Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric Series

By Ng Tze Beng

This article is about the ideas leading to the uniqueness of a convergent trigonometric series. We examine the ideas involved when the limit function of a trigonometric series is Lebesgue integrable. Through the use of Perron's technique, we characterize Lebesgue integrability by Perron's major and minor functions. Through this and de la Vallée-Poussin's majorant and minorant functions and Riemann's idea of passing from the symmetric second derivative to the trigonometric series, an idea now called R-summability, we deduce the uniqueness of Fourier Lebesgue series as stated in Theorem 1. When the trigonometric series need not converge to a Lebegsue integrable function but is everywhere convergent, building on the idea of Perron's major and minor functions but using the second symmetric derivative, James's J-major and J-minor functions are used to introduce the idea of P^2 integral to prove the uniqueness of an everywhere convergent trigonometric series, where the coefficients are now recovered by the P^2 integral.

Introduction

Consider the trigonometric series

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right) \qquad \text{-------------} \quad (A)$$

It is our aim to understand the proof of the uniqueness of the Fourier series of the sum function of (A), that is, the proof of the following fundamental result about Fourier series.

Theorem 1. If the series (A) converges except in an enumerable set E to a finite and integrable function f, then it is the Fourier series of f.

Our approach would be that of Riemann and uses Perron's major and minor functions together with the generalized second derivative.

When the series (A) converges everywhere to a non Lebesgue integrable function f, the coefficients are not recoverable by Lebesgue integration. We present a solution by R. D. James using his P^2 integrals.

We shall elaborate the ideas in three sections. Section A considers characterization of the Lebesgue integral in terms of lower and upper semi-continuous function. Properties of these functions that are useful will be elaborated and the ideas of upper and lower derivates will be introduced to provide a means of obtaining major and minor functions. Section B then considers the idea of the generalized symmetric second derivative of a function, its properties and Riemann's idea in the proof. Section C contains the uniqueness theorems and their proofs. We introduce in this section the idea of R. D. James' P^2 integral and use this to prove the uniqueness of everywhere convergence trigonometric series.

Section A. Semi-continuous Function, Major and Minor Functions and Lebesgue Theory

Lower and upper semi-continuous functions

Let **R*** be the extended real numbers. Suppose f : [a, b] $\to$ **R*** is an extended real valued function.

Let $m_f(x) = \liminf_{\delta \to 0^+} \{ f(t) : t \in B(x,\delta) \cap [a,b] \}$ and

$$M_f(x) = \limsup_{\delta \to 0^+} \{ f(t) : t \in B(x,\delta) \cap [a,b] \}, \text{ where } B(x,\delta) = (x-\delta, x+\delta) \ .$$

Plainly, by the definition of lim inf and lim sup, for all x in [a, b],

$$m_f(x) \le f(x) \le M_f(x)$$

Definition 1. Let c be in [a, b]. The function f is said to be *lower semi-continuous* at c if $m_f(c) = f(c)$. f is said to be *upper semi-continuous* at c if $M_f(c) = f(c)$.

Therefore, a finite valued function is continuous at c if, and only if, it is both lower and upper semi-continuous at c. Note that just having one sided semi-continuity at c does not imply continuity at c.

To work with lower and upper semi-continuous function, it is useful to use an equivalent form of this property. The following theorem gives equivalent definition of lower and upper semi-continuity.

Theorem 2. Suppose $f : [a, b] \to \mathbf{R}^*$ is an extended real valued function. Let c be in $[a, b]$.

(a) f is lower semi-continuous at c if, and only if, for each $\alpha < f(c)$, there exists $\delta > 0$ such that $f(x) > \alpha$ for all $x \in B(c, \delta) \cap [a, b]$.

(b) Suppose $f(c)$ is finite. The function f is lower semi-continuous at c if, and only if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x) > f(c) - \varepsilon \text{ for all } x \in B(c, \delta) \cap [a, b].$$

(c) f is upper semi-continuous at c if, and only if, for each $\alpha > f(c)$, there exists $\delta > 0$ such that $f(x) < \alpha$ for all $x \in B(c, \delta) \cap [a, b]$.

(b) Suppose $f(c)$ is finite. The function f is upper semi-continuous at c if, and only if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x) < f(c) + \varepsilon \text{ for all } x \in B(c, \delta) \cap [a, b].$$

Proof. We prove only parts (a) and (b). Parts (c) and (d) are similarly proved.

(a) If $f(c) = -\infty$, then we have nothing to prove as plainly, $m_f(c) = f(c) = -\infty$. If $m_f(c) = f(c) = \infty$, then by definition of $m_f(c) = \infty$, for each $\alpha < f(c) = \infty$, there exists $\delta > 0$ such that $f(x) > \alpha$ for all $x \in B(c, \delta) \cap [a, b]$. If $f(c)$ is finite, then by definition of $m_f(c) = f(c)$ for each $\alpha < f(c)$, there exists $\delta > 0$ such that $f(x) > \alpha$ for all $x \in B(c, \delta) \cap [a, b]$.

(b) If $f(c)$ is finite, then take $\alpha = f(c) - \varepsilon < f(c)$. Part (b) then follows from part (a).

The next property concerns extreme values of a semi-continuous function on $[a, b]$.

Theorem 3. Suppose $f: [a, b] \to \mathbf{R}^*$ is an extended real valued function.

(i) If f is lower semi-continuous on $[a, b]$, then f assumes its minimum value.

(ii) If f is upper semi-continuous on $[a, b]$, then f assumes its maximum value.

Proof.

(i) Suppose f is lower semi-continuous on $[a, b]$. Let $m = \inf\{f(x) : x \in [a,b]\}$.

If $m = +\infty$, then f is a constant function taking $+\infty$ and we have nothing to prove.

If $m = -\infty$, then there exists a sequence (x_n) in $[a, b]$, such that $f(x_n) \to -\infty$.

If m is finite, then there exists a sequence (x_n) in $[a, b]$, such that $f(x_n) \to m$.

In both cases, by the Bolzano-Weierstrass Theorem, (x_n) has a convergence subsequence (x_{n_k}). Suppose $x_{n_k} \to d$.

Then if m is finite,

$$m = \inf\{f(x) : x \in [a,b]\} \le f(d) = m_f(d)$$

$$\le \liminf_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(x_{n_k}) = m,$$

and so $f(d) = m$.

If $m = -\infty$, then $\lim_{k\to\infty} f(x_{n_k}) = -\infty$ and

$$f(d) = m_f(d) = \liminf_{\delta\to 0^+} \{f(t) : t \in B(d,\delta) \cap [a,b]\} = -\infty = m.$$

The proof of part (ii) is similar and is omitted.

For finite function f we have the following obvious corollary.

Corollary 4. Suppose $f : [a, b] \to \mathbf{R}$ is a real valued function.

(i) If f is lower semi-continuous on $[a, b]$, then f is bounded below.

(ii) If f is upper semi-continuous on $[a, b]$, then f is bounded above.

Remark. A finite-valued semi-continuous function is a Baire class one function, i.e., it is the pointwise limit of a sequence of continuous function**.**

The next result relates Lebesgue integrable function with semi-continuous functions.

Theorem 5. Suppose $f : [a, b] \to \mathbf{R}^*$ is Lebesgue integrable.

For each $\varepsilon > 0$, there exist functions u and v such that

(i) u is lower semi-continuous on $[a, b]$ and v is upper semi-continuous on $[a, b]$;

(ii) for all x in $[a, b]$, $u(x) > -\infty$, $u(x) \geq f(x) \geq v(x)$ and $v(x) < \infty$;

(iii) u and v are Lebesgue integrable on $[a, b]$ and

$$\int_a^b u - \varepsilon < \int_a^b f < \int_a^b v + \varepsilon \ .$$

To establish Theorem 5 we shall use the following characterization of semi-continuous function.

Theorem 6. Suppose $f : [a, b] \to \mathbf{R}^*$ is an extended real valued function.

(a) f is lower semi-continuous on $[a, b]$ if, and only if, the set

$$\{x \in [a,b] : f(x) \leq k\} = f^{-1}([-\infty,k])$$

is closed for every real number k.

(b) f is upper semi-continuous on $[a, b]$ if, and only if, the set

$$\{x \in [a,b] : f(x) \geq k\} = f^{-1}([k,\infty])$$

is closed for every real number k.

Proof.

We shall prove part (a) only. The proof for part (b) is similar.

Suppose f is lower semi-continuous on $[a, b]$. Let k be any real number. Let

$B = \{x \in [a,b] : f(x) \leq k\} = f^{-1}([-\infty,k])$. If $B = \varnothing$, then B is closed. Now assume $B \neq \varnothing$. Let c be a limit point of B. Then c is in $[a, b]$. We shall show that c is in B, i.e., $f(c) \leq k$. If $f(c) = -\infty$, then obviously c is in B. We now assume that $f(c) > -\infty$. By Theorem 2 part(a), for any $\alpha < f(c)$, there exists δ

> 0 such that $f(x) > \alpha$ for all $x \in B(c, \delta) \cap [a, b]$. Since c is a limit point of B, there exists a point d in $B(c, \delta) \cap B - \{c\}$ such that $\alpha < f(d) \leq k$. This shows that k is an upper bound of the set $(-\infty, f(c))$. Therefore, $k \geq f(c)$. Hence c is in B. This shows that B is closed.

Conversely, suppose B is closed for every real number k. Take c in $[a, b]$. If $f(c) = -\infty$, then $m_f(c) = f(c) = -\infty$ and so f is lower semi-continuous at c. Assume now $f(c) > -\infty$. Let $k < f(c)$. By assumption $B = \{x \in [a,b] : f(x) \leq k\} = f^{-1}([-\infty, k])$ is closed in $[a, b]$ and does not contain the point c. This means c is in the complement of B which is open in $[a, b]$. Therefore, there exists $\delta > 0$ such that $B(c, \delta) \cap [a, b] \subseteq$ Complement of B in $[a, b]$. That is, for all x in $B(c, \delta) \cap [a, b]$, $f(x) > k$. Thus, by Theorem 2 part (a), f is lower semi-continuous at c. This completes the proof of part (a).

Proof of Theorem 5.

We prove the theorem when f is non-negative, bounded and Lebesgue integrable. Suppose for all x in $[a, b]$, $0 \leq f(x) < M$ for some real number M.

Given $\varepsilon > 0$, let $\eta = \dfrac{\varepsilon}{b-a+1} < \varepsilon$. Take an integer N such that $N\eta > M$.

For integer $k > 0$, let $E_k = \{x \in [a,b] : (k-1)\eta \leq f(x) < k\eta\} = f^{-1}([(k-1)\eta, k\eta))\}$.

Since f is Lebesgue integrable, f is measurable and E_k is measurable. Since $E_k \subseteq [a, b]$, the measure of E_k, $m(E_k)$ is finite. It follows that there exists an open set G_k such that $E_k \subseteq G_k$ and

$$m(G_k) < m(E_k) + \frac{2^{-k}}{k} \text{ . ----------- (1)}$$

Let $A_k = G_k \cap [a, b]$. Then A_k is open in $[a, b]$ and the characteristic function χ_{A_k} is lower semi-continuous on $[a, b]$. We deduce this as follows. In view of the fact that $\{x \in [a,b] : \chi_{A_k}(x) \leq \alpha\} = \chi_{A_k}^{-1}([-\infty, \alpha])$ is either empty, all of $[a, b]$ or the complement of A_k in $[a, b]$, which is closed in $[a, b]$, it follows from Theorem 6 part (a) that χ_{A_k} is lower semi-continuous on $[a, b]$. Let

$u = \sum_{k=1}^{N} k\eta\chi_{A_k}$. Then u is lower semi-continuous on $[a, b]$ since a finite sum of lower semi-continuous function is lower semi-continuous. Observe that for $x \in E_k$,

$$u(x) \geq k\ \eta > f(x) \geq (k-1)\ \eta . \qquad \text{----------------} \quad (2)$$

Therefore,

$$\int_a^b u = \sum_{k=1}^{N} k\eta\, m(A_k)$$

$$< \sum_{k=1}^{N} k\eta \left(m(E_k) + \frac{2^{-k}}{k} \right) = \sum_{k=1}^{N} (k-1)\eta m(E_k) + \sum_{k=1}^{N} \eta m(E_k) + \eta \sum_{k=1}^{N} 2^{-k}$$

by (1),

$$< \sum_{k=1}^{N} \int_{E_k} f + \eta(b-a) + \eta < \int_a^b f + \eta(b-a+1) = \int_a^b f + \varepsilon . \qquad \text{------------} \quad (3)$$

Suppose now f is nonnegative but unbounded.

For each integer $n > 0$, let $g_n(x) = \min\{f(x), n\}$. Then g_n is nonnegative and bounded and Lebesgue integrable. Define $f_1 = g_1$, and $f_n = g_n - g_{n-1}$ for $n > 1$.

Plainly, each f_n is nonnegative, bounded and Lebesgue integrable. In particular,

$$f = \sum_{n=1}^{\infty} f_n . \qquad \text{--------------------} \quad (4)$$

By the first part of the proof, that is, inequality (3) with ε replaced by $\frac{\varepsilon}{2^n}$,

we can find lower semi-continuous function u_n on $[a, b]$ such that

$$u_n \geq f_n \text{ on } [a, b] \text{ and } \int_a^b u_n < \int_a^b f_n + \frac{\varepsilon}{2^n} . \qquad \text{--------------} \quad (5)$$

Now let $u = \sum_{n=1}^{\infty} u_n$. Since finite sum of lower semi-continuous functions is lower semi-continuous, the n-th partial sum of the series h_n , is lower semi-continuous. Moreover, h_n is non-negative, bounded and converges pointwise to u (finite or infinitely). Then for any real number k,

$$\{x \in [a,b] : u(x) \le k\} = \bigcap_{n=1}^{\infty} \{x \in [a,b] : h_n(x) \le k\}$$

is closed in $[a, b]$ since each $\{x \in [a,b] : h_n(x) \le k\}$ is closed in $[a, b]$. It follows then from Theorem 6, that u is lower semi-continuous. In particular, from (5) we have, $u = \sum_{n=1}^{\infty} u_n \ge \sum_{n=1}^{\infty} f_n = f$ and by using the Lebesgue Monotone Convergence Theorem,

$$\int_a^b u = \sum_{n=1}^{\infty} \int_a^b u_n < \sum_{n=1}^{\infty} \left(\int_a^b f_n + \frac{\varepsilon}{2^n} \right) = \int_a^b f + \varepsilon .$$

Finally, suppose f is an arbitrary Lebesgue integrable function on $[a, b]$.

For each integer $n > 0$, let now $f_n(x) = \max\{f(x), -n\}$. Plainly,

$|f_n| \le |f|$ for all integer $n \ge 1$ and $f_n \to f$ pointwise on $[a, b]$. Therefore, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f .$$

Thus, given $\varepsilon > 0$, we can choose an integer N so that

$$\int_a^b f_N < \int_a^b f + \frac{\varepsilon}{2} . \qquad \text{-------------------------} \quad (6)$$

By definition of f_N, $f_N + N \ge 0$. So $f_N + N$ is nonnegative and Lebesgue integrable. Therefore, by what we have just proved for nonnegative integrable function, there is a lower semi-continuous function u_N such that

$$u_N \ge f_N + N \text{ and } \int_a^b u_N < \int_a^b (f_N + N) + \frac{\varepsilon}{2} . \text{ -------------} \quad (7)$$

Now, let $u = u_N - N$. Then $u = u_N - N \ge f_N \ge f$. In particular, from (7) we have,

$$\int_a^b u = \int_a^b (u_N - N) < \int_a^b f_N + \frac{\varepsilon}{2}$$

$$< \int_a^b f + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \int_a^b f + \varepsilon \quad \text{by inequality (6).}$$

To find an upper semi-continuous function v for the theorem, we note that if a function f is lower semi-continuous, then $-f$ is upper semi-continuous and use what we have just proved in the following manner. We can find a lower semi-continuous function w for $-f$ satisfying $w \geq -f$ and

$$\int_a^b w < \int_a^b -f + \varepsilon .$$

Now let $v = -w$ and so v is upper semi-continuous on $[a, b]$. Then

$$\int_a^b v = \int_a^b -w > \int_a^b f - \varepsilon .$$

This completes the proof of Theorem 5.

The next idea is to characterize Lebesgue integral in terms of major and minor functions, an idea of Perron which leads to a generalization of the Lebesgue integral. For this we need to bring in the idea of upper and lower derivate of a function.

Definition 7. Suppose F: $[a, b] \to \mathbf{R}$ is a real-valued function. Let c be in $[a, b]$. Then the *upper derivate of F* at c is defined by

$$\overline{D}F(c) = \limsup_{x \to c} \frac{F(x) - F(c)}{x - c}$$

and the lower derivate of F at c is defined by

$$\underline{D}F(c) = \liminf_{x \to c} \frac{F(x) - F(c)}{x - c}.$$

It is easy to see that F is differentiable at c if and only if both $\overline{D}F(c)$ and $\underline{D}F(c)$ are finite and equal.

We state some useful results below, starting with one about continuity and the other about monotonicity.

Theorem 8. Suppose F: $[a, b] \to \mathbf{R}$ is a real-valued function. Let c be in $[a, b]$. If both $\overline{D}F(c)$ and $\underline{D}F(c)$ are finite, then F is continuous at c.

Proof. Let $M = \max\left\{\left|\overline{D}F(c)\right|, \left|\underline{D}F(c)\right|\right\}$. Since $\overline{D}F(c) = \limsup_{x \to c} \frac{F(x) - F(c)}{x - c}$, there exists $\delta_1 > 0$ such that

$$x \in (c - \delta_1, c + \delta_1) - \{c\} \Rightarrow \frac{F(x) - F(c)}{x - c} < \overline{D}F(c) + 1 \le M + 1. \quad \text{------- (8)}$$

Similarly, as $\underline{D}F(c) = \liminf_{x \to c} \frac{F(x) - F(c)}{x - c}$, there exists $\delta_2 > 0$ such that

$$x \in (c - \delta_2, c + \delta_2) - \{c\} \Rightarrow \frac{F(x) - F(c)}{x - c} > \underline{D}F(c) - 1 \ge -M - 1. \quad \text{------- (9)}$$

Take $\delta_3 = \min(\delta_1, \delta_2)$. Then it follows from (8) and (9) that

$$x \in (c - \delta_3, c + \delta_3) - \{c\} \Rightarrow -M - 1 \le \frac{F(x) - F(c)}{x - c} \le M + 1.$$

Thus, $x \in (c - \delta_3, c + \delta_3) - \{c\} \Rightarrow \left|\frac{F(x) - F(c)}{x - c}\right| \le M + 1$

$$\Rightarrow \left|F(x) - F(c)\right| \le (M + 1)\left|x - c\right|.$$

Hence, given ε > 0, take $\delta = \min\left\{\delta_3, \frac{\varepsilon}{M + 1}\right\}$. Then

$$|x - c| < \delta \Rightarrow \left|F(x) - F(c)\right| < \varepsilon.$$

Consequently, F must be continuous at c.

