Application of Riemann Integral
By Ng Tze Beng

Part 1. Arc Length

Take a function $f : [a, b] \to \mathbf{R}$ defined on a closed and bounded interval $[a, b]$. Suppose f is continuous on $[a, b]$. Then the graph of the function f is a curve in $\mathbf{R}^2$ and is usually said to be given by the equation $y = f(x)$. What is the length of this curve? We can consider an estimate of the length by taking points on this curve and take the length of a polygonal curve passing through theses points. To do this we take a partition $\Delta : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ for the interval $[a, b]$. Let P_0 be the point $(x_0, f(x_0))$ and $P_i = (x_i, f(x_i))$. Then the length of the polygonal curve $P_0P_1\ldots P_n$ is an approximation of the arc length P_0P_n.

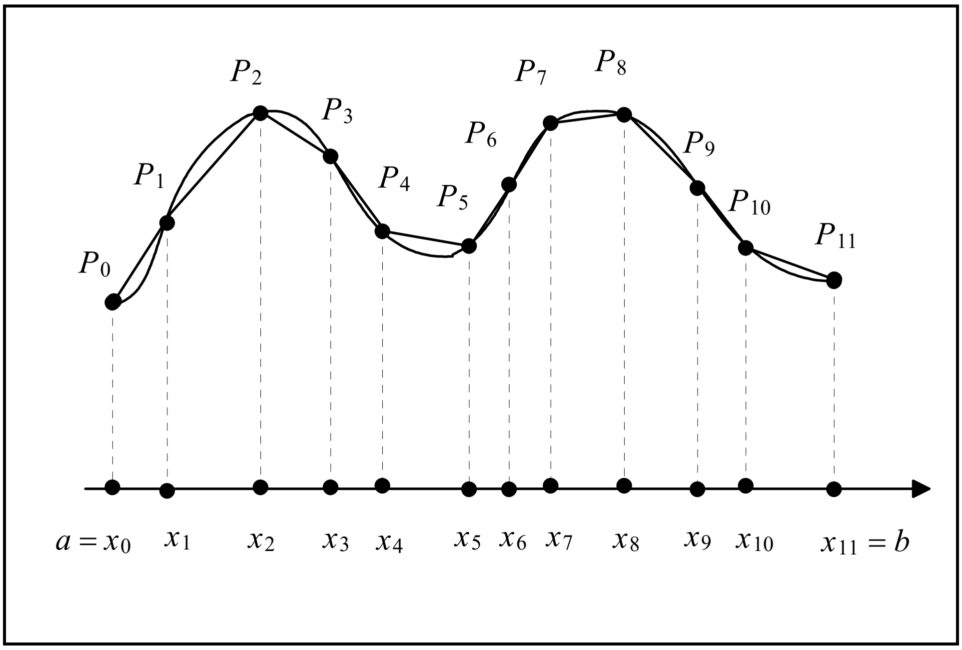

Figure 1. Approximation by polygonal curve, $n = 11$

The length of the polygonal curve $P_0P_1\ldots P_n$ is given by

$$|P_0P_1| + |P_1P_2| + \cdots + |P_{n-1}P_n| \quad \text{or} \quad \sum_{i=1}^{n} |P_{i-1}P_i|.$$

Now, the length of each line segment $|P_{i-1}P_i|$ is the length of the line joining $(x_{i-1}, f(x_{i-1}))$ to $(x_i, f(x_i))$. Thus, by the Pythagorean Theorem,

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}\ .$$

Therefore, the length of the polygonal curve $P_0P_1\ldots P_n$ is given by

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \qquad \text{------------------} \qquad (1)$$

We define the arc length of the curve given by $y = f(x)$, for $a \le x \le b$ to be the limit of all possible polygonal approximation as given by (1), if it exists. If it exists, the curve is called a rectifiable curve, otherwise it is not. Not all continuous curves on a closed and bounded domain are rectifiable.
We may thus write the arc length as

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \qquad \text{-----------------} \qquad (2)$$

Here in (2) as usual, $||\Delta||$ denotes the norm of the partition Δ, i.e., the maximum of all the lengths of the subintervals $[x_{i-1}, x_i]$ as defined by the partition Δ.

The limit (2) does not seem to be easily computable or at all convenient as a calculable process. We shall rewrite (2) under additional condition, in a form that we can apply an integral formula. That is, we shall write (1), if possible as a Riemann sum.

