Intermediate Value Theorem for the Derived Function (Darboux's Theorem)

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The intermediate value theorem for continuous function is a very useful result. A lesser known result about differentiable function is that its derived function also has the intermediate value property.

Theorem 1. Let I be an open interval and suppose $f: I \to \mathbf{R}$ is a differentiable function. Let a, b be two points in I such that $a < b$. Suppose $f'(a) \neq f'(b)$. Then for any value γ strictly between $f'(a)$ and $f'(b)$, there is a point c in (a, b) such that $f'(a) = \gamma$.

Proof. Let us define the following function $g: I \to \mathbf{R}$ by $g(x) = f(x) - \gamma x$ for x in I. Then g is differentiable and $g'(x) = f'(x) - \gamma$. If we can show that g has either a relative maximum or a relative minimum at a point c in (a, b), then we are done. Consider the function $h : [a, b] \to \mathbf{R}$, the restriction of g to the closed interval, $[a, b]$. Then since g is differentiable, g is also continuous on $[a, b]$. Therefore, by the Extreme Value Theorem, the restriction of g, h attains both its maximum and minimum in $[a, b]$. We shall show that at least one of the absolute maximum or absolute minimum occurs in the interior of $[a, b]$. Suppose $h(a)$ is the maximum and $h(b)$ is the minimum. Then for all x in $[a, b]$, $h(x) \leq h(a)$ and $h(x) \geq h(b)$. Hence for all x in $[a, b]$, $f(x) - \gamma x \leq f(a) - \gamma a$, *that is,* $f(x) - f(a) \leq \gamma x - \gamma a$. Therefore, for all x in $[a, b]$ and $x \neq a$, $(f(x) - f(a))/(x - a) \leq \gamma$. Since f is differentiable at a, $f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq \gamma$. Also we have for all x in $[a, b]$, $f(x) - \gamma x \geq f(b) - \gamma b$, *that is,* $f(b) - f(x) \leq \gamma b - \gamma x$. Consequently, for all x in $[a, b)$, $(f(b) - f(x))/(b - x) \leq \gamma$ *since* $b - x >$ *for* $x < b$. Since f is differentiable at b, $f'(b) = \lim_{x \to b^-} \frac{f(b) - f(x)}{b - x} \leq \gamma$. Therefore, we can conclude that $f'(a)$ and $f'(b)$ are both less than or equal to γ, contradicting that γ is strictly between $f'(a)$ and $f'(b)$. Similarly, if $h(a)$ is the minimum and $h(b)$ is the maximum, we can show in like manner that $f'(a)$ and $f'(b)$ are both greater than or equal to γ, giving a contradiction to that γ is strictly between $f'(a)$ and $f'(b)$. We have thus shown that one of the maximum or minimum of h must occur at *a point* c in (a, b). Since $h(c)$ is also a relative extremum and h is differentiable, $h'(c) = f'(c) - \gamma = 0$ and so $f'(c) = \gamma$. (See Theorem 6.1.2, *Calculus, an introduction* page 82.) This completes the proof.

Corollary 2. Let I be an open interval and suppose $f: I \to \mathbf{R}$ is a differentiable function. Then the image of the derived function of f, $f'(I)$ is also an interval.

Proof. The proof is similar to Theorem 4.6.12 of *Calculus, an introduction, page 53* that is the continuous image of an interval is an interval. Let $f'(a) \neq f'(b)$ be in $f'(I)$. We may assume that $f'(a) < f'(b)$. Theorem 1 says that for any γ such that $f'(a) < \gamma < f'(b)$, $\gamma \in f'(I)$. Hence the interval $[f'(a), f'(b)] \subseteq f'(I)$. Therefore, by the usual characterisation of an interval, $f'(I)$ is an interval.

Remark. In the proof of Theorem 1 above, only the right derivative of f at $x = a$ and the left derivative of f at b is used. Thereofore, if the domain of f is an interval I not necessarily open and if the derivative of f is appropriately defined as the right derivative or left derivative at the end point or end points of its domain I, then Theorem 1 holds true when I is just any interval. Consequently, Corollary 2 is also true when the domain of f, I, is replaced by any interval, not necessarily open.