Fourier Series for Even and Odd Functions

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In this note, we discuss slightly more general Fourier cosine and sine series arising from bounded even and odd functions. We present two theorems about even and odd functions. The first is an interesting result about the sum of a Fourier cosine series with non-negative coefficients and the second is about uniform boundedness of the Fourier sine series with non-negative coefficients. We give sufficient conditions for the non-negativity of the Fourier coefficients and state specific convergence results about functions satisfying these sufficient conditions.

A function f defined in $\mathbb{R}$ is *even* when it satisfies $f(x) = f(-x)$ for all x in $\mathbb{R}$. It is *odd* if it satisfies $f(-x) = -f(x)$ for all x in $\mathbb{R}$. These two notions are not mutually exclusive, that is, if f is not even, it need not be odd and vice versa. It is easy to deduce that the Fourier series of a Lebesgue integrable even periodic function is a Fourier cosine series and that of a Lebesgue integrable odd periodic function is a Fourier sine series. In *Fourier Cosine and Sine Series*, I elaborate on the convergence of series with decreasing coefficients or with convex coefficients. By the Riemann-Lebesgue Theorem, the Fourier coefficients converge to 0. Of course, the coefficients can converge to 0 in different ways. A sequence slightly more general than a null decreasing sequence is a null sequence with non-negative terms. Here we examine a larger class of Fourier cosine and sine series with non-negative coefficients and of course, this class includes series with decreasing coefficients.

We begin with the case of even function. The main result is:

Theorem 1. Suppose f is an even, periodic and Lebesgue integrable function of period 2π and is bounded in a small neighbourhood of the origin. Its Fourier series is

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(k\theta). \quad \text{---------------------------} \quad (C)$$

Suppose $a_n \geq 0$ for all integer $n \geq 1$.

Then

(1) $\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k < \infty$.

(2) The Fourier series of *f* converges uniformly to a continuous function *g* and the Fourier series of *f* is also the Fourier series of *g*. Moreover, $g = f$ almost everywhere in $[-\pi, \pi]$.

Proof.

Since *f* is even and Lebesgue integrable, it has a Fourier series, which is a cosine series of the form (C).

We shall make use of the (C,1) means of the Fourier series of *f*.

Let

$$t_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos(kx) \qquad \text{----------------------------- (1)}$$

be the (n+1)-th partial sum of (C). Then the (C,1) mean of *f* is given by

$$\sigma_n(x) = \frac{1}{n+1}\left(t_0(x) + t_1(x) + \cdots + t_n(x)\right) = \frac{1}{n+1}\sum_{k=0}^{n} t_k(x)\,. \qquad \text{------------- (2)}$$

Then we can write the (C,1) mean in terms of the Fejér kernel

$$\sigma_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} t_k(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)K_n(u)du\,, \qquad \text{-------------------------- (3)}$$

where $K_n(u)$ is the Fejér kernel function and

$$K_n(x) = \begin{cases} \dfrac{2}{n+1}\left\{\dfrac{\sin(\frac{1}{2}(n+1)x)}{2\sin(\frac{1}{2}x)}\right\}^2, & x \in [-\pi,\pi]-\{0\} \\ \dfrac{n+1}{2}, & x = 0 \end{cases}\,. \qquad \text{-------------------- (4)}$$

(See *Fourier Cosine and Sine series section 2.4 page 10-11, formula (101) on page 36. Our* $\sigma_n(x)$ here is $\sigma_{n+1}(x)$ there. Also pages 4-6 of *Convergence of Fourier Series*.)

Note that $K_n(x)$ is an even function, $K_n(x) \geq 0$ and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(u)du = 1. \quad \text{------------------------------------} \quad (5)$$

If we can show that the sequence ($\sigma_n(0)$) is bounded, then using this fact we can deduce that $\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k < \infty$.

The proof is simpler, if we assume *f* is bounded, i.e., *f* is bounded in $[-\pi, \pi]$.

Suppose now *f* is bounded in $[-\pi, \pi]$, say $|f(x)| \le M$ for all *x* in $[-\pi, \pi]$ for some real value *M*. Then by using (3), for all *x* in $[-\pi, \pi]$,

$$|\sigma_n(x)| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)K_n(u)du\right| \le \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+u)K_n(u)|du \le M\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(u)du = M ,$$

since $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(u)du = 1$.