The next theorem is a result for a sufficient condition for a function to be increasing.

Theorem 9. Suppose F: $[a, b] \to \mathbf{R}$ is a real-valued function. Suppose $\underline{D}F(x) \ge 0$ for all x in $[a, b]$. Then F is increasing on $[a, b]$.

Proof. We prove the theorem under the condition that $\underline{D}F(x) > 0$ for all x in $[a, b]$. Let $a \le c < d \le b$. We shall show that $F(c) < F(d)$. Now $\underline{D}F(c) > 0$ and so the set $H = \{x \in [c, d]: F(x) > F(c)\}$ is non-empty, for otherwise, $\underline{D}F(c)$ would be less than or equal to 0. H is obviously bounded above by d and so it has a supremum $M \le d$. We claim that $M = d$. Firstly, *M must be in H.* If M does not belong to H, then it is a limit point of H. Since M is a supremum of H, there exists a strictly increasing sequence (a_n) in H such that $a_n \to M$. Now

$$\lim_{n\to\infty} \frac{F(a_n) - F(M)}{a_n - M} \ge \liminf_{n\to\infty} \frac{F(a_n) - F(M)}{a_n - M}$$

$$\ge \liminf_{x\to M} \frac{F(x) - F(M)}{x - M} = \underline{D}F(M) > 0$$

Therefore, there exists an integer N such that $n \ge N$ implies that

$$\frac{F(a_n) - F(M)}{a_n - M} \ge \frac{\underline{D}F(M)}{2} > 0 .$$

Thus, $F(M) > F(a_N) > F(c)$. This shows that M is in H. Now M must be equal to d. If $M < d$, then since $\underline{D}F(M) > 0$, there must be a point x in the interval (M, d) such that $F(x) > F(M)$ for, otherwise $\underline{D}F(M)$ would be less than or equal to 0. Since $F(M) > F(c)$, $F(x) > F(c)$ and so x is in H. This contradicts that M is the supremum of H. Hence $M = d$ and $F(d) > F(c)$. Since c and d are arbitrary, this shows that F is strictly increasing on $[a, b]$.

Suppose now $\underline{D}F(x) \ge 0$ for all x in $[a, b]$. Let $\varepsilon > 0$. Let $G(x) = F(x) + \varepsilon x$ on $[a, b]$. Then $\underline{D}G(x) = \underline{D}F(x) + \varepsilon > 0$ for all x in $[a, b]$, since the derivative of the function x is 1. It follows that for any $d > c$ in $[a, b]$, $G(d) > G(c)$. That is, $F(d) > F(c) + (d - c)\varepsilon$. Since we can choose ε to be arbitrarily small, $F(d) \ge F(c)$. This proves that F is increasing.

The next result below relates semi-continuity with the upper and lower derivates.

Theorem 10. Suppose $f: [a, b] \to \mathbf{R}^*$ is a Lebesgue integrable extended real valued function. Let $F(x) = \int_a^x f$ for x in $[a, b]$. Let $c \in [a, b]$.

(a) If f is lower semi-continuous at c, then $\underline{D}F(c) \geq f(c)$.

(b) If f is upper semi-continuous at c, then $\overline{D}F(c) \leq f(c)$.

Proof.

(a) Suppose f is lower semi-continuous at c. By Theorem 2 part (a), for $\alpha < f(c)$, there exists $\delta > 0$ such that $f(x) > \alpha$ for all $x \in B(c, \delta) \cap [a, b]$. If $f(c) = -\infty$, we have nothing to prove. We assume that $f(c) > -\infty$. For all $x \in B(c, \delta) \cap [a, b] - \{c\}$,

$$\frac{F(x)-F(c)}{x-c} = \frac{1}{x-c}\left\{\int_a^x f - \int_a^c f\right\} = \frac{1}{x-c}\int_c^x f \geq \frac{1}{x-c}\int_c^x \alpha = \alpha \ .$$

Note that the last inequality is obvious if $x > c$. If $x < c$, then

$$\frac{1}{x-c}\int_c^x f = \frac{1}{c-x}\int_x^c f \geq \frac{1}{c-x}\int_x^c \alpha = \alpha \ .$$

This implies that $\underline{D}F(c) \geq \alpha$ for all $\alpha < f(c)$. It follows that $\underline{D}F(c) \geq f(c)$.

(b) Suppose f is upper semi-continuous at c. By Theorem 2 part (c), for $\alpha > f(c)$, there exists $\delta > 0$ such that $f(x) < \alpha$ for all $x \in B(c, \delta) \cap [a, b]$. If $f(c) = \infty$, we have nothing to prove. We assume that $f(c) < \infty$. For all $x \in B(c, \delta) \cap [a, b] - \{c\}$,

$$\frac{F(x)-F(c)}{x-c} = \frac{1}{x-c}\left\{\int_a^x f - \int_a^c f\right\} = \frac{1}{x-c}\int_c^x f \leq \frac{1}{x-c}\int_c^x \alpha = \alpha \ .$$

This implies that $\overline{D}F(c) \leq \alpha$ for all $\alpha > f(c)$. It follows that $\overline{D}F(c) \leq f(c)$.

Major and Minor Functions

We now introduce Perron's major and minor functions.

Definition 11. Suppose $f: [a, b] \to \mathbf{R}^*$ is an extended real valued function.

A real-valued function $U: [a, b] \to \mathbf{R}$ is a *major function of* f on $[a, b]$, if

$\underline{D}U(x) > -\infty$ and $\underline{D}U(x) \geq f(x)$ for all x in $[a, b]$.

A real-valued function V: $[a, b] \to \mathbf{R}$ is a *minor function of f* on $[a, b]$, if $\overline{D}V(x) < \infty$ and $\overline{D}V(x) \le f(x)$ for all x in $[a, b]$.

The next result is a characterization of Lebesgue integrable function in terms of major and minor functions.

We introduce the following notation. If F is a real value function, we denote $F(b) - F(a)$ by F_a^b .

Theorem 12. Suppose f: $[a, b] \to \mathbf{R}^*$ is a measurable extended real valued function. The function f is Lebesgue integrable on $[a, b]$ if, and only if, for each $\varepsilon > 0$, there exist absolutely continuous major and minor functions, U and V of f on $[a, b]$ such that $U(a) = V(a) = 0$ and $U_a^b - V_a^b < \varepsilon$.

Proof. Suppose f is Lebesgue integrable on $[a, b]$. Given $\varepsilon > 0$, by Theorem 5, there exist lower semi-continuous function u and a upper semi-continuous function v such that for all x in $[a, b]$, $u(x) > -\infty$, $u(x) \ge f(x) \ge v(x)$ and $v(x) < \infty$, and u and v are Lebesgue integrable on $[a, b]$ with

$$\int_a^b u - \frac{\varepsilon}{2} < \int_a^b f < \int_a^b v + \frac{\varepsilon}{2} \ . \quad \text{--------------} \quad (10)$$

Let $U(x) = \int_a^x u$ and $V(x) = \int_a^x v$. Then U and V are absolutely continuous finite functions on $[a, b]$. By Theorem 10, $\underline{D}U(x) \ge u(x) > -\infty$ for all x in $[a, b]$, since u is lower semi-continuous on $[a, b]$. Since v is upper semi-continuous on $[a, b]$, $\overline{D}V(x) \le v(x) < \infty$. Observe that $\underline{D}U(x) \ge u(x) \ge f(x)$ and $\overline{D}V(x) \le v(x) \le f(x)$ for all x in $[a, b]$. Hence U is a major function and V is a minor function of f on$[a, b]$. Moreover, it follows from (10) that

$$U_a^b - V_a^b = \int_a^b u - \int_a^b v = \left(\int_a^b u - \int_a^b f\right) - \left(\int_a^b v - \int_a^b f\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Conversely, given any $\varepsilon > 0$, there exist absolutely continuous major and minor functions of f, U and V on $[a, b]$ such that $U_a^b - V_a^b < \varepsilon$. Then since U and V are absolutely continuous, U and V are differentiable almost everywhere on $[a, b]$ and their derivatives are Lebesgue integrable. That is to say, the derivatives U'

and V' exist almost everywhere on [a, b] and U' and V' are Lebesgue integrable. Therefore, $\underline{D}U = U'$ almost everywhere on [a, b] and $\underline{D}U$ is Lebesgue integrable on [a, b]. We also have $\overline{D}V = V'$ almost everywhere on [a, b] and so $\overline{D}V$ is Lebesgue integrable on [a, b]. By definition of major and minor function of f, $\overline{D}V(x) \le f(x) \le \underline{D}U(x)$ for all x in [a, b]. Plainly

$$\int_a^b \underline{D}U - \int_a^b \overline{D}V = \int_a^b U' - \int_a^b V'$$

$$= U_a^b - V_a^b < \varepsilon \text{ by absolute continuity of } U \text{ and } V.$$

Thus, we have shown that given any $\varepsilon > 0$, there exist integrable functions g and h such that $g \le f \le h$ and $\int_a^b h - \int_a^b g < \varepsilon$. It follows that f is Lebesgue integrable on [a, b].

This completes the proof.

Remark.

Suppose U and V are major and minor functions of f on [a, b].

Then $\underline{D}(U - V)(x) \ge \underline{D}U(x) - \overline{D}V(x) \ge f(x) - f(x) = 0$ for all x in [a, b].

Thus, by Theorem 9, $U - V$ is increasing on [a, b]. Therefore, $U_a^b \ge V_a^b$.

If f has at least one major and minor function, then we define the *upper Perron integral* on [a, b] to be $UPf = \inf\{U_a^b : U \text{ a major function of } f\}$ and the *lower Perron integral* to be $LPf = \sup\{V_a^b : V \text{ a minor function of } f\}$. Then we have $LPf \le UPf$. If $LPf = UPf$, then we say f is *Perron integrable* on [a, b]. Theorem 12 then says that any Lebesgue integrable function on [a, b] is Perron integrable.

The next result is a kind of limit convergence theorem for the Lebesgue integral.

Theorem 13. Suppose f: [a, b] $\to$ **R*** is Lebesgue integrable. There are sequences of continuous functions (p_n :[a, b] $\to$**R**) and (P_n :[a, b] $\to$**R**) such that (i) $p_n(a) = P_n(a) = 0$, (ii) $p_n(x) \to \int_a^x f$, $P_n(x) \to \int_a^x f$ uniformly on $[a, b]$ and

$$\overline{D}p_n(x) \le f(x) \le \underline{D}P_n(x),$$

whenever $f(x)$ is finite.

Proof.

Let $\varepsilon = \dfrac{1}{n}$. By Theorem 12, there exist major function U_n and minor function V_n of f on $[a, b]$ such that $\underline{D}U_n(x) > -\infty, \overline{D}V_n(x) < \infty$, $\overline{D}V_n(x) \le f(x) \le \underline{D}U_n(x)$ for all x in $[a, b]$ and $U_n(b) - U_n(a) < V_n(b) - V_n(a) + \dfrac{1}{n}$ and $U_n(a) = V_n(a) = 0$. Moreover, we deduce from the proof of Theorem 12, that the major and minor functions satisfy $U_n(x) \ge \int_a^x f \ge V_n(x)$ and

$$U_n(x) - V_n(x) \le U_n(b) - V_n(b) < \frac{1}{n} \text{ for all } x \text{ in } [a, b].$$

Thus, $0 \le U_n(x) - \int_a^x f \le U_n(x) - V_n(x) < \dfrac{1}{n}$ for all x in $[a, b]$. It follows that $U_n(x) \to \int_a^x f$ uniformly on $[a, b]$. Similarly, we deduce that $V_n(x) \to \int_a^x f$ uniformly on $[a, b]$. Now let $p_n(x) = V_n(x)$ and $P_n(x) = U_n(x)$. Then $p_n(x) \to \int_a^x f$ and $P_n(x) \to \int_a^x f$ uniformly on $[a, b]$. Moreover,

$$\overline{D}p_n(x) = \overline{D}V_n(x) \le f(x) \le \underline{D}U_n(x) = \underline{D}P_n(x).$$

This completes the proof of the theorem.

Remark.

1. Observe that $\underline{D}(U_n(x) - V_n(x)) \ge \underline{D}U_n(x) - \overline{D}V_n(x)) \ge 0$ for all x in $[a, b]$. Therefore, by Theorem 9, $U_n(x) - V_n(x)$ is increasing and nonnegative in $[a, b]$, since $U_n(a) - V_n(a) = 0$.
2. The functions $p_n(x)$ and $P_n(x)$ are also known as *de la Vallée-Poussin's minorant and majorant functions.*

Section B. Riemann's Idea, Symmetric Second Derivative, R-Summability and Convexity.

Let $A_n(x) = a_n \cos(nx) + b_n \sin(nx)$ for $n > 1$ and $A_0(x) = a_0$. We now write the trigonometric series (A) as

$$T(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right) = \frac{1}{2}A_0(x) + \sum_{n=1}^{\infty} A_n(x) \quad \text{-------} \ (A)$$

Observe that $A_n(x) = a_n \cos(nx) + b_n \sin(nx) = \sqrt{a_n^{\ 2} + b_n^{\ 2}} \cos(nx + \mu_n)$, where $\cos(\mu_n) = \dfrac{a_n}{\sqrt{a_n^{\ 2} + b_n^{\ 2}}}$ and $\sin(\mu_n) = -\dfrac{b_n}{\sqrt{a_n^{\ 2} + b_n^{\ 2}}}$. Let $\rho_n = \sqrt{a_n^{\ 2} + b_n^{\ 2}}$.

We derive first a necessary condition for convergence of the series (A).

Theorem 14. If $A_n(x) \to 0$, or in particular, if $T(x)$ converges in a set E of positive measure, then $a_n \to 0$ and $b_n \to 0$.

Proof. It is enough to prove that $\rho_n = \sqrt{a_n^{\ 2} + b_n^{\ 2}} \to 0$. We prove this by contradiction. Suppose $\rho_n \not\to 0$. This means there exists $\varepsilon > 0$ and a subsequence $\left(\rho_{n_k}\right)$ of (ρ_n) such that $\rho_{n_k} > \varepsilon$ for all positive integer k. Note that if $T(x)$ converges in a set E of positive measure, then $A_n(x) \to 0$ in a set E of positive measure.

Since $A_n(x) \to 0$ in E, it follows that $\cos(n_k x + \mu_{n_k}) \to 0$ in E. Since $\cos(n_k x + \mu_{n_k})$ is uniformly bounded by 1, by the Bounded Convergence Theorem,

$$\int_{-\pi}^{\pi} \cos^2(n_k x + \mu_{n_k})\chi_E(x)dx = \int_E \cos^2(n_k x + \mu_{n_k})dx \to 0 \ .$$

But $\cos^2(n_k x + \mu_{n_k}) = \dfrac{1}{2}\left(1 + \cos(2n_k x + 2\mu_{n_k})\right)$ and so

$$\int_E \cos^2(n_k x + \mu_{n_k})dx = \frac{1}{2}m(E) + \frac{1}{2}\int_E \cos(2n_k x + 2\mu_{n_k})dx \ . \quad \text{---------} \ (11)$$

Note that $\cos(2n_k x + 2\mu_{n_k}) = \cos(2n_k x)\cos(2\mu_{n_k}) - \sin(2n_k x)\sin(2\mu_{n_k})$.
Therefore,

$$\int_E \cos(2n_k x + 2\mu_{n_k})dx$$

$$= \cos(2\mu_{n_k})\int_{-\pi}^{\pi} \cos(2n_k x)\chi_E(x)dx - \sin(2\mu_{n_k})\int_{-\pi}^{\pi} \sin(2n_k x)\chi_E(x)dx \quad \text{.---- (12),}$$

By the Lebesgue Riemann Theorem,

$$\int_{-\pi}^{\pi} \cos(2n_k x)\chi_E(x)dx \to 0 \text{ and } \int_{-\pi}^{\pi} \sin(2n_k x)\chi_E(x)dx \to 0$$

and as $\cos(2\mu_{n_k})$ and $\sin(2\mu_{n_k})$ are bounded sequences, it follows from (12) that

$\int_E \cos(2n_k x + 2\mu_{n_k})dx \to 0$. Thus, we obtain from (11) that

$$\int_E \cos^2(n_k x + \mu_{n_k})dx \to \frac{1}{2}m(E) \neq 0.$$

This contradicts $\int_E \cos^2(n_k x + \mu_{n_k})dx \to 0$.

Therefore, $\rho_n = \sqrt{a_n{}^2 + b_n{}^2} \to 0$ and so $a_n \to 0$ and $b_n \to 0$.

If we formally integrate the series (A) term by term twice, we shall obtain the following series

$$\frac{1}{4}a_0 x^2 - \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2} = \Phi(x) \quad \text{-------------------------} \quad \text{(U)}$$

Let $\Psi(x) = \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2}$. Then $\Phi(x) = \frac{1}{4}a_0 x^2 - \Psi(x)$.

We have already proved that if $T(x)$ converges or $A_n(x) \to 0$ on a set of positive measure, then $a_n \to 0$ and $b_n \to 0$. Consequently, $A_n(x) \to 0$ uniformly on **R** and (U) gives a very useful series. We state it formally below.

Theorem 15. If $T(x)$ converges in a set E of positive measure, then $\Psi(x) = \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2}$ converges absolutely and uniformly to a continuous function on $\mathbf{R}$, and so $\Phi(x) = \frac{1}{4} a_0 x^2 - \Psi(x)$ converges absolutely and uniformly to a continuous function on $\mathbf{R}$.

Proof. Since the sequence $(A_n(x))$ is uniformly bounded by Theorem 14, it follows by Weierstrass M-test that $\Psi(x) = \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2}$ converges absolutely and uniformly to a continuous function on $\mathbf{R}$. The second statement is now obvious.

To approach the problem of uniqueness, Riemann's idea is to argue backwards from $\Phi(x)$ to $T(x)$ by a process of generalized symmetric second derivative.

The Idea of Symmetric Second Derivative

Definition 16. Suppose g is a finite function, i.e., a real-valued function.

For any real number h, define

$$\Delta_h^2 g(x) = g(x+h) + g(x-h) - 2g(x).$$

If the limit $\lim_{h\to 0} \frac{\Delta_h^2 g(x)}{h^2}$ exists, then this limit is called the *generalized* (*symmetric*) *second derivative of* g at x. We denote this by $D_2 g(x)$. That is,

$$D_2 g(x) = \lim_{h\to 0} \frac{\Delta_h^2 g(x)}{h^2}.$$

We now describe some properties of symmetric second derivative and some of its variants.

