All About Lim Sup and Lim Inf

By Ng Tze Beng

The use of sequences in analysis is very important in understanding the concepts of continuity, differentiability, compactness and connectedness. Lim sup and lim inf often play an important investigative role in the proofs of many results in analysis. One classic result is that a bounded real sequence converges if and only if its lim sup and lim inf are the same.

To understand a sequence well in one way is to take into consideration all possible limit, including both $-\infty$ and $+\infty$, that its subsequence can tend to. Though a sequence may or may not tend to a limit, the limits of its subsequence can reveal some of its behaviour.

Suppose (a_n) is a real sequence, which is not necessarily bounded. Therefore, it may have a subsequence tending to $-\infty$ or $+\infty$.

Let L be the set of subsequence limits of (a_n). That is ,

$$L = \{x \in [-\infty, +\infty] : \exists \text{ a subsequence } (a_{n_k}) \text{ such that } a_{n_k} \to x \text{ as } k \to \infty\} .$$

We define $\limsup_{n\to\infty} a_n$ to be the *supremum* of L and $\liminf_{n\to\infty} a_n$ to be the *infimum* of L. Here we define *supremum* of L to be the least upper bound of L if $+\infty$ is not in L and $+\infty$ if $+\infty$ is in L. Likewise we define *infimum* of L to be the greatest lower bound of L if $-\infty$ is not in L and $-\infty$ if $-\infty$ is in L. We may drop the symbol "$n\to\infty$" from the notation and simply write lim sup a_n for $\limsup_{n\to\infty} a_n$ and lim inf a_n for $\liminf_{n\to\infty} a_n$.

Note that L is not empty. This is because if (a_n) is bounded, then by the Bolzano-Weierstrass Theorem, (a_n) has a convergent subsequence $\left(a_{n_k}\right)$ such that $a_{n_k} \to y$ for some y in **R** and so $y \in L$. If (a_n) is unbounded then it is unbounded above or unbounded below or both. If (a_n) is unbounded above, then it has a subsequence $\left(a_{n_k}\right)$ with $a_{n_k} \to +\infty$ and so $+\infty \in L$. If (a_n) is unbounded below, then it has a subsequence $\left(a_{n_k}\right)$ with $a_{n_k} \to -\infty$ and so $-\infty \in L$. Thus L is not empty.

The first result that we have about L is:

Proposition 1. With notation as above, both $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ are in L. That is to say, *infimum* of L and *supremum* of L are in L.

Proof. We shall prove that sup L is in L. The proof for inf L is similar.

Let sup $L = \alpha$.

Suppose $\alpha = -\infty$. Then this implies that $+\infty \notin L$ and any real number x is an upper bound for L. This means L contains no real number. Since L is not empty, $L = \{-\infty\}$ and so $\alpha \in L$. Furthermore this means if $a_{n_k} \to x$, then $x = -\infty$. It follows that $a_n \to -\infty$. This is because if $a_n \not\to -\infty$, then there exists a subsequence $\left(a_{n_k}\right)$ which is bounded below. Then $\left(a_{n_k}\right)$ has a subsequence $\left(a_{n_{k(j)}}\right)$ such that $a_{n_{k(j)}} \to x > -\infty$ and so $x \in L$ contradicting $L = \{-\infty\}$.

Suppose now $-\infty < \alpha < \infty$. Then $\infty \notin L$. If L is finite, then we have nothing to prove, since sup L = max $L = \alpha \in L$. If L is infinite and α is not a limit point of L, then α must be in L. This is because if α is not in L and α is not a limit point of L, then there exists an $\varepsilon > 0$, such that $(\alpha-\varepsilon, \alpha+\varepsilon) \cap L = \varnothing$. Then since α = sup L, and if $\alpha \notin L$, then $\alpha-\varepsilon$ would be an upper bound for L and so $\alpha-\varepsilon \geq$ sup $L = \alpha$ which is absurd. So now we assume that α is a limit point of L.

By the definition of supremum there exists a strictly increasing sequence (α_i) in L such that $\alpha_i \to \alpha$.

We can deduce this as follows.

Since α is a limit point of L and an upper bound of L, there exists α_1 in L such that $\alpha - 1 < \alpha_1 < \alpha$. Then there exists α_2 in L such that

$$\alpha_1 \leq \alpha - \min\left(\frac{1}{2}, \alpha - \alpha_1\right) < \alpha_2 < \alpha \ .$$

In this way, we can inductively find α_n in L such that

$$\alpha_{n-1} \le \alpha - \min\left(\frac{1}{n}, \alpha - \alpha_{n-1}\right) < \alpha_n < \alpha \, .$$

Then there exists α_{n+1} in L such that

$$\alpha_n \le \alpha - \min\left(\frac{1}{n+1}, \alpha - \alpha_n\right) < \alpha_{n+1} < \alpha \quad .$$

Thus for all integer $n \ge 1$, $\alpha_{n+1} > \alpha_n$ and $|\alpha_n - \alpha| < \frac{1}{n}$. This means (α_i) is a strictly increasing sequence in L converging to α .

But each α_n is the limit of a subsequence $\left(a_{n_k}\right)$ of (a_n). We are now going to define a subsequence of (a_n) that converges to α .

Starting with α_1 . There exists an integer N_1 such that

$$k \ge N_1 \Rightarrow \left|a_{1_k} - \alpha_1\right| < \frac{1}{2}\left|\alpha_2 - \alpha_1\right| . \qquad \text{---------------------------} \quad (1)$$

There exists an integer N_2 such that

$$k \ge N_2 \Rightarrow \left|a_{2_k} - \alpha_2\right| < \frac{1}{2}\min\left(\left|\alpha_2 - \alpha_1\right|, \left|\alpha_3 - \alpha_2\right|\right) . \quad \text{----------------------} \ (2)$$

There exists an integer Nj such that

$$k \ge N_j \Rightarrow \left|a_{j_k} - \alpha_j\right| < \frac{1}{2}\min\left(\left|\alpha_j - \alpha_{j-1}\right|, \left|\alpha_{j+1} - \alpha_j\right|\right) . \ \text{-------------------} \ (3)$$

Choose $k(1) \ge N_1$ so that $\left|a_{1_{k(1)}} - \alpha_1\right| < \frac{1}{2}\left|\alpha_2 - \alpha_1\right|$ and so $a_{1_{k(1)}} < \alpha_2$.

