}t^2}{1+t^4}\tan^{-1}\left(\sqrt{2}tx-1\right)+C\ ,$$

$$\int\frac{2t^3}{(1+t^4)(1+\sqrt{2}tx+t^2x^2)}dx=\int\frac{2t}{1+t^4}\frac{1}{\left(x+\frac{1}{\sqrt{2}t}\right)^2+\frac{1}{2t^2}}dx=\frac{2\sqrt{2}t^2}{1+t^4}\tan^{-1}\left(\sqrt{2}tx+1\right)+C$$ and

$$\int\frac{4t^3}{(1+t^4)(1+x^2)}dx=\frac{4t^3}{1+t^4}\tan^{-1}(x)+C$$

Therefore,

$$\int_0^b\frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}dx=\left[\frac{1}{\sqrt{2}}\left(\frac{1+t^2}{1+t^4}\right)\ln\left(\frac{1-\sqrt{2}tx+t^2x^2}{1+\sqrt{2}tx+t^2x^2}\right)+\sqrt{2}\left(\frac{1+t^2}{1+t^4}\right)\left(\tan^{-1}\left(\sqrt{2}tx-1\right)+\tan^{-1}\left(\sqrt{2}tx+1\right)\right)\right]_0^b$$

$$+\left[-\frac{2\sqrt{2}t^2}{1+t^4}\tan^{-1}\left(\sqrt{2}tx-1\right)-\frac{2\sqrt{2}t^2}{1+t^4}\tan^{-1}\left(\sqrt{2}tx+1\right)+\frac{4t^3}{1+t^4}\tan^{-1}(x)\right]_0^b$$

$$\int_0^b\frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}dx=\frac{1}{\sqrt{2}}\left(\frac{1+t^2}{1+t^4}\right)\ln\left(\frac{1-\sqrt{2}tb+t^2b^2}{1+\sqrt{2}tb+t^2b^2}\right)+\sqrt{2}\left(\frac{1+t^2}{1+t^4}\right)\left(\tan^{-1}\left(\sqrt{2}tb-1\right)+\tan^{-1}\left(\sqrt{2}tb+1\right)\right)$$

$$-\frac{1}{\sqrt{2}}\left(\frac{1+t^2}{1+t^4}\right)\ln(1)-\sqrt{2}\left(\frac{1+t^2}{1+t^4}\right)\left(\tan^{-1}(-1)+\tan^{-1}(1)\right)$$

$$-\frac{2\sqrt{2}t^2}{1+t^4}\tan^{-1}\left(\sqrt{2}tb-1\right)-\frac{2\sqrt{2}t^2}{1+t^4}\tan^{-1}\left(\sqrt{2}tb+1\right)+\frac{4t^3}{1+t^4}\tan^{-1}(b)$$

$$+\frac{2\sqrt{2}t^2}{1+t^4}\tan^{-1}(-1)+\frac{2\sqrt{2}t^2}{1+t^4}\tan^{-1}(1)$$

$$=\frac{1}{\sqrt{2}}\left(\frac{1+t^2}{1+t^4}\right)\ln\left(\frac{1-\sqrt{2}tb+t^2b^2}{1+\sqrt{2}tb+t^2b^2}\right)+\sqrt{2}\left(\frac{1+t^2}{1+t^4}\right)\left(\tan^{-1}\left(\sqrt{2}tb-1\right)+\tan^{-1}\left(\sqrt{2}tb+1\right)\right)$$

$$-\frac{2\sqrt{2}t^2}{1+t^4}\tan^{-1}\left(\sqrt{2}tb-1\right)-\frac{2\sqrt{2}t^2}{1+t^4}\tan^{-1}\left(\sqrt{2}tb+1\right)+\frac{4t^3}{1+t^4}\tan^{-1}(b)$$

Therefore, for $K \geq t > 0$,

$$\int_0^\infty \frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}dx = 0+\sqrt{2}\left(\frac{1+t^2}{1+t^4}\right)\left(\frac{\pi}{2}+\frac{\pi}{2}\right)-\frac{2\sqrt{2}t^2}{1+t^4}\frac{\pi}{2}-\frac{2\sqrt{2}t^2}{1+t^4}\frac{\pi}{2}+\frac{4t^3}{1+t^4}\frac{\pi}{2}$$

$$=\sqrt{2}\left(\frac{1+t^2}{1+t^4}\right)\pi-\frac{2\sqrt{2}t^2}{1+t^4}\pi+\frac{2t^3}{1+t^4}\pi=\frac{\sqrt{2}\pi}{1+t^4}\left(\sqrt{2}t^3-t^2+1\right). \quad \text{-------------- (1)}$$

$$=\frac{\sqrt{2}\pi}{1+t^4}\left(\sqrt{2}t^3-sign(t)(t^2-1)\right)$$

Obviously, when $t = 0$, $f_t(x)=f(x,t)=0$ is the zero constant function. We have thus $\int_0^\infty \frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}dx=\frac{\sqrt{2}\pi}{1+t^4}\left(\sqrt{2}t^3-sign(t)(t^2-1)\right)$.

Therefore, $F'(t)=\int_0^\infty \frac{\partial f}{\partial t}(x,t)dx=\frac{\sqrt{2}\pi}{1+t^4}\left(\sqrt{2}t^3-sign(t)(t^2-1)\right)$ for $t > 0$.

We shall now find an antiderivative of $\frac{\sqrt{2}\pi}{1+t^4}\left(\sqrt{2}t^3-sign(t)(t^2-1)\right)$ for $t > 0$.

