Change of Variables Theorems
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Suppose g: $[a, b] \to \mathbf{R}$ is a function and $f: [c, d] \to \mathbf{R}$ is another function. Suppose the range of g is contained in $[c, d]$. The change of variable formula is a formula of the following form

$$\int_a^x f(g(t))g'(t)dx = \int_{g(a)}^{g(x)} f(t)dt \text{ -------------------------------- (A)}$$

for any x in the interval $[a, b]$. This formula is usually used to find the integral $\int_a^x H(t))dt$ whenever it is possible to express the integrand $H(t)$ in the form of $f($ g(t))g' (t), where f is a function whose primitive is known or f is equal almost everywhere to the derivative of an absolutely continuous function. The change of variable fomula (A) requires that f be Lebesgue integrable on the domain containing the range of g and that the function $f($ g(t))g' (t) be Lebesgue integrable on $[a, x]$ for every x in $[a, b]$. Now assume that f is Lebesgue integrable on $[c, d]$ and define $F : [c, d] \to \mathbf{R}$ by $F(x) = \int_c^x f(t)dt$. Then we can write (A) as

$$\int_a^x f(g(t))g'(t)dx = F(g(x)) - F(g(a)) \text{ -------------------------- (B)}$$

for every x in $[a, b]$. Thus, (B) is equivalent to (A). Assume that the function $f($ g(t))g' (t) is Lebesgue integrable on $[a, b]$, hence on $[a, x]$ for every x in $[a, b]$. By virtue of the left hand side of (B) being an indefinite integral of a Lebesgue integrable function, the composite function, $F \circ$ g, is absolutely continuous on $[a, b]$. Conversely, suppose that $F \circ$ g is absolutely continuous on $[a, b]$. Then $F \circ$ g is of bounded variation and so has finite derivatives almost everywhere on $[a, b]$. F, being an indefinite integral of a Lebesgue integrable function, is absolutely continuous and so is an N function and has finite derivatives almost everywhere on $[c, d]$. If g has finite derivative almost everywhere on $[a, b]$, then by Theorem 3 below, $(F \circ \text{g})'(x) = f(\text{g}(x))\,\text{g}'(x)$ almost every where on $[a, b]$. Consequently, since $F \circ$ g is absolutely continuous on $[a, b]$, $f(\text{g}(x))\text{g}'(x)$ is Lebesgue integrable on $[a, b]$ and (B) holds.

Thus we have deduced the following theorem.

Theorem 1. Suppose g: $[a, b] \to \mathbf{R}$ is a function having finite derivatives almost everywhere on $[a, b]$ and $f: [c, d] \to \mathbf{R}$ is a Lebesgue integrable function such that the range of g is contained in $[c, d]$. Let $F : [c, d] \to \mathbf{R}$ be defined by $F(x) = \int_c^x f(t)dt$. Then $f(\text{g}(x))\,\text{g}'(x)$ is Lebesgue integrable on $[a, b]$ *and* $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx$, if and only if, $F \circ$ g is absolutely continuous on $[a, b]$.

Remark. In the proof of the converse of Theorem 1, we make use of the fact that g has finite derivatives almost everywhere on $[a, b]$. Question arises if there are functions f and g not having finite derivatives almost everywhere on $[a, b]$ such that $F \circ$ g is absolutely continuous on $[a, b]$ but $(F \circ \text{g})'(x) \neq f(\text{g}(x))\,\text{g}'(x)$ almost every where on $[a, b]$.

We shall need the next technical result regarding critical points in the inverse image of a set of measure zero.

Theorem 2. Suppose g has derivatives (finite or infinite) on a set E with $m(g(E)) = 0$. Then $g' = 0$ almost everywhere on E.

Proof.
Let $B = \{ t \in E : |g'(t)| > 0\}$. Let $C_n = \{ t \in B : |g'(t)| \geq 1/n \}$ and

$B_n = \{ t \in B : |g(s) - g(t)| \geq |s - t|/n , |s - t| < 1/n \}$ for positive integers n.