Choose $k(2) \ge N_2$ so that $2_{k(2)} > 1_{k(1)}$ and

$$\left|a_{2_{k(2)}} - \alpha_2\right| < \frac{1}{2}\min\left(\left|\alpha_2 - \alpha_1\right|, \left|\alpha_3 - \alpha_2\right|\right) \text{ and so } \alpha_1 < a_{2_{k(2)}} < \alpha_3 .$$

----------------- (4)

In this way we successively choose $k(n) \ge N_n$ so that $n_{k(n)} > (n-1)_{k(n-1)}$ and

$$\left|a_{n_{k(n)}} - \alpha_n\right| < \frac{1}{2}\min\left(\left|\alpha_n - \alpha_{n-1}\right|, \left|\alpha_{n+1} - \alpha_n\right|\right) \text{ and so } \alpha_{n-1} < a_{n_{k(n)}} < \alpha_{n+1} .$$

Thus the sequence $(a_{n_{k(n)}})$ is a subsequence of (a_n) and since $\alpha_{n-1} < a_{n_{k(n)}} < \alpha_{n+1}$, by the Squeeze Theorem $a_{n_{k(n)}} \to \alpha$. Thus $\alpha \in L$.

The gist of the above argument is to find successive points of (a_n) between α_{n-1} and α_{n+1} indexed by increasing index numbers.

Suppose now $\alpha = +\infty$.

If L is finite, then since $\alpha = \sup L$, α must be in L and we have nothing to prove.

If L is infinite and bounded above and if $+\infty$ is not in L, then $\sup L \neq +\infty$.

Thus if L is infinite and $L - \{+\infty\}$ is bounded above, then $\sup L = +\infty = \alpha$ implies that α must be in L.

So we are left with the case L is infinite and $L - \{+\infty\}$ is not bounded above.

Therefore, there is a strictly increasing sequence (α_i) in L such that $\alpha_i \to +\infty$.

Now each α_n is the limit of a subsequence $\left(a_{n_k}\right)$ of $\left(a_n\right)$. Following the same construction as before when $-\infty < \alpha < \infty$, we obtain $(a_{n_{k(n)}})$, a subsequence of (a_n) satisfying $\alpha_{n-1} < a_{n_{k(n)}} < \alpha_{n+1}$ for $n \geq 2$. Since $\alpha_i \to +\infty$, $a_{n_{k(n)}} \to +\infty$. Hence $\alpha = +\infty \in L$.

With Proposition 1, it is then easy to prove the following:

Theorem 2. A real sequence (a_n) has a limit (including $\pm\infty$) if and only if

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n .$$

Proof. Suppose (a_n) has a limit. If the limit is a real number, then all subsequence must converge to the same limit say α. Hence

$$L = \{x \in [-\infty, +\infty] : \exists \text{ a subsequence } (a_{n_k}) \text{ such that } a_{n_k} \to x \text{ as } k \to \infty\} = \{\alpha\} .$$

Consequently $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \{\alpha\}$.

If $a_n \to +\infty$, then any subsequence must also tend to $+\infty$. Hence $L = \{+\infty\}$ and so $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \{+\infty\}$. Similarly, If $a_n \to -\infty$, then any subsequence must tend to $-\infty$. Hence $L = \{-\infty\}$ and so $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \{-\infty\}$.

Conversely, if $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$, then L consists of just one element, say α.

We claim that $a_n \to \alpha$.

Suppose $a_n \not\to \alpha$.

If α is a real number, then by negating the definition of convergence, there exists an $\varepsilon > 0$ and a subsequence (a_{n_k}) such that $|a_{n_k} - \alpha| \geq \varepsilon$ for all positive integer k. Then (a_{n_k}) will have a subsequence $(a_{n_{k(i)}})$ that tends to β. Plainly $\beta \neq \alpha$. Thus $\beta \in L$ contradicting that $L = \{\alpha\}$.

If $\alpha = +\infty$, then since $a_n \not\to \alpha$, (a_n) has a subsequence (a_{n_k}) that is bounded above. (a_{n_k}) is either bounded below or unbounded below. Thus (a_{n_k}) either has a subsequence $(a_{n_{k(i)}})$ that tends to a finite value β or tends to $-\infty$. Plainly this sequence is also a subsequence of (a_n). Thus β or $-\infty \in L$ contradicting $L = \{+\infty\}$.

If $\alpha = -\infty$, then since $a_n \not\to \alpha$, (a_n) has a subsequence (a_{n_k}) that is bounded below. Then (a_{n_k}) is either bounded above or unbounded above. Thus (a_{n_k}) either has a subsequence $(a_{n_{k(i)}})$ that tends to a finite value β or tends to $+\infty$. Plainly this sequence is also a subsequence of (a_n). Thus β or $+\infty \in L$ contradicting $L = \{-\infty\}$.

Hence $a_n \to \alpha$.

This completes the proof of Theorem 2.

Lim sup and lim inf may be defined in different ways.

The next theorem gives equivalent condition for the definition of lim sup of a sequence.

Theorem 3. Suppose (a_n) is a real sequence. The following statements are equivalent.

(1) $\limsup_{n\to\infty} a_n = \alpha$.

(2) $\lim_{n\to\infty} \sup\{a_k : k \geq n\} = \alpha$.

(3) For any subsequence (a_{n_k}), $a_{n_k} \to x$ as $k \to \infty \Rightarrow x \leq \alpha$ and there exists a subsequence (a_{n_k}) such that $a_{n_k} \to \alpha$.

