Second Mean Value Theorem for Integrals

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This article is about the Second Mean Value Theorem for Integrals. This theorem, first proved by Hobson in its most generality and with extension by Dixon, is very useful and almost indispensable in many of the arguments in the convergence problem of Fourier series. We shall present a proof along the line taken by Dixon and Hobson.

The Second Mean Value Theorem for Integrals (SMVT)

Statement of the Theorem

Suppose the function f is Lebesgue integrable on $[a, b]$ and g: $[a, b] \to \mathbb{R}$ is monotone.

Then

(i) $$\int_a^b f(x)g(x)dx = g(a)\int_a^C f(x)dx + g(b)\int_C^b f(x)dx \quad \text{-----------} \quad \text{(M)}$$

for some C with $a \le C \le b$;

(ii) (M) holds with $a < C < b$ *except* in some trivial cases where $g(x)$ is constant in the open interval (a, b);

(iii) (M) holds with g(a) and g(b) replaced by A and B respectively so that

the function $h(x) = \begin{cases} A, & x = a, \\ g(x), & a < x < b, \\ B, & x = b \end{cases}$ is monotone; i.e.

$$\int_a^b f(x)g(x)dx = A\int_a^C f(x)dx + B\int_C^b f(x)dx$$

for some C with $a \le C \le b$; $A \le \lim_{x \to a^+} g(x), B \ge \lim_{x \to b^-} g(x)$ if g is increasing and $A \ge \lim_{x \to a^+} g(x), B \le \lim_{x \to b^-} g(x)$ if g is decreasing. C can be taken in (a, b) *except* in some trivial cases where $g(x)$ is constant in the open interval (a, b).

Note that we say a function g is *decreasing* if $x > y \Rightarrow g(x) \le g(y)$. It is said to be *increasing* if $x > y \Rightarrow g(x) \ge g(y)$.

An immediate consequence is the following:

Corollary (Bonnet Mean Value Theorem). Suppose f is Lebesgue integrable on $[a, b]$ and g: $[a, b] \to \mathbb{R}$ is monotone.

(i) If g is non-negative, decreasing and greater than or equal to 0, for $A \ge \lim_{x \to a^+} g(x)$ there exists C such that $a \le C \le b$ and

$$\int_a^b f(x)g(x)dx = A\int_a^C f(x)dx.$$

(ii) If g is non-negative, increasing and greater than or equal to 0, for $B \ge \lim_{x \to b^-} g(x)$ there exists C such that $a \le C \le b$ and

$$\int_a^b f(x)g(x)dx = B\int_C^b f(x)dx.$$

(iii) If g is not constant in (a, b), then the point C in Part (i) and (ii) is in (a, b). More precisely, except for some trivial cases when g is constant in (a, b), the point C in Part (i) and (ii) is in (a, b).

Proof. For Part (i), apply SMVT Part (iii) with $B = 0$. For Part (ii), apply SMVT Part (iii) with $A = 0$. Part (iii) follows from SMVT Part (iii).

Note that we abbreviate the Second Mean Value Theorem for Integrals by SMVT.

Definition 1.

Let $[a, b]$ be a closed and bounded interval. A finite function g: $[a, b] \to \mathbb{R}$ is said to be a *step function*, if there exists a partition of $[a, b]$, $a = x_0 < x_1 < \ldots < x_n = b$, such that g is constant on each open interval (x_{i-1}, x_i), $i = 1, 2, \ldots, n$.

We shall state the following approximation theorem of Lebesgue integrable function without proof. This follows from the approximation of Lebesgue integrable function by simple function and the approximation of characteristic function of a measurable subset of finite measure by step function. The proofs may be found in Royden's *Real Analysis.*

Theorem 2. Suppose g is Lebesgue integrable on $[a, b]$. For any $\varepsilon > 0$, there exists a step function ϕ on $[a, b]$ such that $\int_a^b |g(t)-\phi(t)|\,dt < \varepsilon$.

Our approach is to prove Part (i) of SMVT for step function and then use the approximation theorem to pass to a general Lebesgue integrable function.

The first thing we observe is that Part (i) of SMVT holds true when f is a constant function.

Lemma 3. Suppose f is a constant function on (a, b) and g: $[a, b] \to \mathbb{R}$ is decreasing and $g(a) > g(b)$. Then Part (i) of SMVT holds true.

Proof.