It is easy to see that $B = \cup C_n$. Note that for each x in C_n, there exists k such that $x \in B_k$. This is because if x is in C_n then either $|g'(x)| \geq 1/n$ when $g'(x)$ is finite, or $|g'(x)|$ is infinite. If $g'(x)$ is finite, then there exists $\delta > 0$ such that

$$\left| \left| \frac{g(s)-g(x)}{s-x} \right| - |g'(x)| \right| < \frac{1}{2n}$$

for $0 < |s - x| < \delta$. Take any integer k such that $k > 2n$ and $1/k < \delta$. Then we have

$$0 < |s - x| < 1/k \Rightarrow \left| \frac{g(s)-g(x)}{s-x} \right| > |g'(x)| - \frac{1}{2n} \geq \frac{1}{2n} \geq \frac{1}{k}.$$

This means that $x \in B_k$. If $|g'(x)|$ is infinite, then there exists $\delta > 0$ such that

$$\left| \frac{g(s)-g(x)}{s-x} \right| > 1$$

for $0 < |s - x| < \delta$. In this case, just take any positive integer k such that $1/k < \delta$. Then we have

$$0 < |s - x| < 1/k \Rightarrow \left| \frac{g(s)-g(x)}{s-x} \right| > 1 \geq \frac{1}{k}.$$

Hence, $x \in B_k$. This implies that $B = \cup C_n \subseteq \cup B_n \subseteq B$ and so $B = \cup B_n$. . We shall show that the measure of B is 0 by showing that the measure of B_n is zero. Fix an integer n and consider any interval I of length $1/n$ and its intersection with B_n, $A = I \cap B_n$. We claim that the measure of A is zero. Since $m(g(E)) = 0$, $m(g(B)) = 0$ and so $m(g(A)) = 0$. Given any $\varepsilon > 0$, cover $g(A)$ by a countable union of disjoint interval I_k such that $g(A) = \cup I_k$ and $\sum m(I_k) < \varepsilon$. Let $A_k = g^{-1}(I_k) \cap A$. Then $A = \cup A_k$ and

$$m(A) \leq \Sigma\, m(A_k) \leq \Sigma \sup\{|s-t| : s,t \in A_k\}. \quad \text{------------------} \quad (1)$$

Note here that $\sup\{|s-t| : s,t \in A_k\}$ exists because A_k is bounded as A is bounded. Observe that $A_k \subseteq I \cap B_n \subseteq I$ and I is an interval of length less than $1/n$ and so for any s, t in A_k, $|s - t| < 1/n$. Thus, by the definition of B_n, for s, t in A_k, $|s - t| \leq n\, |g(s) - g(t)|$. Hence,

$$\sup\{|s-t| : s,t \in A_k\} \leq n \sup\{|g(s)-g(t)| : s,t \in A_k\} \leq n\, m(I_k)$$

as $g(A_k) \subseteq I_k$. It then follows from (1) that

$$m(A) \leq n\, \Sigma\, m(I_k) < n\varepsilon.$$

Since ε is arbitrary, we conclude that $m(A) = 0$. We can cover B_n by a countable number of non-overlapping intervals I, each of length $< 1/n$. Thus, by the above argument, the measure of B_n is less than the sum of measure of sets of measure zero and so is of measure zero. It follows that the measure of B is 0. Therefore, $g' = 0$ almost everywhere on E.

Theorem 3. Suppose F has finite derivatives almost everywhere on $[c, d]$ and g and $F \circ$ g have finite derivatives almost everywhere on $[a, b]$. It is assumed that the range of g is contained in $[c, d]$. Suppose F is an N-function, i.e., F maps sets of measure zero to sets of measure zero. Then $(F \circ \text{g})' = (f \circ \text{g})\, \text{g}'$ almost everywhere on $[a, b]$, where $F' = f$ almost everywhere on $[c, d]$, that is to say, the chain rule holds almost everywhere on $[a, b]$.

Proof. Let $Z = \{x \in [c, d]: F'(x)$ does not exist or $F'(x) = \pm \infty$ or $F'(x) \neq f(x)\}$.
Since F has finite derivative almost everywhere on $[c, d]$, the set $\{x \in [c, d]: F'(x)$ does not exist or $F'(x) = \pm \infty\}$ has measure zero. Also as $F' = f$ almost everywhere on $[c, d]$, the set $\{x \in [c, d]: F'(x) \neq f(x)\}$ has measure zero. Consequently, the measure of Z, $m(Z) = 0$.