For $t > 0$, $\frac{\sqrt{2}\pi}{1+t^4}\left(\sqrt{2}t^3-sign(t)(t^2-1)\right)=\frac{\pi}{2}\left(\frac{4t^3}{1+t^4}\right)+\sqrt{2}\pi\frac{1-t^2}{1+t^4}$

$$=\frac{\pi}{2}\left(\frac{4t^3}{1+t^4}\right)-\sqrt{2}\pi\frac{\sqrt{2}t-1}{2(1-\sqrt{2}t+t^2)}+\sqrt{2}\pi\frac{\sqrt{2}t+1}{2(1+t+t^2)}$$

$$=\frac{\pi}{2}\left(\frac{4t^3}{1+t^4}\right)-\frac{\pi}{2}\left(\frac{2t-\sqrt{2}}{1-\sqrt{2}t+t^2}\right)+\frac{\pi}{2}\left(\frac{2t+\sqrt{2}}{1+\sqrt{2}t+t^2}\right).$$

Therefore, for $t > 0$,

$$\int\frac{\sqrt{2}\pi}{1+t^4}\left(\sqrt{2}t^3-sign(t)(t^2-1)\right)dt=\frac{\pi}{2}\ln\left(1+t^4\right)-\frac{\pi}{2}\ln\left(1-\sqrt{2}t+t^2\right)+\frac{\pi}{2}\ln\left(1+\sqrt{2}t+t^2\right)+C$$

$$=\frac{\pi}{2}\ln\left(1+\sqrt{2}t+t^2\right)+\frac{\pi}{2}\ln\left(1-\sqrt{2}t+t^2\right)-\frac{\pi}{2}\ln\left(1-\sqrt{2}t+t^2\right)+\frac{\pi}{2}\ln\left(1+\sqrt{2}t+t^2\right)+C$$

$$=\pi\ln\left(1+\sqrt{2}t+t^2\right)+C \qquad \text{------------------------------------ (2)}$$

It follows that, for $t > 0$, $F(t)=\pi\ln\left(1+\sqrt{2}t+t^2\right)+C$.

Note that $F(t)=\int_0^\infty f(x,t)dx=\int_0^\infty f_t(x)dx$, where $f_t(x)=f(x,t)=\frac{\ln(1+t^4x^4)}{1+x^2}$.

$f_t(x)$ tends to the zero constant function as t tends to 0 on the right. Since

$$|f_t(x)| = \frac{\ln(1+t^4x^4)}{1+x^2} \le \frac{\ln(1+K^4x^4)}{1+x^2} \le 2\frac{\ln(1+K^2x^2)}{1+x^2} \le 2\frac{\ln(K^2+K^2x^2)}{1+x^2} = 2\frac{2\ln(K)+\ln(1+x^2)}{1+x^2},$$

and $2\frac{2\ln(K)+\ln(1+x^2)}{1+x^2}$ is Lebesgue integrable on $[0,\infty)$, by the Lebesgue Dominated Convergence Theorem, $F(t)=\int_0^\infty f_t(x)dx$ tends to 0. Hence,

$0 = \lim_{t\to 0^+} F(t) = \pi \lim_{t\to 0^+} \ln\left(1+\sqrt{2}t+t^2\right) + C = C$ implies that $C = 0$ and so

$$\int_0^\infty \frac{\ln(1+t^4x^4)}{1+x^2}dx = F(t) = \pi\ln\left(1+\sqrt{2}t+t^2\right) \text{ for } t \ge 0.$$

Thus, $\int_0^\infty \frac{\ln(1+x^4)}{1+x^2}dx = F(1) = \pi\ln\left(1+\sqrt{2}+1\right) = \pi\ln\left(2+\sqrt{2}\right)$. Therefore,

$\int_{-\infty}^\infty \frac{\ln(1+x^4)}{1+x^2}dx = 2\pi\ln\left(2+\sqrt{2}\right)$.

(2) Using $f(x,t) = \frac{\ln(t^4+x^4)}{1+x^2}$

Next, we are going to use another function to apply the differentiation under the integral sign.

Let $f(x,t) = \frac{\ln(t^4+x^4)}{1+x^2}$ for all (x, t) in $(0,\infty)\times[0,K]$ with K >1. Then

$\frac{\partial f}{\partial t}(x,t) = \frac{4t^3}{(t^4+x^4)(1+x^2)}$ for all (x, t) in $(0,\infty)\times[0,K]$.

Observe that for (x, t) in $(0,\infty)\times[0,K]$,

$$\frac{\ln(x^4)}{1+x^2} \le f(x,t) = \frac{\ln(t^4+x^4)}{1+x^2} \le \frac{\ln(K^4+K^4x^4)}{1+x^2} \le \frac{\ln(K^4(1+x^2)^2)}{1+x^2} = \frac{4\ln(K)+2\ln(1+x^2)}{1+x^2}.$$

Since $\int_0^\infty \frac{\ln(x^4)}{1+x^2}dx$ and $\int_0^\infty \frac{4\ln(K)+2\ln(1+x^2)}{1+x^2}dx$ are both convergent, $\int_0^\infty f(x,t)dx$ is convergent for all t in [0, K].

Let $G(t) = \int_0^\infty \frac{\partial f}{\partial t}(x,t)dx = \int_0^\infty \frac{4t^3}{(t^4+x^4)(1+x^2)}dx$. Note that this is well defined for all t in [k, K], since $\int_0^\infty \frac{4t^3}{(t^4+x^4)(1+x^2)}dx \le \int_0^\infty \frac{4t^4/t}{(t^4+x^4)(1+x^2)}dx \le \int_0^\infty \frac{4/k}{1+x^2}dx$ for all t in [k, K] with $0<k<K$, K >1, and $\int_0^\infty \frac{4/k}{1+x^2}dx$ is finite. Since $\int_0^\infty \frac{4/k}{1+x^2}dx$ is not dependent on t and the integral $\int_0^\infty \frac{4/k}{1+x^2}dx$ is convergent, this shows that $\int_0^\infty \frac{4t^3}{(t^4+x^4)(1+x^2)}dx$ converges uniformly to $G(t)$ with respect to t in $[k, K]$.

Define $F(t)=\int_0^\infty f(x,t)dx=\int_0^\infty \frac{\ln(t^4+x^4)}{1+x^2}dx$ for t in $[k, K]$. Then by Theorem 60 of *Chapter 14 Improper integral and Lebesgue integral*, $F'(t)=\int_0^\infty \frac{\partial f}{\partial t}(x,t)dx$ for t in $[k, K]$. Since this is true for any $k > 0$, it follows hat $F'(t)=\int_0^\infty \frac{\partial f}{\partial t}(x,t)dx$ for t in $(0, K]$.