Suppose $f(x)$ is a constant function, g is decreasing, $g(a)=1$ and $g(b)=0$. Suppose $f(x) = M$ in (a, b). Since $1 = g(a) \geq g(x) \geq g(b) = 0$,

$$b-a \geq \int_a^b g(x)dx \geq 0.$$

Therefore, either $M(b-a) \geq \int_a^b f(x)g(x)dx = M\int_a^b g(x)dx \geq 0$

or $$M(b-a) \leq \int_a^b f(x)g(x)dx = M\int_a^b g(x)dx \leq 0.$$

Hence for some $0 \leq t \leq 1$,

$$\int_a^b f(x)g(x)dx = tM(b-a) = \int_a^{a+t(b-a)} Mdt = \int_a^{a+t(b-a)} f(t)dt.$$

Note that $a \leq a+t(b-a) \leq b$ and so we can take C to be equal to $a+t(b-a)$ and we get

$\int_a^b f(x)g(x)dx = tM(b-a) = \int_a^{a+t(b-a)} Mdt = g(a)\int_a^C f(t)dt$. This proves the case when *f* is a constant function in (*a, b*).

In general, let $\hat{g}(x) = \dfrac{g(x)-g(b)}{g(a)-g(b)}$ for *x* in [*a*, *b*]. Then $\hat{g}$ is decreasing, $\hat{g}(a)=1$ and $\hat{g}(b)=0$. By what we have just proved, there exists *C* such that *a* ≤ *C* ≤ *b* and

$$\int_a^b f(x)\hat{g}(x)dx = \int_a^C f(x)dx \,. \text{ -------------------------- } (1)$$

Consequently, $\int_a^b f(x)g(x)dx - g(b)\int_a^b f(x)dx = (g(a)-g(b))\int_a^C f(x)dx$

and Part (i) of SMVT holds.

The following technical lemma is used to combine the intervals on which Part (i) of SMVT holds.

Lemma 4. Assume that *g* is decreasing on [*a*, *b*]. If Part (i) of SMVT is true for the subintervals [α, β] and [β, γ], then it is true for the interval [α, γ].

Proof. As Part (i) of SMVT is true for the interval [α, β], we have

$$\int_\alpha^\beta f(x)g(x)dx = g(\alpha)\int_\alpha^C f(x)dx + g(\beta)\int_C^\beta f(x)dx \quad \text{------------ } (2)$$

for some $\alpha \le C \le \beta$. Similarly, we get

$$\int_\beta^\gamma f(x)g(x)dx = g(\beta)\int_\beta^D f(x)dx + g(\gamma)\int_D^\gamma f(x)dx \quad \text{-------------- } (3)$$

for some $\beta \le D \le \gamma$.

We can write (2) and (3) as follows:

$$\int_\alpha^\beta f(x)g(x)dx = g(\alpha)\int_\alpha^C f(x)dx + g(\beta)\left(\int_\alpha^\beta f(x)dx - \int_\alpha^C f(x)dx\right) \text{ --------- } (4)$$

$$\int_\beta^\gamma f(x)g(x)dx = g(\beta)\left(\int_\alpha^D f(x)dx - \int_\alpha^\beta f(x)dx\right) + g(\gamma)\left(\int_\alpha^\gamma f(x)dx - \int_\alpha^D f(x)dx\right)$$

-------- (5)

Thus, adding (4) and (5) gives

$$\int_{\alpha}^{\gamma} f(x)g(x)dx$$

$$=\left(g(\alpha)-g(\beta)\right)\int_{\alpha}^{C} f(x)dx+\left(g(\beta)-g(\gamma)\right)\int_{\alpha}^{D} f(x)dx+g(\gamma)\int_{\alpha}^{\gamma} f(x)dx \text{. ----- (6).}$$

If $g(\alpha)=g(\gamma)$, then g is constant on $[\alpha, \gamma]$. Then from (6) we obtain

$$\int_{\alpha}^{\gamma} f(x)g(x)dx=g(\gamma)\int_{\alpha}^{\gamma} f(x)dx=g(\alpha)\int_{\alpha}^{C} f(x)dx+g(\gamma)\int_{C}^{\gamma} f(x)dx$$

for any C with $\alpha<C<\gamma$.

Now, assume $g(\alpha)>g(\gamma)$.

Observe that

$$\left(\frac{g(\alpha)-g(\beta)}{g(\alpha)-g(\gamma)}\right)\int_{\alpha}^{C} f(x)dx+\left(\frac{g(\beta)-g(\gamma)}{g(\alpha)-g(\gamma)}\right)\int_{\alpha}^{D} f(x)$$

is of the form $\ell\int_{\alpha}^{C} f(x)dx+m\int_{\alpha}^{D} f(x)$ with $\ell, m\geq 0$ and $\ell+m=1$.

