Limit of the Lebesgue Stieltjes Integral and A Change of Variable

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If a function is defined by a Lebegsue Stieltjes integral and turns out that it is of bounded variation, then we can define another Lebesgue Stieltjes integral with it as the integrator. More precisely, suppose $\phi: I \to \mathbb{R}$ is an absolutely continuous function on the closed and bounded interval $I = [a,b]$, where $a < b$. Suppose g is a Borel measurable function on *I*. Define $\Phi: I \to \mathbb{R}$, by $\Phi(x) = \int_a^x g d\lambda_\phi$ where λ_ϕ is the Lebesgue Stieljes measure associated with the function ϕ. Since ϕ is absolutely continuous on *I*, by Theorem 26 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*", $\Phi(x) = \int_a^x g d\lambda_\phi = \int_a^x g(y)\phi'(y)dy$. It follows that Φ is absolutely continuous on *I* and therefore of bounded variation. Suppose $f: I \to \mathbb{R}$ is a Borel measurable function. Then

$$\int_a^x f d\lambda_\Phi = \int_a^x f(y)\Phi'(y)dy = \int_a^x f(y)g(y)\phi'(y)dy = \int_a^x f(y)g(y)d\lambda_\phi .$$

Hence, we have proved:

Theorem 1. Suppose $\phi: I \to \mathbb{R}$ is an absolutely continuous function on the closed and bounded interval $I = [a,b]$, where $a < b$. Suppose g is a Borel measurable function on *I*. Define $\Phi: I \to \mathbb{R}$, by $\Phi(x) = \int_a^x g d\lambda_\phi$ where λ_ϕ is the Lebesgue Stieljes measure associated with the function ϕ. Suppose $f: I \to \mathbb{R}$ is a Borel measurable function. Then

$$\int_a^x f d\lambda_\Phi = \int_a^x f(y)g(y)d\lambda_\phi .$$

If a sequence of functions of bounded variation, whose total variation is uniformly bounded converges to a function, is the limiting function also of bounded variation and if so, does the sequence of Lebesgue Stieltjes integrals defined using the given sequence of functions as integrators converges to the Lebesgue Stieltjes integral with the limiting function as integrator? The answer is "yes". We state this answer as Theorem 2 below.

Theorem 2. Suppose (g_n) is a sequence of functions defined on the closed and bounded interval $[a, b]$ whose total variations are uniformly bounded by a constant K, i.e., $V(g_n,[a,b]) \le K$, for all positive integer n and suppose the sequence (g_n) converges to a finite function g at every point of $[a,b]$. Let f be a continuous function on $[a,b]$. Then g is of finite variation and

$\lim_{n\to\infty}\int_a^b f d\lambda_{g_n} = \int_a^b f d\lambda_g$.

Proof.

We note that since the function f is continuous on $[a, b]$, the Riemann Stieltjes integrals of the function f with integrators g and g_k exist and are equal to their respective Lebesgue Stieltjes integrals. Moreover, since f is continuous on $[a, b]$, for any $a \le d < e < f \le b$, the Rieman Stieltjes integrals,

$$RS\int_d^e f d\lambda_g + RS\int_e^f f d\lambda_g = RS\int_d^f f d\lambda_g \text{ and } RS\int_d^e f d\lambda_{g_k} + RS\int_e^f f d\lambda_{g_k} = RS\int_d^f f d\lambda_{g_k} .$$

We shall show that the limiting function g is of finite variation.

Let $P: a = x_0 < x_1 < \cdots < x_n = b$ be a partition of the closed interval $[a, b]$. Given any $\varepsilon > 0$, since the sequence (g_m) converges to g pointwise, for each $0 \le k \le n$, there exists an integer $N_k > 0$ such that for $0 \le k \le n$,

$$m \ge N_k \Rightarrow |g_m(x_k) - g(x_k)| < \frac{\varepsilon}{2n} . \qquad \text{-------------------------- (1)}$$

Let $N = \max\{N_k : 0 \le k \le n\}$

Then it follows from (1) that for $0 \le k \le n$

$$m \ge N \Rightarrow |g_m(x_k) - g(x_k)| < \frac{\varepsilon}{2n} . \qquad \text{------------------ (2)}$$