Now for each i,

$$\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} = \sqrt{\left(1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2\right)}\,(x_i - x_{i-1})$$

Suppose now f is also differentiable on (a, b), then by the Mean Value Theorem, for each $1 \le i \le n$ there exists η_i in (x_{i-1}, x_i) such that

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(\eta_i)$$

because f is continuous on $[x_{i-1}, x_i]$ and differentiable on (x_{i-1}, x_i). Thus, the length of the polygonal curve $P_0P_1\ldots P_n$ is given by

$$\sum_{i=1}^{n} \sqrt{1 + (f'(\eta_i))^2}\,(x_i - x_{i-1}) = \sum_{i=1}^{n} \sqrt{1 + (f'(\eta_i))^2}\,\Delta x_i \,. \quad \text{-------------------} \quad (4)$$

The expression (4) is then a Riemann sum for the function $\sqrt{1 + (f'(x))^2}$ with respect to the partition $\Delta : a = x_0 < x_1 < x_2 < \ldots < x_n = b$. Therefore, if $\sqrt{1 + (f'(x))^2}$ is Riemann integrable on $[a, b]$, then the limit (2) is then the Riemann integral

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx. \qquad \text{-------------------------------------} \quad (5)$$

If $f'(x)$ is defined and Riemann integrable on $[a, b]$, then $\sqrt{1 + (f'(x))^2}$ is Riemann integrable on $[a, b]$ and the arc length is given by the integral formula (5).

Remark.

1. Limit (2) exists, if and only if, there exists a constant $K > 0$ such that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le K$$

for any partition $\Delta : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ for $[a, b]$.

Observe that we have the following inequality,

$$\begin{aligned}\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| &\le \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \le \sum_{i=1}^{n} |x_i - x_{i-1}| + \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &\le b - a + \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.\end{aligned}$$

If the limit (2) exists, then by the first part of the inequality, $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$ is bounded by $\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$. Conversely, if $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le K$ for any partition $\Delta : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ for $[a, b]$, by the second part of the inequality, $\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$ is bounded by $K + b - a$ and so the limit exists.

This condition is the definition of f being of ***bounded variation*** on $[a, b]$. (Ref: my article on the Calculus web, *Monotone function, Function of bounded variation and the Fundamental Theorem of Calculus.*)

2. If $f'(x)$ is defined and $(f'(x))^2$ is Riemann integrable on $[a, b]$, then $\sqrt{1+(f'(x))^2}$ is Riemann integrable on $[a, b]$. (Ref: Theorem 2 in my article on Calculus web, Composition and Riemann integrability.) In this case the arc length is given by the integral formula (5). This is true, in particular, when f' is defined and continuous on $[a, b]$.

3. If f is ***absolutely continuous*** on $[a, b]$, even though $f'(x)$ may not be defined everywhere on $[a, b]$, the arc length formula (5) still holds but with Riemann integral replaced by Lebesgue integral. (For the definition of absolute continuity, see my article "*Change of Variable or substitution in Riemann and Lebesgue Integration*"). The definition of Lebesgue integral is more advanced and a good reference will be the book "*Real analysis*" by Royden. It is enough to note that some Lebesgue integral is given by an improper Riemann integral. (see Example 2 below). Also, if f is differentiable everywhere on $[a, b]$ and f' is Lebesgue integrable, then f is absolutely continuous on $[a, b]$.

4. In general, if f is continuous and of *bounded variation*, then the arc length is given by the arc length formula (5), with Riemann integral replaced by Lebesgue integral plus another term which is the total variation of a ***singular*** function g associated with f. Indeed g is given by the function $f(x) - F(x)$, where $F(x) = \int_a^x f'(x)dx$ and the integral here is the Lebesgue integral.

Example 1. Let $f(x) = x^2$. Then the graph of f is the parabola with equation $y = x^2$. The length L of the part of the parabola from (0,0) to (1,1) is given by the Riemann integral,

$$L = \int_0^1 \sqrt{1+(f'(x))^2}\, dx = \int_0^1 \sqrt{1+4x^2}\, dx .$$

Now
$$\begin{aligned}\int \sqrt{1+4x^2}\, dx &= \int \sqrt{1+\tan^2(\theta)}\, dx, \text{ where } x = \frac{1}{2}\tan(\theta),\\ &= \int \sqrt{1+\tan^2(\theta)}\,\frac{1}{2}\sec^2(\theta)d\theta = \frac{1}{2}\int \sec^3(\theta)d\theta\\ &= \frac{1}{4}(\tan(\theta)\sec(\theta) + \ln|\tan(\theta)+\sec(\theta)|) + C\\ &= \frac{1}{4}(2x\sqrt{1+4x^2} + \ln|2x+\sqrt{1+4x^2}|) + C.\end{aligned}$$

Therefore, $L = \int_0^1 \sqrt{1+4x^2}\, dx = \left[\frac{1}{4}(2x\sqrt{1+4x^2} + \ln|2x+\sqrt{1+4x^2}|)\right]_0^1 = \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(2+\sqrt{5})$.
Note that in this example, the derived function f' is continuous on [0, 1].

