Riemann Summable Everywhere Series, Two Special Cosine Series and Abel Summable Series

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In this note, we present a result about trigonometric series, which are everywhere Riemann summable to a bounded function. We present two well-known special cosine series, one that is divergent but Riemann summable to zero except at multiples of 2π and another which converges dominatedly to an unbounded Lebsgue integrable function. In the later, we sum the series using complex variable technique. We show also that under certain condition on the trigonometric series, Riemann summability implies Abel summability.

The first result we discuss is an observation that for a trigonometric series whose coefficients are uniformly bounded, R-summability everywhere to a bounded function implies that the trigonometric series is the Fourier series of that function. Although a necessary condition for the trigonometric series to be a Fourier series is that the coefficients must tend to 0, we need not use this fact in the following proof. Theorem 35 of my article *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric Series* states that if the trigonometric series converges except in an enumerable set *E* to a bounded function *f*, then it is the Fourier series of *f*. If there is no exceptional set *E, i.e., E* is empty, we may use, instead of Theorem 35 cited, the result stated above to make the same conclusion. In this case, the trigonometric series converges everywhere to *f* and so it is R-summable to *f* everywhere and we can conclude that it is the Fourier series of *f*.

Suppose $T(\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(n\theta)+b_n\sin(n\theta)\right)$ is a trigonometric series. Letting $A_n(\theta)=a_n\cos(n\theta)+b_n\sin(n\theta)$, we can write $T(\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(\theta)$. Suppose $T(\theta)$ is Riemann summable or R-summable to a bounded function $f(\theta)$ *for all* θ in $[-\pi,\pi]$. Suppose that the series $\sum_{n=1}^{\infty}\left(\frac{a_n\cos(nx)+b_n\sin(nx)}{n^2}\right)=\sum_{n=1}^{\infty}\frac{A_n(x)}{n^2}$ is the Fourier series of a continuous function Ψ in $[-\pi,\pi]$. This supposition is always satisfied if the coefficients a_n, b_n are bounded.

Then we have:

Theorem 1. Suppose the trigonometric series $T(\theta)$ satisfies the condition stated above. Then $T(\theta)$ is the Fourier series of *f*.

Proof. Since $\sum_{n=1}^{\infty}\left(\frac{a_n\cos(nx)+b_n\sin(nx)}{n^2}\right)=\sum_{n=1}^{\infty}\frac{A_n(x)}{n^2}$ is the Fourier series of a continuous function Ψ, for integer $n \geq 1$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\Psi(\theta)\cos(n\theta)d\theta=\frac{a_n}{n^2} \qquad \text{----------------------------- (1)}$$

and $$\frac{1}{\pi}\int_{-\pi}^{\pi}\Psi(\theta)\sin(n\theta)d\theta=\frac{b_n}{n^2}. \qquad \text{----------------------------- (2).}$$

Since $T(\theta)$ is Riemann summable to $f(\theta)$, there exists $\delta > 0$, such that for all $0<|h|<\delta$,

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(\theta)\frac{\sin^2(nh)}{(nh)^2}$$

is convergent and

$$\lim_{h\to 0}\left(\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(\theta)\frac{\sin^2(nh)}{(nh)^2}\right)=f(\theta). \qquad \text{------------------------- (3).}$$

Now, for $h \neq 0$, $\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2}=\frac{\Psi(\theta+2h)+\Psi(\theta-2h)-2\Psi(\theta)}{4h^2}$ is a continuous function in θ. Observe that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\Psi(\theta+2h)\cos(n\theta)d\theta=\frac{1}{\pi}\int_{-\pi+2h}^{\pi+2h}\Psi(u)\cos(nu-2nh)du$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\Psi(\theta)\cos(n\theta-2nh)d\theta, \text{ by periodicity}$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\Psi(\theta-2h)\cos(n\theta)d\theta=\frac{1}{\pi}\int_{-\pi-2h}^{\pi-2h}\Psi(u)\cos(nu+2nh)du$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\Psi(\theta)\cos(n\theta+2nh)d\theta.$$

Hence, for $n \geq 1$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2}\cos(n\theta)d\theta$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\Psi(\theta)\frac{\cos(n\theta+2nh)+\cos(n\theta-2nh)-2\cos(n\theta)}{4h^2}d\theta$$

$$=-\frac{1}{\pi}\int_{-\pi}^{\pi}\Psi(\theta)\cos(n\theta)\frac{\sin^2(nh)}{h^2}d\theta=-\frac{1}{\pi}\int_{-\pi}^{\pi}\Psi(\theta)\cos(n\theta)d\theta\frac{\sin^2(nh)}{h^2}$$

$$=-a_n\frac{\sin^2(nh)}{(nh)^2} \quad . \qquad \text{---} \quad (4).$$

Plainly, $\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2}d\theta=0$.

Similarly, we can deduce that for integer $n\geq 1$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2}\sin(n\theta)d\theta$$

$$=-\frac{1}{\pi}\int_{-\pi}^{\pi}\Psi(\theta)\sin(n\theta)\frac{\sin^2(nh)}{h^2}d\theta=-\frac{1}{\pi}\int_{-\pi}^{\pi}\Psi(\theta)\sin(n\theta)d\theta\frac{\sin^2(nh)}{h^2}$$

$$=-b_n\frac{\sin^2(nh)}{(nh)^2} \quad . \qquad \text{---} \quad (5).$$

This shows that the Fourier series of $\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2}$ is given by

$$-\sum_{n=1}^{\infty}\left(a_n\cos(n\theta)+b_n\cos(n\theta)\right)\frac{\sin^2(nh)}{(nh)^2}=-\sum_{n=1}^{\infty}A_n(\theta)\frac{\sin^2(nh)}{(nh)^2}\text{ . ----------------- (6)}$$

Since $\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(\theta)\frac{\sin^2(nh)}{(nh)^2}$ is convergent for $0<|h|<\delta$, so $-\sum_{n=1}^{\infty}A_n(\theta)\frac{\sin^2(nh)}{(nh)^2}$ is convergent for $0<|h|<\delta$ and as $\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2}$ is continuous, for $\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2}$,

$$\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2}=-\sum_{n=1}^{\infty}A_n(\theta)\frac{\sin^2(nh)}{(nh)^2}\text{ . ------------------------- (7)}$$

But $\lim_{h\to 0}\left(\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(\theta)\frac{\sin^2(nh)}{(nh)^2}\right)=f(\theta)$ and so,

$$\lim_{h\to 0}\left(\sum_{n=1}^{\infty}A_n(\theta)\frac{\sin^2(nh)}{(nh)^2}\right)=f(\theta)-\frac{1}{2}a_0 \ .$$

It follows then from (7) that

$$\lim_{h\to 0}\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2}=\frac{1}{2}a_0-f(\theta) \ . \qquad \text{------------------------} \ (8)$$

Define $\Phi(\theta)=\frac{1}{4}a_0\theta^2-\Psi(\theta).$ Then Φ is continuous and

$$\frac{\Delta_{2h}^2\Phi(\theta)}{4h^2}=\frac{1}{2}a_0-\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2} . \qquad \text{-----------------------} \ (9)$$

Therefore, by virtue of (8),

$$\lim_{h\to 0}\frac{\Delta_{2h}^2\Phi(\theta)}{4h^2}=\frac{1}{2}a_0-\lim_{h\to 0}\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2}=f(\theta) .$$

This means

$$D_2\Phi(\theta)=f(\theta) \text{ for all } \theta \text{ in } [-\pi,\pi] . \qquad \text{-------------} \ (10)$$

Since f is bounded in $[-\pi,\pi]$, $\left|D_2\Phi(\theta)\right|=\left|f(\theta)\right|\le M$ for some positive real number *M*. And so $-M\le D_2\Phi(\theta)\le M$ for all θ in $[-\pi,\pi]$.