Observe that $f(x,t)\to\frac{\ln(x^4)}{1+x^2}$ as $t\to 0^+$ for $x > 0$ and

$\left|\frac{\ln(t^4+x^4)}{1+x^2}\right|\le\max\left\{\frac{2\ln(K^2(1+x^2))}{1+x^2},\frac{\left|\ln(x^4)\right|}{1+x^2}\right\}$ for $0 < t \le K$ and $x\in(0,\infty)$.

Let $h(x)=\begin{cases}\frac{2\ln(K^2(1+x^2))}{1+x^2}, \text{ if } x>1\\ \frac{\left|\ln(x^4)\right|+2\ln(2)+4\ln(K)}{1+x^2}, \text{ if } 0<x\le 1\end{cases}$. Then $\left|\frac{\ln(t^4+x^4)}{1+x^2}\right|\le h(x)$ for all $x > 0$ and $0 < t \le K$. The function $h(x)$ is Lebesgue integrable on $(0,\infty)$. Therefore, since $f_t(x)=f(x,t)=\frac{\ln(t^4+x^4)}{1+x^2}$ for $0 < t \le K$ is dominated by an integrable function $h(x)$, and each $f_t(x)$ is Lebesgue integrable on $(0, \infty)$, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{t\to 0^+}\int_0^\infty f_t(x)dx=\int_0^\infty\frac{\ln(x^4)}{1+x^2}dx=4\int_0^\infty\frac{\ln(x)}{1+x^2}dx=4\int_0^{\frac{\pi}{2}}\ln(\tan(x))dx=0.$$

It follows that F is continuous at 0.

We shall now find an anti-derivative of $\frac{4t^3}{(t^4+x^4)(1+x^2)}$ with respect to x.

$$\frac{4t^3}{(t^4+x^4)(1+x^2)}=\frac{4t^3}{(x^2+\sqrt{2}tx+t^2)(x^2-\sqrt{2}tx+t^2)(1+x^2)}$$

$$=-\frac{\sqrt{2}(1+t^2)x}{(1+t^4)(t^2-\sqrt{2}tx+x^2)}+\frac{2t}{(1+t^4)(t^2-\sqrt{2}tx+x^2)}$$

$$+\frac{\sqrt{2}(1+t^2)x}{(1+t^4)(x^2+\sqrt{2}tx+t^2)}+\frac{2t}{(1+t^4)(x^2+\sqrt{2}tx+t^2)}+\frac{4t^3}{(1+t^4)(1+x^2)}.$$

$$\frac{\sqrt{2}(1+t^2)x}{(1+t^4)(x^2-\sqrt{2}tx+t^2)}=\frac{\sqrt{2}(1+t^2)}{1+t^4}\cdot\frac{\frac{2x-\sqrt{2}t}{2}+\frac{\sqrt{2}t}{2}}{x^2-\sqrt{2}tx+t^2}$$

$$=\frac{1}{2}\frac{\sqrt{2}(1+t^2)}{(1+t^4)}\cdot\frac{2x-\sqrt{2}t}{x^2-\sqrt{2}tx+t^2}+\frac{(1+t^2)t}{(1+t^4)}\cdot\frac{1}{x^2-\sqrt{2}tx+t^2}$$

$$=\frac{\sqrt{2}}{2}\left(\frac{1+t^2}{1+t^4}\right)\cdot\frac{2x-\sqrt{2}t}{x^2-\sqrt{2}tx+t^2}+\frac{(1+t^2)t}{1+t^4}\cdot\frac{1}{\left(x-\frac{\sqrt{2}t}{2}\right)^2+\frac{t^2}{2}},$$

$$\frac{\sqrt{2}(1+t^2)x}{(1+t^4)(x^2+\sqrt{2}tx+t^2)}=\frac{\sqrt{2}(1+t^2)}{1+t^4}\cdot\frac{\frac{2x+\sqrt{2}t}{2}-\frac{\sqrt{2}t}{2}}{x^2+\sqrt{2}tx+t^2}$$

$$=\frac{1}{2}\frac{\sqrt{2}(1+t^2)}{(1+t^4)}\cdot\frac{2x+\sqrt{2}t}{x^2+\sqrt{2}tx+t^2}-\frac{(1+t^2)t}{(1+t^4)}\cdot\frac{1}{x^2+\sqrt{2}tx+t^2}$$

$$=\frac{\sqrt{2}}{2}\left(\frac{1+t^2}{1+t^4}\right)\cdot\frac{2x+\sqrt{2}t}{x^2+\sqrt{2}tx+t^2}-\frac{(1+t^2)t}{1+t^4}\cdot\frac{1}{\left(x+\frac{\sqrt{2}t}{2}\right)^2+\frac{1}{2t^2}}.$$

Therefore, for $t > 0$,

$$\int-\frac{\sqrt{2}(1+t^2)x}{(1+t^4)(x^2-\sqrt{2}tx+t^2)}\,dx+\int\frac{\sqrt{2}(1+t^2)x}{(1+t^4)(x^2+\sqrt{2}tx+t^2)}dx$$

$$=\frac{1}{\sqrt{2}}\left(\frac{1+t^2}{1+t^4}\right)\ln\left(\frac{x^2+\sqrt{2}tx+t^2}{x^2-\sqrt{2}tx+t^2}\right)-\sqrt{2}\,\frac{(1+t^2)}{1+t^4}\left(\tan^{-1}\left(\frac{\sqrt{2}x}{t}-1\right)+\tan^{-1}\left(\frac{\sqrt{2}x}{t}+1\right)\right)+C.$$

$$\int\frac{2t}{(1+t^4)(t^2-\sqrt{2}tx+x^2)}\,dx=\int\frac{2t}{1+t^4}\,\frac{1}{\left(x-\frac{t}{\sqrt{2}}\right)^2+\frac{t^2}{2}}\,dx=\frac{2\sqrt{2}}{1+t^4}\tan^{-1}\left(\frac{\sqrt{2}x}{t}-1\right)+C\ ,$$