Example 2. Consider the unit circle defined by $x^2 + y^2 = 1$. The arc length L of the minor arc from (0,1) to (1,0) is a quarter of the circumference of the circle. We shall calculate this using our integral formula. Let $f(x) = \sqrt{1-x^2}$. Then the length of L is given by the integral,

$$L = \int_0^1 \sqrt{1+(f'(x))^2}\, dx \quad , \text{where } f'(x) = \frac{-x}{\sqrt{1-x^2}}.$$

Note that $f'(x)$ is not defined at $x = 1$. The integral above is an improper integral. The arc length is thus given by

$$\begin{aligned}L &= \int_0^1 \sqrt{1+\frac{x^2}{1-x^2}}\, dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}dx\\ &= \lim_{t\to 1^-}[\sin^{-1}(x)]_0^t = \lim_{t\to 1^-}\sin^{-1}(t) = \sin^{-1}(1) = \frac{\pi}{2}.\end{aligned}$$

An explanation is in order here. We can also think of the minor arc from (0,1) to (1,0), as the limit of the arc from (0, 1) to $(t, \sqrt{1-t^2})$ as t tends to 1 from the left. Thus the arc length L is then the limit of the arc length of the minor arc from (0, 1) to $(t, \sqrt{1-t^2})$ which is given by $\int_0^t \frac{1}{\sqrt{1-x^2}}dx$ since f' is continuous on $[0, t]$ for $0 < t < 1$. (See Remark 2.)

Note also that f' is Lebesgue integrable on [0, 1] and f can be expressed as an indefinite (Lebesgue) integral of f' and so f is absolutely continuous and by remark 3, the arc length is given by the integral formula (5) involving the Lebesgue integral of $\frac{1}{\sqrt{1-x^2}}$ on [0, 1], which is the improper integral of $\frac{1}{\sqrt{1-x^2}}$ on [0,1] and equals $\frac{\pi}{2}$.

Part 2. Volume of solid of revolution.

Suppose $f: [a, b] \to \mathbf{R}$ is defined on a closed and bounded interval $[a, b]$. We take the region bounded by the curve $y = f(x)$, the x-axis , the lines $x = a$ and $x = b$ and form the solid obtained by revolving this region about the x-axis. This solid is called the solid of revolution.

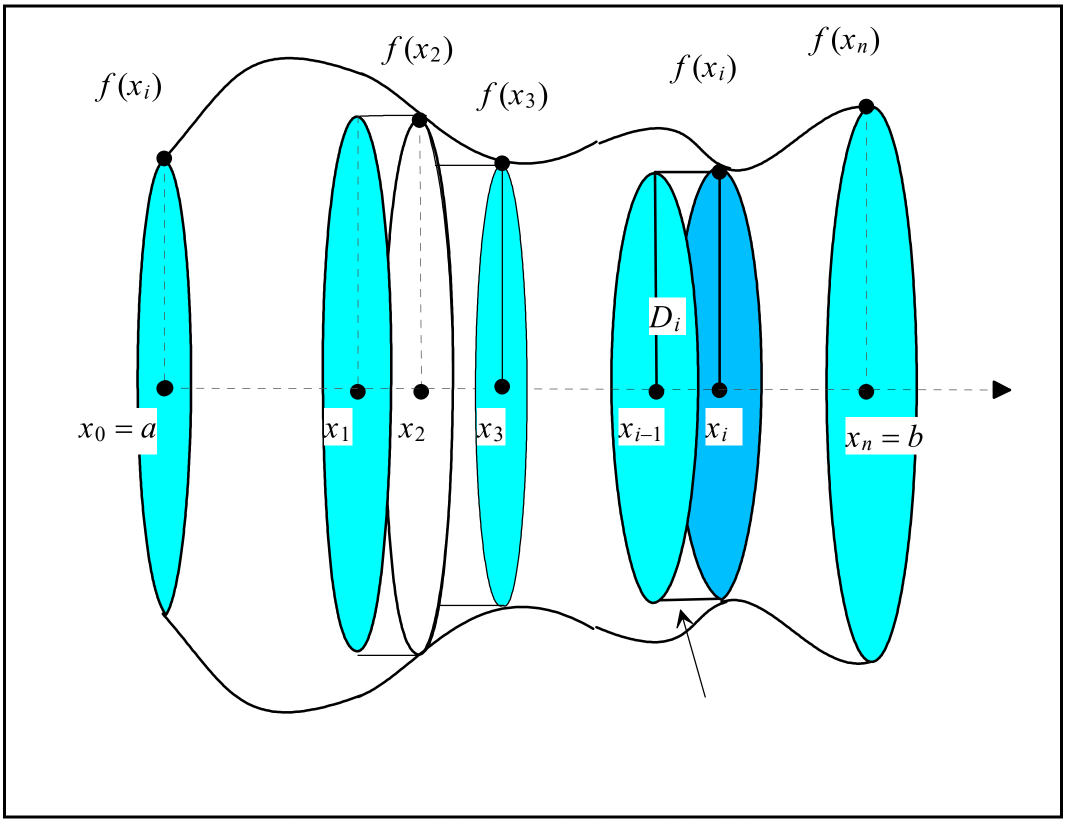

Figure 2 Solid of revolution with disks

As in part 1, we take a partition $\Delta : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ for the interval $[a, b]$. For $1 \le i \le n$, consider the solid disk D_i or cylinder formed by revolving the rectangular area of width $(x_i - x_{i-1})$ and length $|f(x_i)|$. This is a circular disk or solid cylinder of radius $|f(x_i)|$ and thickness $(x_i - x_{i-1})$. Therefore, the volume of this disk D_i is $\pi\,(f(x_i))^2(x_i - x_{i-1})$. Hence, an approximation to the volume of the solid of revolution is the sum of the volumes of these solid disks D_i ,$1 \le i \le n$. This is