Since Φ is continuous on the whole of $\mathbb{R}$ and $\underline{D}_2\Phi(\theta)=D_2\Phi(\theta)\le M$, by Theorem 25 in *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric Series,* for all θ and $h\ne 0$ $\frac{\Delta_h^2\Phi(\theta)}{h^2}\le M$. Similarly, since $\overline{D}_2\Phi(\theta)=D_2\Phi(\theta)\ge -M$, by Theorem 25 of *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric Series*, for all θ and $h\ne 0$ in $\mathbb{R}$, $\frac{\Delta_h^2\Phi(\theta)}{h^2}\ge -M$. It follows that $\left|\frac{\Delta_h^2\Phi(\theta)}{h^2}\right|\le M$ for all θ and all $h\ne 0$. This means that $R_h(\theta)=\frac{\Delta_{2h}^2\Phi(\theta)}{4h^2}$ is uniformly bounded in θ and all $h\ne$

0. Since $\lim_{h\to 0} R_h(\theta) = \lim_{h\to 0} \frac{\Delta_{2h}^2 \Phi(\theta)}{4h^2} = f(\theta)$ and $R_h(\theta)$ is continuous for each $h \neq 0$, it follows that f is Lebesgue integrable.

We also have that

$R_h(\theta)\cos(n\theta) = \frac{\Delta_{2h}^2 \Phi(\theta)}{4h^2}\cos(n\theta)$ is uniformly bounded and

$$R_h(\theta)\cos(n\theta) = \frac{\Delta_{2h}^2 \Phi(\theta)}{4h^2}\cos(n\theta) \to f(\theta)\cos(n\theta) \text{ boundedly,}$$

and so, by the Lebesgue Bounded Convergence Theorem, for integer $n \geq 0$,

as $h \to 0$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} R_h(\theta)\cos(n\theta)d\theta \to \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)d\theta \ . \quad \text{-------------} \quad (11)$$

Similarly, also by the Lebesgue Bounded Convergence Theorem, for integer $n \geq 1$, as $h \to 0$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} R_h(\theta)\sin(n\theta)d\theta \to \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta . \quad \text{----------------} \ (12)$$

Recall $R_h(\theta) = \frac{\Delta_{2h}^2 \Phi(\theta)}{4h^2} = \frac{1}{2}a_0 - \frac{\Delta_{2h}^2 \Psi(\theta)}{4h^2}$ so that by (4) for integer $n \geq 1$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} R_h(\theta)\cos(n\theta)d\theta$$

$$= \frac{a_0}{2\pi}\int_{-\pi}^{\pi}\cos(n\theta)d\theta - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\Delta_{2h}^2 \Psi(\theta)}{4h^2}\cos(n\theta)d\theta$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\Delta_{2h}^2 \Psi(\theta)}{4h^2}\cos(n\theta)d\theta$$

$$= a_n\left(\frac{\sin(nh)}{nh}\right)^2, \text{ by (4),}$$

$\to a_n$ as $h \to 0$ --------------------- (13)

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} R_h(\theta)\sin(n\theta)d\theta$$

$$= \frac{a_0}{2\pi}\int_{-\pi}^{\pi}\sin(n\theta)d\theta - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2}\sin(n\theta)d\theta$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\Delta_{2h}^2\Psi(\theta)}{4h^2}\sin(n\theta)d\theta$$

$$= b_n\left(\frac{\sin(nh)}{nh}\right)^2, \text{ by (5),}$$

$\to b_n$ as $h \to 0$. ------------------------ (14)

Observe that $\frac{1}{\pi}\int_{-\pi}^{\pi} R_h(\theta)d\theta = a_0$.

It follows from (11) and (13) that $\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)d\theta = a_n$ and from (12) and (14) that $\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta = b_n$. Also, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)d\theta = \lim_{h\to 0}\frac{1}{\pi}\int_{-\pi}^{\pi} R_h(\theta)d\theta = a_0.$$

This shows that $T(\theta)$ is the Fourier series of $f(\theta)$.

This completes the proof.

Note that if $T(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(n\theta) + b_n\sin(n\theta)\right)$ converges everywhere to a bounded function f, then the coefficients (a_n) and (b_n) are bounded and the series $\sum_{n=1}^{\infty}\left(\frac{a_n\cos(nx) + b_n\sin(nx)}{n^2}\right) = \sum_{n=1}^{\infty}\frac{A_n(x)}{n^2}$ converges uniformly to a continuous function and is the Fourier series of its sum function. Since R-summability is regular, $T(\theta)$ is R-summable to f everywhere. Thus, the condition of Theorem 1 is satisfied. Hence, $T(\theta)$ is the Fourier series of f.

Lemma 2. The condition that $T(\theta)$ be Riemann summable or R-summable to a bounded function $f(\theta)$ *for all* θ in $[-\pi, \pi]$ cannot be relaxed even for a single point in $[-\pi, \pi]$.

The proof is by way of the following counter example.

Consider the cosine series

(C) $$\frac{1}{2} + \cos(\theta) + \cos(2\theta) + \cdots .$$

Note that $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$ converges uniformly and absolutely to a continuous function and hence it is the Fourier series of its sum function. (C) is plainly not a Fourier series.

Lemma 3. (C) is Riemann summable to 0 for θ in $[-\pi, \pi] - \{0\}$ but not at $\theta = 0$.

Proof.

To show that (C) is R-summable to 0 for θ in $[-\pi, \pi] - \{0\}$, we consider the following even function for each h such that $0 < h \leq \frac{\pi}{2}$

$$J(\theta) = \begin{cases} \pi \dfrac{2h - \theta}{4h^2}, & \text{if } 0 \leq \theta \leq 2h \\ 0 \quad \text{if} \quad 2h \leq \theta \leq \pi \end{cases} \qquad \text{------------ (15)}$$

$J(-\theta) = J(\theta)$ for θ in $[0, \pi]$. J is extended to the whole of $\mathbb{R}$ by periodicity.

Then we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} J(\theta)d\theta = \frac{2}{\pi}\int_{0}^{2h} J(\theta)d\theta = \frac{1}{2h^2}\int_{0}^{2h} (2h - \theta)d\theta = 1 \qquad \text{--------------- (16)}$$

and for integer $n \geq 1$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} J(\theta)\cos(n\theta)d\theta = \frac{2}{\pi}\int_0^{2h} J(\theta)\cos(n\theta)d\theta = \frac{1}{2h^2}\int_0^{2h}(2h-\theta)\cos(n\theta)d\theta$$

$$= \frac{1}{h}\int_0^{2h}\cos(n\theta)d\theta - \frac{1}{2h^2}\int_0^{2h}\theta\cos(n\theta)d\theta$$

$$= \frac{1}{nh}\sin(2nh) - \left[\frac{1}{2h^2}\frac{\sin(n\theta)}{n}\theta\right]_0^{2h} + \frac{1}{2nh^2}\int_0^{2h}\sin(n\theta)d\theta$$

$$= \frac{1}{nh}\sin(2nh) - \frac{1}{nh}\sin(2nh) - \frac{1}{2nh^2}\left[\frac{\cos(n\theta)}{n}\right]_0^{2h}$$

$$= -\frac{1}{2n^2h^2}(\cos(2nh)-1) = \frac{\sin^2(nh)}{(nh)^2}. \qquad \text{-------------------------------- (17)}$$

Hence, the Fourier series of $J(\theta)$ is given by

$$\frac{1}{2} + \sum_{n=1}^{\infty}\cos(n\theta)\frac{\sin^2(nh)}{(nh)^2}. \qquad \text{---------------------------- (18)}$$

Since $J(\theta)$ is continuous and (18) is convergent for $0 < h \leq \frac{\pi}{2}$, the Fourier series of $J(\theta)$ tends to $J(\theta)$.

Hence, we may write

$$\frac{1}{2} + \sum_{n=1}^{\infty}\cos(n\theta)\frac{\sin^2(nh)}{(nh)^2} = J(\theta).$$

Thus, for $2h \leq \theta \leq \pi$,

$$\frac{1}{2} + \sum_{n=1}^{\infty}\cos(n\theta)\frac{\sin^2(nh)}{(nh)^2} = 0.$$

That is to say, for all $0 < h \leq \frac{\theta}{2} \leq \frac{\pi}{2}$, $\frac{1}{2} + \sum_{n=1}^{\infty}\cos(n\theta)\frac{\sin^2(nh)}{(nh)^2} = 0$ and so for any $0 < \theta \leq \pi$, $\frac{1}{2} + \sum_{n=1}^{\infty}\cos(n\theta)\frac{\sin^2(nh)}{(nh)^2} = 0$ for $h \leq \frac{\theta}{2}$.

Letting h tends to 0 we have for $0 < \theta \le \pi$, $\lim_{h\to 0}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\cos(n\theta)\frac{\sin^2(nh)}{(nh)^2}\right)=0$.

