Integration By Parts
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Integration by parts is a formula often used for computing Riemann integrals or definite integrals of the Lebesgue type. We shall start from the usual antiderivative form of the formula.

Suppose F is an antiderivative of f and G is an antiderivative of g. Then we have that $F' = f$ and $G' = g$. Then the product formula for differentiation gives us that

$$(F\,G)' = F'G + F\,G' = f\,G + F\,g. \quad \text{------------------ (1)}$$

From here we consider additional condition on f and g to give a formula for the the relation between antiderivative of $f\,G$ and that of $F\,g$. At this point if we assume that f and g are continuous then both $f\,G$ and $F\,g$ are continuous. Therefore, $f\,G$ and $F\,g$ have antiderivatives. Hence from (1) we can write

$$f\,G = (F\,G)' - F\,g.$$

Thus, any antiderivative of $f\,G$ is of the form $F\,G$ − antiderivative of $F\,g$. This is seen as follows. Suppose H is an antiderivative of $F\,g$. Then $H' = F\,g = (F\,G)' - f\,G$ so that $(F\,G)' - H' = f\,G$. This means $K = F\,G - H$ is an antiderivative of $f\,G$. On the other hand if K is an antiderivative of $f\,G$, then $K' = f\,G = (F\,G)' - F\,g$ so that $(FG - K)' = (F\,G)' - K' = F\,g$ and so $H = FG - K$ is an antiderivative of $F\,g$. Therefore, $K = FG - H$. Thus, we have

$$\int f(x)G(x)dx = F(x)G(x) - \int F(x)g(x)dx.$$

Since the functions f, G, F and g are all continuous and so by the Fundamental Theorem of Calculus, the Riemann integral,

$$\int_a^b f(x)G(x)dx = [F(x)G(x)]_a^b - \int_a^b F(x)g(x)dx.$$

Another way to derive this is as follows. Since $(F\,G)'$ is continuous and so is Riemann integral, by Darboux Theorem $\int_a^b (F\cdot G)'(x)dx = [F(x)G(x)]_a^b$. But on the other hand the Riemann integral is also given by $\int_a^b f(x)G(x)dx + \int_a^b F(x)g(x)dx$. Thus we have, $\int_a^b f(x)G(x)dx + \int_a^b F(x)g(x)dx = [F(x)G(x)]_a^b$ and it follows that $\int_a^b f(x)G(x)dx = [F(x)G(x)]_a^b - \int_a^b F(x)g(x)dx$. We have thus proved the following theorem.

Theorem 1. Suppose F is an antiderivative of f on $[a, b]$ and G is an antiderivative of g on $[a, b]$. Suppose further that both f and g are continuous on $[a, b]$. Then the following formula for Riemann integral holds for any x in $[a, b]$.

$$\int_a^x f(t)G(t)dt = [F(t)G(t)]_a^x - \int_a^x F(t)g(t)dt.$$

Actually the formula is also true for Lebesgue integrals if we have the equivalent form of the Fundamental Theorem of Calculus for Lebesgue integrals. So some additional condition needs to be put on the functions involved. If the function F is *absolutely continuous* on $[a, b]$, then F is differentiable almost everywhere and the Lebesgue integral $\int_a^x F'(t)dt = F(x) - F(a) = [F(t)]_a^b$ for each x in $[a, b]$. (For the definition of absolute continuity, see the article "Change of Variable or Substitution in Riemann Integration".) The following is a classical Theorem.

Theorem 2. The function F is absolutely continuous on [a, b] if and only if F' is Lebesgue integrable and for each x in [a, b], the Lebesgue integral $\int_a^x F'(t)dt = F(x) - F(a) = [F(t)]_a^x$.

[For a proof see Theorem 29.15 of "Principles of Real Analysis by Aliprantis and Burkinshaw or Theorem 9.4 of "Introduction to Measure and Integration" by S.J. Taylor.]

Remark 3. Some explanation is in order here. If F is absolutely continuous on [a, b], then F is of bounded variation so that F is then the difference of two increasing functions. Since any increasing function is differentiable almost everywhere, F is differentiable almost every where on [a, b]. Hence it makes sense to speak of the Lebesgue integral of F' because F' is defined almost everywhere on [a, b].