$$\pi\,(f(x_1))^2(x_1 - x_0) + \pi\,(f(x_2))^2(x_2 - x_1) + \cdots + \pi\,(f(x_n))^2(x_n - x_{n-1})$$
$$= \sum_{i=1}^{n} \pi\,(f(x_i))^2(x_i - x_{i-1}) = \sum_{i=1}^{n} \pi\,(f(x_i))^2 \Delta x_i .$$

Therefore, the volume of the solid of revolution is the limit,

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\,(f(x_i))^2 \Delta x_i \,. \quad \text{-------------------------} \quad (6)$$

Here as usual $||\Delta||$ denotes the norm of the partition Δ, i.e., the maximum of all the lengths of the subintervals $[x_{i-1}, x_i]$ as defined by the partition Δ. Note that $\sum_{i=1}^{n} \pi\,(f(x_i))^2 \Delta x_i$ is a Riemann sum for the function $\pi\,(f(x))^2$ with respect to the partition Δ. It follows that if $(f(x))^2$ is Riemann integrable on $[a, b]$, then the volume of the solid of revolution is given by the Riemann integral,

$$\int_a^b \pi(f(x))^2 dx\,. \quad \text{------------------------------} \quad (7).$$

Of course, if $f(x)$ is Riemann integrable on $[a, b]$, then $(f(x))^2$ is Riemann integrable on $[a, b]$. In particular, if f is continuous on $[a, b]$, it is then Riemann integrable on $[a, b]$ and so the volume of the solid of revolution is given by (7).

Example 3. Let $y = x^2$ be the parabola. Then the volume of the solid of revolution obtained by rotating the region bounded by the parabola, the x-axis and the line $x = 1$ is given by

$$\int_a^b \pi(f(x))^2 dx\,, \text{ where } f(x) = x^2\,,\ a = 0, b = 1,$$

$$= \int_0^1 \pi x^4 dx = \pi\left[\frac{1}{5}x^5\right]_0^1 = \frac{\pi}{5}\,.$$

Part 3. Area of surface of revolution.

Suppose $f: [a, b] \to \mathbf{R}$ is defined on a closed and bounded interval $[a, b]$. Suppose f is a non-negative function. Then the surface of revolution is obtained by rotating the graph of f or $y = f(x)$ about the x-axis. Take a partition $\Delta : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ for the interval $[a, b]$. We can approximate the area of this surface of revolution by the sum total of the area of bands B_i obtained by rotating the line segment joining $(x_{i-1}, f(x_{i-1}))$ to $(x_i, f(x_i))$ for $i = 1$ to $i = n$. Each of these bands is a frustum of a cone. We shall first determine the area of a frustum of a cone before we examine the approximation.