Since the function $H(x)=\int_{\alpha}^{x} f(t)dt$ is continuous, there exists E with $C\leq E\leq D$ such that

$$\left(\frac{g(\alpha)-g(\beta)}{g(\alpha)-g(\gamma)}\right)\int_{\alpha}^{C} f(x)dx+\left(\frac{g(\beta)-g(\gamma)}{g(\alpha)-g(\gamma)}\right)\int_{\alpha}^{D} f(x)=\int_{\alpha}^{E} f(x).$$

Thus

$$\left(g(\alpha)-g(\beta)\right)\int_{\alpha}^{C} f(x)dx+\left(g(\beta)-g(\gamma)\right)\int_{\alpha}^{D} f(x)dx=\left(g(\alpha)-g(\gamma)\right)\int_{\alpha}^{E} f(x)dx.$$

It follows then from (6),

$$\int_{\alpha}^{\gamma} f(x)g(x)dx=\left(g(\alpha)-g(\gamma)\right)\int_{\alpha}^{E} f(x)dx+g(\gamma)\int_{\alpha}^{\gamma} f(x)dx$$

$$=g(\alpha)\int_{\alpha}^{E} f(x)dx+g(\gamma)\int_{E}^{\gamma} f(x)dx.$$

Thus Part (i) of SMVT holds on the interval $[\alpha, \gamma]$.

This completes the proof of Lemma 4.

Proof of the Second Mean Value Theorem Part (i).

We prove Part (i) of SMVT when g is decreasing. The case when g is increasing follows by considering $-g$, since $-g$ is decreasing.

If $g(a) = g(b)$, then g is constant, say $g(x) = K$, then we can take C to be any value in (a, b), since trivially,

$$\int_a^b f(x)g(x)dx = \int_a^b Kf(x)dx = g(a)\int_a^C f(x)dx + g(b)\int_C^b f(x)dx .$$

We assume now $g(a) > g(b)$.

Let $\hat{g}(x) = \dfrac{g(x)-g(b)}{g(a)-g(b)}$ for x in $[a, b]$. Then $\hat{g}$ is decreasing, $\hat{g}(a) = 1$ and $\hat{g}(b) = 0$.

Let $F(x) = \int_a^x f(t)dt$ for x in $[a, b]$. Then F is continuous on $[a, b]$.

It is sufficient to show that there exists C such that $a \le C \le b$ and

$$\int_a^b f(x)\hat{g}(x)dx = F(C) . \quad \text{-------------------------} \quad (7)$$

This is because if (7) holds, then

$$\int_a^b f(x)\hat{g}(x)dx = \int_a^b f(x)\frac{g(x)-g(b)}{g(a)-g(b)}dx = F(C) = \int_a^C f(x)dx .$$

And we have

$$\int_a^b f(x)g(x)dx - g(b)\int_a^b f(x)dx = \left(g(a)-g(b)\right)\int_a^C f(x)dx$$

and so

$$\int_a^b f(x)g(x)dx = g(a)\int_a^C f(x)dx + g(b)\left(\int_a^b f(x)dx - \int_a^C f(x)dx\right)$$

$$= g(a)\int_a^C f(x)dx + g(b)\int_C^b f(x)dx .$$

Thus Part (i) of SMVT follows.

Note that with $\hat{g}(a)=1$ and $\hat{g}(b)=0$, (7), i.e.,

$$\int_a^b f(x)\hat{g}(x)dx = F(C) = \int_a^C f(x)dx = \hat{g}(a)\int_a^C f(x)dx + \hat{g}(b)\int_C^b f(x)dx$$

is a special case of the theorem.

Case 1. f is a constant function on (a, b).

By Lemma 3, when $f(x)$ is a constant function on (a, b), Part (i) of SMVT holds.

Case 2. f is a step function, g is decreasing.

Since f is a step function, there is a partition of $[a, b]$, $a = x_0 < x_1 < \dots < x_n = b$, such that f is constant on each open interval (x_{i-1}, x_i), i= 1, 2, … , n.

By Case 1, we know that Part (i) of SMVT is true on each subinterval $[x_{i-1}, x_i]$, i= 1, 2, … , n. Since there is only a finite number of subintervals in the partition, by Lemma 4, starting from the first interval, we can extend to the whole interval $[a, b]$ for Part (i) of SMVT to hold. Thus Part (i) of SMVT is true for step functions.

Case 3. General f.