The first step towards proving Theorem 1 is the following:

Theorem 17 (Riemann). Suppose $T(x)$ is a trigonometric series, where $a_n \to 0$ and $b_n \to 0$. Let $\Phi(x) = \frac{1}{4}a_0 x^2 - \Psi(x)$ as in (U). If $T(\theta)$ converges to $f(\theta)$, then $D_2\Phi(\theta) = f(\theta)$.

Proof. Suppose $T(\theta)$ converges to $f(\theta)$.

Define $R_h(\theta) = \dfrac{\Delta_{2h}^2 \Phi(\theta)}{4h^2}$. The limit of $R_h(\theta)$ is $D_2\Phi(\theta)$.

Observe that $R_h(\theta) = \dfrac{\Delta_{2h}^2 \Phi(\theta)}{4h^2} = \dfrac{\Phi(\theta+2h)+\Phi(\theta-2h)-2\Phi(\theta)}{4h^2}$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(\theta)\left(\frac{\sin(nh)}{nh}\right)^2 \quad \text{-------------------------- (13)}$$

after applying the addition formula and summing.

We want to prove that $R_h(\theta) \to T(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(\theta)$.

Let $s_n(\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{n} A_n(\theta)$ for $n \geq 1$. Then $s_n(\theta) \to T(\theta)$.

We now introduce some notation, write for $n = 0$, $\left(\dfrac{\sin(nh)}{nh}\right)^2 = 1$.

Let $\ell_n = \left(\dfrac{\sin(nh)}{nh}\right)^2$ for $n \geq 1$, and $\ell_0 = 1$.

Let Γ_n be the partial sum of $R_h(\theta)$ define by

$$\Gamma_n = \sum_{k=0}^{n} A_k(\theta)\left(\frac{\sin(kh)}{kh}\right)^2 . \quad \text{------------------------ (14)}$$

Here we let $A_0(\theta) = \frac{1}{2}a_0$ instead of a_0 only for the proof of this theorem.

By Abel's summation formula,

$$\Gamma_n = \sum_{k=0}^{n-1} s_k \Delta\ell_k + \ell_n s_n = \sum_{k=0}^{n} s_k \Delta\ell_k + \ell_{n+1} s_n, \quad \text{-------------------} \quad (15)$$

where $\Delta\ell_k = \ell_k - \ell_{k+1}$ and $s_k = s_k(\theta)$.

Since $s_n(\theta) \to T(\theta)$ and $\ell_n = \left(\dfrac{\sin(nh)}{nh}\right)^2 \to 0$ for each $h \neq 0$ as $n \to \infty$,

$$\Gamma_n \to \sum_{k=0}^{\infty} s_k \Delta\ell_k = \sum_{k=0}^{\infty} s_k \left(\left(\frac{\sin(kh)}{kh}\right)^2 - \left(\frac{\sin((k+1)h)}{(k+1)h}\right)^2 \right) \text{ for } h \neq 0.$$

Therefore, for $h \neq 0$,

$$R_h(\theta) = \frac{\Delta_{2h}^2 \Phi(\theta)}{4h^2} = \sum_{k=0}^{\infty} s_k \left(\left(\frac{\sin(kh)}{kh}\right)^2 - \left(\frac{\sin((k+1)h)}{(k+1)h}\right)^2 \right). \text{ --------- } (16)$$

Since $s_n(\theta) \to T(\theta) = s$, we may write $s_k = s + e_k$ and $e_k \to 0$.

Then for fixed $h \neq 0$,

$$R_h(\theta) = \sum_{k=0}^{\infty} (s + e_k) \Delta\ell_k \ ,$$

$$\Gamma_n - s = \sum_{k=0}^{n} s_k \Delta\ell_k + \ell_{n+1} s_n - s = \sum_{k=0}^{n} (s + e_k) \Delta\ell_k + \ell_{n+1} s_n - s$$

$$= \sum_{k=0}^{n} e_k \Delta\ell_k + \ell_{n+1} e_n \ .$$

For $0 < N < n$,

$$|\Gamma_n - s| \le \left| \sum_{k=0}^{N-1} e_k \Delta\ell_k \right| + \left| \sum_{k=N}^{n} e_k \Delta\ell_k \right| + |\ell_{n+1}||e_n|$$

$$\le \left| \sum_{k=0}^{N-1} e_k \Delta\ell_k \right| + \sum_{k=N}^{n} |e_k| \left| \int_{kh}^{(k+1)h} \left(\frac{d}{dt} \frac{\sin^2(t)}{t^2} \right) dt \right| + |\ell_{n+1}||e_n|$$

$$\le \left| \sum_{k=0}^{N-1} e_k \Delta\ell_k \right| + \max_{k \ge N} |e_k| \sum_{k=N}^{n} \left| \int_{kh}^{(k+1)h} \left(\frac{d}{dt} \frac{\sin^2(t)}{t^2} \right) dt \right| + \max_{k \ge n} |e_k|, \text{ --------- } (17)$$

since $|\ell_k| \leq 1$.

Let $G(t)$ be the derivative of $\dfrac{\sin^2(t)}{t^2}$ for $t \neq 0$. We then obtain from (17), for $h \neq 0$,

$$\left|\Gamma_n - s\right| \leq \left|\sum_{k=0}^{N-1} e_k \Delta\ell_k\right| + \max_{k\geq N}|e_k| \sum_{k=N}^{n} \int_{kh}^{(k+1)h} |G(t)|dt + \max_{k\geq n}|e_k|$$

$$\leq \left|\sum_{k=0}^{N-1} e_k \Delta\ell_k\right| + \max_{k\geq N}|e_k| \int_{Nh}^{(n+1)h} |G(t)|dt + \max_{k\geq n}|e_k|.$$

Then by passage to the limit, we have, since $\max_{k\geq n}|e_k| \to 0$

$$\left|R_h(\theta) - s\right| \leq \left|\sum_{k=0}^{N-1} e_k \Delta\ell_k\right| + \max_{k\geq N}|e_k| \int_{Nh}^{\infty} |G(t)|dt. \quad \text{-------------} \quad (18).$$

Note that for $t \neq 0$, $G(t) = \dfrac{d}{dt}\dfrac{\sin^2(t)}{t^2} = \dfrac{t\sin(2t) - 2\sin^2(t)}{t^3}$ and so for $t > \pi$,

$|G(t)| \leq \dfrac{1}{t^2} + \dfrac{2}{t^3}$. It follows from this inequality that $\int_{\pi}^{\infty} |G(t)|dt < \infty$. Observe that $\lim_{t\to 0}|G(t)| = 0$. Therefore, $\int_0^{\pi} |G(t)|dt < \infty$. Indeed $\int_0^{\pi} |G(t)|dt = 1$. Hence, $\int_0^{\infty} |G(t)|dt = C < \infty$.

It follows from (18) that

$$\left|R_h(\theta) - s\right| \leq \left|\sum_{k=0}^{N-1} e_k \Delta\ell_k\right| + \max_{k\geq N}|e_k| \int_0^{\infty} |G(t)|dt = \left|\sum_{k=0}^{N-1} e_k \Delta\ell_k\right| + \max_{k\geq N}|e_k| C.$$

Given $\varepsilon > 0$ we may choose sufficiently large N so that $\max_{k\geq N}|e_k| < \dfrac{\varepsilon}{2C}$ since $\max_{k\geq n}|e_k| \to 0$. Thus for this value of N we have

$$\left|R_h(\theta) - s\right| < \left|\sum_{k=0}^{N-1} e_k \Delta\ell_k\right| + \frac{\varepsilon}{2}. \qquad \text{---------------} \quad (19)$$

Since $\lim_{h\to 0}\Delta\ell_k = \lim_{h\to 0}\left(\left(\frac{\sin(kh)}{kh}\right)^2 - \left(\frac{\sin((k+1)h)}{(k+1)h}\right)^2\right) = 0$ for each k = 0,1,.., N-1,

$\lim_{h\to 0}\left|\sum_{k=0}^{N-1} e_k \Delta\ell_k\right| = 0$. It follows that there exists δ>0 so that for 0 <|h| < δ,

$\left|\sum_{k=0}^{N-1} e_k \Delta\ell_k\right| < \frac{\varepsilon}{2}$.

Therefore, it follows from (19) that for 0 <|h| < δ,

$$|R_h(\theta) - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

We can thus conclude that

$$R_h(\theta) \to s = T(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(\theta) .$$

Observe that we have proved a more general result concerning R summability.

Definition 18. In honour of Riemann, if a series $u_0 + \sum_{n=1}^{\infty} u_n \left(\frac{\sin(nh)}{nh}\right)^2 \to s$

as $h \to 0$, we say the series $\sum_{n=0}^{\infty} u_n$ is R- *summable* to the sum s.

Thus, Theorem 17 states that if the trigonometric series $T(\theta)$ satisfies $a_n \to 0$ and $b_n \to 0$ and converges to s at θ, then it is R- summable to s at θ.

We have actually proved the regularity of R-summability. We state the result as follows.

Theorem 19. If $\sum_{n=0}^{\infty} u_n$ converges to the sum s, then the series is R- summable to the sum s.

(The proof of Theorem 19 is almost exactly the same as in Theorem 17 except for appropriate change in notation.)

Now we investigate some properties of the symmetric second derivative and its relation to convexity.

Definition 20.

Let $\overline{D}_2 g(x) = \limsup_{h \to 0} \frac{\Delta_h^2 g(x)}{h^2}$ and $\underline{D}_2 g(x) = \liminf_{h \to 0} \frac{\Delta_h^2 g(x)}{h^2}$.

If $\Delta_h^2 g(x) = o(h)$ or equivalently $\lim_{h \to 0} \frac{\Delta_h^2 g(x)}{h} = 0$, then g is said to be *smooth at x*. g is said to be *smooth* in a set in an interval if it is smooth at every point in the set.

Note that if g is differentiable at x, then g is smooth at x.

Theorem 21. Suppose g is continuous in (a, b) and $\overline{D}_2 g > 0$ in (a, b) except perhaps in an enumerable set E. If E *is empty,* then g is convex. If E is not empty and g is smooth in E, then g is convex in (a, b).

Proof. Suppose $\overline{D}_2 g(x) > 0$ in (a, b) except for x in E.

Note that g is *convex on* (a, b) if for any $\alpha < \beta$ in (a, b),

$$g(x) \leq \frac{(\beta - x)g(\alpha) + (x - \alpha)g(\beta)}{\beta - \alpha} \quad \text{for all } x \text{ in } [\alpha, \beta].$$

g is *concave on* (a, b) if $-g$ is convex *on* (a, b).

Suppose on the contrary that g is not convex. Then there is an interval $[\alpha, \beta]$ in (a, b) such that

$$g(x) > \frac{(\beta - x)g(\alpha) + (x - \alpha)g(\beta)}{\beta - \alpha}$$

for some x in (α, β). That is to say, the function $d(x) = g(x) - mx - n$, where

$$m=\frac{g(\beta)-g(\alpha)}{\beta-\alpha} \text{ and } n=\frac{\beta g(\alpha)-\alpha g(\beta)}{\beta-\alpha}$$

is sometime positive. Note that $d(\alpha) = d(\beta) = 0$. Since g is continuous, d is also continuous. Hence by the Extreme Value Theorem, there exists x_0 in (α, β) such that $d(x_0)$ is the absolute maximum of $d|_{[\alpha,\beta]}$. In particular, $d(x_0) > 0$. Therefore, for sufficiently small h so that $[x_0-h, x_0+h] \subseteq(\alpha, \beta)$

$$\frac{\Delta_h^2 g(x_0)}{h^2}=\frac{\Delta_h^2 d(x_0)}{h^2}=\frac{d(x_0+h)+d(x_0-h)-2d(x_0)}{h^2}\leq 0.$$

It follows that $\overline{D}_2 g(x_0)\leq 0$. If the exceptional set E is empty, then this would contradict $\overline{D}_2 g(x_0)>0$. If the exceptional set E is non-empty, then $x_0 \in E$.

We shall show that E is non-enumerable.

Let $k = d(x_0) > 0$. Since d is continuous at x_0 there exists $\delta > 0$ such that for all x in $(x_0-\delta, x_0+\delta)$, $d(x) > d(x_0)- k/4 = 3k/4 > 0$. Therefore, for x in $(x_0-\delta, x_0+\delta)$,

$$g(x)-(m+h)x-n=d(x)-hx>\tfrac{3}{4}k-hx\ .$$

Let $\delta_2=\dfrac{k}{2(\max(|\alpha|,|\beta|)+1)}$.

If $|h|<\dfrac{k}{2(\max(|\alpha|,|\beta|)+1)}=\delta_2$, then for all x in $[\alpha, \beta]$

$$|hx|<\frac{k|x|}{2(\max(|\alpha|,|\beta|)+1)}<\frac{k}{2}$$

and for x in $(x_0-\delta, x_0+\delta)$,

$$g(x)-(m+h)x-n=d(x)-hx>\frac{3}{4}k-hx>\frac{1}{4}k>0. \text{ --------- } (20)$$

This means that for all sufficiently small μ near m, *i.e.*, for $|\mu - m| < \delta_2$,

$$g(x)-\mu x-n>0$$

for x in $(x_0-\delta, x_0+\delta)$.

Let $d_\mu(x) = g(x) - \mu x - n$. Then d_μ also has positive absolute maximum in $[\alpha, \beta]$. Observe that $d_\mu(x) = g(x) - mx - n + (m-\mu)x = d(x) - (m-\mu)x$.

Hence $d_\mu(\alpha) = d(\alpha) - (m-\mu)\alpha = 0 - (m-\mu)\alpha = (\mu - m)\alpha$ and $d_\mu(\beta) = d(\beta) - (m-\mu)\beta = 0 - (m-\mu)\beta = (\mu - m)\beta$. Let x_μ be the absolute maximizer of d_μ in $[\alpha, \beta]$. Observe that

$d_\mu(x_\mu) \geq d_\mu(x_0) = d(x_0) - (m-\mu)x_0 = k - (m-\mu)x_0 > \frac{1}{2}k > |(m-\mu)\alpha| \geq d_\mu(\alpha)$

and similarly note that $d_\mu(x_\mu) > |(m-\mu)\beta| \geq d_\mu(\beta)$. Hence $x_\mu \in (\alpha, \beta)$.

Let $\Gamma = \{x \in [\alpha, \beta] : d_\mu(x) = d_\mu(x_\mu)\}$. Then Γ is non-empty and is contained in $[\alpha, \beta]$. Let $\tilde{x}_\mu = \sup \Gamma$. Then $\tilde{x}_\mu \in [\alpha, \beta]$. Since $d_\mu(x_\mu) > d_\mu(\beta), d_\mu(\alpha)$, $\tilde{x}_\mu \in (\alpha, \beta)$ and $d_\mu(\tilde{x}_\mu) = d_\mu(x_\mu)$. It follows as in the case for x_0 that $\overline{D}_2 g(\tilde{x}_\mu) = \overline{D}_2 d_\mu(\tilde{x}_\mu) \leq 0$. Hence $\tilde{x}_\mu \in E$.

We shall show next that g is differentiable at $\tilde{x}_\mu$ and that $g'(\tilde{x}_\mu) = \mu$.

Note that $\tilde{x}_\mu$ is a maximizer of d_μ in $[\alpha, \beta]$, we have then for sufficiently small positive h,

$$\frac{d_\mu(\tilde{x}_\mu + h) - d_\mu(\tilde{x}_\mu)}{h} \leq 0 \text{ and } \frac{d_\mu(\tilde{x}_\mu - h) - d_\mu(\tilde{x}_\mu)}{-h} \geq 0.$$

It follows that the right upper derivate of d_μ at $\tilde{x}_\mu$,

$$D^+ d_\mu(\tilde{x}_\mu) = \limsup_{h \to 0^+} \frac{d_\mu(\tilde{x}_\mu + h) - d_\mu(\tilde{x}_\mu)}{h} \leq 0$$ and the lower left derivate

$$D_- d_\mu(\tilde{x}_\mu) = \liminf_{h \to 0^+} \frac{d_\mu(\tilde{x}_\mu - h) - d_\mu(\tilde{x}_\mu)}{-h} \geq 0.$$

Since g is smooth in E, d_μ is also smooth in E, i.e., for x in E, $\Delta_h^2 d_\mu(x) = o(h)$, or

$$\lim_{h \to 0} \frac{\Delta_h^2 d_\mu(x)}{h} = \lim_{h \to 0} \frac{d_\mu(x+h) + d_\mu(x-h) - 2d_\mu(x)}{h} = 0. \text{ -------------- (21)}$$

Thus

$$\lim_{h\to 0}\frac{\Delta_h^2 d_\mu(\tilde{x}_\mu)}{h}=\lim_{h\to 0}\frac{d_\mu(\tilde{x}_\mu+h)+d_\mu(\tilde{x}_\mu-h)-2d_\mu(\tilde{x}_\mu)}{h}=0$$

and so given any $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |h| < \delta$ implies

$$\left|\frac{d_\mu(\tilde{x}_\mu+h)+d_\mu(\tilde{x}_\mu-h)-2d_\mu(\tilde{x}_\mu)}{h}\right|=\left|\frac{d_\mu(\tilde{x}_\mu+h)-d_\mu(\tilde{x}_\mu)}{h}-\frac{d_\mu(\tilde{x}_\mu-h)-d_\mu(\tilde{x}_\mu)}{-h}\right|$$

$$< \varepsilon .$$

This means, for $0 < |h| < \delta$,

$$\frac{d_\mu(\tilde{x}_\mu-h)-d_\mu(\tilde{x}_\mu)}{-h}-\varepsilon<\frac{d_\mu(\tilde{x}_\mu+h)-d_\mu(\tilde{x}_\mu)}{h}<\frac{d_\mu(\tilde{x}_\mu-h)-d_\mu(\tilde{x}_\mu)}{-h}+\varepsilon .$$

It follows that

$$\limsup_{h\to 0^+}\frac{d_\mu(\tilde{x}_\mu-h)-d_\mu(\tilde{x}_\mu)}{-h}-\varepsilon\le\limsup_{h\to 0^+}\frac{d_\mu(\tilde{x}_\mu+h)-d_\mu(\tilde{x}_\mu)}{h}\quad\text{and}$$

$$\limsup_{h\to 0^+}\frac{d_\mu(\tilde{x}_\mu+h)-d_\mu(\tilde{x}_\mu)}{h}\varepsilon\le\limsup_{h\to 0^+}\frac{d_\mu(\tilde{x}_\mu-h)-d_\mu(\tilde{x}_\mu)}{-h}+\varepsilon\ .$$

Since ε is arbitrary,

$$D^+d_\mu(\tilde{x}_\mu)=\limsup_{h\to 0^+}\frac{d_\mu(\tilde{x}_\mu+h)-d_\mu(\tilde{x}_\mu)}{h}$$

$$=\limsup_{h\to 0^+}\frac{d_\mu(\tilde{x}_\mu-h)-d_\mu(\tilde{x}_\mu)}{-h}=D^-d_\mu(\tilde{x}_\mu).$$

Hence, $\overline{D}d_\mu(\tilde{x}_\mu)=D^+d_\mu(\tilde{x}_\mu)\le 0.$

Similarly by using limit *inferior* we can show that

$$\underline{D}d_\mu(\tilde{x}_\mu)=D_-d_\mu(\tilde{x}_\mu)\ge 0.$$

Therefore, $0\le\underline{D}d_\mu(\tilde{x}_\mu)\le\overline{D}d_\mu(\tilde{x}_\mu)\le 0$ and so $\underline{D}d_\mu(\tilde{x}_\mu)=\overline{D}d_\mu(\tilde{x}_\mu)=0$ and d_μ is differentiable at $\tilde{x}_\mu$ with $d_\mu{}'(\tilde{x}_\mu)=0$.