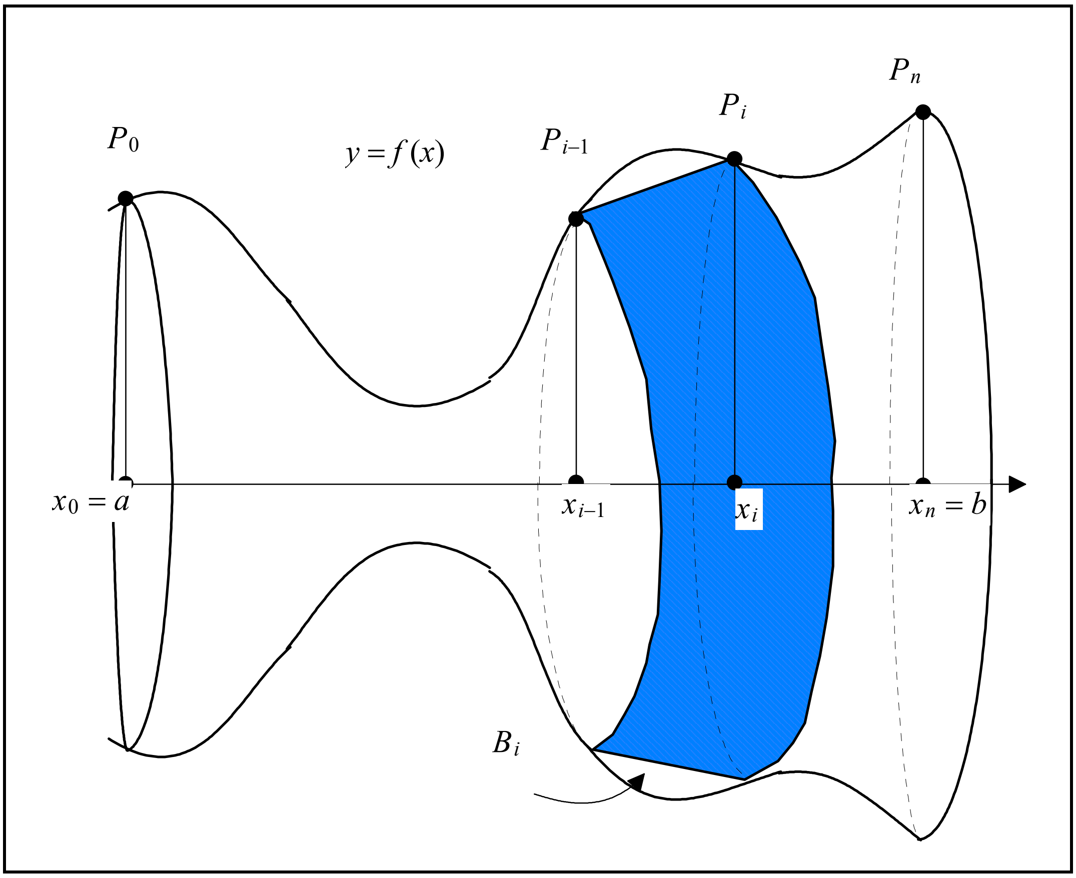

Figure 3. Surface of Revolution with band B_i shown

Figure 4 shows a cone with a band B. If we slice the cone along a slant edge VQ we get the band in a sector of a disk with radius VQ which is ($l + l_2$) and subtended by an angle θ as shown in Figure 5.