Suppose the function f is Lebesgue integrable on $[a, b]$. Then by Theorem 2, for any integer $n \geq 1$, there exists a step function ϕ_n such that

$$\int_a^b |\phi_n(t) - f(t)|dt < \frac{1}{n}. \text{ ---------------------------- } (8)$$

Therefore, for all x in $[a, b]$, $\left|\int_a^x \phi_n(t)dt - \int_a^x f(t)dt\right| \leq \int_a^x |\phi_n(t) - f(t)|dt < \frac{1}{n}$.

Hence,

$$F(x) - \frac{1}{n} = \int_a^x f(t)dt - \frac{1}{n} \leq \int_a^x \phi_n(t)dt \leq \int_a^x f(t)dt + \frac{1}{n} = F(x) + \frac{1}{n}. \text{ ------- } (9)$$

Since F is continuous on $[a, b]$, by the Extreme Value Theorem, there exists

α, β in $[a, b]$, such that $F(\alpha) = \inf\{F(x) : x \in [a,b]\} = m$ and $F(\beta) = \sup\{F(x) : x \in [a,b]\} = M$. It follows then from (9) that

$$m - \frac{1}{n} \le \int_a^x \phi_n(t)dt \le M + \frac{1}{n} \ . \quad \text{---------------------} \quad (10)$$

Since Part (i) of SMVT holds for step function, for each $n \ge 1$, there exists C_n such that $a \le C_n \le b$ and

$$\int_a^b \phi_n(x)\hat{g}(x)dx = \int_a^{C_n} \phi_n(x)dx \, . \quad \text{---------------------------} \quad (11)$$

Therefore, it follows from (10) that for $n \ge 1$,

$$m - \frac{1}{n} \le \int_a^b \phi_n(x)\hat{g}(x)dx \le M + \frac{1}{n} . \quad \text{------------------} \ (12)$$

Now

$$\left| \int_a^b \hat{g}(x)f(x)dx - \int_a^b \hat{g}(x)\phi_n(x)dx \right|$$

$$\le \int_a^b |\hat{g}(x)||f(x) - \phi_n(x)|dx \le \int_a^b |f(x) - \phi_n(x)|dx < \frac{1}{n} \quad \text{by (8).}$$

Therefore,

$$\int_a^b \hat{g}(x)\phi_n(x)dx - \frac{1}{n} \le \int_a^b \hat{g}(x)f(x)dx \le \int_a^b \hat{g}(x)\phi_n(x)dx + \frac{1}{n} . \ \text{-----} \ (13)$$

We deduce from (12) and (13) that

$$m - \frac{2}{n} \le \int_a^b \hat{g}(x)f(x)dx \le M + \frac{2}{n} \ . \quad \text{---------------} \ (14)$$

Therefore, letting n tends to ∞, we have

$$m \le \int_a^b \hat{g}(x)f(x)dx \le M$$

and by the Intermediate Value Theorem, there exists C lying between α and β and hence in $[a, b]$ such that

$$F(C) = \int_a^b \hat{g}(x)f(x)dx \ .$$

Thus, we have proved (7) and Part (i) of SMVT follows.

Proof of Part (ii) of the Second Mean Value Theorem.

We prove Part (ii) of SMVT when g is decreasing. The case when g is increasing follows by considering $-g$.

By part (i), $\int_a^b \hat{g}(x)f(x)dx = F(C)$ for some C in $[a, b]$.

If C lies in (a, b), then part (ii) follows.

If $F(C) \neq F(a) = 0$, then $\int_a^b \hat{g}(x)f(x)dx \neq 0$, i.e.,

$$\int_a^b g(x)f(x)dx \neq g(b)\int_a^b f(x)dx.$$

If $F(C) \neq F(b)$, then $\int_a^b \hat{g}(x)f(x)dx \neq \int_a^b f(x)dx$, i.e.,

$$\int_a^b g(x)f(x)dx \neq g(b)\int_a^b f(x)dx + (g(a)-g(b))\int_a^b f(x)dx = g(a)\int_a^b f(x)dx.$$

Therefore, Part (ii) of SMVT will hold unless $F(C) = F(a) = 0$ or $F(C) = F(b)$ and for all x in (a, b), $F(x) \neq F(a)$ and $F(x) \neq F(b)$.

We shall show that under this condition g must be constant in (a, b).

The condition implies that $F(x) \neq F(C)$ for all x in (a, b). It follows then by continuity that $G(x) = F(x) - F(C)$ is of constant sign. Note that $F(C) = 0$ or $F(b)$.

Our approach will be to show that for any open interval (c, d) in $[a, b]$, g is constant.