By definition of d_μ, $d_\mu{}'(\tilde{x}_\mu) = g'(\tilde{x}_\mu) - \mu$ and so $g'(\tilde{x}_\mu) = \mu$. Thus, for each μ in the interval, $(m-\delta_2, m+\delta_2)$, we can associate an element $\tilde{x}_\mu$ in E such that $g'(\tilde{x}_\mu) = \mu$. Therefore, there are as many elements in E as there are in $(m-\delta_2, m+\delta_2)$. This means that E contains a set which is non-denumerable and so E is a non-denumerable set. This contradicts that E is denumerable. It follows that g must be convex.

Theorem 22. Suppose g is continuous in (a, b) and $\overline{D}_2 g \geq 0$ in (a, b) except perhaps in an enumerable set E. If *E is empty,* then g is convex in (a, b). If E is non-empty and g is smooth in E, then g is convex in (a, b).

Proof. For each integer $n > 0$, let $g_n(x) = g(x) + \frac{1}{2n}x^2$. Then $\overline{D}_2 g_n(x) = \overline{D}_2 g(x) + \frac{1}{n} > 0$ for all x except for x in E. If g is smooth in E, g_n is also smooth in E. Therefore, by Theorem 21, each g_n is convex in (a, b). Since g is the limit of g_n, g is also convex in (a, b).

For concavity we have the following result.

Theorem 23. Suppose g is continuous in (a, b) and $\underline{D}_2 g \leq 0$ in (a, b) except perhaps in an enumerable set E. If E is empty, then g is concave in (a, b). If E is non-empty and g is smooth, then g is concave in (a, b).

Proof. Let $h = -g$. Then $\overline{D}_2 h = \overline{D}_2(-g) = -\underline{D}_2 g \geq 0$. Note that h is continuous in (a, b) and smooth in E if E is non-empty, since g is. Therefore, by Theorem 22, h is convex and so g is concave in (a, b).

Corollary 24. Suppose g is continuous in (a, b) and $D_2 g = 0$ in (a, b) except perhaps in an enumerable set E. *If E is empty,* then g is linear in (a, b). If E is non-empty and g is smooth in E, then g is linear in (a, b).

Proof. If $D_2 g = 0$, then $\overline{D}_2 g = \underline{D}_2 g = 0$. Then by Theorem 22 and Theorem 23 g is both concave and convex in (a, b). Therefore, for any interval $[\alpha, \beta]$ in (a, b) g is a linear function on $[\alpha, \beta]$ and the derivative g' is a constant in $[\alpha, \beta]$.

By letting α tends to a and β tends to b, we conclude that g' is a constant function in (a, b). It follows that g is a linear function.

Theorem 25. Suppose g is continuous in (a, b) and $\overline{D}_2 g \geq c$ in (a, b) except perhaps in an enumerable set E in which g is smooth if E is non-empty. Then $\frac{\Delta_h^2 g(x)}{h^2} \geq c$ for $a < x - h < x + h < b$;

Suppose g is continuous in (a, b) and $\underline{D}_2 g \leq c$ in (a, b) except perhaps in an enumerable set E in which g is smooth if E is non-empty. Then $\frac{\Delta_h^2 g(x)}{h^2} \leq c$ for $a < x - h < x + h < b$.

Proof. Let $p(x) = g(x) - \frac{1}{2}cx^2$. Then p is continuous in (a, b) and smooth in E if E is non-empty. Then $\underline{D}_2 p = \underline{D}_2 g - c \geq 0$ in (a, b) except perhaps in E. Therefore, by Theorem 22, p is convex in (a, b). Since p is convex in (a, b), for any x and h such that $a < x - h < x + h < b$, $\Delta_h^2 p(x) \geq 0$. But $\Delta_h^2 p(x) = \Delta_h^2 g(x) - ch^2 \geq 0$ and so $\Delta_h^2 g(x) \geq ch^2$. Hence $\frac{\Delta_h^2 g(x)}{h^2} \geq c$ for $a < x - h < x + h < b$. Similarly, if $\underline{D}_2 g(x) \leq c$ in (a, b) except perhaps in an enumerable set E in which g is smooth, then $\overline{D}_2(-g) \geq -\underline{D}_2 g \geq -c$ and so by what we have just proved, for $a < x - h < x + h < b$, $\frac{\Delta_h^2(-g)(x)}{h^2} = -\frac{\Delta_h^2 g(x)}{h^2} \geq -c$ and it follows that $\frac{\Delta_h^2 g(x)}{h^2} \leq c$.

Our next result is about smoothness of the function $\Phi(x)$ obtained by formal integration of a trigonometric series $T(x)$ twice. To get back to the trigonometric series using the symmetric second derivative we need to use the smoothness of $\Phi(x)$.

Theorem 26. Suppose $T(x)$ is a trigonometric series that converges in a set E of positive measure or $a_n \to 0$ and $b_n \to 0$. Then the function

$$\Phi(x) = \frac{1}{4}a_0 x^2 - \sum_{n=1}^{\infty} \frac{A_n(x)}{n^2}$$

obtained by formally integrating the trigonometric series $T(x)$ twice, is continuous and smooth on the whole of **R**. That is to say,

$$\frac{\Delta_h^2 \Phi(x)}{h} \to 0 \text{ as } h \to 0 \text{ or } \Delta_h^2 \Phi(x) = o(h).$$

Proof. We shall show that $\dfrac{\Delta_{2h}^2 \Phi(x)}{2h} \to 0$. The theorem then follows. From

(13) , $$R_h(\theta) = \frac{\Delta_{2h}^2 \Phi(\theta)}{4h^2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(\theta)\left(\frac{\sin(nh)}{nh}\right)^2$$

We shall write the summation in three parts. Since $A_n(x) \to 0$ uniformly on **R**, given $\varepsilon > 0$, there exists integer N such that $n > N$ implies that $|A_n(x)| < \varepsilon$ for all x in **R**.

The first part is

$$I = \frac{1}{2}a_0 + \sum_{n=1}^{N} A_n(\theta)\left(\frac{\sin(nh)}{nh}\right)^2 .$$

The second part is chosen according to h. Given any h, with $|h| < \pi$, let s be the integer part of $\dfrac{\pi}{|h|}$. Then we have $1 \le s \le \dfrac{\pi}{|h|} < s+1$. Thus $s|h| \le \pi < (s+1)|h|$.

Part (II) is given by

$$II = \sum_{k=N+1}^{N+s} A_k(\theta)\left(\frac{\sin(kh)}{kh}\right)^2 ,$$

Part (III) is given by

$$III = \sum_{k=N+s+1}^{\infty} A_k(\theta)\left(\frac{\sin(kh)}{kh}\right)^2 .$$

Then we have $\dfrac{\Delta_{2h}^2\Phi(\theta)}{4h^2} = I + II + III$ ---------------------------- (22)

Observe that for any θ, $2h\,I \to 0$ as $h \to 0$ since $h\left(\dfrac{\sin(nh)}{nh}\right)^2 \to 0$.

Since $|A_n(x)| < \varepsilon$ for all x and for all $n > N$, $|II| \le \sum_{k=N+1}^{N+s} |A_k(\theta)| \le \sum_{k=N+1}^{N+s} \varepsilon = s\varepsilon$ and so

$$|hII| \le sh\varepsilon \le \pi\varepsilon . \text{ ----------------- } (23)$$

Observe that

$$|III| \le \sum_{k=N+s+1}^{\infty} \frac{|A_k(\theta)|}{k^2h^2} \le \varepsilon\frac{1}{h^2}\sum_{k=N+s+1}^{\infty}\frac{1}{k^2} \le \varepsilon\frac{1}{h^2}\frac{1}{N+s},$$

since $\sum_{k=1}^{\infty}\dfrac{1}{(s+k)^2} \le \int_s^{\infty}\dfrac{1}{x^2}dx = \dfrac{1}{s}$ for $s \ge 1$.

Therefore, since $(s+N)|h| \ge (s+1)|h| > \pi$, $|III| \le \dfrac{\varepsilon}{\pi}\dfrac{1}{|h|}$. Hence,

$$|IIIh| \le \frac{\varepsilon}{\pi} . \text{ -------------------------------- } (24)$$

It follows then using (23) and (24) that

$$\left|\frac{\Delta_{2h}^2\Phi(\theta)}{2h}\right| = \left|2h\frac{\Delta_{2h}^2\Phi(\theta)}{4h^2}\right| \le 2|hI| + 2|hII| + 2|hIII| < 2|hI| + 2\pi\varepsilon + \frac{2\varepsilon}{\pi} .$$

Since $2h\,I \to 0$ as $h \to 0$, there exists $\delta > 0$ so that for $0 < |h| < \delta$, we have $|2hI| < \varepsilon$. Hence for $0 < |h| < \delta$,

$$\left|\frac{\Delta_{2h}^2\Phi(\theta)}{2h}\right| < \varepsilon + 2\pi\varepsilon + \frac{2\varepsilon}{\pi} = \varepsilon(1 + 2\pi + \frac{2}{\pi}) .$$

Since ε is arbitrarily chosen, this shows that $\lim_{h\to 0}\dfrac{\Delta_{2h}^2\Phi(\theta)}{2h}=0$. Consequently

$\lim_{h\to 0}\dfrac{\Delta_h^2\Phi(\theta)}{h}=0$ for any θ.

This completes the proof,

Incidentally, we have proved the following theorem.

Theorem 27. If (u_n) is a sequence that converges to 0, then

$$h\sum_{n=1}^{\infty}u_n\left(\frac{\sin(nh)}{nh}\right)^2\to 0 \text{ as } h\to 0.$$

The next result is a technical result aims at expressing the difference of actually integrating a function f twice and the function obtained by formally integrating the Fourier series of f twice term by term.

Theorem 28. Suppose $f(x)$ is finite except in an enumerable set E and integrable in (a, b). Suppose $g(x)$ is continuous in (a, b) and smooth in E when E is non-empty and that

$$\underline{D}_2 g(x)\le f(x)\le \overline{D}_2 g(x)$$

for all x in (a, b) not in E. Let $J(x)=\int_a^x\left(\int_a^t f(u)du\right)dt$ be the repeated integral of f. Then $g(x) - J(x)$ is linear in (a, b).

Proof. We shall employ de La Vallee Poussin majorant and minorant functions. Since f is Lebesgue integrable in (a, b), by Theorem 13, there are sequences of continuous functions ($p_n :[a, b]\to\mathbf{R}$) and ($P_n :[a, b]\to\mathbf{R}$) such that (i) $p_n(a) = P_n(a) = 0$, (ii) $p_n(x)\to\int_a^x f$, $P_n(x)\to\int_a^x f$ uniformly in $[a, b]$ and for x not in E,

$$\overline{D}p_n(x)\le f(x)\le \underline{D}P_n(x).$$

Let $q_n(x)=\int_a^x p_n(t)dt$ and $Q_n(x)=\int_a^x P_n(t)dt$. Since $p_n(x)\to\int_a^x f$, $P_n(x)\to\int_a^x f$ uniformly in [a, b], $q_n(x)\to J(x)$ and $Q_n(x)\to J(x)$ uniformly in [a, b]. Note that both $q_n(x)$ and $Q_n(x)$ are differentiable in [a, b]. By the Cauchy Mean Value Theorem,

$$\frac{\Delta_h^2 Q_n(x)}{h^2}=\frac{Q_n(x+h)+Q_n(x-h)-2Q_n(x)}{h^2}=\frac{Q_n'(x+\tilde{h})-Q_n'(x-\tilde{h})}{2\tilde{h}}$$

for some $\tilde{h}$ between 0 and h. There exists $0<\theta<1$ such that $\tilde{h}=h\theta$.

Hence,

$$\frac{\Delta_h^2 Q_n(x)}{h^2}=\frac{Q_n'(x+h\theta)-Q_n'(x-h\theta)}{2h\theta}=\frac{P_n(x+h\theta)-P_n(x-h\theta)}{2h\theta}$$

$$=\frac{1}{2}\frac{P_n(x+h\theta)-P_n(x)}{h\theta}+\frac{1}{2}\frac{P_n(x-h\theta)-P_n(x)}{-h\theta}\,. \quad \text{--------------} \quad (25)$$

Therefore, it follows from (25) that

$$\liminf_{h\to 0}\frac{\Delta_h^2 Q_n(x)}{h^2}\geq\liminf_{h\to 0}\frac{1}{2}\frac{P_n(x+h)-P_n(x)}{h}+\liminf_{h\to 0}\frac{1}{2}\frac{P_n(x-h)-P_n(x)}{-h}$$

$$=\frac{1}{2}\underline{D}P_n(x)+\frac{1}{2}\underline{D}P_n(x)=\underline{D}P_n(x).$$

This means

$$\underline{D_2}Q_n(x)\geq\underline{D}P_n(x)\geq f(x), \qquad \text{----------------} \quad (26)$$

except for x in E, since $\underline{D}P_n(x)\geq f(x)$.

Let $k(x)=J(x)-g(x)$, $K_n(x)=Q_n(x)-g(x)$ and $k_n(x)=q_n(x)-g(x)$. Then

$K_n(x)\to k(x)$ and $k_n(x)\to k(x)$ uniformly in (a, b).

Note that since $J(x)$, $q_n(x)$ and $Q_n(x)$ are differentiable and g is continuous in (a, b), $k(x)$, $K_n(x)$ and $k_n(x)$ are all continuous in (a, b). If E is non-empty, since g is smooth in E, $k(x)$, $K_n(x)$ and $k_n(x)$ are all smooth in E.

To proceed further we use the following inequality for supremum and infimum:

For any two functions $u(x)$ and $v(x)$ continuous in $(x_0-\delta, x_0+\delta)-\{x_0\}$ for some $\delta > 0$,

$$\inf_{x\in(x_0-\delta,x_0+\delta)-\{x_0\}}\{u(x)+v(x)\} \le \sup_{x\in(x_0-\delta,x_0+\delta)-\{x_0\}}\{u(x)\} + \inf_{x\in(x_0-\delta,x_0+\delta)-\{x_0\}}\{v(x)\}$$

$$\le \sup_{x\in(x_0-\delta,x_0+\delta)-\{x_0\}}\{u(x)+v(x)\}$$

provided $\sup_{x\in(x_0-\delta,x_0+\delta)-\{x_0\}}\{u(x)\} + \inf_{x\in(x_0-\delta,x_0+\delta)-\{x_0\}}\{v(x)\}$ is not of the form $\infty - \infty$

so that,

$$\liminf_{x\to x_0}\{u(x)+v(x)\} \le \limsup_{x\to x_0}\{u(x)\} + \liminf_{x\to x_0}\{v(x)\} \le \limsup_{x\to x_0}\{u(x)+v(x)\}$$

---------------------- (27)

Then using (27), except for x in E,

$$\overline{D}_2 K_n(x) \ge \underline{D}_2 Q_n(x) + \overline{D}_2(-g(x)) \ge \underline{D}_2 Q_n(x) - \underline{D}_2 g(x) \ge f(x) - f(x) = 0\,.$$

Therefore, by Theorem 22, $K_n(x)$ is convex in (a, b). Hence $k(x)$ being the limit of $K_n(x)$ is also convex in (a, b).

Similarly, we can deduce that for some $0 < \theta < 1$

$$\frac{\Delta_h^2 q_n(x)}{h^2} = \frac{1}{2}\frac{q_n(x+h\theta)-q_n(x)}{h\theta} + \frac{1}{2}\frac{q_n(x-h\theta)-q_n(x)}{-h\theta}\,.$$

From this we can deduce as above for $Q_n(x)$ that

$$\overline{D}_2 q_n(x) \le \overline{D}_2 p_n(x) \le f(x)\,. \quad \text{------------------} \quad (27)$$

Therefore, for x not in E,

$$\underline{D}_2 k_n(x) \le \overline{D}_2 k_n(x) \le \overline{D}_2 q_n(x) + \underline{D}_2(-g)(x) = \overline{D}_2 q_n(x) - \overline{D}_2 g(x)$$

$$\le f(x) - f(x) = 0\,.$$

Therefore, by Theorem 23, $k_n(x)$ is concave in (a, b). Hence $k(x)$ being the limit of $k_n(x)$ is also concave in (a, b). Thus $k(x)= J(x) – g(x))$ is both convex and concave in (a, b) and so is linear in (a, b). This completes the proof.

Integrating A Fourier Series Formally.

Suppose $T(x)$ is the Fourier series of a Lebesgue integrable function f. Then by the Riemman Lebesgue Theorem, its Fourier coefficients, a_n and b_n satisfy $a_n \to 0$ and $b_n \to 0$. We shall show that by formally integrating the Fourier series term by term, we obtain a uniformly convergent series converging to the integral of f. The Fourier series need not converge and the series so obtained is always uniformly convergent.

The special series $S(x)=\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$ plays a role in this investigation. Note that $S(x)$ converges uniformly in any closed interval free from multiples of 2π and converges boundedly to the function $J(x)$ defined by,

$$J(x)=\begin{cases}\frac{1}{2}(\pi-x), & 0<x<2\pi,\\ 0, & x=0\end{cases}$$

and extended to whole of **R** by periodicity. That it converges boundedly is by Theorem 14 of my note on *Fourier cosine and sine series*. That $S(x)$ converges uniformly in any closed interval free from multiples of 2π may be deduced by using Dirichlet's Test. See Theorem 9 in *Fourier cosine and sine series*.