Suppose $1 \le k \le n$. Then for $m \ge N$

$$\begin{aligned}|g(x_k) - g(x_{k-1})| &= |g(x_k) - g_m(x_k) + g_m(x_k) - g_m(x_{k-1}) + g_m(x_{k-1}) - g(x_{k-1})| \\ &\le |g(x_k) - g_m(x_k)| + |g_m(x_k) - g_m(x_{k-1})| + |g_m(x_{k-1}) - g(x_{k-1})| \\ &\le \frac{\varepsilon}{2n} + |g_m(x_k) - g_m(x_{k-1})| + \frac{\varepsilon}{2n} = \frac{\varepsilon}{n} + |g_m(x_k) - g_m(x_{k-1})| .\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{n}|g(x_k) - g(x_{k-1})| &\le \sum_{k=1}^{n}\left(\frac{\varepsilon}{n} + |g_m(x_k) - g_m(x_{k-1})|\right) = \varepsilon + \sum_{k=1}^{n}|g_m(x_k) - g_m(x_{k-1})| \\ &\le \varepsilon + V(g_m,[a,b]) \le \varepsilon + K .\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $V(g,[a,b]) \le K$. Hence, g is of bounded variation.

We shall now show that $\lim_{n\to\infty} \int_a^b f d\lambda_{g_n} = \int_a^b f d\lambda_g$. For the rest of the proof, all integrals are Riemann Stieltjes integrals.

We shall take an appropriate partition of the interval $[a, b]$ and split the integrals into integrals on each sub intervals of the partition.

We note that since f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$. Therefore, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{3K} . \quad \text{--------------------- (3)}$$

Let $P: a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$ such that $\|P\| = \max\{|x_k - x_{k-1}| : 1 \le k \le n\} < \delta$.

Then

$$\int_a^b f\, d\lambda_g = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x) d\lambda_g = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \big(f(x)-f(x_k)\big)\, d\lambda_g + \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x_k) d\lambda_g . \quad \text{---------(4)}$$

And for each positive integer m,

$$\int_a^b f\, d\lambda_{g_m} = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x) d\lambda_{g_m} = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \big(f(x)-f(x_k)\big)\, d\lambda_{g_m} + \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x_k) d\lambda_{g_m} . \quad \text{--------- (5)}$$

(5) – (4) gives:

$$\begin{aligned}
\int_a^b f\, d\lambda_{g_m} - \int_a^b f\, d\lambda_g &= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \big(f(x)-f(x_k)\big)\, d\lambda_{g_m} - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \big(f(x)-f(x_k)\big)\, d\lambda_g \\
&\quad + \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x_k) d\lambda_{g_m} - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x_k) d\lambda_g \\
&= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \big(f(x)-f(x_k)\big)\, d\lambda_{g_m} - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \big(f(x)-f(x_k)\big)\, d\lambda_g \\
&\quad + \sum_{k=1}^{n} f(x_k)\big(g_m(x_k)-g_m(x_{k-1})\big) - \sum_{k=1}^{n} f(x_k)\big(g(x_k)-g(x_{k-1})\big) \\
&= \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \big(f(x)-f(x_k)\big)\, d\lambda_{g_m} - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \big(f(x)-f(x_k)\big)\, d\lambda_g \\
&\quad + \sum_{k=1}^{n} f(x_k)\big(g_m(x_k)-g(x_k)\big) - \sum_{k=1}^{n} f(x_k)\big(g_m(x_{k-1})-g(x_{k-1})\big).
\end{aligned}$$

Then, $\left|\int_a^b f\,d\lambda_{g_m} - \int_a^b f\,d\lambda_g\ \right| \le \sum_{k=1}^{n}\left|\int_{x_{k-1}}^{x_k}\left(f(x)-f(x_k)\right)d\lambda_{g_m}\right| + \sum_{k=1}^{n}\left|\int_{x_{k-1}}^{x_k}\left(f(x)-f(x_k)\right)d\lambda_g\right|$

$$+\sum_{k=1}^{n}\left|f(x_k)\right|\left|g_m(x_k)-g(x_k)\right| + \sum_{k=1}^{n}\left|f(x_k)\right|\left|g_m(x_{k-1})-g(x_{k-1})\right| .$$