$$\int\frac{2t}{(1+t^4)(x^2+\sqrt{2}tx+t^2)}\,dx=\int\frac{2t}{1+t^4}\,\frac{1}{\left(x+\frac{t}{\sqrt{2}}\right)^2+\frac{t^2}{2}}\,dx=\frac{2\sqrt{2}}{1+t^4}\tan^{-1}\left(\frac{\sqrt{2}x}{t}+1\right)+C\ \text{ and}$$

$$\int\frac{4t^3}{(1+t^4)(1+x^2)}\,dx=\frac{4t^3}{1+t^4}\tan^{-1}(x)+C.$$

$$\int_0^b\frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}\,dx=\left[\frac{1}{\sqrt{2}}\left(\frac{1+t^2}{1+t^4}\right)\ln\left(\frac{x^2+\sqrt{2}tx+t^2}{x^2-\sqrt{2}tx+t^2}\right)-\sqrt{2}\,\frac{(1+t^2)}{1+t^4}\left(\tan^{-1}\left(\frac{\sqrt{2}x}{t}-1\right)+\tan^{-1}\left(\frac{\sqrt{2}x}{t}+1\right)\right)\right]_0^b$$

$$+\left[\frac{2\sqrt{2}}{1+t^4}\tan^{-1}\left(\frac{\sqrt{2}x}{t}-1\right)+\frac{2\sqrt{2}}{1+t^4}\tan^{-1}\left(\frac{\sqrt{2}x}{t}+1\right)+\frac{4t^3}{1+t^4}\tan^{-1}(x)\right]_0^b$$

$$\int_0^b \frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}dx=\frac{1}{\sqrt{2}}\left(\frac{1+t^2}{1+t^4}\right)\ln\left(\frac{b^2+\sqrt{2}tb+t^2}{b^2-\sqrt{2}tb+t^2}\right)-\sqrt{2}\frac{(1+t^2)}{1+t^4}\left(\tan^{-1}\left(\frac{\sqrt{2}b}{t}-1\right)+\tan^{-1}\left(\frac{\sqrt{2}b}{t}+1\right)\right)$$

$$+\frac{1}{\sqrt{2}}\left(\frac{1+t^2}{1+t^4}\right)\ln(1)-\sqrt{2}\left(\frac{1+t^2}{1+t^4}\right)\left(\tan^{-1}(-1)+\tan^{-1}(1)\right)$$

$$+\frac{2\sqrt{2}}{1+t^4}\tan^{-1}\left(\frac{\sqrt{2}b}{t}-1\right)+\frac{2\sqrt{2}}{1+t^4}\tan^{-1}\left(\frac{\sqrt{2}b}{t}+1\right)+\frac{4t^3}{1+t^4}\tan^{-1}(b)$$

$$-\frac{2\sqrt{2}}{1+t^4}\tan^{-1}(-1)+\frac{2\sqrt{2}}{1+t^4}\tan^{-1}(+1)-\frac{4t^3}{1+t^4}\tan^{-1}(0)$$

$$==\frac{1}{\sqrt{2}}\left(\frac{1+t^2}{1+t^4}\right)\ln\left(\frac{b^2+\sqrt{2}tb+t^2}{b^2-\sqrt{2}tb+t^2}\right)-\sqrt{2}\frac{(1+t^2)}{1+t^4}\left(\tan^{-1}\left(\frac{\sqrt{2}b}{t}-1\right)+\tan^{-1}\left(\frac{\sqrt{2}b}{t}+1\right)\right)$$

$$+\frac{2\sqrt{2}}{1+t^4}\tan^{-1}\left(\frac{\sqrt{2}b}{t}-1\right)+\frac{2\sqrt{2}}{1+t^4}\tan^{-1}\left(\frac{\sqrt{2}b}{t}+1\right)+\frac{4t^3}{1+t^4}\tan^{-1}(b).$$

Thus, for $t > 0$,

$$\int_0^\infty \frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}dx=\lim_{b\to\infty}\int_0^b \frac{4t^3x^4}{(1+t^4x^4)(1+x^2)}dx$$

$$=\lim_{b\to\infty}\left[\frac{1}{\sqrt{2}}\left(\frac{1+t^2}{1+t^4}\right)\ln\left(\frac{b^2+\sqrt{2}tb+t^2}{b^2-\sqrt{2}tb+t^2}\right)-\sqrt{2}\frac{(1+t^2)}{1+t^4}\left(\tan^{-1}\left(\frac{\sqrt{2}b}{t}-1\right)+\tan^{-1}\left(\frac{\sqrt{2}b}{t}+1\right)\right)\right.$$

$$\left.+\frac{2\sqrt{2}}{1+t^4}\tan^{-1}\left(\frac{\sqrt{2}b}{t}-1\right)+\frac{2\sqrt{2}}{1+t^4}\tan^{-1}\left(\frac{\sqrt{2}b}{t}+1\right)+\frac{4t^3}{1+t^4}\tan^{-1}(b)\right]$$

$$=\frac{1}{\sqrt{2}}\left(\frac{1+t^2}{1+t^4}\right)\ln(1)-\sqrt{2}\frac{(1+t^2)}{1+t^4}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)+\frac{2\sqrt{2}}{1+t^4}\frac{\pi}{2}+\frac{2\sqrt{2}}{1+t^4}\frac{\pi}{2}+\frac{4t^3}{1+t^4}\frac{\pi}{2}\Bigg]$$

$$=0-\sqrt{2}\frac{(1+t^2)}{1+t^4}\pi+\frac{2\sqrt{2}}{1+t^4}\pi+\frac{2t^3}{1+t^4}\pi$$

$$=0-\sqrt{2}\frac{(1+t^2)}{1+t^4}\pi+\frac{2\sqrt{2}}{1+t^4}\pi+\frac{2t^3}{1+t^4}\pi=\frac{1}{1+t^4}\pi\left(\sqrt{2}-\sqrt{2}t^2+2t^3\right)$$

$$=\frac{\sqrt{2}\pi}{1+t^4}\left(1-t^2+\sqrt{2}t^3\right).$$

Hence, $F'(t)=\int_0^{\infty}\frac{\partial f}{\partial t}(x,t)dx=\frac{\sqrt{2}\pi}{1+t^4}\left(1-t^2+\sqrt{2}t^3\right)$ for t in (0, K]. Since, K is arbitrary, $F'(t)=\frac{\sqrt{2}\pi}{1+t^4}\left(1-t^2+\sqrt{2}t^3\right)$ for $t > 0$.