Consider the preimage $S = g^{-1}(Z)$ of Z. Let $T = [a, b] - S$ be the complement of S in $[a, b]$. Then for any t in T, $g(t) \notin Z$ and so $F'(g(t))$ exists, is finite and $F'(g(t) = f(g(t))$.
Now since g has finite derivatives almost everywhere on $[a, b]$, for any t in $[a, b]$, either $g'(t)$ exists finitely or t belongs to a set of measure zero. Thus, we need to consider only those t in $[a, b]$, where $g'(t)$ exists finitely.

Suppose t is in T and g is differentiable at t (finitely). Then F is differentiable at $g(t)$ and so by the usual chain rule, $(F \circ g)'(t) = F'(g(t))\, g'(t) = f(g(t))\, g'(t)$. This means $(F \circ g)' = (f \circ g)\, g'$ almost everywhere on T.

Now we consider the chain rule on S. Since $g(S) \subseteq Z$, $m(g(S)) = 0$. Then since g is differentiable (finitely) almost everywhere on S, by considering points in S, where g is differentiable finitely, by Theorem 2, $g' = 0$ almost everywhere on S. Hence $(f \circ g)\, g' = 0$ almost everywhere on S. Since F is an N function, $m(F \circ g(S)) = 0$. As $F \circ g$ has finite derivatives almost everywhere on $[a, b]$, $F \circ g$ is differentiable almost everywhere on S. Therefore, by Theorem 2, $(F \circ g)' = 0$ almost everywhere on S. It follows that $(F \circ g)' = (f \circ g)\, g'\ (=0)$ almost everywhere on S.

Thus, we have shown that $(F \circ g)' = (f \circ g)\, g'$ almost everywhere on S and on T and so $(F \circ g)' = (f \circ g)\, g'$ almost everywhere on $[a, b]$. This completes the proof.

Suppose $f : [c, d] \to \mathbf{R}$ is Lebesgue integrable and the range of g is contained in $[c, d]$. Let $F : [c, d] \to \mathbf{R}$ be defined by $F(x) = \int_c^x f(t)dt$. Then F is absolutely continuous and so is an N function and also a function of bounded variation and thus has finite derivatives almost everywhere on $[c, d]$. If g and $F \circ g$ are of bounded variation, then $(F \circ g)' = (f \circ g)\, g'$ almost everywhere on $[a, b]$ by Theorem 3. We record this conclusion below.

Corollary 4. Suppose $f : [c, d] \to \mathbf{R}$ is Lebesgue integrable, g is a function of bounded variation whose range is contained in $[c, d]$. Let $F : [c, d] \to \mathbf{R}$ defined by $F(x) = \int_c^x f(t)dt$. If $F \circ g$ is of bounded variation, then $(F \circ g)' = (f \circ g)\, g'$ almost everywhere on $[a, b]$.

Note that if $f : [c, d] \to \mathbf{R}$ is Lebesgue integrable, then $F(x) = \int_c^x f(t)dt$ is an N function and has finite derivatives almost everywhere on $[c, d]$. Thus, by Theorem 3, we have the following simple deduction of the chain rule holding almost everywhere on $[a, b]$.

Corollary 5. If g and $F \circ g$ have finite derivatives almost everywhere on $[a, b]$ and F is absolutely continuous, then the chain rule holds almost everywhere on $[a, b]$.

Corollary 6. If g is monotone, $F \circ g$ has finite derivatives almost everywhere on $[a, b]$ and F is absolutely continuous, then the chain rule holds almost everywhere on $[a, b]$.

We have relied on some results concerning absolutely continuous functions. For convenience we state the result here as the next theorem.

Theorem 7. Suppose F is an absolutely continuous function on $[a, b]$.
Then (i) F is a continuous function of bounded variation,
(ii) F is an N function, i.e., it maps sets of measure zero to sets of measure zero,
(iii) F is differentiable (finitely) almost everywhere and F' is Lebesgue integrable.
Moreover, (i) and (ii) implies that F is absolutely continuous (Banach-Zarecki).
If F is continuous and satisfies (ii) and (iii), then F is absolutely continuous.
In particular, F is absolutely continuous, if and only if, it is the indefinite integral of its derivative. Thus, the indefinite integral of a Lebesgue integrable function is always absolutely continuous.