Let M be the maximum value of $\left|f(x)\right|$ on $[a, b]$. Then on account of (3) and Theorem 3 below,

$$\left|\int_a^b f\,d\lambda_{g_m} - \int_a^b f\,d\lambda_g\ \right| \le \frac{\varepsilon}{3K}\sum_{k=1}^{n} V_{g_m}[x_{k-1},x_k] + \frac{\varepsilon}{3K}\sum_{k=1}^{n} V_g[x_{k-1},x_k]$$

$$+M\sum_{k=1}^{n}\left|g_m(x_k)-g(x_k)\right| + M\sum_{k=1}^{n}\left|g_m(x_{k-1})-g(x_{k-1})\right|$$

$$\le \frac{\varepsilon}{3K}V_{g_m}[a,b] + \frac{\varepsilon}{3K}V_g[a,b]$$

$$+M\sum_{k=1}^{n}\left|g_m(x_k)-g(x_k)\right| + M\sum_{k=1}^{n}\left|g_m(x_{k-1})-g(x_{k-1})\right|$$

$$\le \frac{2\varepsilon}{3} + +M\sum_{k=1}^{n}\left|g_m(x_k)-g(x_k)\right| + M\sum_{k=1}^{n}\left|g_m(x_{k-1})-g(x_{k-1})\right| . \text{ ------ (6)}$$

For $0 \le k \le n$, $g_m(x_k) \to g(x_k)$. Therefore, for $0 \le k \le n$ there exists an integer $N_k > 0$ such that

$$m \ge N_k \Rightarrow \left|g_m(x_k)-g(x_k)\right| < \frac{\varepsilon}{6nM} . \text{ -------------------- (7)}$$

Let $N = \max\left\{N_k : 0 \le k \le n\right\}$. It follows from (6) and (7) that

$$m \ge N \Rightarrow \left|\int_a^b f\,d\lambda_{g_m} - \int_a^b f\,d\lambda_g\ \right| < \frac{2\varepsilon}{3} + M\sum_{k=1}^{n}\frac{\varepsilon}{6nM} + M\sum_{k=1}^{n}\frac{\varepsilon}{6nM} = \varepsilon .$$

It follows that $\lim_{n\to\infty}\int_a^b f d\lambda_{g_n} = \int_a^b f d\lambda_g$.

Remark. Theorem 2 is known as Helly's Second Theorem.

Theorem 3. Suppose $g:[a,b]\to\mathbb{R}$ is a function of bounded variation defined on the closed and bounded interval $I=[a,b]$. Suppose $f: I\to\mathbb{R}$ is a continuous function defined on *I*. Then $\left|\int_a^b f d\lambda_g\right| \le M(f)V(g,I)$, where $M(f)$ is the maximum value of $|f|$ on *I* and $V(g,I)$ is the total variation of *g* on *I*.

Proof. $\int_a^b f d\lambda_g = \int_a^b f d\mu_P - \int_a^b f d\mu_N$, where P and N are the positive and negative variation functions of g. Then,

$$\left|\int_a^b f d\lambda_g\right| \le \left|\int_a^b f d\mu_P\right| + \left|\int_a^b f d\mu_N\right| \le \int_a^b |f|\, d\mu_P + \int_a^b |f|\, d\mu_N$$

$$\le \int_a^b M(f) d\mu_P + \int_a^b M(f) d\mu_N = M(f)\big(P([a,b]) + N((la,b])\big) = M(f)V(g,I) .$$

We present another proof of Theorem 2 using the Helly Selection Theorem.

Helly Selection Theorem. A uniformly bounded sequence of increasing functions defined on a closed and bounded interval [a, b] contains a subsequence which converges at every point of [a, b] to an increasing function.

Proof of Theorem 2.

Suppose (g_n) is a sequence of functions defined on the closed and bounded interval [a, b] whose total variations are uniformly bounded by a constant K, i.e., $V(g_n,[a,b]) \le K$. Suppose the sequence (g_n) converges pointwise at every point of [a, b] to a function g on [a, b].