Plainly, for $-\pi \le \theta < 0$, $\lim_{h\to 0}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\cos(n\theta)\frac{\sin^2(nh)}{(nh)^2}\right)=0$.

Hence, by periodicity, $\lim_{h\to 0}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\cos(n\theta)\frac{\sin^2(nh)}{(nh)^2}\right)=0$ for $\pi \le \theta < 2\pi$.

That means (C) is Riemann summable to 0 for θ in $[-\pi, \pi] - \{0\}$ or

(C) is Riemann summable to 0 for θ in $(0, 2\pi)$.

Note that for $0 < h \le \frac{\pi}{2}$, $J(0)=\frac{\pi}{2h}$ and so $\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\sin^2(nh)}{(nh)^2}=\frac{\pi}{2h}=\frac{\pi}{2|h|}$. It follows that $\lim_{h\to 0}\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\sin^2(nh)}{(nh)^2}=\lim_{h\to 0}\frac{\pi}{2|h|}=\infty$. This means (C) is not R-summable at $\theta = 0$.

We now state a slightly more general version of Theorem 1.

Theorem 4. Suppose

$$T(\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(n\theta)+b_n\sin(n\theta)\right)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(\theta)$$

is a trigonometric series. Suppose $T(\theta)$ is Riemann summable to a function $f(\theta)$ *for all* θ in $[-\pi, \pi]$ except for a denumerable subset E in $[-\pi, \pi]$. Suppose f is bounded in $[-\pi, \pi] - E$. Suppose that the series $\sum_{n=1}^{\infty}\left(\frac{a_n\cos(nx)+b_n\sin(nx)}{n^2}\right)=\sum_{n=1}^{\infty}\frac{A_n(x)}{n^2}$ is the Fourier series of a continuous function Ψ in $[-\pi, \pi]$ and that Ψ is smooth at every point in E. Then $T(\theta)$ is the Fourier series of f.

Proof. The proof is almost exactly the same as for Theorem 1, using Theorem 25 in *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric Series*. Here, we use $D_2\Phi(\theta) = f(\theta)$ for all θ in $[-\pi, \pi] - E$ and is bounded in $[-\pi, \pi] - E$ and that Φ is smooth in E since Ψ is smooth in E.

Observe that for the trigonometric series (C),

$$\Phi(\theta) = \frac{1}{4}\theta^2 - \Psi(\theta) = \frac{1}{4}\theta^2 - \sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^2}$$

is not smooth at 0. For $\frac{\Delta_{2h}^2\Psi(0)}{4h^2} = -\sum_{n=1}^{\infty}\frac{\sin^2(nh)}{(nh)^2}$ so that

$$\frac{\Delta_{2h}^2\Phi(0)}{4h^2} = \frac{1}{2} - \frac{\Delta_{2h}^2\Psi(0)}{4h^2} = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{\sin^2(nh)}{(nh)^2}$$

and so, for $0 < h \le \frac{\pi}{2}$, since $\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\sin^2(nh)}{(nh)^2} = \frac{\pi}{2h} = \frac{\pi}{2|h|}$,

$$\frac{\Delta_{2h}^2\Phi(0)}{4h^2} = \frac{\pi}{2|h|}.$$

Thus, $\lim_{h\to 0^+} h\frac{\Delta_{2h}^2\Phi(0)}{4h^2} = \lim_{h\to 0^+} h\frac{\pi}{2|h|} = \frac{\pi}{2} \ne 0$, consequently, Φ is not smooth at 0.

Remark.

If $a_n \to 0$ and $b_n \to 0$, then for the trigonometric series,

$$T(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(n\theta) + b_n\sin(n\theta)\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}A_n(\theta),$$

$\Phi(\theta) = \frac{1}{4}\theta^2 - \sum_{n=1}^{\infty}\frac{A_n(\theta)}{n^2}$ is continuous and smooth in $\mathbb{R}$ by Theorem 26 of *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric Series*. Thus, we may formulate Theorem 4 as follows:

Theorem 5. Suppose $a_n \to 0$ and $b_n \to 0$. Suppose the trigonometric series

$$T(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(n\theta) + b_n\sin(n\theta)\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}A_n(\theta)$$

is Riemann summable to a function $f(\theta)$ *for all* θ in $[-\pi, \pi]$ except for a denumerable subset E in $[-\pi, \pi]$ and f is bounded in $[-\pi, \pi] - E$.

Then $T(\theta)$ is the Fourier series of *f*.

Consider the trigonometric series

(S2) $$\sum_{n=2}^{\infty}\frac{1}{\ln(n)}\sin(n\theta)\ .$$ ------------------------ (19)

Note that we cannot apply Theorem 5 to (S2).

It is known that (S2) is not a Fourier series of any function and it converges everywhere to a non-Lebesgue integrable function *g*. (See Example 4 (1) of my article, *Fourier Cosine and Sine Series*.) Note that *g* is not bounded in $[-\pi, \pi]$.

(S2) converges everywhere and so it satisfies the first condition of Theorem 5, it is Riemann summable everywhere but to an unbounded function.

However, the trigonometric series

(C2) $$\sum_{n=2}^{\infty}\frac{1}{\ln(n)}\cos(n\theta)$$ ---------------------- (20)

converges for all θ except for θ which are multiples of 2 π and is the Fourier series of its sum function. (See Example 4 (1) of my article, *Fourier Cosine and Sine Series*.) (C2) is thus Riemann summable everywhere except for θ which are multiples of 2 π. The sum function is unbounded in $[-\pi, \pi]-\{0\}$.

Thus both (S2) and (C2) are Riemann summable to an unbounded function and so Theorem 5 is not applicable. Observe that since

$$\sum_{n=2}^{\infty}\frac{1}{\ln(n)}r^n \geq \sum_{n=2}^{\infty}\frac{1}{n}r^n = -\ln(1-r)-1 \text{ and } \lim_{r\to 1^-}-\ln(1-r)=\infty,$$

(C2) is not Abel-summable at $\theta = 0$. Moreover, since the sequence $\left(\frac{1}{\ln(n)}\right)$ is convex and tends to 0, by Theorem 2 Part (1) of *Fourier Cosine and Sine Series*) , (C2) converges to a non-negative Lebesgue integrable function and is the Fourier series of its sum function. Since it is not Abel-summable, it is also not Riemann summable. (See Theorem 12 below.)

If we replace the condition of boundedness on *f* in Theorem 5 to integrability, then we have the following:

Theorem 6. Suppose $a_n \to 0$ and $b_n \to 0$. Suppose the trigonometric series

$$T(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(\theta)$$

is Riemann summable to a function $f(\theta)$ *for all* θ in $[-\pi, \pi]$ except for a denumerable subset E in $[-\pi, \pi]$ and f is Lebesgue integrable in $[-\pi, \pi]$.

Then $T(\theta)$ is the Fourier series of f.

Proof. The proof is exactly the same as for Theorem 1 in my article *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric series*. Here, we have $\Phi(\theta) = \frac{1}{4}a_0\theta^2 - \Psi(\theta) = \frac{1}{4}a_0\theta^2 - \sum_{n=1}^{\infty}\frac{A_n(\theta)}{n^2}$ is continuous and smooth in $\mathbb{R}$ and we can compare this with the iterated integral of f. Via this and similar consideration for the Fourier series of f, it allows us to compare $T(\theta)$ with the Fourier series of f as in the proof of Theorem 1 in *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric series*.

Remark.

We can apply Theorem 6 to (C2) since (C2) converges to a Lebesgue integrable function.

For result starting from Abel summability we have the following Theorem of Verblunsky and Zygmund:

Theorem 7 (Verblunsky, Zygmund) Suppose the trigonometric series

$$T(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(\theta)$$

satisfies that a_n and b_n are of $o(n)$. If $T(\theta)$ is Abel summable *everywhere* to a finite and integrable function, then $T(\theta)$ is the Fourier series of f.

We may replace the condition that a_n and b_n be of $o(n)$ by that $a_n \to 0$ and $b_n \to 0$, as this is a necessary condition for $T(\theta)$ to be a Fourier series. The proof is also easier than that for Theorem 7, whose proof would not be given here.