A very good reference for this theorem is the article "*On Absolutely Continuous Functions*" , (The American Mathematical Monthly vol 72 (1963), pp. 831-841) by Dale Varberg.

Now we can prove easily the next theorem.

Theorem 8. Suppose g: $[a, b] \to \mathbf{R}$ is an absolutely continuous function and let $f : [c, d] \to \mathbf{R}$ be a bounded Lebesgue integrable function such that the range of g is contained in $[c, d]$. Then we have the following equality for Lebesgue integrals.

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx .$$

Proof. The function $F : [c, d] \to \mathbf{R}$ defined by $F(x) = \int_c^x f(x)dx$ is absolutely continuous. Since f is bounded, F is also Lipschitz. Therefore, $F \circ g$ is absolutely continuous on $[a, b]$ (see Proposition 21 of my article "*Change of variable or substitution in Riemann and Lebesgue integration*"). Then Theorem 8 follows from Theorem 1.

Remark. Theorem 8 is Theorem 31 of "*Change of Variable or Substitution in Riemann and Lebesgue Integration*", where I have proved this using a weaker version of Theorem 2 and Theorem 3 for absolutely continuous functions instead of functions having only finite derivatives almost everywhere.

If we drop the condition that f be bounded, we then have the following theorem.

Theorem 9. Suppose g: $[a, b] \to \mathbf{R}$ is an absolutely continuous function and $f : [c, d] \to \mathbf{R}$ is a Lebesgue integrable function such that the range of g is contained in $[c, d]$ and $(f \circ g)\, g'$ is Lebesgue integrable on $[a, b]$. Then we have the change of variable formula for Lebesgue integral,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx .$$

Proof. Write f as $f^+ - f^-$, where f^+ and f^- are the positive and negative parts of f, i.e., $f^+(x) = max\{0, f(x)\}$ and $f^-(x) = -min\{0, f(x)\}$. Then f is Lebesgue integrable, if and only if, f^+ and f^- are Lebesgue integrable. For each positive integer n, $f^+_n = \min\{n, f^+\}$ is Lebesgue integrable and f^+_n converges pointwise to f^+ on $[c, d]$. Obviously, each f^+_n is bounded. Similarly, $f^-_n = \min\{n, f^-\}$ is Lebesgue integrable on $[c, d]$ and converges pointwise to f^- on $[c, d]$. It follows that $h_n = f^+_n - f^-_n$ is bounded, Lebesgue integrable and converges pointwise to $f^+ - f^- = f$. Then by Theorem 8, $(h_n \circ g)\, g'$ is Lebesgue integrable on $[a, b]$ and

$$\int_a^b h_n(g(x))g'(x)dx = \int_{g(a)}^{g(b)} h_n(x)dx . \quad \text{---------------------} \quad (1)$$

Since ($h_n \circ$g) g′ converges pointwise to ($f \circ$g) g′ , which is Lebesgue integrable, by the Lebesgue Dominated convergnce Theorem, the left hand side of (1) $\int_a^b h_n(g(x))g'(x)dx$ converges to $\int_a^b f(g(x))g'(x)dx$. Also since h_n converges pointwise to f and f is Lebesgue integrable, by the Lebesgue Dominated Convergence Theorem again, $\int_{g(a)}^{g(b)} h_n(x)dx$ converges to $\int_{g(a)}^{g(b)} f(x)dx$. Consequently, $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx$.

Corollary 10. Suppose g: [a, b] → **R** is an absolutely continuous function and f: [c, d] → **R** is a Lebesgue integrable function such that the range of g is contained in [c, d] and ($f \circ g$) g' is Lebesgue integrable on [a, b]. Then $F \circ$ g is absolutely continuous on [a, b]

Theorem 11. Suppose g: [a, b] → **R** is of bounded variation on [a, b] and f: [c, d] → **R** is a Lebesgue integrable function such that the range of g is contained in [c, d] and $F \circ g$ is absolutely continuous on [a, b], where F is defined as above. Then we have the change of variable formula for Lebesgue integral.

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx .$$

Proof. Since g is of bounded variation on [a, b], g has finite derivatives almost everywhere on [a, b]. The theorem then follows from Theorem 1.

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