Let P_n and N_n be the positive and negative variation of functions g_n. Then

$g_n(x) = g_n(a) + P_n(x) - N_n(x)$ and the total variation function of g_n is given by $V_g(x) = V_g[a,x] = P_n(x) + N_n(x)$. The function $g_n(a) + P_n(x)$ is an increasing function,

Note that $|g_n(a) + P_n(x)| \le |g_n(a)| + P_n(x) + N_n(x) \le |g_n(a)| + V(g_n[a,b]) \le |g_n(a)| + K$. Since the sequence $(g_n(a))$ is convergent, it is bounded, that is there exists a constant $C > 0$ such that $|g_n(a)| \le C$ for all positive integer n. Hence, $|g_n(a) + P_n(x)| \le C + K$ for all positive integer n. Thus, the sequence $(g_n(a) + P_n(x))$ is uniformly bounded. Therefore, by the Helly selection Theorem it has a subsequence $\left(g_{n_k}(a) + P_{n_k}(x)\right)$ which converges pointwise to an increasing function $P^*(x)$. By replacing the sequence (g_n) with the subsequence $\left(g_{n_k}\right)$ we may assume that $g_n(x) = g_n(a) + P_n(x) - N_n(x)$ and the sequence $(g_n(a) + P_n(x))$ converges pointwise to $P^*(x)$. Similarly, since the sequence $(N_n(x))$ is uniformly bounded by *K*, it has a convergent subsequence $\left(N_{n_k}(x)\right)$ converging pointwise to an increasing function $N^*(x)$ on [a, b]. By replacing the sequence (g_n) with the subsequence $\left(g_{n_k}\right)$, we may assume that

$g_n(x) = g_n(a) + P_n(x) - N_n(x)$ converges pointwise to *g*, the sequence $\left(g_n(a) + P_n(x)\right)$ converges pointwise to an increasing function $P^*(x)$ and the sequence $\left(N_n(x)\right)$ converging pointwise to an increasing function $N^*(x)$ on [a, b]. It follows that the limiting function $g(x) = P^*(x) - N^*(x)$ is a function of bounded variation whose total variation is bounded by *C*+2*K*.

If $f:[a,b] \to \mathbb{R}$ is a continuous function, then as in the proof of Theorem 1 but not using Theorem 3, we can show that $\lim_{n\to\infty}\int_a^b f d\mu_{P_n+g(a)} = \int_a^b f d\mu_{P^*}$ and that $\lim_{n\to\infty}\int_a^b f d\mu_{N_n} = \int_a^b f d\mu_{N^*}$. Therefore,

$$\lim_{n\to\infty}\int_a^b f d\lambda_{g_n} = \lim_{n\to\infty}\int_a^b f d\lambda_{P_n+g_n(a)-N_n} = \lim_{n\to\infty}\left(\int_a^b f d\mu_{P_n+g_n(a)} - \int_a^b f d\mu_{N_n}\right)$$

$$= \lim_{n\to\infty}\int_a^b f d\mu_{P_n+g_n(a)} - \lim_{n\to\infty}\int_a^b f d\mu_{N_n} = \int_a^b f d\mu_{P^*} - \int_a^b f d\mu_{N^*} = \int_a^b f d\lambda_{P^*-N^*} = \int_a^b f d\lambda_g .$$

Now we shall investigate the relaxation of the condition of Theorem 1.

We shall do this in stages.

Theorem 4.

Suppose $\phi: I \to \mathbb{R}$ is an increasing function on the closed and bounded interval $I = [a,b]$, where $a < b$. Suppose ϕ is right continuous or left continuous. Suppose *g* is a Borel measurable non-negative function on *I*. Define $\Phi: I \to \mathbb{R}$, by $\Phi(x) = \int_a^x g d\mu_\phi$ where μ_ϕ is the Lebesgue Stieljes measure associated with the function ϕ. Suppose $f:[a,b] \to \mathbb{R}$ is a Borel measurable function. Then

$$\int_a^b f d\mu_\Phi = \int_a^b fg d\mu_\phi$$

Proof. Suppose ϕ is right continuous. Then by Theorem 45 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\Phi(x) = \int_a^x g d\mu_\phi = \int_{\varphi(a)}^{\phi(x)} g \circ \nu(y) dy ,$$

where ν is the generalised left continuous inverse of ϕ defined in Definition 38 of the above cited article. Note that ν is an increasing left continuous function on *J*, where $J=[\phi(a),\phi(b)]$ is the smallest interval containing the image of ϕ.