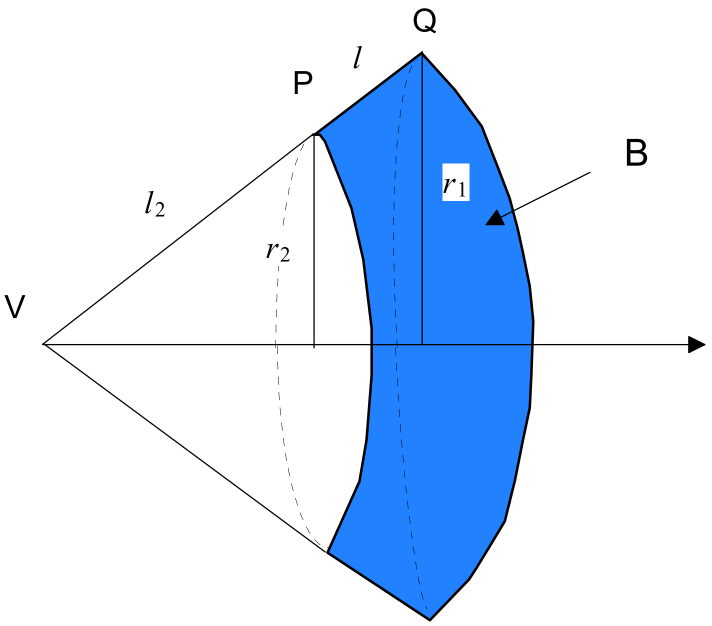

Figure 4 Cone with frustum shown

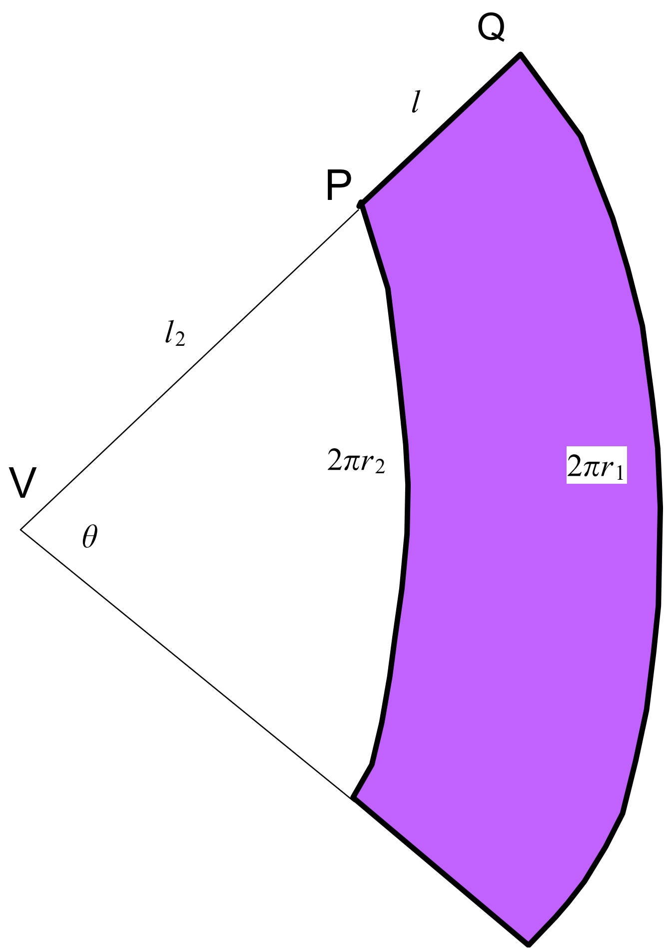

Figure 5

Here the angle $\theta = \dfrac{2\pi r_1}{l+l_2} = \dfrac{2\pi r_2}{l_2}$ and so $l_2 r_1 = r_2(l+l_2)$. Therefore, the area of the band B is given by

$$\frac{\theta}{2\pi}(\pi(l+l_2)^2 - \pi l_2^2) = \pi(l+l_2)r_1 - \pi l_2 r_2 = \pi(lr_1 + r_2(l+l_2)) - l_2 r_2) = \pi l(r_1 + r_2). \text{ ------ } (8)$$

Now we apply formula (8) to our band B_i determined by rotating $P_{i-1}P_i$ about the x-axis, with $r_1 = f(x_i)$, $r_2 = f(x_{i-1})$ and $l = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$. We have that the area of B_i. is given by

$$\pi(f(x_i) + f(x_{i-1}))\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} .$$

Thus, assuming that $f(x) \geq 0$ on $[a, b]$, an approximation to the area of the surface of revolution is,

$$\sum_{i=1}^{n} \text{area of } B_i = \sum_{i=1}^{n} \pi(f(x_i) + f(x_{i-1}))\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$= \sum_{i=1}^{n} \pi(f(x_i) + f(x_{i-1}))\sqrt{\left(1 + (\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}})^2\right)}(x_i - x_{i-1}) \text{ . ----------------- } (9)$$

Suppose f is differentiable on (a, b). Then by the Mean Value Theorem, for each $1 \leq i \leq n$, there exists η_i in (x_{i-1}, x_i) such that

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(\eta_i).$$

And so (9) becomes

$$\sum_{i=1}^{n} \pi(f(x_i) + f(x_{i-1}))\sqrt{(1 + (f'(\eta_i))^2} \, \Delta x_i . \quad \text{-------------------- } (10)$$

If f is continuous and $\sqrt{(1 + (f'(x))^2}$ is Riemann integrable on $[a, b]$, (10) may be "approximated" by

$$\sum_{i=1}^{n} 2\pi f(\eta_i)\sqrt{(1 + (f'(\eta_i))^2} \, \Delta x_i . \text{ ---------------------------- } (11)$$

Thus, (11) is a Riemann sum for the function $2\pi f(x)\sqrt{(1 + (f'(x))^2}$. Since, by assumption, both $f(x)$ and $\sqrt{(1 + (f'(x))^2}$ are Riemann integrable, the function $2\pi f(x)\sqrt{(1 + (f'(x))^2}$ is Riemann integrable. Therefore,

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi f(\eta_i)\sqrt{(1 + (f'(\eta_i))^2} \, \Delta x_i = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2} \, dx .$$

Hence, if we have that $f(x)$ is continuous and non-negative on $[a, b]$ and if either f' is defined and continuous on $[a, b]$ or f' is Riemann integrable on $[a, b]$, then the area of the surface of revolution obtained by rotating the curve $y = f(x)$ about the x-axis is given by

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2} \, dx. \text{ ------------------------- } (12)$$

We shall explain after the following examples what we meant by (10) can be approximated by (11).

Example 4. Find the area of the surface of revolution obtained by rotating the curve $y = x^2$, $0 \leq x \leq 1$, about the x-axis.
The function is $f(x) = x^2$. Obviously, f is continuous on [0, 1], f is differentiable on [0,1] and $f'(x) = 2x$ is continuous on [0, 1]. Therefore, the surface area is given by (12) as,

$$S = \int_0^1 2\pi f(x)\sqrt{1 + (f'(x))^2} \, dx = 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2} \, dx . \text{ ----------------- } (13)$$

Thus, we need to determine the Riemann integral (13).
Note that

$$\int x^2\sqrt{1+4x^2}\,dx = \frac{1}{8}\int \tan^2(\theta)\sec^3(\theta)d\theta, \text{ where } 2x = \tan(\theta).$$

