## Injectivity and Monotonicity of Continuous Function

Have you ever wonder if a function is injective, then it is (strictly) monotonic. Obviously, if we do not assume that the function is continuous, then a simple piecewise function can be constructed to give a counterexample. If continuity is assumed and the domain is an interval, then the answer is affirmative. That means injectivity and (strict) monotonicity is equivalent.

Let us recall some definition. Let $I$ be an interval.

**Definition 1.** A function $f : I \to \mathbf{R}$ is *injective* if for any $x$, $y$ in $I$, $f(x) = f(y)$ implies that $x = y$.

**Definition 2.** A function $f : I \to \mathbf{R}$ is (*strictly*) *monotonic* if it is either (strictly) increasing or (strictly) decreasing. That means for all $x > y$ in $I$, either $f(x) > f(y)$ or $f(x) < f(y)$.

We state our assertion as follows.

**Theorem 3.** If $I$ is an interval and $f : I \to \mathbf{R}$ is continuous and injective, then $f$ is (strictly) monotonic.

First we shall prove the following observation. For the ease of exposition, we shall use 'monotonic' interchangebly with 'strictly monotonic' and 'increasing' with 'strictly increasing'.

**Proposition 4.** Suppose $I$ is an interval and $f : I \to \mathbf{R}$ is continuous and injective. Then for any $x$, $y$ and $z$ in $I$ with $x < y < z$ either $f(x) < f(y) < f(z)$ or $f(x) > f(y) > f(z)$. Hence we have:

(i) If $f(x) < f(y)$ or $f(x) < f(z)$ or $f(y) < f(z)$, then $f(x) < f(y) < f(z)$.
(ii) If $f(x) > f(y)$ or $f(x) > f(z)$ or $f(y) > f(z)$, then $f(x) > f(y) > f(z)$.

**Proof.** Suppose $x < y < z$. Then $(x, y) \cap (y, z) = \varnothing$. Since $f$ is injective, this implies that $f((x, y)) \cap f((y, z)) = \varnothing$. We have then the following possibilities regarding $f(x)$, $f(y)$ and $f(z)$:

Case (1) $f(x) < f(y)$ and $f(y) < f(z)$.
Case (2) $f(x) < f(y)$ and $f(y) > f(z)$.
Case (3) $f(x) > f(y)$ and $f(y) < f(z)$.
Case (4) $f(x) > f(y)$ and $f(y) > f(z)$.

Then by the IntermediateValue Theorem, since $I$ is an interval, we have the following conclusions according to each case above:

(1) $(f(x), f(y)) \subseteq f((x, y))$ and $(f(y), f(z)) \subseteq f((y, z))$;
(2) $(f(x), f(y)) \subseteq f((x, y))$ and $(f(z), f(y)) \subseteq f((y, z))$;
(3) $(f(y), f(x)) \subseteq f((x, y))$ and $(f(y), f(z)) \subseteq f((y, z))$;
(4) $(f(y), f(x)) \subseteq f((x, y))$ and $(f(z), f(y)) \subseteq f((y, z))$.

Case (2) implies that $(f(x), f(y)) \cap (f(z), f(y)) = (\max(f(x), f(z)), f(y)) \neq \varnothing$. But $(f(x), f(y)) \cap (f(z), f(y)) \subseteq f((x, y)) \cap f((y, z)) = \varnothing$ and so $(f(x), f(y)) \cap (f(z), f(y)) = \varnothing$ contradicting $(f(x), f(y)) \cap (f(z), f(y)) \neq \varnothing$. Thus Case (2) is not possible. Similarly, case (3) implies that $(f(y), f(x)) \cap (f(y), f(z)) = (f(y), \min(f(x), f(z))) \neq \varnothing$. But $(f(y), f(x)) \cap (f(y), f(z)) \subseteq f((x, y)) \cap f((y, z)) = \varnothing$ and so $(f(y), f(x)) \cap (f(y), f(z)) = \varnothing$ contradicting $(f(y), f(x)) \cap (f(y), f(z)) \neq \varnothing$. Thus Case (3) is not possible.

Therefore, we are left with cases (1) and (4). That is to say ,$f(x) < f(y) < f(z)$ or $f(x) > f(y) > f(z)$. This completes the proof of the proposition.

**Proof of Theorem 3.**

Suppose for some $x_1$, $x_2$ in $I$ with $x_1 < x_2$ , we have that $f(x_1) < f(x_2)$. We shall show that then $f$ is (strictly) increasing, i.e., for any $y$, $z$ in $I$ with $y < z$ , $f(y) < f(z)$.
If $x_1 = y$ and $x_2 = z$, then we have nothing to show since $f(x_1) < f(x_2)$. If only one of $y$ or $z$ is equal to either $x_1$ or $x_2$ , then by Proposition 4 part (i) $f(y) < f(z)$. It remains to see the same conclusion can be reached when $y$ and $z$ are distinct from both $x_1$ or $x_2$. By the total ordering on **R**, we have the following six possibilities:

Case (1) $y < z < x_1 < x_2$;
Case (2) $y < x_1 < z < x_2$;
Case (3) $y < x_1 < x_2 < z$;
Case (4) $x_1 < y < z < x_2$;
Case (5) $x_1 < y < x_2 < z$;
Case (6) $x_1 < x_2 < y < z$.

For cases (1), (2) and (3), applying Proposition 4 Part (i), we obtained $f(y) < f(x_1)$ using the inequality $y < x_1 < x_2$ and the supposition $f(x_1) < f(x_2)$. Applying Proposition 4 Part (i) again, we have then $f(y) < f(z)$ since $f(y) < f(x_1)$ and either $y < x_1 < z$ or $y < z < x_1$.

For cases (4) and (5) since $x_1 < y < x_2$ and $f(x_1) < f(x_2)$, applying Proposition 4 Part (i), we get $f(y) < f(x_2)$. Then applying Proposition 4 Part (i) again, we get $f(y) < f(z)$ since we now have $f(y) < f(x_2)$ and either $y < z < x_2$ or $y < x_2 < z$.

For case (6) Applying Proposition 4 part (i) gives us $f(x_2) < f(y)$. Therefore, applying again Proposition 4 Part (i), we get $f(y) < f(z)$ since $x_2 < y < z$. Hence $f$ is (strictly) increasing.

Similarly, if we have $f(x_1) > f(x_2)$, we can show that for any $y$, $z$ in $I$ with $y < z$, we have that $f(y) > f(z)$. We only have to reverse the inequality in the images in the above proceeding and use Proposition 4 Part (ii) instead of Part (i). This means that $f$ is (strictly) decreasing.
Therefore, $f$ is (strictly) monotonic. This completes the proof of Theorem 3.

**Example of a discontinuous function which is injective but not monotonic.**

Define $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = x$ if $x$ is rational and $g(x) = -x$ if $x$ is irrational. Then g is not continuous and g is injective but not monotonic.