Let $\Gamma: J\to\mathbb{R}$ be defined by $\Gamma(y)=\int_{\varphi(a)}^{y} g\circ\nu(t)dt$. Then Γ is absolutely continuous and increasing on *J*. Then $\Phi(x)=\Gamma\circ\phi(x)$.

$$\int_a^b fd\mu_\Phi=\int_a^b fd\mu_{\Gamma\circ\phi}.$$

By Theorem 58 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\int_a^b fd\mu_{\Gamma\circ\phi}=\int_{\varphi(a)}^{\phi(b)} f\circ\nu(y)d\mu_\Gamma ,$$

where ν is the generalised left continuous inverse of ϕ.

Since Γ is absolutely continuous,

$$\int_{\varphi(a)}^{\phi(b)} f\circ\nu(y)d\mu_\Gamma=\int_{\varphi(a)}^{\phi(b)} f\circ\nu(y)\cdot\Gamma'(y)dy=\int_{\varphi(a)}^{\phi(b)} f\circ\nu(y)\cdot g\circ\nu(y)dy=\int_a^b f(x)\cdot g(x)d\mu_\phi .$$

Suppose ϕ is left continuous. Then by Theorem 45 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\Phi(x)=\int_a^x gd\mu_\phi=\int_{\varphi(a)}^{\phi(x)} g\circ\eta(y)dy ,$$

where η is the generalised right continuous inverse of ϕ defined in Definition 38 of the above cited article. Note that η is an increasing right continuous function on *J*, where $J=[\phi(a),\phi(b)]$ is the smallest interval containing the image of ϕ.

Let $\Gamma: J\to\mathbb{R}$ be defined by $\Gamma(y)=\int_{\varphi(a)}^{y} g\circ\eta(t)dt$. Then Γ is absolutely continuous and increasing on *J*. Then $\Phi(x)=\Gamma\circ\phi(x)$.

$$\int_a^b fd\mu_\Phi=\int_a^b fd\mu_{\Gamma\circ\phi}.$$

By Theorem 59 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\int_a^b f d\mu_{\Gamma\circ\phi} = \int_{\varphi(a)}^{\phi(b)} f\circ\eta(y) d\mu_\Gamma ,$$

where η is the generalised right continuous inverse of ϕ.

Since Γ is absolutely continuous,

$$\int_{\varphi(a)}^{\phi(b)} f\circ\eta(y) d\mu_\Gamma = \int_{\varphi(a)}^{\phi(b)} f\circ\eta(y)\cdot\Gamma'(y)dy = \int_{\varphi(a)}^{\phi(b)} f\circ\eta(y)\cdot g\circ\eta(y)dy = \int_a^b f(x)\cdot g(x)d\mu_\phi .$$

Remark. The requirement that the function g be non-negative can be lifted. This requirement implies that the function $\Gamma(y)$ is increasing and continuous so that we can apply Theorem 58 or Theorem 59 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*".

Theorem 5.

Suppose $\phi: I\to\mathbb{R}$ is an increasing function on the closed and bounded interval $I=[a,b]$, where $a < b$. Suppose ϕ is right continuous or left continuous. Suppose g is a Borel measurable function on *I*. Define $\Phi: I\to\mathbb{R}$, by $\Phi(x) = \int_a^x g d\mu_\phi$ where μ_ϕ is the Lebesgue Stieljes measure associated with the function ϕ. Suppose $f:[a,b]\to\mathbb{R}$ is a Borel measurable function. Then

$$\int_a^b f d\lambda_\Phi = \int_a^b fg d\mu_\phi$$

Proof.

Suppose ϕ is right continuous.

Then by Theorem 45 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\Phi(x) = \int_a^x g d\mu_\phi = \int_{\varphi(a)}^{\phi(x)} g\circ\nu(y)dy ,$$

where ν is the generalised left continuous inverse of ϕ defined in Definition 38 of the above cited article. Note that ν is an increasing left continuous function on *J*, where $J=[\phi(a),\phi(b)]$ is the smallest interval containing the image of ϕ.