Therefore, the sequence of (C,1) means of the Fourier series is uniformly bounded by *M*. Consequently, $|\sigma_n(0)| \le M$.

Suppose we assume only that *f* is bounded in a small neighbourhood of the origin by *K*. That is, there exists $0 < \delta < \pi$ such that $|f(x)| \le K$ for all *x* in $[-\delta, \delta]$.

$$\sigma_n(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)K_n(u)du = \frac{1}{\pi}\int_{0}^{\pi} f(u)K_n(u)du + \frac{1}{\pi}\int_{-\pi}^{0} f(u)K_n(u)du$$

$= \frac{1}{\pi}\int_{0}^{\pi} f(u)K_n(u)du + \frac{1}{\pi}\int_{0}^{\pi} f(u)K_n(u)du$, since $f(u)$ and $K_n(u)$ are even functions,

$$= \frac{2}{\pi}\int_{0}^{\pi} f(u)K_n(u)du$$

$$= \frac{2}{\pi}\int_{0}^{\delta} f(u)K_n(u)du + \frac{2}{\pi}\int_{\delta}^{\pi} f(u)K_n(u)du . \quad \text{--------------------------------} \quad (6)$$

Now,

$$\int_{\delta}^{\pi} f(u)K_n(u)du = \frac{2}{n+1}\int_{\delta}^{\pi} f(u)\left\{\frac{\sin(\frac{1}{2}(n+1)u)}{2\sin(\frac{1}{2}u)}\right\}^2 du$$

$$= \frac{1}{2(n+1)} \frac{1}{\sin^2(\frac{1}{2}\delta)} \int_\delta^\eta f(u) \sin^2(\tfrac{1}{2}(n+1)u)du \ , \quad \text{----------------------------- (7)}$$

for some $\delta < \eta < \pi$,

by the Second Mean Value Theorem for Integral.

(See *Theorem 11 and Corollary 12* of *Convergence of Fourier Series*.)

Hence, it follows from (7) that

$$\left| \int_\delta^\pi f(u) K_n(u) du \right| \le \frac{1}{2(n+1)} \frac{1}{\sin^2(\frac{1}{2}\delta)} \int_\delta^\eta |f(u)| du \le \frac{1}{2\sin^2(\frac{1}{2}\delta)} \int_0^\pi |f(u)| du \ . \quad \text{------ (8)}$$

Observe that

$$\frac{2}{\pi} \left| \int_0^\delta f(u) K_n(u) du \right| \le \frac{2}{\pi} \int_0^\delta |f(u) K_n(u)| du \le \frac{2}{\pi} \int_0^\delta K K_n(u) du \le K \frac{2}{\pi} \int_0^\pi K_n(u) du = K \ .$$

----------(9)

Therefore, it follows from (6), (8) and (9) that

$$|\sigma_n(0)| \le \left| \frac{2}{\pi} \int_0^\delta f(u) K_n(u) du \right| + \left| \frac{2}{\pi} \int_\delta^\pi f(u) K_n(u) du \right|$$

$$\le K + \frac{2}{\pi} \frac{1}{2\sin^2(\frac{1}{2}\delta)} \int_0^\pi |f(u)| du = K + \frac{1}{\pi \sin^2(\frac{1}{2}\delta)} \int_0^\pi |f(u)| du \ .$$

If we let $M = K + \frac{1}{\pi \sin^2(\frac{1}{2}\delta)} \int_0^\pi |f(u)| du$, then we have $|\sigma_n(0)| \le M$.

So we have proved that $|\sigma_n(0)| \le M$ for all integer $n \ge 0$.

Note that $t_n(0) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k$ is the (n+1)-th partial sum of the series $\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k$.