Now by integration by parts, it can be shown that

$$\begin{aligned}\int \tan^2(\theta)\sec^3(\theta)d\theta &= \int \tan(\theta)\sec^2(\theta)(\sec(\theta)\tan(\theta))d\theta \\ &= \tan(\theta)\sec^3(\theta) - \int \sec(\theta)(\sec^4(\theta)+2\tan^2(\theta)\sec^2(\theta))d\theta \\ &= \tan(\theta)\sec^3(\theta) - \int \sec^5(\theta)d\theta - 2\int \tan^2(\theta)\sec^3(\theta))d\theta \\ &= \tan(\theta)\sec^3(\theta) - \int \sec^3(\theta)(1+\tan^2(\theta)d\theta - 2\int \tan^2(\theta)\sec^3(\theta))d\theta \\ &= \tan(\theta)\sec^3(\theta) - \int \sec^3(\theta)d\theta - 3\int \tan^2(\theta)\sec^3(\theta))d\theta \\ &- \tan(\theta)\sec^3(\theta) - \int \sec^3(\theta)d\theta - 3\int \tan^2(\theta)\sec^3(\theta))d\theta \\ &= \tan(\theta)\sec^3(\theta) - \frac{1}{2}\sec(\theta)\tan(\theta) - \frac{1}{2}\ln|\tan(\theta)+\sec(\theta)| - 3\int \tan^2(\theta)\sec^3(\theta))d\theta.\end{aligned}$$

Therefore,

$$\int \tan^2(\theta)\sec^3(\theta)d\theta = \frac{1}{4}\tan(\theta)\sec^3(\theta) - \frac{1}{8}\sec(\theta)\tan(\theta) - \frac{1}{8}\ln|\tan(\theta)+\sec(\theta)| + C.$$

Thus,

$$\begin{aligned}\int x^2\sqrt{1+4x^2}\,dx &= \frac{1}{32}\tan(\theta)\sec^3(\theta) - \frac{1}{64}\sec(\theta)\tan(\theta) - \frac{1}{64}\ln|\tan(\theta)+\sec(\theta)| + C' \\ &= \frac{1}{16}x(1+4x^2)\sqrt{1+4x^2} - \frac{1}{32}x\sqrt{1+4x^2} - \frac{1}{64}\ln|2x+\sqrt{1+4x^2}\,| + C' \\ &= \frac{1}{4}x^3\sqrt{1+4x^2} + \frac{1}{32}x\sqrt{1+4x^2} - \frac{1}{64}\ln|2x+\sqrt{1+4x^2}\,| + C'.\end{aligned}$$

Hence, the surface area,

$$\begin{aligned}S &= 2\pi\int_0^1 x^2\sqrt{1+4x^2}\,dx \\ &= 2\pi\left[\frac{1}{4}x^3\sqrt{1+4x^2} + \frac{1}{32}x\sqrt{1+4x^2} - \frac{1}{64}\ln|2x+\sqrt{1+4x^2}\,|\right]_0^1 \\ &= \left(\frac{9}{16}\sqrt{5} - \frac{1}{32}\ln(2+\sqrt{5}\,)\right)\pi.\end{aligned}$$

Example 5. Find the area of the surface of revolution obtained by rotating the curve $y = \sqrt{1-x^2}$, $0 \le x \le 1$, about the x-axis.
Let $f(x) = \sqrt{1-x^2}$. Then $f'(x) = \frac{-x}{\sqrt{1-x^2}}$ for $0 \le x < 1$ and f is not differentiable at $x = 1$. Then the surface area is the left limit of the surface area S_t of that part of curve with $0 \le x < t$ rotated 2π radians about the x-axis, as t tends to 1 on the left.
By formula (12), S_t is given as below:

$$\begin{aligned}S_t &= \int_0^t 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi\int_0^t \sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx \\ &= 2\pi\int_0^t dx = 2\pi t.\end{aligned}$$

Therefore, the surface area is $\lim_{t\to 1^-} S_t = \lim_{t\to 1^-} 2\pi t = 2\pi$.
As a consequence the surface are of the unit ball is 4π.

Remark. Why (10) can be "approximated" by (11).

We shall assume that f is continuous and non-negative on $[a, b]$, f is differentiable on $[a, b]$ and f' is Riemann integrable on $[a, b]$. Thus, it follows that $\sqrt{(1+(f'(x))^2}$ is Riemann integrable on $[a, b]$. The curve $y = f(x)$, $a \le x \le b$, is rectifiable and the arc length is given by $L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$. Note that since f is continuous on $[a, b]$, it is uniformly continuous on $[a, b]$. Hence, we have for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|x-y| < \delta_1 \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{8(1+L)}. \qquad \text{-------------} \quad (14)$$

Thus, for each i from 1 to n, if $\|\Delta\| < \delta_1$, we have then that

$$|f(x_i) - f(\eta_i)| < \frac{\varepsilon}{8(1+L)}$$

and also that $|f(x_{i-1}) - f(\eta_i)| < \frac{\varepsilon}{8(1+L)}$. This is because $|x_i - \eta_i| \le |x_i - x_{i-1}| \le \|\Delta\| < \delta_1$ and that $|x_{i-1} - \eta_i| \le |x_i - x_{i-1}| \le \|\Delta\| < \delta_1$. It follows that

$$2f(\eta_i) - \frac{\varepsilon}{4(1+L)} < f(x_i) + f(x_{i-1}) < 2f(\eta_i) + \frac{\varepsilon}{4(1+L)} . \quad \text{----------- (15)}$$