Theorem 8. Suppose the trigonometric series

$$T(\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(n\theta)+b_n\sin(n\theta)\right)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(\theta)$$

satisfies that $a_n \to 0$ and $b_n \to 0$. If $T(\theta)$ is Abel summable *everywhere* to a finite and integrable function f, then $T(\theta)$ is the Fourier series of f.

Before we prove Theorem 8, we state Rajchman's Lemma without proof.

Let $g^*(\theta)=\limsup_{r\to 1^-}\left(\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(n\theta)+b_n\sin(n\theta)\right)r^n\right)=\limsup_{r\to 1^-}\left(\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(\theta)r^n\right)$

and $g_*(\theta)=\liminf_{r\to 1^-}\left(\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(\theta)r^n\right)$.

Lemma 9. (Rajchman's Lemma).

Suppose that the series $\sum_{n=1}^{\infty}\left(\frac{a_n\cos(nx)+b_n\sin(nx)}{n^2}\right)=\sum_{n=1}^{\infty}\frac{A_n(x)}{n^2}$ is the Fourier series of Ψ in $[-\pi, \pi]$. Let $\Phi(\theta)=\frac{1}{4}a_0\theta^2-\Psi(\theta)$. Suppose $\frac{1}{4}a_0\theta^2-\sum_{n=1}^{\infty}\frac{A_n(\theta)}{n^2}$ is Abel summable at θ to $\Phi(\theta)$. Then

$$\underline{D}_2\Phi(\theta)\le g^*(\theta) \quad , \quad g_*(\theta)\le \bar{D}_2\Phi(\theta).$$

For the proof of Lemma 9 see (7.6) *pages 353-354 of Trigonometric Series Volume 1 by Zygmund.*

Proof of Theorem 8

Since $a_n \to 0$ and $b_n \to 0$, $\sum_{n=1}^{\infty} \frac{A_n(\theta)}{n^2}$ converges absolutely and uniformly to a continuous function Ψ in $[-\pi, \pi]$. Hence, $\frac{1}{4}a_0\theta^2 - \sum_{n=1}^{\infty} \frac{A_n(\theta)}{n^2}$ converges uniformly and absolutely to a continuous and smooth function $\Phi(\theta) = \frac{1}{4}a_0\theta^2 - \Psi(\theta)$ (see Theorem 26 in *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric series*).

It follows that $\frac{1}{4}a_0\theta^2 - \sum_{n=1}^{\infty} \frac{A_n(\theta)}{n^2}$ is Abel summable at θ to $\Phi(\theta)$ for all θ.

Since $T(\theta)$ is Abel summable *everywhere* to a finite and integrable function f,
$g^*(\theta) = \limsup_{r \to 1^-} \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(\theta) r^n \right) = g_*(\theta) = \liminf_{r \to 1^-} \left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(\theta) r^n \right) = f(\theta)$ for all θ.
Therefore, by Lemma 9 (Rajchman's Lemma),

$$\underline{D}_2\Phi(\theta) \le f(\theta) \le \bar{D}_2\Phi(\theta)$$

for all θ.

Then by Theorem 28 of *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric series,* $\Phi(\theta) - J(\theta)$, where
$J(\theta) = \int_0^{\theta} \left(\int_0^t f(u)du \right) dt$, is linear. If we take
$\frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \left(\alpha_n \cos(n\theta) + \beta_n \sin(n\theta) \right)$ to be the Fourier series of *f* and

$H(\theta) = \frac{1}{4}\alpha_0\theta^2 - \sum_{n=1}^{\infty} \frac{\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)}{n^2}$, then $H(\theta) - J(\theta)$ is linear.

It follows that

$$L(\theta) = \Phi(\theta) - H(\theta) = \frac{1}{4}(a_0 - \alpha_0)\theta^2 - \sum_{n=1}^{\infty} \left(\frac{(a_n - \alpha_n)\cos(n\theta) + (b_n - \beta_n)\sin(n\theta)}{n^2} \right)$$

is linear.

Hence, as in the proof of Theorem 1 of *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric series*, we deduce that

$a_0 - \alpha_0 = 0$ and $\sum_{n=1}^{\infty}\left(\frac{(a_n-\alpha_n)\cos(n\theta)+(b_n-\beta_n)\sin(n\theta)}{n^2}\right)=C$. By taking the integral on both sides from 0 to 2π we see that $C = 0$. It then follows that $(a_n-\alpha_n)=0$ and $(b_n-\beta_n)=0$ and so $a_n=\alpha_n$ and $b_n=\beta_n$. Thus, $T(\theta)$ is precisely the Fourier series of $f(\theta)$. This completes the proof of Theorem 8.

If $a_n \to 0$ and $b_n \to 0$, then $\Phi(\theta)=\frac{1}{4}a_0\theta^2-\Psi(\theta)$ is continuous and smooth in $\mathbb{R}$. To apply Theorem 28 of *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric series*, we require only that the inequality $\underline{D}_2\Phi(\theta)\le f(\theta)\le \overline{D}_2\Phi(\theta)$ be satisfied for all θ in $[-\pi, \pi]$ except for a denumerable set E in $[-\pi, \pi]$ and that Φ is smooth in E.

Thus, we have a variation of Theorem 8.

Theorem 10. Suppose the trigonometric series

$$T(\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(n\theta)+b_n\sin(n\theta)\right)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(\theta)$$

satisfies that $a_n \to 0$ and $b_n \to 0$. If $T(\theta)$ is Abel summable to a finite value $f(\theta)$ for all θ in $[-\pi, \pi]$ except for a denumerable set E in $[-\pi, \pi]$ and f is Lebesgue integrable in $[-\pi, \pi]$, then $T(\theta)$ is the Fourier series of f.

The proof of Theorem 10 is exactly the same as for Theorem 8.

Remark. Idea of the proof of Theorem 7.

Under the hypothesis of Theorem 7, i.e., a_n and b_n are of $o(n)$ and $T(\theta)$ is Abel summable *everywhere* to a finite and integrable function, $\sum_{n=1}^{\infty}\frac{A_n(\theta)}{n^2}$ need not be uniformly convergent to a continuous function and so we cannot conclude immediately that it is A-summable. However, by the Riesz-Fisher Theorem, $\sum_{n=1}^{\infty}\frac{A_n(\theta)}{n^2}$ is a Fourier series. It can be shown that $T(\theta)$ is Abel summable implies that $\sum_{n=1}^{\infty}\frac{A_n(\theta)}{n^2}$ is A-summable to $H(\theta)$. It is harder to show that $H(\theta)$ is continuous everywhere. Once this is established then $\sum_{n=1}^{\infty}\frac{A_n(\theta)}{n^2}$ is the Fourier series of $H(\theta)$ by Theorem 8 and $\frac{1}{4}a_0\theta^2-\sum_{n=1}^{\infty}\frac{A_n(\theta)}{n^2}$ is A-summable everywhere to the continuous function $\frac{1}{4}a_0\theta^2-H(\theta)$. We can then deduce that $T(\theta)$ is the Fourier series of f by using Rajchman's Lemma as in the proof of Theorem 8.

Next consider the cosine series:

(C3) $$\cos(\theta)+\frac{\cos(2\theta)}{2}+\frac{\cos(3\theta)}{3}\cdots \qquad \text{-------------} \quad (21)$$

Theorem 11. (C3) converges pointwise and dominatedly to the continuous function $\ln\left(\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right)$ for θ in $(0, 2\pi)$. (C3) is the Fourier series of $\ln\left(\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right)$ and converges to $\ln\left(\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right)$ in the L^1 norm.

Proof.

Consider the complex valued function $f(z)=Log(1-z)$, where *Log* is the principal complex logarithmic function. Since 1 – *z* maps the open unit disc $D=\{a:|z|<1\}$ into the analytic domain of the principal logarithmic function, *f* is analytic in the open unit disk and in particular, by the chain rule, for all *z* in *D*,

$$f'(z)=-\frac{1}{1-z}=-(1+z+z^2+\cdots)\ . \qquad \text{--------------------} \quad (22)$$

And so for *z* in *D*,

$$f(z)=\int_{[0\to z]}f'(\zeta)d\zeta=-\int_{[0\to z]}\sum_{n=0}^{\infty}\zeta^n d\zeta=-\sum_{n=0}^{\infty}\int_{[0\to z]}\zeta^n d\zeta$$

$$= -\sum_{n=1}^{\infty} \frac{z^n}{n} \ . \qquad \text{--------------------------------} \quad (23)$$

Let $z = re^{i\theta}$ for $0 \le r < 1$. Substitute this value of z in (23) we get

$$f(re^{i\theta}) = -\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} r^n - i\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} r^n \ . \quad \text{--------------} \quad (24)$$

Hence, the real part of $f(re^{i\theta})$ is given by

$$-\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} r^n \ . \qquad \text{------------------------} \quad (25)$$

That is to say, the real part of $Log(1-re^{i\theta})$ is given by (25).