As before, the antiderivative of $\frac{\sqrt{2}\pi}{1+t^4}\left(1-t^2+\sqrt{2}t^3\right)$ is given by $\pi\ln\left(1+\sqrt{2}t+t^2\right)+C$; It follows that $F(t)=\pi\ln\left(1+\sqrt{2}t+t^2\right)+C$ for t in (0, K]. Since $\lim_{t\to o^+}F(t)=\lim_{t\to 0^+}\int_0^{\infty}f_t(x)dx=0$, $C = 0$ and so we have $F(t)=\pi\ln\left(1+\sqrt{2}t+t^2\right)$. Thus, $\int_0^{\infty}\frac{\ln(1+x^4)}{1+x^2}dx=F(1)=\pi\ln\left(1+\sqrt{2}+1\right)=\pi\ln\left(2+\sqrt{2}\right)$.

Use of Fubini's Theorem

This would involve double integral.

First of all, note that $\ln(1+x^4)=\int_0^1\frac{4x^4t^3}{1+x^4t^4}dt$. Therefore,

$$\int_0^{\infty}\frac{\ln(1+x^4)}{1+x^2}dx=\int_0^{\infty}\frac{1}{1+x^2}\left(\int_0^1\frac{4x^4t^3}{1+x^4t^4}dt\right)dx=\int_0^{\infty}\left(\int_0^1\frac{4x^4t^3}{1+x^4t^4}\frac{1}{1+x^2}dt\right)dx .$$

Let $f(x,t)=\frac{4x^4t^3}{1+x^4t^4}\cdot\frac{1}{1+x^2}$ for $(x,t)\in[0,\infty)\times[0,1]$. Then $f(x,t)$ is non-negative and continuous on $[0,\infty)\times[0,1]$. We have already shown that $\int_0^{\infty}\frac{\ln(1+x^4)}{1+x^2}dx$ is convergent.

For any $b > 0$, $f(x,t)$ is continuous on $[0,b]\times[0,1]$. Therefore by, Fubini's Theorem, $\int_0^b\left(\int_0^1\frac{4x^4t^3}{1+x^4t^4}\frac{1}{1+x^2}dt\right)dx=\int_0^1\left(\int_0^b\frac{4x^4t^3}{1+x^4t^4}\frac{1}{1+x^2}dx\right)dt$. Hence,

$$\int_0^{\infty}\left(\int_0^1\frac{4x^4t^3}{1+x^4t^4}\frac{1}{1+x^2}dt\right)dx=\lim_{b\to\infty}\int_0^b\left(\int_0^1\frac{4x^4t^3}{1+x^4t^4}\frac{1}{1+x^2}dt\right)dx=\lim_{b\to\infty}\int_0^1\left(\int_0^b\frac{4x^4t^3}{1+x^4t^4}\frac{1}{1+x^2}dx\right)dt .$$

Since $\frac{4x^4t^3}{1+x^4t^4}\frac{1}{1+x^2}$ is non-negative and $\int_0^{\infty}\frac{4x^4t^3}{1+x^4t^4}\frac{1}{1+x^2}dx$ is convergent for all t,

$$\int_0^{\infty}\left(\int_0^1\frac{4x^4t^3}{1+x^4t^4}\frac{1}{1+x^2}dt\right)dx=\int_0^1\left(\int_0^{\infty}\frac{4x^4t^3}{1+x^4t^4}\frac{1}{1+x^2}dx\right)dt.$$ Now by (1)

$$\int_0^\infty \frac{4x^4t^3}{1+x^4t^4}\frac{1}{1+x^2}\,dx = \frac{\sqrt{2}\pi}{1+t^4}\left(\sqrt{2}t^3 - t^2 + 1\right)$$ and so

$$\int_0^\infty \left(\int_0^1 \frac{4x^4t^3}{1+x^4t^4}\frac{1}{1+x^2}\,dt\right)dx = \int_0^1 \frac{\sqrt{2}\pi}{1+t^4}\left(\sqrt{2}t^3 - t^2 + 1\right)dt = \left[\pi \ln\left(1+\sqrt{2}t + t^2\right)\right]_0^1 = \pi \ln\left(2+\sqrt{2}\right).$$

Hence, $\int_0^\infty \frac{\ln(1+x^4)}{1+x^2}\,dx = \pi \ln\left(2+\sqrt{2}\right)$.

The next method is the hardest to apply mainly of the problem with intermediate integrals' convergence with respect to limit.

Using $f(x,t) = \frac{\ln(1+tx+x^2)}{1+x^2}$

Note that $\frac{\ln(1+x^4)}{1+x^2} = \frac{\ln(1+\sqrt{2}x+x^2)+\ln(1-\sqrt{2}x+x^2)}{1+x^2}$ and so

$$\int_{-\infty}^\infty \frac{\ln(1+x^4)}{1+x^2}\,dx = \int_{-\infty}^\infty \frac{\ln(1+\sqrt{2}x+x^2)}{1+x^2} + \int_{-\infty}^\infty \frac{\ln(1-\sqrt{2}x+x^2)}{1+x^2}.$$

Observe that $\int_{-\infty}^\infty \frac{\ln(1-\sqrt{2}x+x^2)}{1+x^2} = \int_{-\infty}^\infty \frac{\ln(1+\sqrt{2}x+x^2)}{1+x^2}$ by a change of variable argument and $\int_{-\infty}^\infty \frac{\ln(1+x^4)}{1+x^2}\,dx = 2\int_{-\infty}^\infty \frac{\ln(1+\sqrt{2}x+x^2)}{1+x^2}$. We need only to evaluate $\int_{-\infty}^\infty \frac{\ln(1+\sqrt{2}x+x^2)}{1+x^2}$. For this we shall use the function,

$$f(x,t) = \frac{\ln(1+tx+x^2)}{1+x^2} \text{ for } (x,t)\in(-\infty,\infty)\times[1,2].$$

Now, $\frac{\partial f}{\partial t}(x,t) = \frac{x}{(1+tx+x^2)(1+x^2)}$ for $(x,t)\in(-\infty,\infty)\times[1,2]$.