Therefore,

$$\begin{aligned} &2f(\eta_i)\sqrt{(1+(f'(\eta_i))^2} - \frac{\varepsilon}{4(1+L)}\sqrt{(1+(f'(\eta_i))^2} \\ &< (f(x_i) + f(x_{i-1}))\sqrt{(1+(f'(\eta_i))^2} \\ &< 2f(\eta_i)\sqrt{(1+(f'(\eta_i))^2} + \frac{\varepsilon}{4(1+L)}\sqrt{(1+(f'(\eta_i))^2} . \end{aligned}$$

And so we have,

$$\begin{aligned} &2\sum_{i=1}^{n} f(\eta_i)\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i - \frac{\varepsilon}{4(1+L)}\sum_{i=1}^{n}\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i \\ &< \sum_{i=1}^{n}(f(x_i) + f(x_{i-1}))\sqrt{(1+(f'(\eta_i))^2} \\ &< 2\sum_{i=1}^{n} f(\eta_i)\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i + \frac{\varepsilon}{4(1+L)}\sum_{i=1}^{n}\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i . \quad \text{---------------- (16)} \end{aligned}$$

Since $L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$, there exists $\delta_2 > 0$ such that

$$\|\Delta\| < \delta_2 \Rightarrow \frac{L}{2} = L - \frac{L}{2} < \sum_{i-1}^{n}\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i < L + \frac{L}{2} = \frac{3L}{2} .$$

Therefore, $$\frac{\varepsilon}{4(1+L)}\sum_{i-1}^{n}\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i < \frac{3L\varepsilon}{8(1+L)} < \frac{\varepsilon}{2} . \quad \text{--------------------- (17)}$$

Thus, if $\|\Delta\| < \delta = \min(\delta_1, \delta_2)$, we have then from (16) and (17) that

$$\begin{aligned} 2\sum_{i-1}^{n} f(\eta_i)\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i - \frac{\varepsilon}{2} &< \sum_{i-1}^{n}(f(x_i) + f(x_{i-1}))\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i \\ &< 2\sum_{i-1}^{n} f(\eta_i)\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i + \frac{\varepsilon}{2} . \quad \text{---------------- (18)} \end{aligned}$$

This means that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition $\Delta : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ for $[a, b]$ with $\|\Delta\| < \delta$,

$$\left|\sum_{i-1}^{n}(f(x_i) + f(x_{i-1}))\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i - 2\sum_{i-1}^{n} f(\eta_i)\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i\right| < \frac{\varepsilon}{2}$$

Hence, $\sum_{i-1}^{n}(f(x_i) + f(x_{i-1}))\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i$ can be "approximated" by $2\sum_{i-1}^{n} f(\eta_i)\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i$.
Consequently, $\sum_{i-1}^{n}\pi(f(x_i) + f(x_{i-1}))\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i$ can be "approximated" by $2\pi\sum_{i-1}^{n} f(\eta_i)\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i$.
Now we proceed to show that the integral formula (12) gives the surface area.
By our assumption that f is continuous and differentiable on $[a, b]$ and f' is Riemann integrable on $[a, b]$, the function $2f(x)\sqrt{1+(f'(x))^2}$ is Riemann integrable on $[a, b]$. Hence the integral exists. Suppose

$\int_a^b 2f(x)\sqrt{1+(f'(x))^2}\,dx = M$. Then by the definition of the Riemann integral, for any ε > 0, there exists a $\delta_3 > 0$ such that for any partition $\Delta : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ for $[a, b]$,

$$\|\Delta\| < \delta_3 \Rightarrow M - \frac{\varepsilon}{2} < \sum_{i-1}^{n} 2f(\eta_i)\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i < M + \frac{\varepsilon}{2} \text{ ------------------ } (19)$$

Now we take $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then by (18) and (19), for any partition $\Delta : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ for $[a, b]$,

$$\|\Delta\| < \delta \Rightarrow M - \varepsilon < \sum_{i-1}^{n} (f(x_i) + f(x_{i-1}))\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i < M + \varepsilon,$$

or equivalently,

$$\|\Delta\| < \delta \Rightarrow \left| \sum_{i-1}^{n} (f(x_i) + f(x_{i-1}))\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i - M \right| < \varepsilon.$$

This means

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (f(x_i) + f(x_{i-1}))\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i = M = \int_a^b 2f(x)\sqrt{1+(f'(x))^2}\,dx.$$

Thus, $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi(f(x_i) + f(x_{i-1}))\sqrt{(1+(f'(\eta_i))^2}\,\Delta x_i = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx.$