Thus, if we assume that both F and G are absolutely continuous, then $F \cdot G$ is also *absolutely continuous*. In particular, $(F \cdot G)' = F'G + F\,G' = f\,G + F\,g$ almost everywhere, where $F' = f$ and $G' = g$ almost everywhere. Then because $F \cdot G$ is absolutely continuous on [a, b], we have then the Lebesgue integral $\int_a^x (F\cdot G)'(t)dt = [F(t)G(t)]_a^b$ for any x in [a, b]. On the other hand since $F' = f$ almost everywhere and F' is Lebesgue integrable, f is also Lebesgue integrable and thus the product $f\,G$ is Lebesgue integrable because G is also continuous. Similarly, we deduce that $F\,g$ is also Lebesgue integrable. Therefore, since $(F \cdot G)' = f\,G + F\,g$ almost everywhere, for any x in [a, b], the Lebesgue integral

$$\int_a^x (F\cdot G)'(t)dt = \int_a^x f(t)G(t)dt + \int_a^x F(t)g(t)dt = [F(t)G(t)]_a^x$$

and so the formula $\int_a^x f(t)G(t)dt = [F(t)G(t)]_a^x - \int_a^x F(t)g(t)dt$ holds for Lebesgue integrals. We have thus proved the following result.

Theorem 4. Suppose F and G are absolutely continuous on [a, b]. Suppose $F' = f$ almost every where and $G' = g$ almost everywhere on [a, b]. Then for any x in [a, b], the following formula for Lebesgue integral holds for any x in [a, b].

$$\int_a^x f(t)G(t)dt = [F(t)G(t)]_a^x - \int_a^x F(t)g(t)dt.$$

When do we know a function is absolutely continuous on [a, b]? Besides checking the definition are there other ways of deciding this? Continuity alone does not necessarily imply absolute continuity nor does differentiability. We need the equivalence of the Darboux Theorem for Lebesgue integral. The following is true but is a hard Theorem in analysis which may be proved using the Vitali-Caratheodory Theorem or approximation of Lebesgue integrable functions by semicontinuous functions.

Theorem 5. Suppose F is differentiable (everywhere) on [a, b] and that the derivative F' is Lebesgue integrable. Then for any x in [a, b] the Lebesgue integral,

$$\int_a^x F'(t)dt = F(x) - F(a) = [F(t)]_a^x.$$

[Ref. Theorem 8.21 in *Real and Complex Analysis* by Walter Rudin.]

Remark 6. 1. Theorem 5 says that if F is differentiable on [a, b] and that the derivative F' is Lebesgue integrable, then F is absolutely continuous. Note that not every differentiable function has a Lebesgue integrable derived function. Indeed there are functions which are every where differentiable on [a, b] but whose derived functions are not Lebesgue integrable. An example is given in Example 2.1.2.5 in *Theorems and Counterexamples in Mathematics* by Gelbaum and Olmsted. Thus, the hypothesis that F' be Lebesgue integrable is necessary. Under the condition of the theorem it is also necessary that F be differentiable everywhere.

For there are functions F differentiable almost everywhere and for which the Lebesgue integral $\int_a^b F'(t)dt < F(b) - F(a)$ so that F is not absolutely continuous on $[a, b]$. Take F to be the Cantor function C_0 which is differentiable almost everywhere on [0, 1] with $C_0{}'(x) = 0$ almost everywhere and $C_0(x) > 0$ for all $0 < x \leq 1$. Then
$0 = \int_0^x C_0'(t)dt < C_0(x) - C_0(0) = C_0(x)$ for $0 < x \leq 1$.

If F is not differentiable everywhere, in addition to F' being Lebesgue integrable, various additional conditions may be imposed to give an absolutely continuous F. (See my article, *"When is a continuous functions on a closed and bounded interval be of bounded variation, absolutely continuous? The answer and application to generalized change of variable for Lebesgue integral*".)

2. Note that if F is differentiable (everywhere) on $[a, b]$ and the derived function F' is bounded on $[a, b]$, then F is absolutely continuous and so F' is Lebesgue integrable. For such a function, Theorem 2 applies to conclude that the Lebesgue integral
$\int_a^x F'(t)dt = F(x) - F(a) = [F(t)]_a^x$ for any x in $[a, b]$.