Consider the following expansion: for $t \neq 0$,

$$\frac{x}{(1+tx+x^2)(1+x^2)} = \frac{1}{t}\frac{x}{(1+x^2)} - \frac{1}{t}\frac{x}{(1+tx+x^2)}.$$

Therefore, for $0<t<2$,

$$\int \frac{x}{(1+tx+x^2)(1+x^2)}\,dx = \frac{1}{t}\tan^{-1}(x) - \frac{1}{t}\frac{2}{\sqrt{4-t^2}}\tan^{-1}\left(\frac{2x+t}{\sqrt{4-t^2}}\right) + C$$

For $t = 0$, $\frac{\partial f}{\partial t}(x,0) = \frac{x}{(1+x^2)^2}$ so that $\int \frac{\partial f}{\partial t}(x,0)dx = \int \frac{x}{(1+x^2)^2}\,dx = -\frac{1}{2}\frac{1}{(1+x^2)} + C$.

Therefore, for $0<t<2$,

$$\int_0^{\infty}\frac{\partial f}{\partial t}(x,t)dx=\int_0^{\infty}\frac{x}{(1+tx+x^2)(1+x^2)}dx=\frac{\pi}{2t}-\frac{1}{t}\frac{2}{\sqrt{4-t^2}}\frac{\pi}{2}+\frac{1}{t}\frac{2}{\sqrt{4-t^2}}\tan^{-1}\left(\frac{t}{\sqrt{4-t^2}}\right)+C.$$

and

$$\int_{-\infty}^{0}\frac{\partial f}{\partial t}(x,t)dx=\int_{-\infty}^{0}\frac{x}{(1+tx+x^2)(1+x^2)}dx=\frac{\pi}{2t}-\frac{1}{t}\frac{2}{\sqrt{4-t^2}}\frac{\pi}{2}-\frac{1}{t}\frac{2}{\sqrt{4-t^2}}\tan^{-1}\left(\frac{t}{\sqrt{4-t^2}}\right)+C.$$

Thus, for $0<t<2$,

$$\int_{-\infty}^{\infty}\frac{x}{(1+tx+x^2)(1+x^2)}dx=\frac{\pi}{t}-\frac{1}{t}\frac{2\pi}{\sqrt{4-t^2}}\quad.$$

For $t = 2$, $\dfrac{x}{(1+2x+x^2)(1+x^2)}=\dfrac{1}{2}\dfrac{1}{(1+x^2)}-\dfrac{1}{2}\dfrac{1}{(1+x)^2}$,

$$\int_{-\infty}^{\infty}\frac{\partial f}{\partial t}(x,2)dx=\left[\frac{1}{2}\tan^{-1}(x)+\frac{1}{2}\frac{1}{(1+x)}\right]_{-\infty}^{\infty}=\frac{\pi}{2}.$$

For $0 < k < 1$ and $k < K < 2$ $k\le t\le K$,

$$\frac{1}{t}\tan^{-1}(x)-\frac{1}{t}\frac{2}{\sqrt{4-t^2}}\tan^{-1}\left(\frac{2x+t}{\sqrt{4-t^2}}\right)\le\frac{1}{k}\tan^{-1}(x)+\frac{1}{k}\frac{2}{\sqrt{4-K^2}}\tan^{-1}\frac{2x+K}{\sqrt{4-K^2}}\ ,$$

so that

$$\int_{-\infty}^{\infty}\left(\frac{1}{t}\tan^{-1}(x)-\frac{1}{t}\frac{2}{\sqrt{4-t^2}}\tan^{-1}\left(\frac{2x+t}{\sqrt{4-t^2}}\right)\right)dx\le\int_{-\infty}^{\infty}\left(\frac{1}{k}\tan^{-1}(x)+\frac{1}{k}\frac{2}{\sqrt{4-K^2}}\tan^{-1}\frac{2x+K}{\sqrt{4-K^2}}\right)dx.$$

This implies that the integral $\int_0^{\infty}\frac{\partial f}{\partial t}(x,t)dx$ converges uniformly with respect to t in $[k, K]$. It follows that the function, $F(t)=\int_{-\infty}^{\infty}f(x,t)dx=\int_{-\infty}^{\infty}\frac{\ln(1+tx+x^2)}{1+x^2}dx$ is differentiable with respect to t in $(0,2)$ and

$$F'(t)=\int_{-\infty}^{\infty}\frac{\partial f}{\partial t}(x,t)dx=\frac{\pi}{t}-\frac{1}{t}\frac{2\pi}{\sqrt{4-t^2}}.$$

Now, $\int\frac{1}{t\sqrt{4-t^2}}dt=\frac{1}{2}\ln\left(\frac{2-\sqrt{4-t^2}}{t}\right)+C$ for t in $(0,2)$ and so

$$F(t)=\pi\ln(t)-\pi\ln\left(\frac{2-\sqrt{4-t^2}}{t}\right)+C\ \text{ for } t \text{ in } (0,2). \quad \text{-------------------- (3)}$$

Now, $f_t(x) = f(x,t) = \dfrac{\ln(1+tx+x^2)}{1+x^2}$ tends to $f_2(x) = \dfrac{\ln(1+2x+x^2)}{1+x^2}$ as t tends to 2 from below.

Note that $|f(x,t)| = \dfrac{\ln(1+tx+x^2)}{1+x^2} \le \dfrac{\ln(1+2x+x^2)}{1+x^2} = 2\dfrac{\ln(1+x)}{1+x^2}$ for $x \ge 0$ and $1 \le t \le 2$.

Since $\dfrac{\ln(1+x)}{1+x^2}$ is Lebesgue integrable on $[0,\infty)$, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{t\to 2^-}\int_0^\infty f(x,t)dx = \lim_{t\to 2^-}\int_0^\infty \frac{\ln(1+tx+x^2)}{1+x^2}dx = \int_0^\infty \frac{\ln(1+2x+x^2)}{1+x^2}dx . \text{ --------- (1)}$$

For $1 \le t \le 2$ and $x \le 0$, $1+2x+x^2 \le 1+tx+x^2 \le 1+x+x^2$.