Let $\Gamma : J \to \mathbb{R}$ be defined by $\Gamma(y) = \int_{\varphi(a)}^{y} g \circ \nu(t)dt$. Then Γ is absolutely continuous on *J* and so is a function of bounded variation. Then $\Phi(x) = \Gamma \circ \phi(x)$ is a function of bounded variation.

$$\int_a^b f d\lambda_\Phi = \int_a^b f d\lambda_{\Gamma \circ \phi} .$$

By Theorem 64 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*", as Γ is absolutely continuous,

$$\int_a^b f d\lambda_{\Gamma \circ \phi} = \int_{\varphi(a)}^{\phi(b)} f \circ \nu(y) \cdot \Gamma'(y) d\lambda_\Gamma ,$$

where ν is the generalised left continuous inverse of ϕ.

Therefore, $\int_a^b f d\lambda_{\Gamma \circ \phi} = \int_{\varphi(a)}^{\phi(b)} f \circ \nu(y) \cdot g \circ \nu(y) dy = \int_a^b f(x) \cdot g(x) d\mu_\phi$.

Suppose ϕ is left continuous.

Then by Part (ii) of Theorem 45 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*",

$$\Phi(x) = \int_a^x g d\mu_\phi = \int_{\varphi(a)}^{\phi(x)} g \circ \eta(y) dy ,$$

where η is the generalised right continuous inverse of ϕ defined in Definition 38 of the above cited article. Note that η is an increasing right continuous function on *J*, where $J = [\phi(a), \phi(b)]$ is the smallest interval containing the image of ϕ.

Let $\Gamma : J \to \mathbb{R}$ be defined by $\Gamma(y) = \int_{\varphi(a)}^{y} g \circ \eta(t)dt$. Then Γ is absolutely continuous on *J* and so is a function of bounded variation. Then $\Phi(x) = \Gamma \circ \phi(x)$ is a function of bounded variation.

$$\int_a^b f d\lambda_\Phi = \int_a^b f d\lambda_{\Gamma \circ \phi} .$$

By Theorem 64 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*" and as Γ is absolutely continuous,

$$\int_a^b f d\lambda_{\Gamma \circ \phi} = \int_{\varphi(a)}^{\phi(b)} f \circ \eta(y) \cdot \Gamma'(y) d\lambda_\Gamma = \int_{\varphi(a)}^{\phi(b)} f \circ \eta(y) \cdot g \circ \eta(y) dy = \int_a^b f(x) \cdot g(x) d\mu_\phi ,$$

We extend the result of Theorem 5, when the function $\phi : I \to \mathbb{R}$ is a function of bounded variation.

Corollary 6.

Suppose $\phi : I \to \mathbb{R}$ is a function of bounded variation on the closed and bounded interval $I = [a,b]$, where $a < b$. Suppose ϕ is right continuous or left continuous. Suppose g is a Borel measurable function on I. Define $\Phi : I \to \mathbb{R}$, by $\Phi(x) = \int_a^x g d\lambda_\phi$ where λ_ϕ is the Lebesgue Stieljes measure associated with the function ϕ. Suppose $f : [a,b] \to \mathbb{R}$ is a Borel measurable function. Then

$$\int_a^b f d\lambda_\Phi = \int_a^b fg d\lambda_\phi .$$

Proof. Let V_ϕ be the total variation of ϕ. Then V_ϕ and $V_\phi - \phi$ are both increasing functions.