Then we have the following formulation of a weaker version of Theorem 4.

Theorem 7. Suppose f is Lebesgue integrable and has an antiderivative F on $[a, b]$. Suppose also that g is Lebesgue integrable and has an antiderivative G on $[a, b]$. Then the following formula for the Lebesgue integral holds for each x in $[a, b]$.

$$\int_a^x f(t)G(t)dt = [F(t)G(t)]_a^x - \int_a^x F(t)g(t)dt.$$

Proof. By Theorem 5, F and G are absolutely continuous on $[a, b]$. Therefore, the theorem follows from Theorem 4.

An often used form of the formula is the involvement of indefinite integrals. The following result gives the connection between indefinite integral and absolutely continuous function.

Theorem 8. Suppose f is Lebesgue integrable on $[a, b]$. Then the function F defined on $[a, b]$ by $F(x) = F(a) + \int_a^x f(t)dt$ is absolutely continuous on $[a, b]$, differentiable almost everywhere on $[a, b]$ and $F'(x) = f(x)$ almost everywhere on $[a, b]$.

[For a proof of Theorem 8, refer to Theorem 9.3 of "*Introduction to Measure and Integration*" by S.J. Taylor or Theorem 5.10 and Theorem 5.14 of "*Real Analysis*" by Royden.]

Any function of the form $F(x) = F(a) + \int_a^x f(t)dt$ is called an *indefinite integral* of f. We can recast Theorem 4 as follows.

Theorem 9. Suppose f and g are Lebesgue integrable on $[a, b]$. Suppose F and G are indefinite integrals of f and g respectively on $[a, b]$. Then for any x in $[a, b]$, the following formula for Lebesgue integral holds for any x in $[a, b]$.

$$\int_a^x f(t)G(t)dt = [F(t)G(t)]_a^x - \int_a^x F(t)g(t)dt.$$

Proof. Immediate from Theorem 4 since by Theorem 8, $F' = f$ and $G' = g$ almost everywhere on $[a, b]$.

Remark. Theorem 2 and Theorem 9 gives the following characterization of absolutely continuous function. *Every absolutely continuous function on* $[a, b]$ *is the indefinite integral of its derivative.*

Theorem 7 specialises to the following version for Riemann integrals.

Theorem 10. Suppose f is Riemann integrable and has an antiderivative F on $[a, b]$. Suppose also that g is Riemann integrable and has an antiderivative G on $[a, b]$. Then the following formula for the Riemann integral holds for each x in $[a, b]$.

$$\int_a^x f(t)G(t)dt = [F(t)G(t)]_a^x - \int_a^x F(t)g(t)dt.$$

Proof. Observe that both fG and Fg are Riemann integrable on $[a, b]$. This is because the product of a continuous function and a Riemann integrable function is also Riemann integrable. Therefore, by Theorem 7, the Riemann integral

$$\text{(Riemann integral)} \int_a^x f(t)G(t)dt = \text{(Lebesgue)}\int_a^x f(t)G(t)dt$$
$$= [F(t)G(t)]_a^x - \text{(Lebesgue)}\int_a^x F(t)g(t)dt = [F(t)G(t)]_a^x - \text{(Riemann)}\int_a^x F(t)g(t)dt.$$

If antiderivative is not what you seek or is not readily available, the following version for indefinite integral holds too. This is particularly of use when any one of f or g is Riemann integrable but has no antiderivative.

Theorem 11. Suppose f and g are Riemann integrable on $[a, b]$. Suppose F and G are indefinite integrals of f and g respectively on $[a, b]$. Then for any x in $[a, b]$, the following formula for Riemann integral holds for any x in $[a, b]$.

$$\int_a^x f(t)G(t)dt = [F(t)G(t)]_a^x - \int_a^x F(t)g(t)dt.$$

Proof. Immediate from Theorem 9.

Remark. Theorem 10 can be proved directly using Darboux Theorem since by the product rule, $F(x)G(x)$ there is differentiable and it's derivative is Riemann integrable. Note that in general, indefinite integral need not be differentiable and so Theorem 11 can be considered as a slight generalization of Theorem 10.