(4) For any $\beta > \alpha$, the set $\{k : a_k \geq \beta\}$ is finite and for any $\beta < \alpha$, the set $\{k : a_k \geq \beta\}$ is infinite.

Before we prove Theorem 3, the following is a useful fact.

Proposition 4. Suppose (a_n) is a real sequence. For each integer $k \geq 1$, let $\beta_k = \sup\{a_n : n \geq k\}$. Then there is always a subsequence (a_{n_k}) of (a_n) such that $a_{n_k} \to \lim_{n\to\infty} \beta_n$.

Proof.

Let $\lim_{n\to\infty} \beta_n = \beta$.

If $\beta = +\infty$, then $\beta_n = +\infty$ for all positive integer n because if for some k, $\beta_k < +\infty$, then for all $n \geq k$, $\beta_n \leq \beta_k < \infty$ and so $\beta = \lim_{n\to\infty} \beta_n \leq \beta_k < +\infty$ contradicting $\beta = +\infty$. Thus if $\beta = +\infty$, then (a_n) is not bounded above and so there exists a subsequence (a_{n_k}) of (a_n) such that $a_{n_k} \to +\infty = \beta$.

Assume now $\beta < +\infty$.

By definition of supremum, there exists $n_1 \geq 1$, such that

$$\beta_1 \geq a_{n_1} > \beta_1 - 1 \quad ,$$

Next there exists $n_2 \geq n_1 + 1 > n_1$, such that

$$\beta_{n_1+1} \geq a_{n_2} > \beta_{n_1+1} - \frac{1}{n_1+1}.$$

We define a_{n_k} successively as follows.

If a_{n_k} is defined, then there exists $n_{k+1} \geq n_k + 1 > n_k$ such that

$$\beta_{n_k+1} \geq a_{n_{k+1}} > \beta_{n_k+1} - \frac{1}{n_k+1}.$$

Note that $n_k \geq k$.

Therefore, by the Squeeze Theorem,

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} \beta_{n_k+1} = \lim_{n\to\infty} \beta_n = \beta,$$

since $\lim_{k\to\infty}\left(\beta_{n_k+1} - \frac{1}{n_k+1}\right) = \lim_{k\to\infty} \beta_{n_k+1} = \lim_{n\to\infty} \beta_n$.

This completes the proof.

Proof of Theorem 3.

(1) $\Rightarrow$ (2)

For each integer $k \geq 1$, let $\beta_k = \sup\{a_n : n \geq k\}$. Then $\beta_k \geq a_k$.

If $\alpha = +\infty$, then (a_n) is not bounded above. This is because if (a_n) is bounded above, say by $K > 0$, then any subsequence is also bounded above by K. So the limit of any subsequence is also bounded above by K. Therefore, $\alpha \leq K$. This contradicts $\alpha = +\infty$. Consequently, $\beta_n = +\infty$ for all integer $n \geq 1$ and we have $\lim_{n\to\infty} \beta_n = \alpha$.

Now we assume $\alpha < +\infty$.

Then $\lim_{n\to\infty} \beta_n < +\infty$. This is because if $\lim_{n\to\infty} \beta_n = +\infty$, then by Proposition 4, , there exists a subsequence (a_{n_k}) of (a_n) such that $a_{n_k} \to +\infty$. Therefore, α =lim sup $a_n = +\infty$ contradicting $\alpha < +\infty$.

By Proposition 1, there exists a subsequence (a_{n_k}) of (a_n) such that $a_{n_k} \to \alpha$.

Therefore, $\lim_{k\to\infty}\beta_{n_k} \geq \lim_{k\to\infty} a_{n_k} = \alpha$. Since (β_n) is a decreasing sequence, it always has a limit and so the subsequence (β_{n_k}) tends to the same limit. Hence

$$\lim_{n\to\infty}\beta_n = \lim_{k\to\infty}\beta_{n_k} \geq \lim_{k\to\infty} a_{n_k} = \alpha .$$

By Proposition 4, there is a subsequence (a_{n_k}) of (a_n) such that $a_{n_k} \to \lim_{n\to\infty}\beta_n$.

Thus $\lim_{n\to\infty}\beta_n = \lim_{k\to\infty} a_{n_k} \leq \limsup a_n = \alpha$.

Hence $\lim_{n\to\infty}\beta_n = \alpha$.

(2) ⇒ (3)

Since $\beta_{n_k} \geq a_{n_k}$, if $a_{n_k} \to x$ as $k \to \infty$, then $\alpha = \lim_{n\to\infty}\beta_n = \lim_{k\to\infty}\beta_{n_k} \geq \lim_{k\to\infty} a_{n_k} = x$. By Proposition 4, there exists a subsequence (a_{n_k}) such that $a_{n_k} \to \alpha$.

(3) ⇒ (4)

Assume (3)

By (3) if $\alpha = -\infty$, any subsequence that has a limit must have the limit equal to $-\infty$. This implies that $a_n \to -\infty = \alpha$ and we have nothing to prove since plainly for any $\beta > \alpha$ the set $M=\{k : a_k \geq \beta\}$ is finite.

Suppose $\alpha > -\infty$.

If $\alpha = +\infty$, then since there does not exist $\beta > \alpha$ we have nothing to prove for the statement: set $M = \{k : a_k \geq \beta\}$ is finite. By (3), there exists a subsequence (a_{n_k}) such that $a_{n_k} \to \alpha = +\infty$. Thus for any $\beta < +\infty = \alpha$, there exists an integer N such that $k \geq N \Rightarrow a_{n_k} \geq \beta$ and so the set $\{k : a_{n_k} \geq \beta\}$ is infinite and that means $\{k : a_k \geq \beta\}$ is infinite too.

Assume now $-\infty < \alpha < +\infty$.