Hence,

$$\operatorname{Re} Log(1-re^{i\theta}) = -\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} r^n \ .$$

Since $-\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n}$ is convergent for θ not a multiple of 2π, by Abel's theorem,

$$-\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} = \lim_{r \to 1^-} \operatorname{Re} Log(1-re^{i\theta}) \quad \text{or}$$

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} = -\lim_{r \to 1^-} \operatorname{Re} Log(1-re^{i\theta}) \ . \qquad \text{------------------} \quad (26)$$

But $\operatorname{Re} Log(1-re^{i\theta}) = \ln(|1-re^{i\theta}|) = \frac{1}{2}\ln(1+r^2-2r\cos(\theta))$ and so

$$\lim_{r \to 1^-} \operatorname{Re} Log(1-re^{i\theta}) = \lim_{r \to 1^-} \frac{1}{2}\ln(1+r^2-2r\cos(\theta))$$

$$= \frac{1}{2}\ln(2-2\cos(\theta)) = \frac{1}{2}\ln(4\sin^2(\tfrac{\theta}{2})) = \ln(2\sin(\tfrac{\theta}{2}))$$

for θ in $(0, 2\pi)$.

Hence, it follows from (26) that for θ in $(0, 2\pi)$,

$$\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n}=-\ln(2\sin(\tfrac{\theta}{2}))=\ln\left(\frac{1}{2\sin(\frac{\theta}{2})}\right).$$

We shall show that the convergence is dominated by a Lebesgue integrable function.

Let $t_n(\theta)=\sum_{k=1}^{n}\frac{\cos(k\theta)}{k}$ be the n-th partial sum of (C3). Let $a_n=\frac{1}{n}$. Then

we can write

$$t_n(\theta)=\sum_{k=1}^{n}a_k\cos(k\theta)\,. \quad \text{---------------------------} \quad (27)$$

Using Abel's summation formula with $a_0=0$, we have

$$t_n(\theta)=\sum_{k=0}^{n-1}D_k(\theta)\Delta a_k+a_nD_n(\theta)\,, \quad \text{----------------------} \quad (28)$$

where

$$D_k(\theta)=\frac{1}{2}+\sum_{j=1}^{k}\cos(j\theta)=\frac{\sin((k+\frac{1}{2})\theta)}{2\sin(\frac{1}{2}\theta)} \quad \text{-------------------------} \ (29)$$

is the Dirichlet kernel. (See (8), (9) and (16) of my article *Fourier Cosine and Sine Series*.)

Applying the summation formula again gives

$$t_n(\theta)=\sum_{k=0}^{n-2}(k+1)K_k(\theta)\Delta^2a_k+nK_{n-1}(\theta)\Delta a_{n-1}+a_nD_n(\theta)\,, \quad \text{----------------} \quad (30)$$

where $K_k(\theta)=\frac{1}{k+1}\sum_{j=0}^{k}D_j(\theta)=\frac{2}{k+1}\left\{\frac{\sin(\frac{1}{2}(k+1)\theta)}{2\sin(\frac{1}{2}\theta)}\right\}^2$ is the Fejér kernel. (See (14), (15) and (19) of my article *Fourier Cosine and Sine Series*.)

For a fixed θ in (0, 2π),

$$a_nD_n(\theta)=\frac{1}{n}\frac{\sin((n+\frac{1}{2})\theta)}{2\sin(\frac{1}{2}\theta)}\to 0 \quad \text{as} \quad n\to\infty\ . \qquad \text{---------------} \quad (31)$$

Note that $|D_n(\theta)|\le\frac{1}{2}+n$ so that $|a_nD_n(\theta)|\le 2$. ----------------- (32)

Observe too that

$$nK_{n-1}(\theta)\Delta a_{n-1} = 2\left\{\frac{\sin(\frac{1}{2}n\theta)}{2\sin(\frac{1}{2}\theta)}\right\}^2 \Delta a_{n-1} \to 0 \text{ as } n \to \infty \quad \text{---------------------} (33)$$

and $|nK_{n-1}(x)| = \left|\sum_{j=0}^{n-1} D_j(x)\right| \le \sum_{j=0}^{n-1} |D_j(x)| \le \frac{n}{2} + \frac{n-1}{2}n = \frac{1}{2}n^2$ so that for $n \ge 2$,

$$|nK_{n-1}(\theta)\Delta a_{n-1}| \le \frac{1}{2}n^2 \frac{1}{n(n-1)} < 1. \quad \text{----------------------------} (34)$$

Observe that, for $k \ge 1$,

$$(k+1)K_k(\theta)\Delta^2 a_k = 2\left\{\frac{\sin(\frac{1}{2}(k+1)\theta)}{2\sin(\frac{1}{2}\theta)}\right\}^2 \Delta^2 a_k \ge 0. \quad \text{---------------------} (35)$$

For $k = 0$, $K_0(\theta)\Delta^2 a_0 = D_0(\theta)(\Delta a_0 - \Delta a_1) = \frac{1}{2}(-1-\frac{1}{2}) = -\frac{3}{4}$ and so, for $n \ge 3$,

$$|t_n(\theta)| \le \sum_{k=1}^{n-2}(k+1)K_k(\theta)\Delta^2 a_k + 3 + \frac{3}{4}. \quad \text{------------------------------} (36)$$

Plainly, $|t_n(\theta)| \le 3\frac{3}{4}$ for $n = 1$ and 2.

If we define $b_0 = \frac{3}{2}$ and $b_n = a_n$ for $n \ge 1$, then $\Delta^2 b_0 = 0$ and $\Delta^2 b_n = \Delta^2 a_n$ for $n \ge 1$ and

$$\sum_{k=1}^{n-2}(k+1)K_k(\theta)\Delta^2 a_k = \sum_{k=0}^{n-2}(k+1)K_k(\theta)\Delta^2 b_k. \quad \text{--------------------} (37)$$

It follows that

$$|t_n(\theta)| \le \sum_{k=0}^{n-2}(k+1)K_k(\theta)\Delta^2 b_k + \frac{15}{4} \le \sum_{k=0}^{\infty}(k+1)K_k(\theta)\Delta^2 b_k + \frac{15}{4}. \quad \text{-------} (38)$$

Since each term of the series $\sum_{k=0}^{\infty}(k+1)K_k(\theta)\Delta^2 b_k$ is non-negative, by Bepo Levi Theorem, $\sum_{k=0}^{n}(k+1)K_k(\theta)\Delta^2 b_k$ converges to a Lebesgue integrable function $g(\theta)$ in $(0, 2\pi)$.

Indeed, since (b_n) is a decreasing null convex sequence, $\sum_{k=0}^{\infty}(k+1)\Delta^2 b_k = b_0$ and

$$\int_0^{\pi} g(\theta)d\theta = \sum_{k=0}^{\infty}(k+1)\Delta^2 b_k \int_0^{\pi} K_k(\theta)d\theta = \sum_{k=0}^{\infty}(k+1)\Delta^2 b_k \frac{\pi}{2} = \frac{\pi}{2} b_0 = \frac{3}{4}\pi . \quad \text{---- (39)}$$

This shows that the partial sums of (C3) are dominated by $g(\theta)$ +15/4. Hence, (C3) converges dominatedly in (0, 2π). It follows that (C3) is the Fourier series of $\ln\left(\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right)$ and that (C3) converges to $\ln\left(\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right)$ in the L^1 norm.

Remark.