If $x \le -2$, then $x^2+2x = x(x+2) \ge 0$ and so $x^2+2x+1 \ge 1$. Thus, for $x \le -2$ and $1 \le t \le 2$,

$$|f(x,t)| = \frac{\ln(1+tx+x^2)}{1+x^2} \le \frac{\ln(1+x+x^2)}{1+x^2} \le \frac{\ln(1+x^2)}{1+x^2} .$$

Since $\dfrac{\ln(1+x^2)}{1+x^2}$ is Lebesgue integrable on $(-\infty,-2)$, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{t\to 2^-}\int_{-\infty}^{-2} f(x,t)dx = \lim_{t\to 2^-}\int_{-\infty}^{-2} \frac{\ln(1+tx+x^2)}{1+x^2}dx = \int_{-\infty}^{-2} \frac{\ln(1+2x+x^2)}{1+x^2}dx \quad \text{--------- (2)}$$

If $-1 \le x \le 0$, then $\ln(1+x+x^2) \le 0$ and so

$$\frac{\ln(1+2x+x^2)}{1+x^2} \le f(x,t) = \frac{\ln(1+tx+x^2)}{1+x^2} \le \frac{\ln(1+x+x^2)}{1+x^2} \le 0 .$$

Hence, $|f(x,t)| = -\dfrac{\ln(1+tx+x^2)}{1+x^2} \le -\dfrac{\ln(1+2x+x^2)}{1+x^2}$. Since $-\dfrac{\ln(1+2x+x^2)}{1+x^2}$ is Lebesgue integrable on $[-1,0]$, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{t\to 2^-}\int_{-1}^{0} f(x,t)dx = \lim_{t\to 2^-}\int_{-1}^{0} \frac{\ln(1+tx+x^2)}{1+x^2}dx = \int_{-1}^{0} \frac{\ln(1+2x+x^2)}{1+x^2}dx . \text{ ---------- (3)}$$

Now we examine the integral over $[-2,-1]$.

For $t_1 < t_2$ in [1, 2] and $x < 0$, $xt_1 > xt_2$ so that $\ln(1+t_1x+x^2) \ge \ln(1+t_2x+x^2)$ and

$$f_{t_1}(x) = f(x,t_1) = \frac{\ln(1+t_1x+x^2)}{1+x^2} \ge \frac{\ln(1+t_2x+x^2)}{1+x^2} = f_{t_2}(x)$$

Therefore, $-f_t(x)$ is increasing in t with domain [–2, –1] and t in [1,2]. Note that each $-f_t(x)$ is an increasing function in x in [–2, –1]. This follows from the fact that $-\ln(1+tx+x^2)$ is an increasing function in x in [–2, –1] for any t in [1,2]. Therefore, $-f_t(x)+f_1(-2)=-f_t+\frac{\ln(3)}{5}$ is non-negative and monotone increasing in t in [1,2]. Therefore, by the Monotone Convergence Theorem,

$$\lim_{t\to 2^-}\left(-\int_{-2}^{-1} f_t(x)dx+\int_{-2}^{-1}\frac{\ln(3)}{5}dx\right)=\int_{-2}^{-1}\left(-f_2(x)+\frac{\ln(3)}{5}\right)dx .$$

It follows that

$$\lim_{t\to 2^-}\int_{-2}^{-1} f(x,t)dx=\lim_{t\to 2^-}\int_{-2}^{-1} f_t(x)dx=\int_{-2}^{-1} f_2(x)dx=\int_{-2}^{-1}\frac{\ln(1+2x+x^2)}{1+x^2}dx . \text{ ------------(4)}$$

Hence, combining (1) to (4), we get

$$\lim_{t\to 2^-}\int_{-\infty}^{\infty} f(x,t)dx=\int_{-\infty}^{\infty}\frac{\ln(1+2x+x^2)}{1+x^2}dx .$$

Thus, $\lim_{t\to 2^-}F(t)=\lim_{t\to 2^-}\int_{-\infty}^{\infty} f(x,t)dx=F(2)$.

We shall now evaluate F(2).

$$\int_0^{\infty} f_2(x)dx=\int_0^{\infty} f(x,2)dx=\int_0^{\infty}\frac{\ln\left((1+x)^2\right)}{1+x^2}dx=2\int_0^{\infty}\frac{\ln(1+x)}{1+x^2}dx$$

$$=2\int_0^{\frac{\pi}{2}}\ln\left(1+\tan(y)\right)dy=2\int_0^{\frac{\pi}{2}}\ln\left(\sin(y)+\cos(y)\right)dy-2\int_0^{\frac{\pi}{2}}\ln\left(\cos(y)\right)dy$$

$$=2\int_0^{\frac{\pi}{2}}\ln\left(\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin(y)+\frac{1}{\sqrt{2}}\cos(y)\right)\right)dy-2\left(-\frac{\pi}{2}\ln(2)\right) ,$$

since $\int_0^{\frac{\pi}{2}}\ln\left(\cos(y)\right)dy=-\frac{\pi}{2}\ln(2)$,

$$=2\int_0^{\frac{\pi}{2}}\ln\left(\sqrt{2}\right)dy+2\int_0^{\frac{\pi}{2}}\ln\left(\sin\left(y+\frac{\pi}{4}\right)\right)dy+\pi\ln(2)$$

$$=\frac{3\pi}{2}\ln(2)+2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\ln\left(\sin(x)\right)dx=\frac{3\pi}{2}\ln(2)+2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ln\left(\sin(x)\right)dx+2\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\ln\left(\sin(x)\right)dx$$

$$=\frac{3\pi}{2}\ln(2)+2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ln\left(\sin(x)\right)dx+2\int_0^{\frac{\pi}{4}}\ln\left(\sin\left(\frac{\pi}{2}+x\right)\right)dx$$