Suppose there exists $\beta > \alpha$ such that the set $M=\{k : a_k \geq \beta\}$ is infinite. By ordering the points in M, we may index M by an increasing function say $c(n)$. That is, $M = \{c(k) : k \text{ an integer } \geq 1\}$. Then $(a_{c(k)})$ is a subsequence of (a_n) that is bounded below by β. Hence $(a_{c(k)})$ has a subsequence $(a_{c(k(j))})$ such that

$a_{c(k(j))} \to x$ and $x \geq \beta > \alpha$. Now $\left(a_{c(k(j))}\right)$ is also a subsequence of (a_n). But by assumption (3), $x \leq \alpha$. We have thus arrived at a contradiction. Therefore, for all $\beta > \alpha$ the set $M = \{k : a_k \geq \beta\}$ is finite.

By (3), there exists a subsequence $\left(a_{n_k}\right)$ such that $a_{n_k} \to \alpha$.

Therefore for any $\beta < \alpha$, there exists integer K such that

$$k \geq K \Rightarrow \left|a_{n_k} - \alpha\right| < \left|\alpha - \beta\right| \Rightarrow a_{n_k} > \alpha - \left|\alpha - \beta\right| = \beta .$$

Hence the set $\{k : a_{n_k} \geq \beta\}$ is infinite. Consequently, the set $\{k : a_k \geq \beta\}$ is infinite.

(4) $\Rightarrow$ (1)

Suppose $\alpha = +\infty$.

Then (4) says for any $\beta < \alpha$, the set $\{k : a_k \geq \beta\}$ is infinite.

Thus the sequence (a_n) is not bounded above. Therefore, it has a subsequence $\left(a_{n_k}\right)$ such that $a_{n_k} \to \alpha = +\infty$. Hence

$$\alpha = +\infty \in L = \{x \in [-\infty, +\infty] : \exists \text{ a subsequence } (a_{n_k}) \text{ such that } a_{n_k} \to x \text{ as } k \to \infty\} .$$

Thus $\limsup\limits_{n \to \infty} a_n = \limsup a_n = \sup L = +\infty = \alpha$.

Suppose $\alpha = -\infty$. Then (4) says:

For any $\beta > \alpha = -\infty$, the set $\{k : a_k \geq \beta\}$ is finite.

Thus for any $\beta > \alpha = -\infty$, the set $\{k : a_k < \beta\}$ is infinite. This means the sequence (a_n) is not bounded below. Therefore it has a subsequence $\left(a_{n_k}\right)$ such that $a_{n_k} \to \alpha = -\infty$. Hence $\alpha = -\infty \in L$. We claim that $L = \{-\infty\}$. We deduce this as follows. Suppose a subsequence $\left(a_{n_k}\right)$ has a limit $x > -\infty$. That is, $a_{n_k} \to x$. Plainly $x \neq +\infty$ since the set $\{k : a_k \geq 0\}$ is finite. Hence $+\infty > x > -\infty$. So there exists integer N such that

$$k \geq N \Rightarrow \left|a_{n_k} - x\right| < 1 \Rightarrow x - 1 < a_{n_k} < x + 1 .$$

Consequently the set $\{k : a_{n_k} \geq x-1\}$ is infinite and so the set $\{k : a_k \geq x-1\}$ is infinite contradicting that $\{k : a_k \geq x-1\}$ is finite by assumption (4).

Hence $\limsup_{n\to\infty} a_n = \limsup a_n = \sup L = -\infty = \alpha$.

Now we assume $-\infty < \alpha < +\infty$.

First of all by taking $\beta = \alpha + 1$, the set $\{k : a_k \geq \beta\}$ is finite implies that (a_n) is bounded above.

Suppose $a_{n_k} \to x$.

We shall first show that $x \leq \alpha$.

If $x = -\infty$, then plainly $x \leq \alpha$ and there is nothing to show.

Thus we may assume that $-\infty < x < +\infty$.

If $x > \alpha$, then there exists K such that

$$k \geq K \Rightarrow \left|a_{n_k} - x\right| < \frac{1}{2}\left|x-\alpha\right| \Rightarrow x - \frac{1}{2}\left|x-\alpha\right| < a_{n_k} < x + \frac{1}{2}\left|x-\alpha\right| \Rightarrow a_{n_k} > \frac{1}{2}(x+\alpha) > \alpha \text{ .}$$

Taking $\beta = \frac{1}{2}(x+\alpha) > \alpha$. We have then the set $\{k : a_k \geq \beta\}$ is infinite. But this contradicts that $\{k : a_k \geq \beta\}$ is finite. Hence $x \leq \alpha$.

We shall next construct a subsequence (a_{n_k}) such that $a_{n_k} \to \alpha$.

For each integer $n \geq 1$, let $B_n = \{k : a_k \geq \alpha - \frac{1}{n}\}$. Then by assumption (4) each B_n is an infinite set.

Take any n_1 in B_1. Since B_2 is infinite, we can choose n_2 in B_2 such that $n_2 > n_1$. Inductively we define the sequence (a_{n_k}) as follows:

If n_k in B_k is defined, then take $n_{k+1} > n_k$ such that $n_{k+1} \in B_{k+1}$.

Hence (a_{n_k}) is a subsequence of (a_n) such that $a_{n_k} \geq \alpha - \frac{1}{k}$. Thus (a_{n_k}) is bounded below by $\alpha - 1$. (a_{n_k}) is also bounded above as (a_n) is bounded above.

Hence $\left(a_{n_k}\right)$ is bounded and so by the Bolzano Weierstrass Theorem, $\left(a_{n_k}\right)$ has a convergent subsequence $\left(a_{n_{k(j)}}\right)$ such that $a_{n_{k(j)}} \to y$ as $j \to \infty$.

We have already shown that $y \le \alpha$.

Now $a_{n_{k(j)}} \ge \alpha - \dfrac{1}{k(j)}$ and so $y = \lim\limits_{j\to\infty} a_{n_{k(j)}} \ge \lim\limits_{j\to\infty}\left(\alpha - \dfrac{1}{k(j)}\right) = \alpha$ since $k(j) \to \infty$.