(1) Since the convergence of (C3) is dominated by a Lebesgue integrable function, the limit function $\ln\left(\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right)$ is Lebesgue integrable on [0, 2π]. A direct calculation shows that $\ln\left(\frac{1}{\sin(\theta)}\right)$ is Lebesgue integrable on [0, π] and that $\int_0^{\pi/2}\ln\left(\frac{1}{\sin(\theta)}\right)d\theta = \frac{\pi}{2}\ln(2)$, from which we can deduce that $\ln\left(\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right) = \ln\left(\frac{1}{\sin\left(\frac{\theta}{2}\right)}\right) - \ln(2)$ is Lebesgue integrable and that

$$\int_0^{\pi}\ln\left(\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right)d\theta = \int_0^{\pi}\ln\left(\frac{1}{\sin\left(\frac{\theta}{2}\right)}\right)d\theta - \pi\ln(2)$$

$$= 2\int_0^{\pi/2}\ln\left(\frac{1}{\sin(u)}\right)du - \pi\ln(2) = 0.$$

This confirms that the constant term of the Fourier series is 0.

Observe that $\int_{\pi/2}^{\pi}\ln\left(\frac{1}{\sin(\theta)}\right)d\theta = \lim_{t\to\pi^-}\int_{\pi/2}^{t}\ln\left(\frac{1}{\sin(\theta)}\right)d\theta$

$$= \lim_{t\to\pi^-}\int_{\pi/2}^{t}\ln\left(\frac{1}{\sin(\pi-\theta)}\right)d\theta = -\lim_{t\to 0^+}\int_{\pi/2}^{t}\ln\left(\frac{1}{\sin(u)}\right)du = \lim_{t\to 0^+}\int_{t}^{\pi/2}\ln\left(\frac{1}{\sin(u)}\right)du$$

$$= \int_0^{\pi/2} \ln\left(\frac{1}{\sin(\theta)}\right) d\theta = \frac{\pi}{2}\ln(2).$$

(2) Note that adding $\frac{3}{4}$ to (C3) gives

(C4) $$\frac{3}{4} + \cos(\theta) + \frac{\cos(2\theta)}{2} + \frac{\cos(3\theta)}{3} \cdots$$

and this is the trigonometric series $\frac{1}{2}b_0 + \sum_{n=1}^{\infty} b_n \cos(n\theta)$, where (b_n) is given as above and is a convex sequence. By Theorem 2 of my article *Fourier Cosine and Sine Series,* (C4) converges except at $\theta = 0$ in [– $\pi/2$, $\pi/2$] to a Lebesgue integrable function and is the Fourier series of its sum function and that the convergence is also in the L^1 norm. It follows that (C3) also converges to a Lebesgue integrable function and is the Fourier series of its sum function and that the convergence is also in the L^1 norm. Indeed (C4) converges to

$\frac{3}{4} + \ln\left(\frac{1}{2|\sin\left(\frac{\theta}{2}\right)|}\right)$ in [– $\pi/2$, $\pi/2$] except at $\theta = 0$.

Note that $\sum_{k=0}^{\infty}(k+1)K_k(\theta)\Delta^2 b_k$ converges to $\frac{3}{4} + \ln\left(\frac{1}{2|\sin\left(\frac{\theta}{2}\right)|}\right)$ in (0, 2π) and so

$g(\theta) = \frac{3}{4} + \ln\left(\frac{1}{2|\sin\left(\frac{\theta}{2}\right)|}\right)$ in (0, 2π). Indeed, the partial sums of (C3) are

dominated by $\frac{9}{2} + \ln\left(\frac{1}{2|\sin\left(\frac{\theta}{2}\right)|}\right)$ in (0, 2π).

(3) Knowing that (C3) converges almost everywhere in (0, 2π) to

$\ln\left(\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right)$, which is Lebegsue integrable in [0, 2π], an application of

Theorem 1 in my article *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric series,* concludes that (C3) is the

Fourier series of $\ln\left(\frac{1}{2|\sin\left(\frac{\theta}{2}\right)|}\right)$. We may apply also either Theorem 6 or

Theorem 10 to make the same conclusion.

(4) We may use complex analytic function in the unit disk and Abel's Theorem to sum a trigonometric series as above. Suppose

$$T(\theta)=\frac{1}{2}a_0+\sum_{k=1}^{\infty}\left(a_k\cos(k\theta)+b_k\sin(k\theta)\right)=\frac{1}{2}a_0+\sum_{k=1}^{\infty}A_k(\theta)$$

is a trigonometric series. Let $F(z)=c_0+2\sum_{k=1}^{\infty}c_k z^k$, where b_0 = 0 and

$c_n=\frac{1}{2}(a_n-ib_n)$. Then if we write $z=re^{i\theta}$,

$u(r,\theta)=\frac{1}{2}a_0+\sum_{k=1}^{\infty}r^kA_k(\theta)=\frac{1}{2}(F(z)+\overline{F(z)})=\operatorname{Re}F(z)$. Thus, if $T(\theta)$ is convergent at θ,

$$\lim_{r\to1^-}u(r,\theta)=\lim_{r\to1^-}\left(\frac{1}{2}a_0+\sum_{k=1}^{\infty}r^kA_k(\theta)\right)=\lim_{r\to1^-}\operatorname{Re}F(re^{i\theta})=T(\theta).$$

If $F(z)$ has a closed formula and $\lim_{r\to1^-}\operatorname{Re}F(re^{i\theta})=\operatorname{Re}F(e^{i\theta})$, then

$T(\theta)=\operatorname{Re}F(e^{i\theta})$.

For (C3), we can take $F(z)=\sum_{n=1}^{\infty}\frac{z^n}{n}$ and a closed formula for F(z) is

$F(z)=-Log(1-z)$ and for θ in (0, 2π) we obtain $T(\theta)=\operatorname{Re}F(e^{i\theta})$ as the answer above.

The conjugate series of (C3) is

$$-\sin(\theta)-\frac{\sin(2\theta)}{2}-\frac{\sin(3\theta)}{3}\cdots$$

and the imaginary part of $F(re^{i\theta})$ is $\sum_{k=1}^{\infty}r^k\frac{1}{k}\sin(k\theta)$ and we obtain for θ in (0, 2π),

$$\sin(\theta)+\frac{\sin(2\theta)}{2}+\frac{\sin(3\theta)}{3}\cdots=-\operatorname{Im}Log(1-e^{i\theta})$$

$$=-Arg\,(1-e^{i\theta})=\frac{1}{2}(\pi-\theta).$$

Hence, the conjugate series converges to – $h(\theta)$ in $[0, 2\pi]$, where

$$h(\theta) = \begin{cases} \frac{1}{2}(\pi - \theta), & 0 < \theta < 2\pi \\ 0 & \theta = 0, \quad 2\pi \end{cases} .$$

We now turn to some relation between Riemann summability and Abel summability.

Theorem 12. Suppose F is a periodic Lebesgue integrable function of period 2π. Suppose $T(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(n\theta) + b_n\sin(n\theta)\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}A_n(\theta)$ is the Fourier series of F. If F has a generalized second symmetric derivative $D_2\,F(\theta)$ at θ, then the second derived series of the Fourier series of F at θ,

$$\sum_{k=1}^{\infty} -k^2\left(a_k\cos(k\theta) + b_k\sin(k\theta)\right) = -\sum_{k=1}^{\infty}k^2 A_k(\theta),$$

is A-summable to $D_2F(\theta)$.

Proof.

Consider

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}A_k(\theta)r^k = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}\left(a_k\cos(k\theta) + b_k\sin(k\theta)\right)r^k$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}P(r,t-\theta)F(t)dt, \qquad \text{-------------- (40)}$$

where $P(r,x) = \dfrac{1-r^2}{2(1+r^2-2\,r\cos(x))}$ is the Poisson kernel.

(See (14) of my article *Abel summability of Fourier series and its Derived series*.)