Suppose ϕ is right continuous. It follows that V_ϕ is right continuous. Hence, $\phi_1 = V_\phi$ and $\phi_2 = V_\phi - \phi$ are both right continuous. Note that $\phi = \phi_1 - \phi_2$ Then $\Phi(x) = \int_a^x g d\lambda_\phi = \int_a^x g d\mu_{\phi_1} - \int_a^x g d\mu_{\phi_2}$ is a difference of two functions of bounded variation and so is of bounded variation. Let $\Phi_1(x) = \int_a^x g d\mu_{\phi_1}$ and $\Phi_2(x) = \int_a^x g d\mu_{\phi_2}$.
Hence,

$$\int_a^b f d\lambda_\Phi = \int_a^b f d\lambda_{\Phi_1 - \Phi_2} = \int_a^b f d\lambda_{\Phi_1} - \int_a^b f d\lambda_{\Phi_2} . \text{ ------------------ } (1)$$

By Theorem 5, $\int_a^b f d\lambda_{\Phi_1} = \int_a^b fg d\mu_{\phi_1}$ and $\int_a^b f d\lambda_{\Phi_2} = \int_a^b fg d\mu_{\phi_2}$. It follows from (1) that

$$\int_a^b f d\lambda_\Phi = \int_a^b f d\lambda_{\Phi_1 - \Phi_2} = \int_a^b fg d\mu_{\phi_1} - \int_a^b fg d\mu_{\phi_2} = \int_a^b fg d\lambda_{\phi_1 - \phi_2} = \int_a^b fg d\lambda_\phi .$$

Suppose ϕ is left continuous. It follows $\phi_1 = V_\phi$ and $\phi_2 = V_\phi - \phi$ are both left continuous. It follows similarly as above that $\int_a^b f d\lambda_\Phi = \int_a^b fg d\lambda_\phi$.

More generally we have

Corollary 7.

Suppose $\phi : I \to \mathbb{R}$ is a function of bounded variation on the closed and bounded interval $I = [a,b]$, where $a < b$. Suppose ϕ is the difference or sum of two increasing functions ϕ_1 and ϕ_2. Suppose ϕ_1 and ϕ_2 are both right continuous or left continuous or ϕ_1 is right continuous and ϕ_2 is left continuous or ϕ_1 is left continuous and ϕ_2 is right continuous. Suppose *g* is a Borel measurable function on *I*. Define $\Phi : I \to \mathbb{R}$, by $\Phi(x) = \int_a^x g d\lambda_\phi$ where λ_ϕ is the Lebesgue Stieljes measure associated with the function ϕ. Suppose $f : [a,b] \to \mathbb{R}$ is a Borel measurable function. Then

$$\int_a^b f d\lambda_\Phi = \int_a^b fg d\lambda_\phi .$$

The proof of Corollary 7 is similar to that of Corollary 6 and is omitted.

We now only require the function $\phi : I \to \mathbb{R}$ to be of bounded variation.

Theorem 8. Suppose $\phi : I \to \mathbb{R}$ is a function of bounded variation on the closed and bounded interval $I = [a,b]$, where $a < b$. Suppose *g* is a Borel measurable function on *I*. Define $\Phi : I \to \mathbb{R}$, by $\Phi(x) = \int_a^x g d\lambda_\phi$ where λ_ϕ is the Lebesgue Stieltjes measure associated with the function ϕ. Suppose $f : [a,b] \to \mathbb{R}$ is a Borel measurable function. Then

$$\int_a^b f d\lambda_\Phi = \int_a^b fg d\lambda_\phi .$$

Proof. As detailed in the proof of Corollary 62 of "*Lebesgue Stieltjes Measure, de La Vallée Poussin's Decomposition, Change of Variable, Integration by Parts for Lebesgue Stieltjes Integrals*", an increasing function on the interval [*a*, *b*] can be decomposed as a sum of increasing continuous function, increasing right continuous function and an increasing left continuous function. More precisely, an increasing function ϕ on [*a*, *b*] can be written as

$$\phi = \Phi_{ac} + \Phi_c + \Phi_{ls} + \Phi_{rs} ,$$

where Φ_{ac} is an absolutely continuous increasing function with $\Phi_{ac}'(x) = \phi'(x)$ almost everywhere on [*a*, *b*], Φ_c is a continuous increasing singular function, i.e., $\Phi_c'(x) = 0$ almost everywhere, Φ_{ls} is a right continuous increasing saltus type function and Φ_{rs} is a left continuous increasing function. Let $\Phi_a = \Phi_{ac} + \Phi_c$. Then Φ_a is an increasing continuous function. Thus, $\phi = \Phi_a + \Phi_{ls} + \Phi_{rs}$.