Thus $y = \alpha$. This means that $\alpha \in L$. Hence, since α is also an upper bound of L, lim sup a_n = sup $L = \alpha$.

This completes the proof of Theorem 3.

We have similar results for lim inf.

Proposition 5. Suppose (a_n) is a real sequence. For each integer $k \ge 1$, let $\beta_k = \inf\{a_n : n \ge k\}$. Then there is always a subsequence $\left(a_{n_k}\right)$ of (a_n) such that $a_{n_k} \to \lim\limits_{n\to\infty} \beta_n$.

Theorem 6. Suppose (a_n) is a real sequence. The following statements are equivalent.

(1) $\liminf\limits_{n\to\infty} a_n = \alpha$.

(2) $\lim\limits_{n\to\infty} \inf\{a_k : k \ge n\} = \alpha$.

(3) For any subsequence $\left(a_{n_k}\right)$, $a_{n_k} \to x$ as $k \to \infty \Rightarrow x \ge \alpha$ and there exists a subsequence $\left(a_{n_k}\right)$ such that $a_{n_k} \to \alpha$.

(4) For any $\beta < \alpha$, the set $\{k : a_k \le \beta\}$ is finite and for any $\beta > \alpha$, the set $\{k : a_k \le \beta\}$ is infinite.

Properties of lim sup and lim inf

The following results are useful inequalities which may come in handy.

Proposition 7. Suppose (a_n) and (b_n) are two real sequences.

Then whenever lim sup a_n + lim sup b_n is defined, i.e., lim sup a_n + lim sup b_n is not of the form $(+\infty) + (-\infty)$ or $(-\infty) + (+\infty)$,

$$\limsup (a_n + b_n) \le \limsup a_n + \limsup b_n .$$

Proof.

Suppose lim sup $(a_n + b_n) = \gamma$, lim sup $a_n = \alpha$ and lim sup $b_n = \beta$.

If $\alpha = +\infty$ or $\beta = +\infty$, since $\alpha + \beta$ is not of the form $(+\infty) + (-\infty)$ or $(-\infty) + (+\infty)$, we have nothing to prove.

We shall assume that $-\infty < \alpha , \beta < +\infty$.

By proposition 1, there exists a subsequence $\left(a_{n_k} + b_{n_k}\right)$ such that $a_{n_k} + b_{n_k} \to \gamma$.

The sequence $\left(a_{n_k}\right)$ has a subsequence $\left(a_{n_{k(j)}}\right)$ such that $a_{n_{k(j)}} \to x \le \alpha < +\infty$. It follows that $a_{n_{k(j)}} + b_{n_{k(j)}} \to \gamma$.

But $\left(b_{n_{k(j)}}\right)$ has a subsequence $\left(b_{n_{k(j(\ell))}}\right)$ such that $b_{n_{k(j(\ell))}} \to y \le \beta < +\infty$. We then have $a_{n_{k(j(\ell))}} + b_{n_{k(j(\ell))}} \to \gamma$. But $\left(a_{n_{k(j(\ell))}}\right)$being a subsequence of $\left(a_{n_{k(j)}}\right)$, $a_{n_{k(j(\ell))}} \to x$. Hence $\gamma = \lim\limits_{\ell\to\infty}\left(a_{n_{k(j(\ell))}} + b_{n_{k(j(\ell))}}\right) = \lim\limits_{\ell\to\infty} a_{n_{k(j(\ell))}} + \lim\limits_{\ell\to\infty} b_{n_{k(j(\ell))}} = x + y$.

It follows that $\gamma = x + y \le \alpha + \beta = \limsup a_n + \limsup b_n$. This means lim sup $(a_n + b_n) \le$ lim sup a_n + lim sup b_n .

If $\alpha = -\infty$, then $a_n \to -\infty$. As in the above proceeding, by proposition 1, there exists a subsequence $\left(a_{n_k} + b_{n_k}\right)$ such that $a_{n_k} + b_{n_k} \to \gamma$.

Since $a_n \to -\infty$, $a_{n_k} \to -\infty$.

Since lim sup $b_n = \beta \neq +\infty$, (b_n) is bounded above and so $a_{n_k} + b_{n_k} \to -\infty$. Hence $\gamma = -\infty$. By convention $\alpha + \beta = -\infty$. What this means is that if $\alpha = -\infty$, then we have nothing to prove.

Similarly, if $\beta = -\infty$, then since $\alpha \neq +\infty$, $\gamma = \alpha + \beta = -\infty$ and we have nothing to prove.

This completes the proof.

For lim inf we have the following:

Proposition 8. Suppose (a_n) and (b_n) are two real sequences.

Then whenever lim inf a_n + lim inf b_n is defined, i.e., lim inf a_n + lim inf b_n is not of the form $(+\infty) + (-\infty)$ or $(-\infty) + (+\infty)$,

$$\liminf (a_n + b_n) \geq \liminf a_n + \liminf b_n .$$

The proof of Proposition 8 is similar to that of Proposition 7.

Next we have:

Theorem 9. Suppose (a_n) and (b_n) are two real sequences. Then

$$\liminf (a_n + b_n) \leq \liminf a_n + \limsup b_n \leq \limsup (a_n + b_n)$$

whenever lim inf a_n + lim sup b_n is not of the form form $(+\infty) + (-\infty)$ or $(-\infty) + (+\infty)$.

Proof.

Let $\alpha = \liminf a_n$ and $\beta = \limsup b_n$.

Suppose $\alpha = -\infty$ and $\beta < +\infty$. Then (a_n) is not bounded below and (b_n) is bounded above. Thus $(a_n + b_n)$ is not bounded below. Therefore, there exists a subsequence $\left(a_{n_k}+b_{n_k}\right)$ such that $\left(a_{n_k}+b_{n_k}\right)\to -\infty$. Hence lim inf $(a_n + b_n) = -\infty$. By convention $\alpha + \beta = -\infty$ and so we have nothing to prove.