Let $a < c < d < b$. We apply Part (i) of SMVT to each of the three intervals, $[a, c]$, $[c, d]$ and $[d, b]$. We obtain

$$\int_a^c f(x)\hat{g}(x)dx = \hat{g}(a)\int_a^{C_1} f(x)dx + \hat{g}(c)\int_{C_1}^c f(x)dx,$$

$$\int_c^d f(x)\hat{g}(x)dx = \hat{g}(c)\int_c^{C_2} f(x)dx + \hat{g}(d)\int_{C_2}^d f(x)dx \text{ and}$$

$$\int_d^b f(x)\hat{g}(x)dx = \hat{g}(d)\int_d^{C_3} f(x)dx + \hat{g}(b)\int_{C_3}^b f(x)dx. \qquad \text{--------------- (15)}$$

Now writing (15) in terms of $F(x)$ and noting that $\hat{g}(a) = 1$ and $\hat{g}(b) = 0$, we get:

$$\int_a^c f(x)\hat{g}(x)dx = F(C_1) + \hat{g}(c)\big(F(c) - F(C_1)\big),$$

$$\int_c^d f(x)\hat{g}(x)dx = \hat{g}(c)\big(F(C_2) - F(c)\big) + \hat{g}(d)\big(F(d) - F(C_2)\big) \text{ and}$$

$$\int_d^b f(x)\hat{g}(x)dx = \hat{g}(d)\big(F(C_3) - F(d)\big), \qquad \text{------------------------- (16)}$$

for some $a \le C_1 \le c \le C_2 \le d \le C_3 \le b$.

Adding all three equations in (16) we obtain

$$\int_a^b f(x)\hat{g}(x)dx = F(C_1)\big(1 - \hat{g}(c)\big) + F(C_2)\big(\hat{g}(c) - \hat{g}(d)\big) + F(C_3)\hat{g}(d). \text{----- (17)}$$

Since $\int_a^b \hat{g}(x)f(x)dx = F(C)$, we get from (17) that

$$\big(F(C_1) - F(C)\big)\big(1 - \hat{g}(c)\big)$$

$$+\big(F(C_2) - F(C)\big)\big(\hat{g}(c) - \hat{g}(d)\big) + \big(F(C_3) - F(C)\big)\hat{g}(d) = 0. \qquad \text{----- (18)}$$

Note that since $\hat{g}(x)$ is decreasing on $[a, b]$,

$$1 - \hat{g}(c) \ge 0, \;\; \hat{g}(c) - \hat{g}(d) \ge 0, \;\; \hat{g}(d) \ge \hat{g}(b) = 0.$$

If $G(x) = F(x) - F(C) > 0$ in (a, b), then since F is continuous on $[a, b]$, $G(x) \ge 0$ on $[a, b]$. If $G(x) = F(x) - F(C) < 0$ in (a, b), then $G(x) \le 0$ on $[a, b]$.

Therefore, each of the three terms in (18) is of the same sign or equal to 0. Hence each of the three terms in (18) must vanish. Consequently, since $F(C_2) - F(C) \ne 0$ for C_2 is in (a, b), $\hat{g}(c) - \hat{g}(d) = 0$. This implies that $g(c) = g(d)$.

Therefore, g must be constant on $[c, d]$. Since $[c, d]$ is any interval in (a, b), this implies that g is constant in (a, b).

This proves Part (ii) of SMVT for decreasing g.

Proof of Part (iii) Second Mean Value Theorem.

Suppose f is Lebesgue integrable on $[a, b]$ and g: $[a, b] \to \mathbb{R}$ is monotone.

Consider the function $h(x) = \begin{cases} A, & x = a, \\ g(x), & a < x < b, \\ B, & x = b \end{cases}$ which is monotone on $[a, b]$.

This means $A \leq \lim\limits_{x \to a^+} g(x), B \geq \lim\limits_{x \to b^-} g(x)$ if g is increasing and $A \geq \lim\limits_{x \to a^+} g(x), B \leq \lim\limits_{x \to b^-} g(x)$ if g is decreasing.

Therefore, applying Part (i) of SMVT to h we get

$$\int_a^b f(x)g(x)dx = \int_a^b f(x)h(x)dx = h(a)\int_a^C f(x)dx + h(b)\int_C^b f(x)dx$$

$$= A\int_a^C f(x)dx + B\int_C^b f(x)dx \, ,$$

for some C with $a \leq C \leq b$.

If g is not constant on (a, b), then the point C can be taken to be in (a, b).

More precisely, by Part (ii) of SMVT, C can be taken to be in (a, b) except for some trivial cases when g is constant in (a, b).

This completes the proof of the Second Mean Value Theorem for Integrals.