Kestelman Change of Variable Theorem for Riemann Integral
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The change of variable formula for Riemann integral always hold when the substitution function is an indefinite Riemann integral of a Rieman integrable function. The primitive of a bounded Riemann integrable function is such a function. In my article "*Change of Variable or Substitution in Riemann and Lebesgue Integration*", I gave a general but not the most general version of the change of variable theorem for Lebesgue integral (Theorem 31). We shall use this theorem to prove Kestelman formulation of the change of variable for Riemann integral. I reproduce Theorem 31 here for convenience as Theorem 1.

Theorem 1. Suppose g: $[a, b] \to \mathbf{R}$ is an absolutely continuous function and let $f : [c, d] \to \mathbf{R}$ be a bounded Lebesgue integrable function such that the range of g is contained in $[c, d]$. Then we have the following equality for Lebesgue integrals.

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx .$$

Note that if we define $F : [c, d] \to \mathbf{R}$ by $F(x) = \int_c^x f(x)dx$, then $F \circ$ g is absolutely continuous on $[a, b]$.

There are a few generalizations or variations of Theorem 1. One version is to drop the boundedness requirement for f but this caters mainly to Lebesgue integral. Another is to drop the absolute continuity of g.

Theorem 2. Suppose g: $[a, b] \to \mathbf{R}$ is an absolutely continuous function and $f : [c, d] \to \mathbf{R}$ is a Lebesgue integrable function such that the range of g is contained in $[c, d]$ and $(f \circ g)\, g'$ is Lebesgue integrable on $[a, b]$. Then we have the change of variable formula for Lebesgue integral,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx .$$

Theorem 3. Suppose g: $[a, b] \to \mathbf{R}$ is of bounded variation on $[a, b]$ and $f : [c, d] \to \mathbf{R}$ is a Lebesgue integrable function such that the range of g is contained in $[c, d]$ and $F \circ g$ is absolutely continuous on $[a, b]$, where F is defined as above. Then we have the change of variable formula for Lebesgue integral,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx .$$

Observe that in all three theorems, the absolute continuity of $F \circ g$ is a necessary condition.

Proof of Theorem 2 and Theorem 3 can be found in the article by James Serrin and Dale Varberg, *A General Chain Rule for Derivatives and the Change of Variables Formula for the Lebesgue Integral*, *The American Mathematical Monthly, vol 76 (1969) 514-520.* (Both can be deduced from Theorem 3 there.) K.G. Johnson in *Discontinuous Functions of Bounded Variation and a New Change of Variable Theorem For a Lebesgue Integral, Duke Math J vol 36 (1969) 117-124*, gave a different proof of Theorem 3 using a slight generalization of the Banach Indicatrix function.

When we apply these theorems to Riemann integral, then the functions f and $(f \circ g)\, g'$ *must necessarily be bounded.*

We state Kestelman version of the change of variable theorem for Riemann integral.

Theorem 4 (Kestelman). Suppose h is Riemann integrable on the closed and bounded interval $[a, b]$ and H: $[a, b] \to \mathbf{R}$ is an indefinite integral of h, i.e., $H(x) = H(a) + \int_a^x h(t)dt$ for x in $[a, b]$. Suppose f is a function Riemann integrable on $H([a, b])$. Then we have the change of variable formula for Riemann integral,

$$\int_a^b f(H(x))h(x)dx = \int_{H(a)}^{H(b)} f(x)dx \,. \qquad \text{--------------------------} \quad (*)$$

Proof.
Since H is absolutely continuous and hence continuous, $H([a, b])$ is a closed and bounded interval. Let $H([a, b]) = [c, d]$. If $f: [c, d] \to \mathbf{R}$ is Riemann integrable and therefore bounded, then the indefinite integral F of f is absolutely continuous and Lipschitz and so $F \circ H$ is absolutely continuous on $[a, b]$. Therefore, by Theorem 1, we have the following equality as Lebesgue integrals.

$$\int_a^b f(H(x))h(x)dx = \int_{H(a)}^{H(b)} f(x)dx \,. \qquad \text{-----------------------------} \quad (A)$$

The right hand side of (A) is also a Riemann integral. Thus, it is sufficient to show that the integral on the left hand side is also a Riemann integral. Note here that Theorem 2 and 3 also give the same conclusion since $F \circ H$ is absolutely continuous and so is of bounded variation and also that $(F \circ H)' = (f \circ H)\, h$ almost everywhere on $[a, b]$. Kestelman gave a proof of Theorem 4 without using Lebesgue integral but Riemann integral and only the concept of a null set or a set of measure zero. However, we shall simply use (A) and show that $(f \circ H)\, h$ is Riemann integrable by showing that its set of discontinuity is of measure zero.

That f is Riemann integrable on $H([a, b]) = [c, d]$ implies, by Lebesgue Theorem, *that* f is continuous on $[c, d]$ except on a set E of measure zero in $[c, d]$. Since H is continuous on $[a, b]$, it follows that $f \circ H$ is continuous on $[a, b] - H^{-1}(E)$. Again, by Lebesgue Theorem, h is continuous except on a subset A of measure zero in $[a, b]$ since h is Riemann integrable on $[a, b]$.

Moreover, $H'(x) = h(x)$ for x not in A. Hence, $(f \circ H)\, h$ is continuous except on the subset $H^{-1}(E) \cup A$. Let $K = \{x \in [a, b]: H'(x) \neq 0\}$. Then by Proposition 30 of *Change of Variable in Riemann and Lebesgue integration,* the measure of $H^{-1}(E) \cap K$ is zero.

We now claim that $(f \circ H)\, h$ is continuous on $H^{-1}(E) - (A \cup K) = (H^{-1}(E) - A) \cap (H^{-1}(E) - K)$. This is deduced as follows. Take any x in $H^{-1}(E) - (A \cup K)$, then x is in $H^{-1}(E) - A$ and *so* h is continuous at x. As x is also in $H^{-1}(E) - K$, $H'(x) = h(x) = 0$. Since $f \circ H$ is bounded, it follows that $(f \circ H)\, h$ is continuous at x. A quick way to see this is to take a sequence (x_n) such that $x_n \to x$. Since h is continuous at x, $h(x_n) \to h(x) = 0$. Note that $(f \circ H(x_n))$ is a bounded sequence and so $f \circ H(x_n).\, h(x_n) \to 0 = f \circ H(x).\, h(x)$. This means that $(f \circ H)\, h$ is continuous at x.

Since A and $H^{-1}(E) \cap K$ are of measure zero, the measure $m((H^{-1}(E) \cap K) \cup A)) = 0$. Thus, as $(f \circ H)\, h$ is continuous on $H^{-1}(E) - (A \cup K)$ and outside $H^{-1}(E) \cup A$, $(f \circ H)\, h$ is continuous except perhaps on $H^{-1}(E) \cap (K \cup A)) \cup A = H^{-1}(E) \cap K \cup A$, which is a set of measure zero. It follows that $(f \circ H)\, h$ is Riemann integrable and so the change of variable formula (*) follows.