Suppose $\alpha = +\infty$ and $\beta > -\infty$. Then lim sup $a_n = +\infty$ and so $a_n \to \infty$. By Proposition 1, there exists a subsequence $\left(b_{n_k}\right)$ such that $b_{n_k} \to \beta > -\infty$. Hence $\left(b_{n_k}\right)$ is bounded below. Now $\left(a_{n_k}\right)$ is not bounded above since $a_n \to \infty$. Thus $\left(a_{n_k}+b_{n_k}\right)$ is not bounded above consequently it has a subsequence $\left(a_{n_{k(j)}}+b_{n_{k(j)}}\right)$ such that $a_{n_{k(j)}}+b_{n_{k(j)}} \to +\infty$ as $j\to\infty$. Since $\left(a_{n_{k(j)}}+b_{n_{k(j)}}\right)$ is a subsequence of (a_n+b_n), lim sup $(a_n + b_n) = +\infty$ and we have nothing to prove.

Suppose $\beta = +\infty$ and $\alpha > -\infty$. Then (b_n) is not bounded above and (a_n) is bounded below. Thus $(a_n + b_n)$ is not bounded above. Therefore, there exists a subsequence $\left(a_{n_k}+b_{n_k}\right)$ such that $\left(a_{n_k}+b_{n_k}\right)\to +\infty$. Hence lim sup $(a_n + b_n) = +\infty$. By convention $\alpha + \beta = +\infty$ and so we have nothing to prove.

Suppose $\beta = -\infty$ and $\alpha < +\infty$. Then lim sup $b_n = -\infty$ and so $b_n \to -\infty$. By Proposition 1, there exists a subsequence $\left(a_{n_k}\right)$ such that $a_{n_k} \to \alpha < +\infty$. Hence $\left(a_{n_k}\right)$ is bounded above. Now $\left(b_{n_k}\right)$ is not bounded below since $b_n \to -\infty$. Thus $\left(a_{n_k}+b_{n_k}\right)$ is not bounded below consequently it has a subsequence $\left(a_{n_{k(j)}}+b_{n_{k(j)}}\right)$ such that $a_{n_{k(j)}}+b_{n_{k(j)}} \to -\infty$ as $j\to\infty$. Since $\left(a_{n_{k(j)}}+b_{n_{k(j)}}\right)$ is a subsequence of (a_n+b_n), lim inf $(a_n + b_n) = -\infty$ and we have nothing to prove.

Now we assume $-\infty < \alpha, \beta < +\infty$.

By Proposition 1, there exists a subsequence $\left(b_{n_k}\right)$ such that $b_{n_k} \to \beta$. Take the subsequence $\left(a_{n_k}\right)$. Then since α = lim inf $a_n > -\infty$, (a_n) is bounded below and so $\left(a_{n_k}\right)$ is bounded below and has a subsequence $\left(a_{n_{k(j)}}\right)$ such that $a_{n_{k(j)}} \to x$

for some x. Plainly $x \geq \alpha > -\infty$. Since $b_{n_k} \to \beta$, $b_{n_{k(j)}} \to \beta$. Therefore, $a_{n_{k(j)}} + b_{n_{k(j)}} \to x + \beta$. Hence by the definition of lim sup, $x + \beta \leq \limsup(a_n + b_n)$. But $\alpha + \beta \leq x + \beta$ and so $\liminf a_n + \limsup b_n = \alpha + \beta \leq \limsup(a_n + b_n)$.

Now we prove the left side of the inequality similarly. By Proposition 1, there exists a subsequence (a_{n_k}) such that $a_{n_k} \to \alpha$. Consider the subsequence (b_{n_k}) of (b_n). Since $\beta = \limsup b_n < +\infty$, (b_n) is bounded above and so (b_{n_k}) is bounded above and has a subsequence $(b_{n_{k(j)}})$ such that $b_{n_{k(j)}} \to y$ for some y. By definition of $\limsup b_n$, $y \leq \beta < +\infty$. Therefore, $a_{n_{k(j)}} + b_{n_{k(j)}} \to \alpha + y$. Hence by the definition of $\liminf(a_n + b_n)$, $\alpha + y \geq \liminf(a_n + b_n)$. But $\alpha + \beta \geq \alpha + y$ and so $\liminf(a_n + b_n) \leq \alpha + y \leq \alpha + \beta = \liminf a_n + \limsup b_n$.

This completes the proof.

The next result is a useful fact for manipulating sequence.

Proposition 10. Suppose (a_n) is a real sequence.

Then (1) $\limsup(-a_n) = -\liminf a_n$ and

(2) $\liminf(-a_n) = -\limsup a_n$.

Here we use the convention that $-(-\infty) = +\infty$ and $-(+\infty) = -\infty$.

Proof. This is a consequence of the fact that for a set A of real numbers,

$\sup(-A) = -\inf A$ and $\inf(-A) = -\sup A$.

We shall use the equivalent definition (2) given by Theorem 3.

Suppose $\limsup(-a_n) = \alpha$. Let $\alpha_k = \sup\{-a_n : n \geq k\}$. Then $\alpha_k \to \alpha$.

$\alpha_k = \sup\{-a_n : n \geq k\} = -\inf\{a_n : n \geq k\}$. Since $\inf\{a_n : n \geq k\} \to \liminf a_n$,

$\alpha_k \to -\liminf a_n$. This means $\limsup(-a_n) = \alpha = -\liminf a_n$.

Similarly,

$$\liminf(-a_n) = \lim_{k\to\infty}\inf\{-a_n : n \geq k\} = \lim_{k\to\infty}\left(-\sup\{a_n : n \geq k\}\right)$$

$$= -\lim_{k\to\infty}\left(\sup\{a_n : n \geq k\}\right) = -\limsup a_n \ .$$

The next result gives some insight as to why the Cauchy Hadamard formula for the radius of convergence of a power series is more useful than plain old ratio test.