We shall need the partial derivative of the Poisson kernel. Note that

$$\frac{\partial}{\partial t}P(r,t) = \frac{-r(1-r^2)\sin(t)}{(1+r^2-2\,r\cos(t))^2}.$$

Then

$$\frac{\partial}{\partial\theta}u(r,\theta)=\frac{\partial}{\partial\theta}\frac{1}{\pi}\int_{-\pi}^{\pi}P(r,t-\theta)F(t)dt=-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial t}P(r,t-\theta)F(t)dt$$

and so

$$\frac{\partial^2}{\partial\theta^2}u(r,\theta)=-\frac{\partial}{\partial\theta}\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial t}P(r,t-\theta)F(t)dt=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial^2}{\partial t^2}P(r,t-\theta)F(t)dt\,. \quad \text{--- (41)}$$

It follows that the second derived series of $T(\theta)$ is Abel summable at θ, if, and only if, the limit $\lim_{r\to 1^-}\frac{\partial^2}{\partial\theta^2}u(r,\theta)=\lim_{r\to 1^-}\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial^2}{\partial t^2}P(r,t-\theta)F(t)dt$ exists.

$$\frac{\partial^2}{\partial\theta^2}u(r,\theta)=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial^2}{\partial t^2}P(r,t-\theta)F(t)dt$$

$$=\frac{1}{\pi}\int_{-\pi-\theta}^{\pi-\theta}\frac{\partial^2}{\partial t^2}P(r,s)F(s+\theta)ds$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial^2}{\partial t^2}P(r,s)F(s+\theta)ds\text{, by periodicity,}$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\frac{\partial^2}{\partial t^2}P(r,s)F(s+\theta)ds+\frac{1}{\pi}\int_{-\pi}^{0}\frac{\partial^2}{\partial t^2}P(r,s)F(s+\theta)ds$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\frac{\partial^2}{\partial t^2}P(r,s)F(s+\theta)ds+\frac{1}{\pi}\int_{0}^{\pi}\frac{\partial^2}{\partial t^2}P(r,-s)F(-s+\theta)ds\,. \quad \text{----- (42)}$$

But $\frac{\partial^2}{\partial t^2}P(r,t)=-r(1-r^2)\frac{(1+r^2)\cos(t)+2r\cos^2(t)-4r}{(1+r^2-2\,r\cos(t))^3}$ so that $\frac{\partial^2}{\partial t^2}P(r,-s)=\frac{\partial^2}{\partial t^2}P(r,s)$ and so it follows from (42) that

$$\frac{\partial^2}{\partial\theta^2}u(r,\theta)=\frac{1}{\pi}\int_{0}^{\pi}\frac{\partial^2}{\partial t^2}P(r,s)F(s+\theta)ds+\frac{1}{\pi}\int_{0}^{\pi}\frac{\partial^2}{\partial t^2}P(r,s)F(-s+\theta)ds$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\big(F(\theta+s)+F(\theta-s)\big)ds \quad \text{-------------------------------- (43)}$$

$$= -\frac{1}{\pi}\int_0^{\pi} r(1-r^2)\frac{(1+r^2)\cos(t)+2r\cos^2(t)-4r}{(1+r^2-2\,r\cos(t))^3}\big(F(\theta+s)+F(\theta-s)\big)ds \text{ . --- (44)}$$

$$\frac{1}{\pi}\int_0^{\pi}\frac{\partial^2}{\partial t^2}P(r,s)\sin^2(s)ds$$

$$= \left[\frac{1}{\pi}\sin^2(s)\frac{\partial}{\partial t}P(r,s)\right]_{s=0}^{s=\pi} - \frac{1}{\pi}\int_0^{\pi} 2\sin(s)\cos(s)\frac{\partial}{\partial t}P(r,s)ds$$

$$= -\frac{1}{\pi}\int_0^{\pi}\sin(2s)\frac{\partial}{\partial t}P(r,s)ds \text{, by integration by parts,}$$

$$= \left[-\frac{1}{\pi}\sin(2s)P(r,s)\right]_{s=0}^{s=\pi} + \frac{1}{\pi}\int_0^{\pi} 2\cos(2s)P(r,s)ds \text{, by integration by parts,}$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\cos(2s)P(r,s)ds$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{\cos(2s)(1-r^2)}{1+r^2-2r\cos(s)}ds$$

$$= \frac{1-r^2}{\pi}\int_0^{\pi}\frac{\cos(2s)}{1+r^2-2r\cos(s)}ds = \frac{1-r^2}{\pi}\frac{\pi r^2}{1-r^2} = r^2, \qquad \text{--------------- (45)}$$

$$\text{since } \int_0^{\pi}\frac{\cos(2s)}{1+r^2-2r\cos(s)}ds = \frac{\pi r^2}{1-r^2}.$$

Hence,

$$\frac{\partial^2}{\partial\theta^2}u(r,\theta) - D_2F(\theta) == \frac{1}{\pi}\int_0^{\pi}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\big(F(\theta+s)+F(\theta-s)\big)ds - D_2F(\theta)$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\big(F(\theta+s)+F(\theta-s)\big)ds - \frac{1}{\pi}\int_0^{\pi}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\sin^2(s)\frac{D_2F(\theta)}{r^2}ds$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\left(F(\theta+s)+F(\theta-s)-\sin^2(s)\frac{D_2F(\theta)}{r^2}\right)ds. \quad \text{------------------ (46)}$$

Now, observe that $\frac{1}{\pi}\int_0^{\pi}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)ds=\frac{1}{\pi}\left[\frac{\partial}{\partial t}P(r,s)\right]_{s=0}^{s=\pi}=0$.

And so we can write

$$\frac{\partial^2}{\partial\theta^2}u(r,\theta)-D_2F(\theta)$$

$$=\frac{1}{\pi}\int_0^{\pi}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\left(F(\theta+s)+F(\theta-s)-2F(\theta)-\sin^2(s)\frac{D_2F(\theta)}{r^2}\right)ds\ . \qquad \text{---------------} \quad (47)$$

Now, we shall examine the bounds for $\frac{\partial^2}{\partial t^2}P(r,s)$.

Recall $\frac{\partial^2}{\partial t^2}P(r,s)=-r(1-r^2)\frac{(1+r^2)\cos(s)+2r\cos^2(s)-4r}{(1+r^2-2\,r\cos(s))^3}$

$$=-r(1-r^2)\left(\frac{\cos(s)}{(1+r^2-2\,r\cos(s))^2}-\frac{4r\sin^2(s)}{(1+r^2-2\,r\cos(s))^3}\right). \qquad \text{-------------} \ (48)$$

Let $0<\delta<\frac{\pi}{2}$.

Then for $\delta\le s<\pi/2$ and $0\le r<1$,

$$\left|\frac{\partial^2}{\partial t^2}P(r,s)\right|\le r(1-r^2)\left(\frac{1}{\sin^4(\delta)}+\frac{4r}{\sin^4(\delta)}\right)\le r(1-r^2)(1+4r)\left(\frac{1}{\sin^4(\delta)}\right)$$

while for $\pi/2\le s\le\pi$ and $0\le r<1$,

$$\left|\frac{\partial^2}{\partial t^2}P(r,s)\right|\le r(1-r^2)(1+4r)\,.$$

Hence for $\delta\le s\le\pi$ and $0<r<1$,

$$\left|\frac{\partial^2}{\partial t^2}P(r,s)\right|\le r(1-r^2)(1+4r)\left(\frac{1}{\sin^4(\delta)}\right)\ . \qquad \text{---------------------} \quad (49)$$

It follows then for $0<\delta<\pi/2$ and $0<r<1$,

$$\frac{1}{\pi}\left|\int_{\delta}^{\pi}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\left(F(\theta+s)+F(\theta-s)-2F(\theta)-\sin^2(s)\frac{D_2F(\theta)}{r^2}\right)ds\right|$$

$$\leq \frac{r(1-r^2)(1+4r)}{\sin^4(\delta)}\frac{1}{\pi}\int_{\delta}^{\pi}\left(|F(\theta+s)|+|F(\theta-s)|+2|F(\theta)|+\frac{|D_2F(\theta)|}{r^2}\right)ds$$

$$\leq \frac{r(1-r^2)(1+4r)}{\sin^4(\delta)}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}2|F(s)|ds+2|F(\theta)|+\frac{|D_2F(\theta)|}{r^2}\right) \text{ . ------------------- (50)}$$

It follows from (50) that

$$\lim_{r\to 1^-}\frac{1}{\pi}\left|\int_{\delta}^{\pi}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\left(F(\theta+s)+F(\theta-s)-2F(\theta)-\sin^2(s)\frac{D_2F(\theta)}{r^2}\right)ds\right|=0.$$

Hence,

$$\lim_{r\to 1^-}\frac{1}{\pi}\int_{\delta}^{\pi}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\left(F(\theta+s)+F(\theta-s)-2F(\theta)-\sin^2(s)\frac{D_2F(\theta)}{r^2}\right)ds=0 \text{ . ----- (51)}$$