Suppose

$$T(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)=\frac{1}{2}A_0(x)+\sum_{n=1}^{\infty}A_n(x),$$

is the Fourier series of the Lebesgue integrable function f. Consider the series obtained by formally integrating the series

$$\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)=\sum_{n=1}^{\infty}A_n(x)$$

term by term, that is,

$$\sum_{n=1}^{\infty}\int_0^x\left(a_n\cos(nt)+b_n\sin(nt)\right)dt=\sum_{n=1}^{\infty}\int_0^x A_n(t)dt$$

$$=\sum_{n=1}^{\infty}\left(\frac{a_n\sin(nx)}{n}-\frac{b_n\cos(nx)}{n}+\frac{b_n}{n}\right)=\sum_{n=1}^{\infty}\frac{b_n}{n}-\sum_{n=1}^{\infty}\frac{b_n\cos(nx)-a_n\sin(nx)}{n}.$$

Let $B_n(x) = b_n \cos(nx) - a_n \sin(nx)$ for integer $n \geq 1$. Then the above series is given by

$$W(x) = \sum_{n=1}^{\infty} \frac{b_n}{n} - \sum_{n=1}^{\infty} \frac{B_n(x)}{n}. \qquad \text{----------------------- (28)}$$

We shall show that this series $W(x)$ is uniformly convergent and converges to the function $F(x) = \int_0^x f(t)dt - \frac{1}{2}a_0 x$.

By definition of the Fourier coefficient b_n ,

$$\sum_{m=1}^{n} \frac{b_m}{m} = \sum_{m=1}^{n} \frac{1}{m}\left(\frac{1}{\pi}\int_0^{2\pi} f(x)\sin(mx)dx\right) = \frac{1}{\pi}\int_0^{2\pi}\left(f(x)\sum_{m=1}^{n}\frac{\sin(mx)}{m}\right)dx$$

$$\to \frac{1}{\pi}\int_0^{2\pi}\left(f(x)J(x)\right)dx = \frac{1}{\pi}\int_0^{2\pi}\left(f(x)\frac{1}{2}(\pi - x)\right)dx$$

by the Lebesgue Dominated Convergence Theorem as deduced below.

Since $S(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \to J(x)$ boundedly, $\left|\sum_{m=1}^{n} \frac{\sin(mx)}{m}\right| \leq K$ for some real number K and for all integer $n \geq 1$. Therefore, $\left|f(x)\sum_{m=1}^{n}\frac{\sin(mx)}{m}\right| \leq K|f(x)|$ and since $K\,|f|$ is Lebesgue integrable, we invoke the Lebesgue Dominated Convergence Theorem to give the above statement. Hence, we have $\sum_{n=1}^{\infty}\frac{b_n}{n}$ is always convergent and

$$\sum_{n=1}^{\infty}\frac{b_n}{n} = \frac{1}{\pi}\int_0^{2\pi}\left(f(x)\frac{1}{2}(\pi - x)\right)dx.$$

Hence we have proved the following theorem.

Theorem 29. If the function f is Lebesgue integrable and represented by the Fourier series $T(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right) = \frac{1}{2}A_0(x) + \sum_{n=1}^{\infty}A_n(x)$, then $\sum_{n=1}^{\infty}\frac{b_n}{n}$ is convergent and $\sum_{n=1}^{\infty}\frac{b_n}{n} = \frac{1}{\pi}\int_0^{2\pi}\left(f(x)\frac{1}{2}(\pi - x)\right)dx$.

Consider now $F(x)=\int_0^x f(t)dt-\frac{1}{2}a_0x=\int_0^x\left(f(t)-\frac{1}{2}a_0\right)dt$ on $[0, 2\pi]$. Note that $F(0) = F(2\pi) = 0$. Observe that F is absolutely continuous on $[0, 2\pi]$ and periodic on **R** with period 2π.

Then $\frac{1}{\pi}\int_0^{2\pi}F(x)\cos(nx)dx=\frac{1}{\pi}\left[F(x)\frac{\sin(nx)}{n}\right]_0^{2\pi}-\frac{1}{\pi}\int_0^{2\pi}\left(f(x)-\frac{1}{2}a_0\right)\frac{\sin(nx)}{n}dx$,

by integration by parts,

$$=-\frac{1}{\pi}\int_0^{2\pi}f(x)\frac{\sin(nx)}{n}dx+\frac{1}{2\pi}a_0\int_0^{2\pi}\frac{\sin(nx)}{n}dx=-\frac{1}{n}b_n$$

and $\frac{1}{\pi}\int_0^{2\pi}F(x)\sin(nx)dx=\frac{1}{\pi}\left[-F(x)\frac{\cos(nx)}{n}\right]_0^{2\pi}+\frac{1}{\pi}\int_0^{2\pi}\left(f(x)-\frac{1}{2}a_0\right)\frac{\cos(nx)}{n}dx$

$$=\frac{1}{\pi}\int_0^{2\pi}f(x)\frac{\cos(nx)}{n}dx-\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}a_0\frac{\cos(nx)}{n}dx=\frac{1}{n}a_n,$$

by integration by parts.

Hence the Fourier coefficients for $F(x)$ are $(-\frac{1}{n}b_n\ ,\ \frac{1}{n}a_n\ )$ for $n\geq 1$.

$$\frac{1}{2\pi}\int_0^{2\pi}F(x)dx=\frac{1}{2\pi}\left[F(t)t\right]_0^{2\pi}-\int_0^{2\pi}F'(t)tdt=-\int_0^{2\pi}\left(f(t)-\frac{1}{2}a_0\right)tdt$$

by integration by parts

$$=-\frac{1}{2\pi}\int_0^{2\pi}f(t)tdt+\frac{1}{2\pi}a_0\int_0^{2\pi}\frac{1}{2}tdt=-\frac{1}{2\pi}\int_0^{2\pi}f(t)tdt+\frac{1}{2\pi}a_0\frac{(2\pi)^2}{4}$$

$$=-\frac{1}{2\pi}\int_0^{2\pi}f(t)tdt+\frac{1}{2\pi}\frac{1}{\pi}\int_0^{2\pi}f(t)dt\frac{(2\pi)^2}{4}=\frac{1}{2\pi}\int_0^{2\pi}(\pi-t)f(t)dt=\sum_{n=1}^{\infty}\frac{b_n}{n}$$ by Theorem 29.

Hence the Fourier series of $F(x)$ is given by

$$W(x)=\sum_{n=1}^{\infty}\frac{b_n}{n}-\sum_{n=1}^{\infty}\frac{b_n\cos(nx)-a_n\sin(nx)}{n}=\sum_{n=1}^{\infty}\frac{b_n}{n}-\sum_{n=1}^{\infty}\frac{B_n(x)}{n}\quad .$$

We may invoke the theory of Fourier series of a continuous function of bounded variation that its Fourier series converges to the function.

For now we shall show its convergence directly by a simple device of shifting the function itself.

Fix a θ in $[0, 2\pi]$. Let $g(t) = f(\theta+t)$. Here f is defined outside $[0, 2\pi]$ by periodicity. Then g is Lebesgue integrable since f is. We shall consider the Fourier coefficients of $g(t)$.

$$\frac{1}{\pi}\int_0^{2\pi} g(t)\cos(nt)dt = \frac{1}{\pi}\int_0^{2\pi} f(t+\theta)\cos(nt)dt = \frac{1}{\pi}\int_\theta^{2\pi+\theta} f(u)\cos(n(u-\theta))du$$

$$= \cos(n\theta)\frac{1}{\pi}\int_\theta^{2\pi+\theta} f(u)\cos(nu)du + \sin(n\theta)\frac{1}{\pi}\int_\theta^{2\pi+\theta} f(u)\sin(nu)du$$

$$= a_n\cos(n\theta) + b_n\sin(n\theta) = A_n(\theta)$$

and

$$\frac{1}{\pi}\int_0^{2\pi} g(t)\sin(nt)dt = \frac{1}{\pi}\int_0^{2\pi} f(t+\theta)\sin(nt)dt = \frac{1}{\pi}\int_\theta^{2\pi+\theta} f(u)\sin(n(u-\theta))du$$

$$= \cos(n\theta)\frac{1}{\pi}\int_\theta^{2\pi+\theta} f(u)\sin(nu)du - \sin(n\theta)\frac{1}{\pi}\int_\theta^{2\pi+\theta} f(u)\cos(nu)du$$

$$= b_n\cos(n\theta) - a_n\sin(n\theta) = B_n(\theta).$$

Therefore, the Fourier coefficients of $g(t)$ is given by $(A_n(\theta), B_n(\theta))$.

It follows then by Theorem 29 that $\sum_{n=1}^{\infty}\frac{B_n(\theta)}{n}$ is convergent and converges to

$$\frac{1}{\pi}\int_0^{2\pi}\left(g(t)\frac{1}{2}(\pi-t)\right)dt = \frac{1}{\pi}\int_0^{2\pi}\left(f(t+\theta)\frac{1}{2}(\pi-t)\right)dt = \frac{1}{\pi}\int_0^{2\pi} F'(t+\theta)\frac{1}{2}(\pi-t)dt,$$

since $\frac{1}{\pi}\int_0^{2\pi}\frac{a_0}{2}(\pi-t)dt = 0$,

$$= \frac{1}{2\pi}\left[F(t+\theta)(\pi-t)\right]_0^{2\pi} + \frac{1}{2\pi}\int_0^{2\pi} F(t+\theta)dt$$

$$= \frac{1}{2\pi}\left(F(\theta+2\pi)(-\pi) - F(\theta)\pi\right) + \frac{1}{2\pi}\int_0^{2\pi} F(t+\theta)dt$$

$$= -F(\theta) + \frac{1}{2\pi}\int_0^{2\pi} F(t+\theta)dt. \qquad \text{----------------------------- (29)}$$

Now,

$$\frac{1}{2\pi}\int_0^{2\pi} F(t+\theta)dt = \frac{1}{2\pi}\int_\theta^{2\pi+\theta} F(u)du = \frac{1}{2\pi}\int_0^{2\pi} F(u)du\,, \quad \text{by periodicity,}$$

$$= \frac{1}{2\pi}\left[F(u)u\right]_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} F'(u)udu = -\frac{1}{2\pi}\int_0^{2\pi}\left(f(u) - \frac{1}{2}a_0\right)udu$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(\pi - u)f(u)du \quad \text{since } \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}a_0 udu = \frac{1}{2\pi}\int_0^{2\pi}\pi f(u)du = \frac{1}{2}a_0\pi$$

$$= \sum_{n=1}^{\infty}\frac{b_n}{n}, \quad \text{by Theorem 29.}$$

Therefore, it follows from (29),

$$\frac{1}{\pi}\int_0^{2\pi}\left(g(t)\frac{1}{2}(\pi - t)\right)dt = \frac{1}{\pi}\int_0^{2\pi}\left(f(t+\theta)\frac{1}{2}(\pi - t)\right)dt = -F(\theta) + \sum_{n=1}^{\infty}\frac{b_n}{n}$$

$$= -\int_0^{\theta} f(t)dt + \frac{1}{2}a_0\theta + \sum_{n=1}^{\infty}\frac{b_n}{n}$$

Hence,

$$\sum_{n=1}^{\infty}\frac{B_n(\theta)}{n} = -\int_0^{\theta} f(t)dt + \frac{1}{2}a_0\theta + \sum_{n=1}^{\infty}\frac{b_n}{n}.$$

Therefore, the series $W(x) = \sum_{n=1}^{\infty}\frac{b_n}{n} - \sum_{n=1}^{\infty}\frac{b_n\cos(nx) - a_n\sin(nx)}{n} = \sum_{n=1}^{\infty}\frac{b_n}{n} - \sum_{n=1}^{\infty}\frac{B_n(x)}{n}$

converges to $\int_0^x f(t)dt - \frac{1}{2}a_0 x = F(x)$.

We have thus proved the convergence part of the following theorem.

Theorem 30. Suppose *f* is periodic with period 2π and is Lebesgue integrable.

Its Fourier series may be integrated term by term and the integrated series converges uniformly and

$$\int_0^x f(t)dt = \frac{1}{2}a_0 x + \sum_{n=1}^{\infty}\frac{b_n}{n} - \sum_{n=1}^{\infty}\frac{b_n\cos(nx) - a_n\sin(nx)}{n} = \frac{1}{2}a_0 x + \sum_{n=1}^{\infty}\frac{b_n}{n} - \sum_{n=1}^{\infty}\frac{B_n(x)}{n},$$

i.e., the righthand series converges uniformly to $\int_0^x f(t)dt$.

Proof. We have already proved convergence. We now show that the convergence is uniform. It is sufficient to show that the convergence of $\sum_{n=1}^{\infty}\frac{B_n(\theta)}{n}$ is uniform in θ.

Recall that the Fourier coefficients of $g(t) = f(t+\theta)$ is given by $(A_n(\theta), B_n(\theta))$. Therefore,

$$\sum_{n=p}^{q}\frac{B_n(\theta)}{n} = \frac{1}{\pi}\int_0^{2\pi} f(\theta+t)\sum_{n=p}^{n=q}\frac{\sin(nt)}{n}dt\,. \quad \text{-----------------} \quad (30)$$

Note that for any $p \geq 1$, $\left|\sum_{n=1}^{p}\frac{\sin(nt)}{n}\right| \leq 2+\pi = K$. ----------------------- (31)

(See (118) of *Fourier Cosine and Sine Series*.)

$$\left|\sum_{n=p}^{q}\frac{B_n(\theta)}{n}\right| \leq \frac{1}{\pi}\int_0^{2\pi}\left|f(\theta+t)\right|\left|\sum_{n=p}^{n=q}\frac{\sin(nt)}{n}\right|dt \leq \int_0^{2\pi}\left|f(\theta+t)\right|\left|\sum_{n=p}^{n=q}\frac{\sin(nt)}{n}\right|dt\,. \quad \text{-------} \quad (32)$$

$$\int_0^{2\pi}\left|f(\theta+t)\right|\left|\sum_{n=p}^{n=q}\frac{\sin(nt)}{n}\right|dt = \int_0^{\delta}\left|f(\theta+t)\right|\left|\sum_{n=p}^{n=q}\frac{\sin(nt)}{n}\right|dt + \int_{2\pi-\delta}^{2\pi}\left|f(\theta+t)\right|\left|\sum_{n=p}^{n=q}\frac{\sin(nt)}{n}\right|dt$$

$$+\int_{\delta}^{2\pi-\delta}\left|f(\theta+t)\right|\left|\sum_{n=p}^{n=q}\frac{\sin(nt)}{n}\right|dt$$

$$\leq 2K\int_0^{\delta}\left|f(\theta+t)\right|dt + 2K\int_{2\pi-\delta}^{2\pi}\left|f(\theta+t)\right|dt + +\int_{\delta}^{2\pi-\delta}\left|f(\theta+t)\right|\left|\sum_{n=p}^{n=q}\frac{\sin(nt)}{n}\right|dt\,. \quad \text{-----} \quad (33)$$

Given any $\varepsilon > 0$, by the absolute continuity of the Lebesgue integral on an interval, there exists $\delta_1 > 0$ such that for any measurable subset E of measure less than δ_1 , $\int_E |f| < \frac{\varepsilon}{8K}$. Thus take any $0 < \delta < \delta_1$, we have from (32) and (33) that

$$\left|\sum_{n=p}^{q}\frac{B_n(\theta)}{n}\right| \leq \frac{\varepsilon}{2} + \int_{\delta}^{2\pi-\delta}\left|f(\theta+t)\right|\left|\sum_{n=p}^{n=q}\frac{\sin(nt)}{n}\right|dt\,. \quad \text{----------------} \quad (34)$$

Since $\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}$ converges uniformly on the interval $[\delta, 2\pi-\delta]$, it is uniformly Cauchy and so there exists an integer N such that for all integers $q \geq p \geq N$ and for all t in $[\delta, 2\pi-\delta]$,

$$\left|\sum_{n=p}^{n=q}\frac{\sin(nt)}{n}\right| \le \frac{\varepsilon}{2\left(1+\int_0^{2\pi}|f(t)|dt)\right)} \quad . \quad \text{-----------------} \ (35)$$

Thus, it follows from (34) and (35) that for any θ, and for $q \ge p \ge N$,

$$\left|\sum_{n=p}^{q}\frac{B_n(\theta)}{n}\right| \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2\left(1+\int_0^{2\pi}|f(t)|dt)\right)}\int_{\delta}^{2\pi-\delta}|f(\theta+t)|dt$$

$$\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2\left(1+\int_0^{2\pi}|f(t)|dt)\right)}\int_0^{2\pi}|f(\theta+t)|dt \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2\left(1+\int_0^{2\pi}|f(t)|dt)\right)}\int_0^{2\pi}|f(t)|dt < \varepsilon \ .$$

This shows that $\sum_{n=1}^{\infty}\frac{B_n(\theta)}{n}$ is uniformly Cauchy on **R** and so $\sum_{n=1}^{\infty}\frac{B_n(\theta)}{n}$ converges uniformly on **R**. This completes the proof of Theorem 30.

Now, we may use Theorem 30 to investigate the relation of the symmetric second derivative with the double integral of *f*.

Suppose *f* is periodic and integrable on finite interval and *T*(*x*) is its Fourier series.

Let $\phi(\theta,t)=\frac{1}{2}\left(f(\theta+t)+f(\theta-t)\right)$. Then using integration by parts, we have

$$\int_0^x \phi(\theta,t)(2h-t)dt = \left[\int_0^s \phi(\theta,t)dt\left(2h-s\right)\right]_0^x + \int_0^x\left(\int_0^s \phi(\theta,t)dt\right)ds \ . \ \text{-----} \ (36)$$

In view of Theorem 30, we can express the integral on the right side as a series.

Theorem 31. Suppose *f* is periodic and Lebesgue integrable. Let

$$R_h(\theta)=\frac{\Delta_{2h}^2\Phi(\theta)}{4h^2}=\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(\theta)\left(\frac{\sin(nh)}{nh}\right)^2$$

as defined in Theorem 17 using the Fourier series *T*(θ) of *f*. Then

$$R_h(\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(\theta)\left(\frac{\sin(nh)}{nh}\right)^2=\frac{1}{2h^2}\int_0^{2h}\phi(\theta,t)(2h-t)dt \ .$$

Proof. Using (36) we obtain

$$\int_0^{2h}\phi(\theta,t)(2h-t)dt=\left[\int_0^s\phi(\theta,t)dt\left(2h-s\right)\right]_0^{2h}+\int_0^{2h}\left(\int_0^s\phi(\theta,t)dt\right)ds$$

$$=\int_0^{2h}\left(\int_0^s\phi(\theta,t)dt\right)ds. \text{ ------------------------------ } (37)$$

Now $\int_0^s\phi(\theta,t)dt=\int_0^s\frac{1}{2}\left(f(\theta+t)+f(\theta-t)\right)dt=\frac{1}{2}\int_0^s f(\theta+t)dt+\frac{1}{2}\int_0^s f(\theta-t)dt$

$$=\frac{1}{2}\int_0^s f(\theta+t)dt-\frac{1}{2}\int_0^{-s} f(\theta+t)dt. \text{ -------------------- } (38)$$

The Fourier series for $g(t)=f(\theta+t)$ is given by $(A_n(\theta), B_n(\theta))$.