Theorem 11.

Suppose (a_n) is a positive sequence, i.e., a sequence of positive terms. Then

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \left(a_n\right)^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left(a_n\right)^{\frac{1}{n}} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \ .$$

Proof. Note that $\liminf\limits_{n\to\infty} \left(a_n\right)^{\frac{1}{n}} \leq \limsup\limits_{n\to\infty} \left(a_n\right)^{\frac{1}{n}}$ and so we need only prove the remaining two inequalities,

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \left(a_n\right)^{\frac{1}{n}} \text{ and } \limsup_{n\to\infty} \left(a_n\right)^{\frac{1}{n}} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \ .$$

For each integer $n \geq 1$, let $S_n = \left\{ \dfrac{a_{j+1}}{a_j} : j \geq n \right\} = \left\{ \dfrac{a_{n+1}}{a_n}, \dfrac{a_{n+2}}{a_{n+1}}, \cdots \right\}$

If $S_1 = \left\{ \dfrac{a_{j+1}}{a_j} : j \geq n \right\} = \left\{ \dfrac{a_2}{a_1}, \dfrac{a_3}{a_2}, \cdots \right\}$ is not bounded above, then S_n is not bounded above for all integer $n \geq 1$. Hence sup $S_n = +\infty$ for all integer $n \geq 1$. Thus

$\limsup\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \limsup\limits_{n\to\infty} S_n = +\infty$. Plainly, $\limsup\limits_{n\to\infty} \left(a_n\right)^{\frac{1}{n}} \leq \limsup\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = +\infty$

So now we assume that the sequence $\left(\dfrac{a_{n+1}}{a_n} \right)$ or equivalently S_1 is bounded above. It is plainly bounded below by 0 and so it is bounded. Let

$\alpha_n = \sup S_n = \sup\left\{\frac{x_{k+1}}{x_k} : k \geq n\right\}$. Then (α_n) is a decreasing sequence and converges to a real number K and so $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \alpha_n = K$. Therefore, given any $\varepsilon > 0$, there exists an integer N such that $n \geq N \Rightarrow |\alpha_n - K| < \varepsilon$. Hence, for $n \geq N$,

$$\frac{a_{n+1}}{a_n} \leq \alpha_n < K + \varepsilon .$$

It follows that $a_{n+1} \leq (K+\varepsilon)a_n$. Iterating this inequality until N gives,

$$a_n \leq (K+\varepsilon)^{n-N} a_N$$

for $n \geq N$. It then follows that for all integer $n \geq N$.

$$(a_n)^{\frac{1}{n}} \leq (K+\varepsilon)^{1-\frac{N}{n}} (a_N)^{\frac{1}{n}} .$$

Note that both $\left\{(a_N)^{\frac{1}{j}} : j \geq N\right\}$ and $\left\{(K+\varepsilon)^{1-\frac{N}{j}} : j \geq N\right\}$ are bounded since the terms form two convergent sequences.

It follows from the above inequality that for all integer $n \geq N$,

$$\sup\left\{(a_j)^{\frac{1}{j}} : j \geq n\right\} \leq \sup\left\{(K+\varepsilon)^{1-\frac{N}{j}} (a_N)^{\frac{1}{j}} : j \geq n\right\}.$$

Therefore,

$$\limsup_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \sup\left\{(a_j)^{\frac{1}{j}} : j \geq n\right\} \leq \lim_{n\to\infty} \sup\left\{(K+\varepsilon)^{1-\frac{N}{j}} (a_N)^{\frac{1}{j}} : j \geq n\right\} = K + \varepsilon \quad \text{-- (5)}$$

since $(a_N)^{\frac{1}{j}} \to 1$ as $j \to \infty$ and $(K+\varepsilon)^{1-\frac{N}{j}} \to (K+\varepsilon)$ as $j \to \infty$.

Since (5) is true for any $\varepsilon > 0$, we have then

$$\limsup_{n\to\infty} (a_n)^{\frac{1}{n}} \leq K = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} .$$

Now we prove the remaining inequality.

Plainly S_n is bounded below. Thus, by the completeness property of the real numbers, infimum of S_n exists. Let $\beta_n = \inf S_n$, the infimum of S_n. Observe that (β_n) is an increasing sequence**.** By Theorem 3 or equivalent definition of lim inf , $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \beta_n$. Thus $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n}$ is either a real number greater or equal to zero or $+\infty$.

If $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty$, then $\{ \beta_n : n \geq 1 \}$ is not bounded above. It follows that S_n is not bounded above for each integer $n \geq 1$. Hence given any $K > 0$, there exists an integer M such that

$$n \geq M \Rightarrow \frac{a_{n+1}}{a_n} \geq \beta_n > K \ .$$

Therefore, for $n \geq M$, $a_n > a_M K^{n-M}$. Thus, for all $j \geq n \geq M$,

$$(a_j)^{\frac{1}{j}} > (a_M)^{\frac{1}{j}} K^{1-\frac{M}{j}} \geq \inf\left\{(a_M)^{\frac{1}{j}} K^{1-\frac{M}{j}} : j \geq n\right\} .$$

This implies

$$\inf\left\{(a_j)^{\frac{1}{j}} : j \geq n\right\} \geq \inf\left\{(a_M)^{\frac{1}{j}} K^{1-\frac{M}{j}} : j \geq n\right\}. \qquad \text{----------------------} \ \mathbf{(6)}$$

Since $\liminf_{n\to\infty} \left\{(a_M)^{\frac{1}{j}} K^{1-\frac{M}{j}} : j \geq n\right\} = \lim_{n\to\infty} (a_M)^{\frac{1}{n}} K^{1-\frac{M}{n}} = K > 0$**,** there exists an integer N such that for $j \geq N$, $\inf\left\{(a_M)^{\frac{1}{k}} K^{1-\frac{M}{k}} : k \geq j\right\} > \frac{K}{2}$. It follows from (6) that, for $j \geq N$, $\inf\left\{(a_k)^{\frac{1}{k}} : k \geq j\right\} > \frac{K}{2}$. Since K is arbitrary, this shows that the sequence $\left(\inf\left\{(a_k)^{\frac{1}{k}} : k \geq j\right\}\right)$ is not bounded above. Hence

$$\liminf_{n\to\infty} (a_n)^{\frac{1}{n}} = \liminf_{j\to\infty} \left\{(a_k)^{\frac{1}{k}} : k \geq j\right\} = +\infty .$$

Therefore**,** $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \liminf_{n\to\infty} (a_n)^{\frac{1}{n}} = \limsup_{n\to\infty} (a_n)^{\frac{1}{n}} = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty$.