Now, since $D_2F(\theta)=\lim_{h\to 0}\frac{F(\theta+h)+F(\theta-h)-2F(\theta)}{h^2}$,

$$\lim_{h\to 0}\frac{F(\theta+h)+F(\theta-h)-2F(\theta)}{\sin^2(h)}$$

$$=\lim_{h\to 0}\frac{F(\theta+h)+F(\theta-h)-2F(\theta)}{h^2}\frac{h^2}{\sin^2(h)}=D_2F(\theta).$$

Thus, given $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < \delta < \pi/2$ and

$$0<|s|\leq\delta \Rightarrow \left|\frac{F(\theta+s)+F(\theta-s)-2F(\theta)}{\sin^2(s)}-D_2F(\theta)\right|<\varepsilon. \text{ ----------- (52)}$$

For this value of δ ,

$$\left|\frac{1}{\pi}\int_{0}^{\delta}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\left(F(\theta+s)+F(\theta-s)-2F(\theta)-\sin^2(s)\frac{D_2F(\theta)}{r^2}\right)ds\right|$$

$$=\left|\frac{1}{\pi}\int_0^{\delta}\left(\frac{\partial^2}{\partial t^2}P(r,s)\sin^2(s)\right)\left(\frac{F(\theta+s)+F(\theta-s)-2F(\theta}{\sin^2(s)}-D_2F(\theta)+D_2F(\theta)\left(1-\frac{1}{r^2}\right)\right)ds\right|$$

$$\le\frac{1}{\pi}\int_0^{\delta}\left|\frac{\partial^2}{\partial t^2}P(r,s)\sin^2(s)\right|\left|\frac{F(\theta+s)+F(\theta-s)-2F(\theta}{\sin^2(s)}-D_2F(\theta)\right|ds$$

$$+\frac{1}{\pi}\int_0^{\delta}\left|\left(\frac{\partial^2}{\partial t^2}P(r,s)\sin^2(s)\right)\right|ds\left|D_2F(\theta)\right|\left|1-\frac{1}{r^2}\right|$$

$$\le\left(\varepsilon+\left|1-\frac{1}{r^2}\right|\left|D_2F(\theta)\right|\right)\frac{1}{\pi}\int_0^{\delta}\left|\frac{\partial^2}{\partial t^2}P(r,s)\sin^2(s)\right|ds\,.$$

$$\le\left(\varepsilon+\left(\frac{1}{r^2}-1\right)\left|D_2F(\theta)\right|\right)\frac{1}{\pi}\int_0^{\pi}\left|\frac{\partial^2}{\partial t^2}P(r,s)\sin^2(s)\right|ds \quad \text{----------------------------} \quad (53)$$

Now, using (48) we get

$$\frac{\partial^2}{\partial t^2}P(r,s)\sin^2(s)=-r(1-r^2)\left(\frac{\cos(s)\sin^2(s)}{(1+r^2-2\,r\cos(s))^2}-\frac{4r\sin^4(s)}{(1+r^2-2\,r\cos(s))^3}\right)$$

-- (54)

and so,

$$\left|\frac{\partial^2}{\partial t^2}P(r,s)\sin^2(s)\right|\le r(1-r^2)\left(\frac{1}{(1+r^2-2\,r\cos(s))}+\frac{4r}{(1+r^2-2\,r\cos(s))}\right).$$

------------------------------------ (55)

Therefore,

$$\int_0^{\pi}\left|\frac{\partial^2}{\partial t^2}P(r,s)\sin^2(s)\right|ds\le r(1-r^2)(1+4r)\int_0^{\pi}\frac{1}{(1+r^2-2\,r\cos(s))}ds$$

$$\le r(1-r^2)(1+4r)\frac{\pi}{1-r^2}=r(1+4r)\pi\,. \quad \text{-------} \quad (56)$$

Therefore, it follows from (53) and (56) that given $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < \delta < \pi/2$ and for $0 < r < 1$,

$$\frac{1}{\pi}\left|\int_0^{\delta}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\left(F(\theta+s)+F(\theta-s)-2F(\theta)-\sin^2(s)\frac{D_2F(\theta)}{r^2}\right)ds\right|$$

$$\leq\left(\varepsilon+\left(\frac{1}{r^2}-1\right)\left|D_2F(\theta)\right|\right)\frac{1}{\pi}\int_0^{\pi}\left|\frac{\partial^2}{\partial t^2}P(r,s)\sin^2(s)\right|ds\leq\left(\varepsilon+\left(\frac{1}{r^2}-1\right)\left|D_2F(\theta)\right|\right)r(1+4r)$$

$$<5\varepsilon+5\left(\frac{1}{r^2}-1\right)\left|D_2F(\theta)\right|.$$

Hence,

$$\limsup_{r\to1^-}\frac{1}{\pi}\left|\int_0^{\delta}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\left(F(\theta+s)+F(\theta-s)-2F(\theta)-\sin^2(s)\frac{D_2F(\theta)}{r^2}\right)ds\right|\leq5\varepsilon.$$

------------------ (57)

Therefore,

$$\limsup_{r\to1^-}\left|\frac{\partial^2}{\partial\theta^2}u(r,\theta)-D_2F(\theta)\right|$$

$$=\limsup_{r\to1^-}\frac{1}{\pi}\left|\int_0^{\pi}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\left(F(\theta+s)+F(\theta-s)-2F(\theta)-\sin^2(s)\frac{D_2F(\theta)}{r^2}\right)ds\right|$$

$$\leq\limsup_{r\to1^-}\frac{1}{\pi}\left|\int_0^{\delta}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\left(F(\theta+s)+F(\theta-s)-2F(\theta)-\sin^2(s)\frac{D_2F(\theta)}{r^2}\right)ds\right|$$

$$+\limsup_{r\to1^-}\frac{1}{\pi}\left|\int_{\delta}^{\pi}\left(\frac{\partial^2}{\partial t^2}P(r,s)\right)\left(F(\theta+s)+F(\theta-s)-2F(\theta)-\sin^2(s)\frac{D_2F(\theta)}{r^2}\right)ds\right|$$

$\leq5\varepsilon+0=5\varepsilon$, by (57) and (51).

Since ε is arbitrary, $\limsup_{r\to1^-}\left|\frac{\partial^2}{\partial\theta^2}u(r,\theta)-D_2F(\theta)\right|$=0 and so $\lim_{r\to1^-}\left(\frac{\partial^2}{\partial\theta^2}u(r,\theta)-D_2F(\theta)\right)=0$. This means $\lim_{r\to1^-}\frac{\partial^2}{\partial\theta^2}u(r,\theta)=D_2F(\theta)$ and the second derived series of $T(\theta)$ is A-summable at θ to $D_2F(\theta)$.

This proves Theorem 12.

Theorem 13.

Suppose $T(\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(n\theta)+b_n\sin(n\theta)\right)$ is a trigonometric series.

(1) Suppose $T(\theta)$ satisfies the condition of Theorem 1. In particular, it is Riemann summable everywhere to $f(\theta)$. Then $T(\theta)$ is Abel summable everywhere to $f(\theta)$.

(2) Suppose $T(\theta)$ satisfies the condition of Theorem 4 or Theorem 5 or Theorem 6. In particular, it is Riemann summable at θ to $f(\theta)$ except for θ in an enumerable set E. Then $T(\theta)$ is Abel summable at θ to $f(\theta)$ for all θ in $[-\pi, \pi]-E$.

Proof. Recall that in all cases $\sum_{n=1}^{\infty}\left(\frac{a_n\cos(nx)+b_n\sin(nx)}{n^2}\right)=\sum_{n=1}^{\infty}\frac{A_n(x)}{n^2}$ is the Fourier series of a continuous function Ψ. In particular $T(\theta)$ is Riemann summable to $f(\theta)$ implies that $f(\theta)=\frac{1}{2}a_0-D_2\Psi(\theta)$. Hence, $D_2\Psi(\theta)=\frac{1}{2}a_0-f(\theta)$.

Note that the second derived series of $\sum_{n=1}^{\infty}\frac{A_n(x)}{n^2}$ is $-\sum_{n=1}^{\infty}A_n(\theta)=-T(\theta)+\frac{1}{2}a_0$.

By Theorem 12, $-T(\theta)+\frac{1}{2}a_0$ is A-summable to $D_2\Psi(\theta)=\frac{1}{2}a_0-f(\theta)$.

It follows that $T(\theta)$ is A-summable to $f(\theta)$.

This completes the proof.

Remark.

Theorem 13 and Theorem 10 implies Theorem 6.

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