I,e, its Fourier series is

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(A_n(\theta)\cos(nx)+B_n(\theta)\sin(nx)\right).$$

Therefore, by Theorem 30,

$$\int_0^x f(t+\theta)dt=\frac{1}{2}a_0x+\sum_{n=1}^{\infty}\frac{B_n(\theta)}{n}-\sum_{n=1}^{\infty}\frac{B_n(\theta)\cos(nx)-A_n(\theta)\sin(nx)}{n} \text{ ------ } (39)$$

and the series on the right hand side converges uniformly in x. Hence, we have

$$\int_0^{-x} f(t+\theta)dt=-\frac{1}{2}a_0x+\sum_{n=1}^{\infty}\frac{B_n(\theta)}{n}-\sum_{n=1}^{\infty}\frac{B_n(\theta)\cos(nx)+A_n(\theta)\sin(nx)}{n} \text{ .------ } (40)$$

It follows then from (38), (39) and (40) that

$$\int_0^x\phi(\theta,t)dt=\frac{1}{2}a_0x+\sum_{n=1}^{\infty}\frac{A_n(\theta)\sin(nx)}{n}, \text{ ------------------ } (41)$$

and the series on the right hand side of (41) converges uniformly in x for x in **R**.

Therefore, we can integrate (41) term by term, obtaining

$$\int_0^x\left(\int_0^s\phi(\theta,t)dt\right)ds=\frac{1}{4}a_0x^2+\sum_{n=1}^{\infty}\frac{A_n(\theta)}{n^2}-\sum_{n=1}^{\infty}\frac{A_n(\theta)\cos(nx)}{n^2}$$

$$= \frac{1}{4}a_0 x^2 + \sum_{n=1}^{\infty} A_n(\theta) \frac{2\sin^2\left(\frac{nx}{2}\right)}{n^2}. \qquad \text{------------- (42)}$$

Hence, using (37), and (42),

$$\int_0^{2h} \phi(\theta,t)(2h-t)dt = a_0 h^2 + \sum_{n=1}^{\infty} A_n(\theta) \frac{2\sin^2(nh)}{n^2} .$$

It follows that

$$\frac{1}{2h^2}\int_0^{2h} \phi(\theta,t)(2h-t)dt = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(\theta)\left(\frac{\sin(nh)}{nh}\right)^2 = R_h(\theta).$$

This proves Theorem 31.

By Theorem 19, if $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(\theta)$ converges to a value *c*, then the series is *R* summable to *c*, i.e., $R_h(\theta) \to c$ as $h \to 0$. With Theorem 31, we may have a different way of determining *R* summability by using the integral $\frac{1}{2h^2}\int_0^{2h} \phi(\theta,t)(2h-t)dt$.

From (41) we obtain

$$\frac{1}{t}\int_0^{t} \phi(\theta,u)du = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \frac{A_n(\theta)\sin(nt)}{nt}. \qquad \text{------------- (43).}$$

Now, if *f* is Lebesgue integrable, then the function $F(x) = \int_0^x f(t)dt$ is absolutely continuous, differentiable almost everywhere and $F'(x) = f(x)$ almost everywhere. Hence for almost all θ,

$$\lim_{t\to 0} \frac{\int_0^t f(\theta+u)du}{t} = f(\theta) \text{ and } \lim_{t\to 0} \frac{\int_0^t f(\theta-u)du}{t} = f(\theta). \text{ -------- (44)}$$

By using the functions $g(t) = f(\theta+t)$ and $h(t) = f(\theta-t)$ with $G(x) = \int_0^x g(t)dt$ and $H(x) = \int_0^x h(t)dt$, the above is just the statement $G'(0) = g(0) = f(\theta)$ and $H'(0) = h(0) = f(\theta)$. It follows from (44) that for almost all θ,

$$\lim_{t\to 0}\frac{1}{t}\int_0^t \phi(\theta,u)du = f(\theta). \quad \text{--------------------} \quad (45)$$

Theorem 32. Suppose f is periodic and Lebesgue integrable. Then for almost all θ,

$$\lim_{h\to 0}\frac{1}{2h^2}\int_0^{2h} \phi(\theta,t)(2h-t)dt = f(\theta),$$

that is, for almost all θ, $R_h(\theta) \to f(\theta)$. Indeed, if $\lim_{t\to 0}\frac{1}{t}\int_0^t \phi(\theta,u)du = c$, then

$\lim_{h\to 0}\frac{1}{2h^2}\int_0^{2h} \phi(\theta,t)(2h-t)dt = c$ or $R_h(\theta) \to c$.

Proof. In view of (45) it is sufficient to prove that $\lim_{t\to 0}\frac{1}{t}\int_0^t \phi(\theta,u)du = c$ implies $\lim_{h\to 0}\frac{1}{2h^2}\int_0^{2h} \phi(\theta,t)(2h-t)dt = c$.

$$\frac{1}{2h^2}\int_0^{2h} \phi(\theta,t)(2h-t)dt = 2\frac{1}{2h}\int_0^{2h} \phi(\theta,t)dt - \frac{1}{2h^2}\int_0^{2h} \phi(\theta,t)tdt$$

$$\to 2c - \lim_{h\to 0}\frac{1}{2h^2}\int_0^{2h} \phi(\theta,t)tdt \ . \quad \text{---------} \quad (46)$$

$$\frac{1}{2h^2}\int_0^{2h} \phi(\theta,t)tdt = \frac{1}{2h^2}\int_0^{2h} \left(\phi(\theta,t)-c\right)tdt + \frac{1}{2h^2}\int_0^{2h} ctdt$$

$$= \frac{1}{2h^2}\int_0^{2h} \left(\phi(\theta,t)-c\right)tdt + c \quad \text{-------------} \quad (47)$$

We claim that $\lim_{h\to 0}\frac{1}{2h^2}\int_0^{2h}\left(\phi(\theta,t)-c\right)tdt=0$.

$$\frac{1}{2h^2}\int_0^{2h}\left(\phi(\theta,t)-c\right)tdt=\frac{1}{2h^2}\left\{\left[\left(\int_0^s\left(\phi(\theta,t)-c\right)dt\right)s\right]_0^{2h}-\int_0^{2h}\left(\int_0^s\left(\phi(\theta,t)-c\right)dt\right)ds\right\}$$

$$=\frac{1}{2h^2}\left\{\left(\int_0^{2h}\left(\phi(\theta,t)-c\right)dt\right)2h-\int_0^{2h}\left(\int_0^s\left(\phi(\theta,t)-c\right)dt\right)ds\right\}$$

$$=2\frac{1}{2h}\int_0^{2h}\phi(\theta,t)dt-2c-\frac{1}{2h^2}\int_0^{2h}\left(\int_0^s\left(\phi(\theta,t)-c\right)dt\right)ds. \quad \text{--------------- (48)}$$

Now $2\frac{1}{2h}\int_0^{2h}\phi(\theta,t)dt-2c\to 2c-2c=0$. We shall show that

$\frac{1}{2h^2}\int_0^{2h}\left(\int_0^s\left(\phi(\theta,t)-c\right)dt\right)ds\to 0$.

Since $\lim_{s\to 0}\frac{1}{s}\int_0^s\left(\phi(\theta,t)-c\right)dt=0$, given $\varepsilon > 0$, there exists, $\delta > 0$ such that for

$0 < |s| < \delta$, $\left|\frac{1}{s}\int_0^s\left(\phi(\theta,t)-c\right)dt\right|<\varepsilon$ or $\left|\int_0^s\left(\phi(\theta,t)-c\right)dt\right|<\varepsilon|s|$. --------- (49)

Therefore, for $0<|h|<\frac{\delta}{2}$, for $h > 0$,

$$\frac{1}{2h^2}\left|\int_0^{2h}\left(\int_0^s\left(\phi(\theta,t)-c\right)dt\right)ds\right|\le\frac{1}{2h^2}\int_0^{2h}\left|\left(\int_0^s\left(\phi(\theta,t)-c\right)dt\right)\right|ds\le\frac{1}{2h^2}\int_0^{2h}\varepsilon s ds=\varepsilon$$

and for $h < 0$,

$$\frac{1}{2h^2}\left|\int_0^{2h}\left(\int_0^s\left(\phi(\theta,t)-c\right)dt\right)ds\right|\le\frac{1}{2h^2}\int_{2h}^{0}\left|\left(\int_0^s\left(\phi(\theta,t)-c\right)dt\right)\right|ds\le\frac{1}{2h^2}\int_{2h}^{0}\varepsilon|s|ds=\varepsilon.$$

Thus, for $0<|h|<\frac{\delta}{2}$, $\frac{1}{2h^2}\left|\int_0^{2h}\left(\int_0^s\left(\phi(\theta,t)-c\right)dt\right)ds\right|\le\varepsilon$. It follows that

$\frac{1}{2h^2}\int_0^{2h}\left(\int_0^s\left(\phi(\theta,t)-c\right)dt\right)ds\to 0$. Therefore, from (46), (47) and (48) we get

$\frac{1}{2h^2}\int_0^{2h}\phi(\theta,t)(2h-t)dt\to 2c-\lim_{h\to 0}\frac{1}{2h^2}\int_0^{2h}\phi(\theta,t)tdt=2c-c=c$.

Thus, $R_h(\theta)\to c$.

As a consequence of Theorem 32, we have:

Theorem 33. Suppose *f* is periodic and Lebesgue integrable. Then for almost all θ, its Fourier series $T(\theta)$ is R-summable to $f(\theta)$.

We have actually proved the following:

Theorem 33A. If the Fourier series of *f* is Lebesgue summable at θ to c, then it is Riemann summable or R-summable to c.

(For the definition of Lebesgue summability, see page 4 of *Abel-summability of Fourier series and its derived series* and Theorem 20 there.)

Section C. Uniqueness Theorems.

Uniqueness of Fourier and Trigonometric Series

Our first uniqueness theorem concerns trigonometric series.

Theorem 34. If two trigonometric series converge to the same sum except in an enumerable set E, then they are identical. More precisely, if a trigonometric series $T(\theta)$ converges to 0 except in E, then $a_n = 0$ and $b_n = 0$ for all n, i.e., $T(\theta)$ is identically 0.

Proof. Suppose $T(\theta)$ converges to 0 except in E. Then by Theorem 14, $a_n \to 0$ and $b_n \to 0$. It follows then by Theorem 15, $\Psi(\theta) = \sum_{n=1}^{\infty} \frac{A_n(\theta)}{n^2}$ converges absolutely and uniformly to a continuous function on **R** and $\Phi(\theta) = \frac{1}{4} a_0 \theta^2 - \Psi(\theta)$ is continuous on **R**. By Theorem 26, $\Phi(\theta)$ is smooth on the whole of **R**. By Theorem 17, except for θ in E, $D_2\Phi(\theta) = 0$. It then follows from Corollary 24 that $\Phi(\theta)$ is a linear function on **R**.

Suppose $\Phi(\theta) = m\theta + C$. Observe that $\Psi(\theta) = \sum_{n=1}^{\infty} \frac{A_n(\theta)}{n^2}$ is a bounded function on **R** since $a_n \to 0$ and $b_n \to 0$. Therefore, $|\Psi(\theta)| \le K$ for some K and for all θ in **R**.

Thus,

$$\left|\frac{\Phi(\theta)}{\theta}\right| = \left|\frac{1}{4}a_0\theta - \frac{\Psi(\theta)}{\theta}\right| \geq \frac{1}{4}|a_0\theta| - \left|\frac{\Psi(\theta)}{\theta}\right| \geq \frac{1}{4}|a_0\theta| - \frac{K}{|\theta|}.$$

We claim that $a_0 = 0$. If $a_0 \neq 0$, then by the above inequality, since $\frac{1}{4}|a_0\theta| - \frac{K}{|\theta|} \to +\infty$ as $\theta \to +\infty$,

$$\left|\frac{\Phi(\theta)}{\theta}\right| \to +\infty \text{ as } \theta \to +\infty.$$

But $\left|\frac{\Phi(\theta)}{\theta}\right| = \left|m + \frac{C}{\theta}\right| \to |m| < \infty$ as $\theta \to +\infty$. This shows that $a_0 = 0$. Hence,

$\Phi(\theta) = -\Psi(\theta) = m\theta + C$. Since $\Psi(\theta)$ is bounded, $m = 0$. Thus $\Psi(\theta)$ is the constant function $-C$, i.e., $\Psi(\theta) = -C$ for all θ in **R**. Thus $\Psi(\theta) = \sum_{n=1}^{\infty} \frac{A_n(\theta)}{n^2}$ converges uniformly to the constant function $-C$. We may thus integrate $\Psi(\theta)$ term by term, giving

$$-2\pi C = \int_0^{2\pi} (-C)d\theta = \int_0^{2\pi} \Psi(\theta)d\theta = \sum_{n=1}^{\infty} \frac{\int_0^{2\pi} A_n(\theta)d\theta}{n^2} = 0,$$

since $\int_0^{2\pi} A_n(\theta)d\theta = a_n\int_0^{2\pi} \cos(n\theta)d\theta + b_n\int_0^{2\pi} \sin(n\theta)d\theta = 0$. Hence $C = 0$ and so $\Psi(\theta) = 0$ for all θ in **R.** Again, since convergence of $\Psi(\theta)$ to 0 is uniform,

$$\frac{a_n}{n^2} = \frac{1}{\pi}\int_0^{2\pi} \Psi(\theta)\cos(n\theta)d\theta = 0 \text{ and } \frac{b_n}{n^2} = \frac{1}{\pi}\int_0^{2\pi} \Psi(\theta)\sin(n\theta)d\theta = 0.$$

Therefore, $a_n = b_n = 0$ for all $n \geq 1$. We have already shown that $a_0 = 0$. Hence the trigonometric series is identically 0.

Our next result paves the way for the proof of Theorem 1. It states that Theorem 1 is true if the trigonometric series converges except in an enumerable set to a bounded function.

Theorem 35. Suppose the trigonometric series $T(\theta)$ converges except in an enumerable set E to a bounded function $f(\theta)$, then it is the Fourier series of $f(\theta)$.

Proof. Since $T(\theta)$ converges to $f(\theta)$ except for θ in E of zero measure, $T(\theta)$ converges to $f(\theta)$ almost everywhere and so f is measurable. Since f is also bounded, f is Lebesgue integrable on any bounded interval and in particular on $[-\pi, \pi]$. By Theorem 17, except for θ in E, $D_2\Phi(\theta) = f(\theta)$. Suppose $|f(\theta)| \leq M$ for some real number M and for all θ not in E. Then $D_2\Phi(\theta) = f(\theta) \leq M$ except for θ in an enumerable set E. Therefore, $\underline{D}_2\Phi(\theta) = D_2\Phi(\theta) \leq M$ except for θ in E. By Theorem 26, Φ is continuous and smooth on the whole of $\mathbf{R}$. Therefore, by Theorem 25,

$\frac{\Delta_h^2\Phi(\theta)}{h^2} \leq M$ for all θ. Similarly, since $\overline{D}_2\Phi(\theta) = D_2\Phi(\theta) \geq -M$, by Theorem 25, $\frac{\Delta_h^2\Phi(\theta)}{h^2} \geq -M$ for all θ in $\mathbf{R}$. It follows that $\left|\frac{\Delta_h^2\Phi(\theta)}{h^2}\right| \leq M$ for all θ and all $h \neq 0$. This means that $R_h(\theta) = \frac{\Delta_{2h}^2\Phi(\theta)}{4h^2}$ is uniformly bounded in θ and all $h \neq 0$. Recall that $R_h(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(\theta)\left(\frac{\sin(nh)}{nh}\right)^2$. Therefore, by the Lebesgue Bounded Convergence Theorem,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} R_h(\theta)\cos(n\theta)d\theta$$

$$= \frac{a_0}{2\pi}\int_{-\pi}^{\pi}\cos(n\theta)d\theta + \sum_{k=1}^{\infty}\frac{1}{\pi}\left(\int_{-\pi}^{\pi} A_k(\theta)\cos(n\theta)d\theta\right)\left(\frac{\sin(kh)}{kh}\right)^2$$

$$= a_n\left(\frac{\sin(nh)}{nh}\right)^2 \to a_n \text{ as } h \to 0 \text{ and}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} R_h(\theta)\sin(n\theta)d\theta$$

$$= \frac{a_0}{2\pi}\int_{-\pi}^{\pi}\sin(n\theta)d\theta + \sum_{k=1}^{\infty}\frac{1}{\pi}\left(\int_{-\pi}^{\pi} A_k(\theta)\sin(n\theta)d\theta\right)\left(\frac{\sin(kh)}{kh}\right)^2$$

$$= b_n\left(\frac{\sin(nh)}{nh}\right)^2 \to b_n \text{ as } h \to 0.$$

Since $R_h(\theta) \to f(\theta)$ boundedly almost everywhere, $R_h(\theta)\cos(n\theta) \to f(\theta)\cos(n\theta)$ boundedly almost everywhere and so by the Lebesgue Bounded Convergence Theorem,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} R_h(\theta)\cos(n\theta)d\theta \to \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)d\theta.$$

This shows that $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)d\theta$ is the Fourier coefficient of *f*.

Similarly, since $R_h(\theta)\sin(n\theta) \to f(\theta)\sin(n\theta)$ boundedly almost everywhere and so by the Lebesgue Bounded Convergence Theorem,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} R_h(\theta)\sin(n\theta)d\theta \to \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta.$$

Therefore, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta$ is the Fourier coefficient of *f*. Hence $T(\theta)$ is the Fourier series of *f*.

Now we remove the boundedness condition on *f* in Theorem 35. This gives the statement in Theorem 1.

We restate the theorem here.

Theorem 1. If the trigonometric series $T(\theta)$ converges except in an enumerable set E to a finite and integrable function f, then it is the Fourier series of *f*.

Proof. Suppose $T(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right) = \frac{1}{2}A_0(x) + \sum_{n=1}^{\infty} A_n(x)$.