We now assume that $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} < +\infty$. That is, the sequence (β_n) is convergent, or equivalently that it is bounded above. Let $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \beta_n = q$. Then $q \geq 0$. If $q = 0$, we have nothing to prove. Assume now $q > 0$. Then for any $\varepsilon > 0$, there exists a positive integer N such that $n \ge N \Rightarrow |\beta_n - q| < \varepsilon$. Thus for any $\varepsilon >$ 0 such that $0 < \varepsilon < q/2$, $n \ge N \Rightarrow q + \varepsilon > \beta_n > q - \varepsilon > \frac{q}{2} > 0$. It follows that,

$$n \ge N \Rightarrow \frac{a_{n+1}}{a_n} \ge \beta_n > q - \varepsilon > 0 \ .$$

Hence, $n \ge N \Rightarrow a_{n+1} > (q-\varepsilon) a_n > 0$. Therefore, $n \ge N \Rightarrow a_n \ge a_N (q-\varepsilon)^{n-N}$. So taking n-th root we get

$$n \ge N \Rightarrow (a_n)^{\frac{1}{n}} \ge (a_N)^{\frac{1}{n}} (q-\varepsilon)^{1-\frac{N}{n}} \quad \text{--------------------------} \ (7)$$

Hence $\quad n \ge N \Rightarrow \inf\left\{(a_j)^{\frac{1}{j}} : j \ge n\right\} \ge \inf\left\{(a_N)^{\frac{1}{j}} (q-\varepsilon)^{1-\frac{N}{j}} : j \ge n\right\}$.

Therefore, $\liminf_{n\to\infty} \left\{(a_j)^{\frac{1}{j}} : j \ge n\right\} \ge \liminf_{n\to\infty} \left\{(a_N)^{\frac{1}{j}} (q-\varepsilon)^{1-\frac{N}{j}} : j \ge n\right\}$.

But $\liminf_{n\to\infty} \left\{(a_N)^{\frac{1}{j}} (q-\varepsilon)^{1-\frac{N}{j}} : j \ge n\right\} = \lim_{n\to\infty} (a_N)^{\frac{1}{n}} (q-\varepsilon)^{1-\frac{N}{n}} = q - \varepsilon$ for any $0 < \varepsilon < q$.

So we have $\liminf_{n\to\infty} \left\{(a_j)^{\frac{1}{j}} : j \ge n\right\} \ge q - \varepsilon$. Therefore, since ε is arbitrarily small, $\liminf_{n\to\infty} \left\{(a_j)^{\frac{1}{j}} : j \ge n\right\} \ge q$. This means $\liminf_{n\to\infty} (a_n)^{\frac{1}{n}} \ge \liminf_{n\to\infty} \frac{a_{n+1}}{a_n}$.

This completes the proof.

By virtue of Theorem 11 and Theorem 2, an immediate corollary to Theorem 12 is the following:

Corollary 12.

Suppose (a_n) is a positive sequence, i.e., a sequence of positive terms. Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists implies that $\lim_{n\to\infty} (a_n)^{\frac{1}{n}}$ exists and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$.

Remark. It is possible for $\lim_{n\to\infty}(a_n)^{\frac{1}{n}}$ to exist and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ does not exist. See the following example.

Eample.

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series where the coefficients a_n is defined by $a_{2k-1} = \left(\frac{1}{3}\right)^{2k-1}$ for $k \geq 1$ and $a_{2k} = \left(\frac{1}{6}\right)^{2k}$ for $k \geq 0$. Note that $a_n > 0$ for all integer $n \geq 0$. Thus

$$\frac{a_{n+1}}{a_n} = \begin{cases} \dfrac{\left(\frac{1}{6}\right)^{n+1}}{\left(\frac{1}{3}\right)^{n}} & \text{if } n \text{ is odd} \\ \dfrac{\left(\frac{1}{3}\right)^{n+1}}{\left(\frac{1}{6}\right)^{n}} & \text{if } n \text{ is even} \end{cases} = \begin{cases} \dfrac{1}{6}\left(\dfrac{1}{2}\right)^{n} & \text{if } n \text{ is odd} \\ \dfrac{2^n}{3} & \text{if } n \text{ is even} \end{cases}.$$

It follows that for all integer $n \geq 0$, $\inf\left\{\frac{a_{j+1}}{a_j} : j \geq n\right\} = 0$ and $\sup\left\{\frac{a_{j+1}}{a_j} : j \geq n\right\} = +\infty$.

Hence $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ does not exist.

But for integer $n \geq 1$, $(a_n)^{\frac{1}{n}} = \begin{cases} \frac{1}{3}, & \text{if } n \text{ is odd} \\ \frac{1}{6}, & \text{if } n \text{ is even} \end{cases}$. Therefore, for any integer $n \geq 1$, $\sup\left\{(a_j)^{\frac{1}{j}} : j \geq n\right\} = \frac{1}{3}$. It follows that $\limsup_{n\to\infty}(a_n)^{\frac{1}{n}} = \frac{1}{3}$. Hence by the Cauchy Hadamard formula, the radius of convergence of the power series is 3 whereas the ratio test gives no information about the radius of convergence.