Suppose $\phi : I \to \mathbb{R}$ is a function of bounded variation. Then $\phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are increasing functions. Then $\Phi(x) = \int_a^x g d\lambda_\phi = \int_a^x g d\mu_{\phi_1} - \int_a^x g d\mu_{\phi_2}$. Let $\Phi_1(x) = \int_a^x g d\mu_{\phi_1}$ and $\Phi_2(x) = \int_a^x g d\mu_{\phi_2}$. Hence,

$$\int_a^b f d\lambda_\Phi = \int_a^b f d\lambda_{\Phi_1 - \Phi_2} = \int_a^b f d\lambda_{\Phi_1} - \int_a^b f d\lambda_{\Phi_2}.$$

Now $\phi_1 = \Phi_{a,1} + \Phi_{ls,1} + \Phi_{rs,1}$ is a sum of continuous increasing function, left continuous increasing function and right continuous increasing function. Therefore, by Corollary 7, $\int_a^b f d\lambda_{\Phi_1} = \int_a^b fg d\mu_{\phi_1}$. Similarly, we deduce that $\int_a^b f d\lambda_{\Phi_2} = \int_a^b fg d\mu_{\phi_2}$. It follows that

$$\begin{aligned}\int_a^b f d\lambda_\Phi &= \int_a^b f d\lambda_{\Phi_1 - \Phi_2} = \int_a^b f d\lambda_{\Phi_1} - \int_a^b f d\lambda_{\Phi_2} \\ &= \int_a^b fg d\mu_{\phi_1} - \int_a^b fg d\mu_{\phi_2} = \int_a^b fg d\lambda_{\phi_1 - \phi_2} \\ &= \int_a^b fg d\lambda_\phi .\end{aligned}$$

Remark. In the proof of Theorem 8, we have that

$$\phi = \phi_1 - \phi_2,\ \phi_1 = \Phi_{a,1} + \Phi_{ls,1} + \Phi_{rs,1} \text{ and } \phi_2 = \Phi_{a,2} + \Phi_{ls,2} + \Phi_{rs,2}.$$

Thus,

$$\begin{aligned}\phi &= \Phi_{a,1} + \Phi_{ls,1} + \Phi_{rs,1} - \left(\Phi_{a,2} + \Phi_{ls,2} + \Phi_{rs,2}\right) \\ &= \left(\Phi_{a,1} - \Phi_{a,2} + \Phi_{ls,1} - \Phi_{ls,2}\right) + \left(\Phi_{rs,1} - \Phi_{rs,2}\right)\end{aligned}$$

is a sum of right continuous function of bounded variation and left continuous function of bounded variation. Let $\phi_3 = \Phi_{a,1} - \Phi_{a,2} + \Phi_{ls,1} - \Phi_{ls,2}$ and $\phi_4 = \Phi_{rs,1} - \Phi_{rs,2}$. Then $\phi = \phi_3 + \phi_4$. Let $\Phi_3(x) = \int_a^x g d\mu_{\phi_3}$ and $\Phi_4(x) = \int_a^x g d\mu_{\phi_4}$. Then $\Phi(x) = \int_a^x g d\lambda_\phi = \int_a^x g d\lambda_{\phi_3} + \int_a^x g d\lambda_{\phi_4} = \Phi_3(x) + \Phi_4(x)$. By Corollary 6,

$$\int_a^b f d\lambda_{\Phi_3} = \int_a^b fg d\lambda_{\phi_3} \text{ and } \int_a^b f d\lambda_{\Phi_4} = \int_a^b fg d\lambda_{\phi_4}.$$

Therefore,

$$\begin{aligned}\int_a^b f d\lambda_\Phi &= \int_a^b f d\lambda_{\Phi_3 + \Phi_4} = \int_a^b f d\lambda_{\Phi_3} + \int_a^b f d\lambda_{\Phi_4} = \int_a^b fg d\lambda_{\phi_3} + \int_a^b fg d\lambda_{\phi_4} \\ &= \int_a^b fg d\lambda_{\phi_3 + \phi_4} = \int_a^b fg d\lambda_\phi .\end{aligned}$$

This gives another proof of Theorem 8.