$$=\frac{3\pi}{2}\ln(2)+2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ln\left(\sin(x)\right)dx+2\int_0^{\frac{\pi}{4}}\ln\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx$$

$$= \frac{3\pi}{2}\ln(2) + 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\sin(x)\right)dx - 2\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \ln\left(\sin\left(y\right)\right)dy$$

$$= \frac{3\pi}{2}\ln(2) + 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\sin(x)\right)dx \text{ .} \quad \text{-------------------------------- (5)}$$

$$\int_{-1}^{0} f_2(x)dx = \int_{-1}^{0} f(x,2)dx = \int_{-1}^{0} \frac{\ln\left((1+x)^2\right)}{1+x^2}dx = 2\int_{-1}^{0} \frac{\ln\left(1+x\right)}{1+x^2}dx$$

$$= 2\int_{-\frac{\pi}{4}}^{0} \ln\left(1+\tan(y)\right)dy = 2\int_{-\frac{\pi}{4}}^{0} \ln\left(\sin(y)+\cos(y)\right)dy - 2\int_{-\frac{\pi}{4}}^{0} \ln\left(\cos(y)\right)dy$$

$$= 2\int_{-\frac{\pi}{4}}^{0} \ln\left(\sqrt{2}\sin(y+\tfrac{\pi}{4})\right)dy - 2\int_{0}^{\frac{\pi}{4}} \ln\left(\cos(y)\right)dy$$

$$= 2\int_{-\frac{\pi}{4}}^{0} \ln\left(\sqrt{2}\right)dy + 2\int_{-\frac{\pi}{4}}^{0} \ln\left(\sin(y+\tfrac{\pi}{4})\right)dy - 2\int_{0}^{\frac{\pi}{4}} \ln\left(\cos(y)\right)dy$$

$$= \frac{\pi}{4}\ln(2) + 2\int_{0}^{\frac{\pi}{4}} \ln\left(\sin(x)\right)dx - 2\int_{0}^{\frac{\pi}{4}} \ln\left(\cos(x)\right)dx \text{ .} \quad \text{-------------------- (6)}$$

$$\int_{-\infty}^{-1} f_2(x)dx = \int_{-\infty}^{-1} f(x,2)dx = \int_{-\infty}^{-1} \frac{\ln\left((1+x)^2\right)}{1+x^2}dx = -\int_{\infty}^{1} \frac{\ln\left((1-y)^2\right)}{1+y^2}dx = \int_{1}^{\infty} \frac{\ln\left((1-y)^2\right)}{1+y^2}dx$$

$$= 2\int_{1}^{\infty} \frac{\ln\left(y-1\right)}{1+y^2}dy = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\tan(x)-1\right)dx = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\sin(x)-\cos(x)\right)dx - 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\cos(x)\right)dx$$

$$= 2\int_{0}^{\frac{\pi}{4}} \ln\left(\sqrt{2}\sin\left(x\right)\right)dx - 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\cos(x)\right)dx = 2\int_{0}^{\frac{\pi}{4}} \ln\left(\sqrt{2}\right)dx + 2\int_{0}^{\frac{\pi}{4}} \ln\left(\sin\left(x\right)\right)dx - 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\cos(x)\right)dx$$

$$= \frac{\pi}{4}\ln(2) + 2\int_{0}^{\frac{\pi}{4}} \ln\left(\sin\left(x\right)\right)dx - 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\cos(x)\right)dx \text{ .} \quad \text{------------------------------ (7)}$$

Therefore,

$$F(2) = \int_{-\infty}^{\infty} f_2(x)dx = \frac{3\pi}{2}\ln(2) + 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\sin(x)\right)dx + \frac{\pi}{4}\ln(2) + 2\int_{0}^{\frac{\pi}{4}} \ln\left(\sin(x)\right)dx - 2\int_{0}^{\frac{\pi}{4}} \ln\left(\cos(x)\right)dx$$

$$+ \frac{\pi}{4}\ln(2) + 2\int_{0}^{\frac{\pi}{4}} \ln\left(\sin\left(x\right)\right)dx - 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\cos(x)\right)dx$$

$$= 2\pi\ln(2) - 2\int_{0}^{\frac{\pi}{2}} \ln(\cos(x)dx + 4\int_{0}^{\frac{\pi}{2}} \ln(\sin(x)dx$$

$$= 2\pi\ln(2) + 2\int_{0}^{\frac{\pi}{2}} \ln(\cos(x)dx = 2\pi\ln(2) - \pi\ln(2) = \pi\ln(2) \text{ .}$$

It follows that $F(2) = \lim_{t\to 2^-} F(t) = \lim_{t\to 2^-}\left(\pi\ln(t) - \pi\ln\left(\frac{2-\sqrt{4-t^2}}{t}\right)\right) + C = \pi\ln(2) + C$.

Since $F(2)=\pi\ln(2)$, C = 0 and $F(t)=\pi\ln(t)-\pi\ln\left(\frac{2-\sqrt{4-t^2}}{t}\right)$ on [1, 2].

Thus,

$$\int_{-\infty}^{\infty}\frac{\ln(1+\sqrt{2}x+x^2)}{1+x^2}dx=F(\sqrt{2})=\pi\ln(\sqrt{2})-\pi\ln\left(\frac{2-\sqrt{4-2}}{\sqrt{2}}\right)=\pi\ln(2)-\pi\ln\left(2-\sqrt{2}\right)$$

$$=\pi\ln(2)+\pi\ln\left(\frac{1}{2-\sqrt{2}}\right)=\pi\ln(2)+\pi\ln\left(\frac{2+\sqrt{2}}{2}\right)=\pi\ln\left(2+\sqrt{2}\right).$$

It follows that $\int_{-\infty}^{\infty}\frac{\ln(1+x^4)}{1+x^2}dx=2\pi\ln\left(2+\sqrt{2}\right)$ and $\int_{0}^{\infty}\frac{\ln(1+x^4)}{1+x^2}dx=\pi\ln\left(2+\sqrt{2}\right)$.

References.

My Calculus Web at Firebase.com

[1] Mathematical Analysis, An Introduction, *Chapter 14 Improper integral and Lebesgue integral*.

[2] Introduction to Measure Theory