Let $\Phi(x) = \frac{1}{4}a_0x^2 - \sum_{n=1}^{\infty}\frac{A_n(x)}{n^2}$. For the proof we shall compare Φ with the iterated integral of *f*.

By Theorem 17, except for θ in E, $D_2\Phi(\theta) = f(\theta)$. By Theorem 26, Φ is continuous and smooth on the whole of **R**. Let $J(x) = \int_0^x \left(\int_0^t f(u)du \right) dt$ be the repeated integral of f. Then by Theorem 28, $\Phi(x) - J(x)$ is linear in **R**.

The function f is Lebesgue integrable and so it has a Fourier series given by

$$A(x) = \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \left(\alpha_n \cos(nx) + \beta_n \sin(nx) \right).$$

Let $g(x) = \frac{1}{4}\alpha_0 x^2 - \sum_{n=1}^{\infty} \left(\frac{\alpha_n \cos(nx) + \beta_n \sin(nx)}{n^2} \right)$. Then $J(x) - g(x)$ is linear in **R**. We deduce this as follows.

Firstly,

$$J'(x) = \int_0^x f(t)dt$$

$$= \frac{1}{2}\alpha_0 x + \sum_{n=1}^{\infty} \frac{\beta_n}{n} - \sum_{n=1}^{\infty} \frac{\beta_n \cos(nx) - \alpha_n \sin(nx)}{n}$$ by Theorem 30,

$$= g'(x) + \sum_{n=1}^{\infty} \frac{\beta_n}{n}$$, since the differentiated series is uniformly convergent.

Therefore, $J(x) - g(x)$ is linear in **R**. Thus, since $\Phi(x) - J(x)$ is linear in **R**, $\Phi(x) - g(x)$ is linear in **R**. That is to say,

$$L(x) = \frac{1}{4}(a_0 - \alpha_0)x^2 - \sum_{n=1}^{\infty} \left(\frac{(a_n - \alpha_n)\cos(nx) + (b_n - \beta_n)\sin(nx)}{n^2} \right)$$

is linear in **R**.

Suppose $L(x) = Ax - C$. ----------------------------- (50)

Since $\sum_{n=1}^{\infty} \left(\frac{(a_n - \alpha_n)\cos(nx) + (b_n - \beta_n)\sin(nx)}{n^2} \right)$ is uniformly bounded, as in the proof of Theorem 34, $(a_0 - \alpha_0) = 0$ and so $a_0 = \alpha_0$. Thus,

$$Ax - C = -\sum_{n=1}^{\infty} \left(\frac{(a_n - \alpha_n)\cos(nx) + (b_n - \beta_n)\sin(nx)}{n^2} \right). \quad \text{---------- (51)}$$

Since the right hand side of (51) is bounded in **R**, $A = 0$. Hence,

$$C=\sum_{n=1}^{\infty}\left(\frac{(a_n-\alpha_n)\cos(nx)+(b_n-\beta_n)\sin(nx)}{n^2}\right).$$

Thus, the right hand side converges uniformly to a constant function. Then

$2\pi C=\int_0^{2\pi}Cdx=\sum_{n=1}^{\infty}\int_0^{2\pi}\left(\frac{(a_n-\alpha_n)\cos(nx)+(b_n-\beta_n)\sin(nx)}{n^2}\right)dx=0$ and so $C = 0$. It follows that $\sum_{n=1}^{\infty}\left(\frac{(a_n-\alpha_n)\cos(nx)+(b_n-\beta_n)\sin(nx)}{n^2}\right)=0$. By Theorem 34,

$(a_n-\alpha_n)=0$ and $(b_n-\beta_n)=0$ and so $a_n=\alpha_n$ and $b_n=\beta_n$. Thus $T(x)$ is the Fourier series of the limiting function $f(x)$.

This completes the proof.

James P² integral and Convergent Trigonometric series

We now consider the question of recovering the coefficients of a convergent trigonometric series.

Suppose the trigonometric series (A) converges to a finite function *f*. How can we recover the coefficients a_n , b_n ? If *f* is Lebesgue integrable, then by Theorem 1, the series is the Fourier series of *f* and the coefficients are given by the Euler formula,

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos(nx)dx\,,\quad n = 0, 1, 2, \ \ldots\ ,$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin(nx)dx\ ,\quad n = 1, 2, \ \ldots .$$

Suppose *f* is not Lebesgue integrable, how can the coefficients a_n , b_n be determined ? This question has been settled, albeit by different methods and the invention of different types of integration theory, involving a change of the form of the above Fourier formula for the coefficients, by Denjoy, Verblumsky, Marcinkiewicz and Zygmund, Burkill and James in the period 1935-1950.

We shall describe one solution by James using his Perron second integral, which was named in honour of Perron's method of defining Perron integral of a function. This is also called the P^2 integral. This is invented to tackle this problem and the first crucial result is the following theorem which we state before we give a description of the P^2 integral.

Theorem 36. If the trigonometric series (A) converges to a finite function f or equivalently, if f is the pointwise limit of a convergent trigonometric series in $(0, 2\pi)$, then f is necessarily P^2 integrable.

Suppose f is defined in an interval (a, b). We describe James' major and minor functions of f in almost the same fashion as Perron's major and minor functions except we use the generalized symmetric second derivatives and require the functions to vanish at the end points.

Definition 37. Suppose $f: [a, b] \to \mathbf{R}$ is a finite valued function. The pair of real-valued functions $M : [a, b] \to \mathbf{R}$ and $m : [a, b] \to \mathbf{R}$ are called respectively the *J-major and J-minor functions of f* on $[a, b]$, if

(1) M and m are continuous on $[a, b]$, $M(a) = M(b) = m(a) = m(b)=0$,

(2) $\underline{D}_2 M(x) \geq f(x) \geq \overline{D}_2 m(x)$ for x in (a, b) except for a denumerable set E_0 in (a, b),

(3) $\underline{D}_2 M(x) > -\infty$, $\overline{D}_2 m(x) < \infty$ for all x in (a, b) except for a denumerable set E_0 in (a, b),

(4) $M(x)$ and $m(x)$ are smooth for all x in E_0.

Definition 38. The real valued function $f : [a, b] \to \mathbf{R}$ is said to be P^2 integrable over (a, γ, b), where $a < \gamma < b$, if for any $\varepsilon > 0$, there exists a pair of *J-major and J-minor functions of f* , M and m on $[a, b]$, such that $0 \leq m(\gamma) - M(\gamma) \leq \varepsilon$.

We denote by $-J(\gamma)$ the common values
$\sup\{-m(\gamma): m \text{ a } J\text{-minor function of } f\}=\inf\{-M(\gamma): M \text{ a } J\text{-major functions of } f\}$
and define the P^2 integral of f to be $J(\gamma)$, that is to say,

$$\int_{(a,\gamma,b)} f(x)dx = J(\gamma) \ .$$

Note that in the definition we have taken into account that if M and m are *J-major and J-minor functions of f*, then $M - m$ is convex on $[a, b]$. Condition (2) implies that $\overline{D}_2(M - m)(x) \geq \underline{D}_2 M(x) - \overline{D}_2 m(x) \geq f(x) - f(x) = 0$ for x in (a, b) except for a denumerable set so that by Theorem 21, $M - m$ is convex. Then by continuity and the fact that $(M - m)(b) = (M - m)(a) = 0$, $M(x) - m(x) \leq 0$ and so $-m(x) \leq -M(x)$.

We may define f to be P^2 *integrable over* (a, γ, b) if
$\sup\{-m(\gamma) : m \text{ a } J\text{-minor function of } f\} = \inf\{-M(\gamma) : M \text{ a } J\text{-major functions of } f\}$
and denote the negative of the common value by $\int_{(a,\gamma,b)} f(t)dt$.

Remark. The condition (2) of Definition 37 may be replaced by a weaker condition that the inequality be satisfied except for a set of measure zero is shown by James using a non-negative, increasing and absolutely continuous function χ given in Lemma 1, 11.8, page 369 of "Theory of functions by Titchmarsh, E. C. 1932", together with the fact that the indefinite integral of a bounded increasing function is convex.

Theorem 39 (James). The real valued function $f : [a, b] \to \mathbf{R}$ is P^2 integrable over (a, γ, b), where $a < \gamma < b$, if, and only if, for any $\varepsilon > 0$, there exists a pair of functions M and m on $[a, b]$, satisfying

(1) M and m are continuous on $[a, b]$, $M(a) = M(b) = m(a) = m(b) = 0$,

(2) $\underline{D}_2 M(x) \geq f(x) \geq \overline{D}_2 m(x)$ for x in (a, b) except for a set E of measure zero in (a, b),

(3) $\underline{D}_2 M(x) > -\infty$, $\overline{D}_2 m(x) < \infty$ for all x in (a, b) except for a denumerable set E_0 in (a, b),

(4) $M(x)$ and $m(x)$ are smooth for all x in E_0,

and

(5) $0 \leq m(\gamma) - M(\gamma) \leq \varepsilon$.

We shall give a proof of this theorem later.

For the bounds of the value of the P^2 integral over (a, γ, b), we have

Corollary 40. Suppose $f : [a, b] \to \mathbf{R}$ is P^2 integrable over (a, γ, b), where $a < \gamma < b$. Then for any pair of functions M and m satisfying conditions (1) to (5) of Theorem 39, we have

$$M(\gamma) \le \int_{(a,\gamma,b)} f(t)dt \le m(\gamma) .$$

Theorem 41. If $f : [a, b] \to \mathbf{R}$ is P^2 integrable over (a, γ, b) and $f = g$ almost everywhere in (a, b), then g is also P^2 integrable over (a, γ, b) and $\int_{(a,\gamma,b)} f(x)dx = \int_{(a,\gamma,b)} g(x)dx$.

Proof. By Theorem 39, the functions M and m satisfying conditions (1) to (5) of Theorem 39 for the function f also work for g, since $f = g$ almost everywhere in (a, b). It follows then from Theorem 39 that g is also P^2 integrable over (a, γ, b).

Moreover, by Corollary 40, both integrals lie in the interval $(M(\gamma), m(\gamma))$ of length $\le \varepsilon$. Since ε is arbitrary, the integrals must be the same.

The next theorem gives a descriptive definition of the P^2 integral.

Theorem 42. Suppose that $F(x)$ is continuous in $[a, b]$ and that $D_2F(x)$ is defined for all x in (a, b) except for a set E of measure zero, and that $\overline{D}_2F(x)$ and $\underline{D}_2F(x)$ are finite for all x in (a, b) with the possible exception of a denumerable set E_0, where $F(x)$ is smooth. If $f(x) = D_2F(x)$, where $D_2F(x)$ is defined and $f(x) = 0$ elsewhere, then $f(x)$ is P^2 integrable over (a, γ, b) for any γ in (a, b) and

$$\int_{(a,\gamma,b)} f(x)dx = F(\gamma) - \left(\frac{b-\gamma}{b-a}\right)F(a) - \left(\frac{\gamma-a}{b-a}\right)F(b). \quad \text{----------} \quad (52)$$

Proof. Let $M(x) = m(x) = F(x) - \left(\frac{b-x}{b-a}\right)F(a) - \left(\frac{x-a}{b-a}\right)F(b)$. Then $D_2M(x)$ and $D_2m(x) = f(x)$ for all x in (a, b) except for a set of measure zero and $\overline{D}_2 m(x)$ and $\underline{D}_2 M(x)$ are finite with the exception of a denumerable set where M ad m are smooth. It follows that conditions (1) to (5) of Theorem 39 are satisfied. By Corollary 40,

$$\int_{(a,\gamma,b)} f(x)dx = M(\gamma) = m(\gamma) = F(\gamma) - \left(\frac{b-\gamma}{b-a}\right)F(a) - \left(\frac{\gamma-a}{b-a}\right)F(b).$$

Theorem 43. Suppose $F(x)$ and $G(x)$ are two functions satisfying the hypothesis of Theorem 42. Suppose $D_2F(x) = D_2G(x)$ for almost all x in (a, b).

Then for any x in (a, b),

$$F(x) - \left(\frac{b-x}{b-a}\right)F(a) - \left(\frac{x-a}{b-a}\right)F(b) = G(x) - \left(\frac{b-x}{b-a}\right)G(a) - \left(\frac{x-a}{b-a}\right)G(b).$$

In particular, $F(x) - G(x)$ is a linear function in (a, b).

Proof.

Suppose $D_2F(x) = D_2G(x)$ for all x not in E and E is a set of measure zero in (a, b). Let $f(x) = \begin{cases} D_2F(x), & x \in (a,b) - E \\ 0, & x \in E \end{cases}$ and $g(x) = \begin{cases} D_2G(x), & x \in (a,b) - E \\ 0, & x \in E \end{cases}$

Then $f(x) = g(x)$ for x not in E. That is, $f = g$ almost everywhere in (a, b).

By Theorem 42, $f(x)$ is P^2 integrable over (a, x, b) for any x in (a, b) and

$$\int_{(a,x,b)} f(t)dt = F(x) - \left(\frac{b-x}{b-a}\right)F(a) - \left(\frac{x-a}{b-a}\right)F(b).$$

Also by Theorem 42, $g(x)$ is P^2 integrable over (a, x, b) for any x in (a, b) and

$$\int_{(a,x,b)} g(t)dt = G(x) - \left(\frac{b-x}{b-a}\right)G(a) - \left(\frac{x-a}{b-a}\right)G(b).$$

Since $f = g$ almost everywhere in (a, b), by Theorem 41, these two P^2 integrals are the same, i.e.,

$$F(x) - \left(\frac{b-x}{b-a}\right)F(a) - \left(\frac{x-a}{b-a}\right)F(b) = G(x) - \left(\frac{b-x}{b-a}\right)G(a) - \left(\frac{x-a}{b-a}\right)G(b).$$

Hence,

$$F(x)-G(x)=\left(\frac{b-x}{b-a}\right)F(a)+\left(\frac{x-a}{b-a}\right)F(b)-\left(\frac{b-x}{b-a}\right)G(a)-\left(\frac{x-a}{b-a}\right)G(b)$$

is a linear function.

Suppose now the trigonometric series

$$T(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(x)$$

converges everywhere to a function f. Then by Theorem 15, $\Phi(x)=\frac{1}{4}a_0x^2-\sum_{n=1}^{\infty}\frac{A_n(x)}{n^2}=\frac{1}{4}a_0x^2-\sum_{n=1}^{\infty}\frac{a_n\cos(nx)+b_n\sin(nx)}{n^2}$ converges absolutely and uniformly to a continuous function on **R**. By Riemann Theorem (Theorem 17), $D_2\Phi(x)=f(x)$ everywhere. It follows then from Theorem 42 that f is P^2 integrable over $(-2\pi, x, 2\pi)$ for x in $(-2\pi, 2\pi)$ and that

$$\int_{(-2\pi,x,2\pi)}f(t)dt=\Phi(x)-\left(\frac{2\pi-x}{4\pi}\right)\Phi(-2\pi)-\left(\frac{x+2\pi}{4\pi}\right)\Phi(2\pi).$$

In particular,

$$\int_{(-2\pi,0,2\pi)}f(t)dt=\Phi(0)-\frac{1}{2}\Phi(-2\pi)-\frac{1}{2}\Phi(2\pi)$$

$$=-\sum_{n=1}^{\infty}\frac{a_n}{n^2}-\frac{1}{2}\left(\frac{1}{4}a_0(-2\pi)^2-\sum_{n=1}^{\infty}\frac{a_n}{n^2}\right)-\frac{1}{2}\left(\frac{1}{4}a_0(2\pi)^2-\sum_{n=1}^{\infty}\frac{a_n}{n^2}\right)=-\pi^2a_0.$$

Consequently,

$$a_0=-\frac{1}{\pi^2}\int_{(-2\pi,0,2\pi)}f(t)dt.$$

Hence we have,

Theorem 44. If the trigonometric series $T(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)$ converges everywhere to a function f, then f is necessarily P^2 integrable over $(-2\pi, c, 2\pi)$ for any c in the interval $(-2\pi, 2\pi)$ and $a_0=-\frac{1}{\pi^2}\int_{(-2\pi,0,2\pi)} f(t)dt$.

Remark. Theorem 36 of course follows from Theorem 44.

Theorem 45. If the trigonometric series $T(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)$ converges everywhere to a function f, then for $n > 0$,
$a_n=-\frac{1}{\pi^2}\int_{(-2\pi,0,2\pi)} f(t)\cos(nt)dt$ and $b_n=-\frac{1}{\pi^2}\int_{(-2\pi,0,2\pi)} f(t)\sin(nt)dt$.

We shall write $f(x)\cos(nx)$ as the limit of a trigonometric series. We employ the following technique as explained by R L Jefferey in his 1953 lecture on Trigonometric Series to the Royal Society of Canada.

Lemma 46. Suppose K_{-m} , K_0 , K_m are complex numbers such that $K_{-m}+K_0+K_m=0$. Let $\ldots,c_{-n},c_{-n+1,}\ldots,c_0,c_1,\ldots,c_n,\ldots$ be a sequence of real numbers with $c_{-n},\ c_n\to 0$ as $n\to\infty$. Then for each $m > 0$,

$$\lim_{k\to\infty}\sum_{n=-k}^{k}\left(K_{-m}c_{n+m}+K_0c_n+K_mc_{n-m}\right)=0.$$

Proof. Let $S_k=\sum_{n=-k}^{k}\left(K_{-m}c_{n+m}+K_0c_n+K_mc_{n-m}\right)$ for each positive integer k.

Then

$$S_k=\sum_{n=-k}^{k}\left(K_{-m}c_{n+m}+K_0c_n+K_mc_{n-m}\right)$$

$$=K_{-m}\sum_{n=-k}^{k}c_{n+m}+K_0\sum_{n=-k}^{k}c_n+K_m\sum_{n=-k}^{k}c_{n-m}$$

$$=\left(K_{-m}+K_0+K_m\right)\sum_{n=-k+m}^{k-m}c_n+K_{-m}\sum_{n=k-m+1}^{k+m}c_n+K_0\sum_{n=-k}^{-k+m-1}c_n+K_0\sum_{n=k-m+1}^{k}c_n+K_m\sum_{n=-k-m}^{-k+m-1}c_n$$

$$= K_{-m} \sum_{n=k-m+1}^{k+m} c_n + K_0 \left(\sum_{n=-k}^{-k+m-1} c_n + \sum_{n=k-m+1}^{k} c_n \right) + K_m \sum_{n=-k-m}^{-k+m-1} c_n$$

$$= K_{-m} \sum_{n=k-m+1}^{k+m} c_n + K_0 \left(\sum_{n=k-m+1}^{k} c_{-n} + \sum_{n=k-m+1}^{k} c_n \right) + K_m \sum_{n=k-m+1}^{k+m} c_{-n}$$

Each of the above summations is a finite sum of sequences that tends to 0 and so $S_k \to 0$.

We shall apply Lemma 46 to trigonometric series. Observe that

$$T(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2} \right)$$

$$= \frac{1}{2}a_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left((a_n - ib_n)e^{inx} + (a_n + ib_n)e^{-inx}\right)$$

Now define two sequences of real numbers, $a_{-n} = a_n$ and $b_{-n} = -b_n$ for $n > 0$ and $b_0 = 0$. We can now write the partial sum of $T(x)$ as

$$T_k(x) = \frac{1}{2}a_0 + \sum_{n=1}^{k}\left(a_n \cos(nx) + b_n \sin(nx)\right) = \frac{1}{2}\sum_{n=-k}^{k}\left((a_n - ib_n)e^{inx}\right). \quad \text{-------------- (53)}$$

Since $T_k(x)$ converges to $f(x)$ everywhere, $c_n = (a_n - ib_n)e^{inx}$ converges to 0 as n tends to $\pm\infty$. It follows by Lemma 46 that

$$\lim_{k\to\infty} \sum_{n=-k}^{k} \left(K_{-m}(a_{n+m} - ib_{n+m})e^{i(n+m)x} + K_0(a_n - ib_n)e^{inx} + K_m(a_{n-m} - ib_{n-m})e^{i(n-m)x} \right) = 0 .$$

-- (54)

If we now let $K_{-m} = \frac{1}{2}e^{-imx}$, $K_0 = -\frac{1}{2}\left(e^{-imx} + e^{imx}\right)$ and $K_m = \frac{1}{2}e^{imx}$, then

$K_{-m} + K_0 + K_m = 0$. And we have

$$\sum_{n=-k}^{k} \left(K_{-m}(a_{n+m} - ib_{n+m})e^{i(n+m)x} + K_0(a_n - ib_n)e^{inx} + K_m(a_{n-m} - ib_{n-m})e^{i(n-m)x} \right)$$

$$= \sum_{n=-k}^{k} \left(\frac{(a_{n+m} - ib_{n+m})}{2} e^{inx} - 2\cos(mx) \frac{(a_n - ib_n)}{2} e^{inx} + \frac{(a_{n-m} - ib_{n-m})}{2} e^{inx} \right).$$

It follow from (54) that

$$\lim_{k\to\infty} 2\cos(mx) \sum_{n=-k}^{k} \left(\frac{(a_n - ib_n)}{2} e^{inx} \right) = \lim_{k\to\infty} \sum_{n=-k}^{k} \left(\frac{(a_{n+m} - ib_{n+m})}{2} e^{inx} + \frac{(a_{n-m} - ib_{n-m})}{2} e^{inx} \right)$$

$$= \lim_{k\to\infty} \sum_{n=-k}^{k} \left(\frac{(a_{n+m} + a_{n-m})}{2} e^{inx} - i \frac{(b_{n-m} + b_{n+m})}{2} e^{inx} \right). \quad \text{-------------------------- (55)}$$

But from (53), $\lim_{k\to\infty} \sum_{n=-k}^{k} \left(\frac{(a_n - ib_n)}{2} e^{inx} \right) = f(x)$ and so

$\lim_{k\to\infty} 2\cos(mx) \sum_{n=-k}^{k} \left(\frac{(a_n - ib_n)}{2} e^{inx} \right) = 2\cos(mx) f(x)$. Therefore, from (55),

$$\lim_{k\to\infty} \sum_{n=-k}^{k} \left(\frac{(a_{n+m} + a_{n-m})}{4} e^{inx} - i \frac{(b_{n-m} + b_{n+m})}{4} e^{inx} \right) = \cos(mx) f(x)$$

Hence, we have shown that $\cos(mx) f(x)$ is the limit of a convergent trigonometric series

$$\sum_{n=-\infty}^{\infty} \left(\frac{(a_{n+m} + a_{n-m})}{4} e^{inx} - i \frac{(b_{n-m} + b_{n+m})}{4} e^{inx} \right) = \cos(mx) f(x),$$

whose constant term is $\frac{(a_m + a_{-m})}{4} - i \frac{(b_{-m} + b_m)}{4} = \frac{a_m}{2}$. It follows from Theorem 44 that $a_m = -\frac{1}{\pi^2} \int_{(-2\pi,0,2\pi)} f(t)\cos(mt)dt$.

Similarly, if we take $K_{-m} = -\frac{1}{2i} e^{-imx}$, $K_0 = -\frac{1}{2i}\left(e^{imx} - e^{-imx}\right)$ and $K_m = \frac{1}{2i} e^{imx}$, we can deduce that

$$\sum_{n=-\infty}^{\infty} \left(\frac{(b_{n+m} - b_{n-m})}{4} e^{inx} + i \frac{(a_{n+m} - a_{n-m})}{4} e^{inx} \right) = \sin(mx) f(x),$$

where the trigonometric series on the left has constant term equalling

$$\frac{(b_m - b_{-m})}{4} - i\frac{(a_m - a_{-m})}{4} = \frac{b_m}{2}.$$

Thus, by Theorem 44, $b_m = -\frac{1}{\pi^2}\int_{(-2\pi,0,2\pi)} f(t)\sin(mt)dt$.

James' Argument (Proof of Theorem 39)

Because of the use of Theorem 22 to deduce convexity we actually require condition (2) of Definition 37 to hold except for a denumerable set. James' argument is that if there are functions *M* and *m* satisfying conditions (1) to (5) of Theorem 39, then there are *J*- major and *J*-minor functions for *f* satisfying the condition of P^2 integrability for *f*.

Central to the argument is the construction of a non-negative increasing function whose derivative at every point of a set of measure zero is infinite.

Lemma 47 (Titchmarsh). Suppose *E* is a set of measure zero in (*a*, *b*). Then for any ε > 0, there is a non-negative increasing and absolutely continuous function Φ(*x*) such that Φ′(*x*) = + ∞ for every *x* in *E*, Φ(*a*) = 0 and Φ(*b*) < ε .

Proof.

Let $\varepsilon_n = \varepsilon/2^n$ for integer $n \geq 1$. Then $\sum_{n=1}^{\infty}\varepsilon_n = \varepsilon$. By the outer regularity of Lebesgue measure, for each integer $n \geq 1$, there exists an open set U_n such that $E \subseteq U_n$ and the measure of U_n , $m(U_n) < \varepsilon_n$. By taking intersection of these consecutive open sets if need be, we may assume that $U_{n+1} \subseteq U_n$.

Let f_n be the characteristic function of U_n. That is,

$$f_n(x) = \begin{cases} 1, \text{ if } x \in U_n \\ 0, \text{ if } x \notin U_n \end{cases} .$$

Let $\phi_n = f_1 + f_2 + \cdots + f_n = \sum_{k=1}^{n} f_k$. Then (ϕ_n) is an increasing sequence of non-negative functions. In particular, $\phi_n(x) = n$ for *x* in *E*.

Observe that $\int_a^x f_n = \int_{(a,x)\cap U_n} f_n = m\big((a,x)\cap U_n\big) \le m\big(U_n\big) < \varepsilon_n$. It follows that

$$\int_a^x \phi_n = \sum_{k=1}^{n}\int_a^x f_k < \sum_{k=1}^{n} \varepsilon_n < \varepsilon \quad \text{----------------} \quad (56).$$

Therefore, by Bepo Levi ‘s Lemma, ϕ_n converges to a finite function ϕ almost everywhere and in particular,

$$\int_a^x \phi = \lim_{n\to\infty}\int_a^x \phi_n < \varepsilon . \qquad \text{-------------------} \quad (57)$$

Let $\Phi(x) = \int_a^x \phi$ and $\Phi_n(x) = \int_a^x \phi_n$. Note that ϕ_n and ϕ are measurable and Lebesgue integrable and that (Φ_n) is an increasing sequence of absolutely continuous functions converging to an increasing absolutely continuous nonnegative and bounded function Φ.

Observe that since f_k is the characteristic function of U_k, the derivative of

$g_k(x) = \int_a^x f_k$ satisfies $g_k{}'(x) = 1$ for all x in U_k. It follows that

$D\Phi_n(x) = \sum_{k=1}^{n} g_n{}'(x) = n$ for all x in U_n. This means that for any $\delta > 0$, there exists $\delta_1 > 0$ for x in U_n, such that for $0 < |h| < \delta_1$

$$\frac{\Phi_n(x+h) - \Phi_n(x)}{h} > n - \delta \quad \text{-----------------} \quad (58)$$

Consequently, for x in U_n, $0 < |h| < \delta_1$ implies

$$\frac{\Phi(x+h)-\Phi(x)}{h} = \frac{\int_x^{x+h}\phi}{h} \ge \frac{\int_x^{x+h}\phi_n}{h} = \frac{\Phi_n(x+h)-\Phi_n(x)}{h} > n - \delta .$$

Therefore, for x in U_n,

$$\limsup_{h\to 0^+}\frac{\Phi(x+h)-\Phi(x)}{h}, \limsup_{h\to 0^-}\frac{\Phi(x+h)-\Phi(x)}{h} \ge n$$

and

$$\liminf_{h\to 0^+}\frac{\Phi(x+h)-\Phi(x)}{h},\liminf_{h\to 0^-}\frac{\Phi(x+h)-\Phi(x)}{h}\geq n.$$

This means that $D\Phi(x) = \Phi'(x) \geq n$ for any x in U_n. Since $E \subseteq U_n$ for any positive integer n, this implies that $D\Phi(x) = \Phi'(x) = +\infty$ for x in E. Furthermore, by (57), $\Phi(x)=\int_a^x \phi < \varepsilon$. Moreover, $\Phi(a)=0$ and $\Phi(b)<\varepsilon$. This completes the proof.

Proof of Theorem 39. James' Argument.

Since any enumerable set is of measure zero, the condition of P^2 integrability over (a, γ, b), where $a < \gamma < b$, implies conditions (1) to (5) of Theorem 39. So we shall prove the converse.

Take any γ in (a, b). Suppose $f : [a, b] \to \mathbf{R}$ is a finite valued function. Given any $\varepsilon > 0$, by hypothesis, there are functions $M : [a, b] \to \mathbf{R}$ and

$m : [a, b] \to \mathbf{R}$ satisfying

(1) M and m are continuous on $[a, b]$, $M(a) = M(b) = m(a) = m(b)=0$,

(2) $\underline{D}_2 M(x) \geq f(x) \geq \overline{D}_2 m(x)$ for x in (a, b) except for a set E of measure zero,

(3) $\underline{D}_2 M(x) > -\infty$, $\overline{D}_2 m(x) < \infty$ for all x in (a, b) except for a denumerable set E_0 in (a, b),

(4) $M(x)$ and $m(x)$ are smooth for all x in E_0

and

(5) $0 \leq m(\gamma) - M(\gamma) \leq \varepsilon/2$.

By Lemma 47, there is a non-negative increasing and absolutely continuous function $\Phi(x)$ such that $\Phi'(x) = +\infty$ for every x in E, $\Phi(a) = 0$ and

$$0 \leq \Phi(b) < \frac{\varepsilon}{4(\gamma - a)} . \quad \text{------------------------} \quad (59)$$

Define $$\Psi(x)=\int_a^x \Phi - \frac{x-a}{b-a}\int_a^b \Phi .$$

The function $\int_a^x \Phi$ is convex since it is the integral of a bounded continuous increasing function. We can deduce this by showing that this function is mid-point convex and that mid-point convexity for a continuous function implies convexity for the function on (a, b) (Jensen Theorem). Notice that for $x < y$,

$\int_a^{\frac{x+y}{2}} \Phi \leq \frac{1}{2}\left(\int_a^x \Phi + \int_a^y \Phi\right) \Leftrightarrow \int_x^{\frac{x+y}{2}} \Phi \leq \int_{\frac{x+y}{2}}^y \Phi$ and the inequality on the right holds since Φ is non negative and increasing. Hence $\Phi(x)$ is convex and so $\Psi(x)$ is convex. Since $\Psi(a) = \Psi(b) = 0$, it follows that

$$\Psi(\gamma) = \int_a^\gamma \Phi - \frac{\gamma - a}{b-a}\int_a^b \Phi \leq 0. \qquad \text{----------------} \quad (60)$$

Hence,

$$0 \leq -\Psi(\gamma) = \frac{\gamma - a}{b-a}\int_a^b \Phi - \int_a^\gamma \Phi \leq \frac{\gamma - a}{b-a}\int_a^b \Phi < \frac{\gamma - a}{b-a}(b-a)\frac{\varepsilon}{4(\gamma - a)} = \frac{\varepsilon}{4},$$

----------- (61)

by (59).

Observe that Ψ is differentiable in (a, b) since $\int_a^x \Phi$ is differentiable as Φ is continuous.

We now examine the symmetric second derivative of Ψ.

$$\frac{\Delta_h^2 \Psi(x)}{h^2} = \frac{\Psi(x+h) + \Psi(x-h) - 2\Psi(x)}{h^2}$$

$$= \frac{\int_a^{x+h} \Phi + \int_a^{x-h} \Phi - 2\int_a^x \Phi}{h^2} = \frac{\Phi(x+h) - \Phi(x-h)}{2h}$$

for some h between 0 and h by Cauchy Mean Value Theorem

$$= \frac{\Phi(x+\theta h) - \Phi(x-\theta h)}{2\theta h}, \text{ for some } 0 < \theta < 1,$$

$$= \frac{1}{2}\frac{\Phi(x+\theta h) - \Phi(x)}{\theta h} + \frac{1}{2}\frac{\Phi(x-\theta h) - \Phi(x)}{-\theta h}. \qquad \text{--------------} \quad (62)$$

It then follows from (62) that

$$\liminf_{h\to 0}\frac{\Delta_h^2\Psi(x)}{h^2}\geq\frac{1}{2}\liminf_{h\to 0}\frac{\Phi(x+h)-\Phi(x)}{h}+\frac{1}{2}\liminf_{h\to 0}\frac{\Phi(x-h)-\Phi(x)}{-h}$$

$$=\frac{1}{2}\underline{D}\Phi(x)+\frac{1}{2}\underline{D}\Phi(x)=\underline{D}\Phi(x).$$

Therefore, $\underline{D_2}\Psi(x)\geq\underline{D}\Phi(x)\geq 0$. -------------------------------------- (63)

Using Ψ we now define *J* major and minor functions for *f*.

Let $M(x)=M(x)+\Psi(x)$ and $m(x)=m(x)-\Psi(x)$. Note that $M(x)$ and $m(x)$ are continuous on $[a, b]$ and $M(a)=M(b)=m(a)=m(b)=0$.

Since $M(x)$ and $m(x)$ are smooth for all x in E_0 and $\Psi(x)$ is differentiable in (a, b), $M(x)$ and $m(x)$ are smooth for all x in E_0.

Then from the definition of $M(x)$ and $m(x)$, we deduce that

$$\underline{D_2}M(x)\geq\underline{D_2}M(x)+\underline{D_2}\Psi(x) \qquad \text{--------------------} \quad (64)$$

and

$$\overline{D_2}m(x)\leq\overline{D_2}m(x)-\underline{D_2}\Psi(x) \qquad \text{-------------------} \quad (65)$$

It follows from (64), (65) and (63) and condition (3) that $\underline{D_2}M(x)>-\infty$ and $\overline{D_2}m(x)<\infty$ for all x in (a, b) except for a denumerable set E_0 in (a, b).

By conditions (2) and (63), we have then

$$\underline{D_2}M(x)\geq f(x)\geq\overline{D_2}m(x) \text{ for } x \text{ in } (a, b)-E.$$

Now, for x in $E-E_0$, $\underline{D}\Phi(x)=+\infty$ and so by (63),

$$\underline{D_2}\Psi(x)=+\infty. \qquad \text{--------------------------} \quad (66)$$

Since $\underline{D}_2 M(x) > -\infty$ for x in $E - E_0$, it follows that for x in $E - E_0$, by (64) and (66),

$$\underline{D}_2 M(x) = +\infty > f(x).$$

Since $\overline{D}_2 m(x) < \infty$ for x in $E - E_0$, it follows that for x in $E - E_0$, by (65) and (66),

$$\overline{D}_2 m(x) = -\infty < f(x).$$

It then follows that for x *not in* E_0,

$$\underline{D}_2 M(x) > f(x) > \overline{D}_2 m(x).$$

Moreover, from (61) and that $0 \leq m(\gamma) - M(\gamma) < \varepsilon/2$, we obtain

$$0 \leq m(\gamma) - M(\gamma) = m(\gamma) - M(\gamma) - 2\Psi(\gamma) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $M(x)$ and $m(x)$ are J major and minor functions for f satisfying $0 \leq m(\gamma) - M(\gamma) < \varepsilon$. Therefore, by Definition 38, f is P^2 integrable over (a, γ, b).

Proof of Corollary 40.

Suppose $f : [a, b] \to \mathbf{R}$ is P^2 integrable over (a, γ, b), where $a < \gamma < b$. Then by Theorem 39, given any $\varepsilon > 0$, there exists a pair of functions M and m satisfying conditions (1) to (5) of Theorem 39. Let $\delta > 0$ be arbitrarily given. As in the proof of Theorem 39, let $\Psi(x) = \int_a^x \Phi - \frac{x-a}{b-a}\int_a^b \Phi$, where $\Phi(x)$ is a non-negative increasing and absolutely continuous function such that $\Phi'(x) = +\infty$ for every x in E, $\Phi(a) = 0$ and $0 \leq \Phi(b) < \dfrac{\delta}{(\gamma - a)}$.

Let $M(x) = M(x) + \Psi(x)$ and $m(x) = m(x) - \Psi(x)$. We have shown that these functions are J major and minor functions for f. By the definition of P^2 integrability of f over (a, γ, b),

$$M(\gamma) \le \int_{(a,\gamma,b)} f(t)dt \le m(\gamma).$$

Hence, $$M(\gamma) + \Psi(\gamma) \le \int_{(a,\gamma,b)} f(t)dt \le m(\gamma) - \Psi(\gamma).$$

But by (60),

$$0 \le -\Psi(\gamma) = \frac{\gamma - a}{b-a}\int_a^b \Phi - \int_a^\gamma \Phi \le \frac{\gamma - a}{b-a}\int_a^b \Phi < \frac{\gamma - a}{b-a}(b-a)\frac{\delta}{(\gamma - a)} = \delta.$$

It follows that

$$M(\gamma) - \delta \le \int_{(a,\gamma,b)} f(t)dt \le m(\gamma) + \delta.$$

Since $\delta > 0$ is arbitrary, $M(\gamma) \le \int_{(a,\gamma,b)} f(t)dt \le m(\gamma)$. This completes the proof.