Since $a_n \ge 0$ for all integer $n \ge 1$, we only need to show that the sequence $t_n(0)$ is bounded. We show this by showing that $\tilde{t}_n(0) = \sum_{k=1}^{n} a_k$ is bounded. Observe that

$$\frac{1}{2} \tilde{t}_n(0) \le \frac{(n+1)\tilde{t}_n(0)}{2n+1}$$

$$\leq \frac{\tilde{t}_0(0)+\tilde{t}_1(0)+\cdots+\tilde{t}_n(0)+\tilde{t}_{n+1}(0)+\cdots+\tilde{t}_{2n}(0)}{2n+1} = \sigma_{2n}(0) - \frac{1}{2}a_0 \leq M + \frac{1}{2}|a_0| .$$

Hence, $\tilde{t}_n(0) \leq 2M + |a_0|$ for all integer $n \geq 0$.

Therefore, by the Monotone Convergence Theorem, $\tilde{t}_n(0)$ is convergent and so $\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k < \infty$.

It then follows by the Weierstrass M test that the Fourier series of f converges uniformly and absolutely to a continuous function g in $[-\pi, \pi]$. Therefore, the Fourier series of f is also the Fourier series of g. Since g is continuous, by *Theorem 34 of Convergence of Fourier Series*, the sequence of (C, 1) means of the Fourier series of g, which is the same as the (C, 1) means of the Fourier series of f, converges uniformly and boundedly to $g(x)$ in $[-\pi, \pi]$. On the other hand, by *Theorem 36 of Convergence of Fourier Series*, the sequence of (C,1) means of the Fourier series of f converges almost everywhere to $f(x)$.

Hence, $f(x) = g(x)$ almost everywhere in $[-\pi, \pi]$.

Now, we come to the case of odd function.

Theorem 2. Suppose f is an odd, Lebesgue integrable and periodic function of period 2π. Suppose f is bounded and its Fourier series at θ is given by

$$\sum_{k=1}^{\infty} b_k \sin(k\theta), \quad \text{---------------------------} \quad \text{(S)}$$

with $b_n \geq 0$ for all integer $n \geq 1$.

Then

(1) The partial sum $s_n(\theta) = \sum_{k=1}^{n} b_k \sin(k\theta)$ is uniformly bounded.

(2) If f is continuous in $[-\pi, \pi]$, then the Fourier series of f converges uniformly to f.

Proof.

The (C,1) means of the Fourier series of f is given by

$$\sigma_n(x) = \frac{1}{n+1}\left(s_1(x)+\cdots+s_n(x)\right) = \frac{1}{n+1}\sum_{k=1}^{n} s_k(x)$$

$$= \frac{1}{n+1}\sum_{k=1}^{n}(n+1-k)b_k\sin(kx) = \sum_{k=1}^{n}(1-\frac{k}{n+1})b_k\sin(kx)\,. \quad \text{--------------} \quad (10)$$

On the other hand we also know that

$$\sigma_n(x) = \frac{1}{n+1}\sum_{k=1}^{n} s_k(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)K_n(u)du\,.$$

Suppose f is bounded by K, that is, $|f(x)| \le K$ for all x in $[-\pi, \pi]$. Then

$$\left|\sigma_n(x)\right| \le K\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(u)du = K\,. \quad \text{----------------------------} \quad (11)$$

It then follows from (10), that for all integer $n \ge 1$ and x in $[-\pi, \pi]$,

$$\left|\sum_{k=1}^{n}(1-\frac{k}{n+1})b_k\sin(kx)\right| \le K\,. \quad \text{------------------------------} \quad (12)$$

Let $N = 2n$ and $x = \dfrac{\pi}{4n}$.

Then for $1 \le k \le N = 2n$,

$$\sin(kx) \ge \frac{2}{\pi}kx = \frac{k}{2n}\,. \quad \text{------------------------} \quad (13)$$

Since $b_n \ge 0$ for all integer $n \ge 1$ and $1-\dfrac{k}{2n+1} \ge 0$ for $1 \le k \le N = 2n$,

$$\left|\sigma_N(x)\right| = \left|\sigma_{2n}(x)\right| = \left|\sum_{k=1}^{2n}(1-\frac{k}{2n+1})b_k\sin(kx)\right| = \sum_{k=1}^{2n}(1-\frac{k}{2n+1})b_k\sin(kx)$$

$$\ge \sum_{k=1}^{2n}(1-\frac{k}{2n+1})b_k\frac{k}{2n}, \quad \text{-----------------------------------} \quad (14)$$

by (13).

Combining (11) and (14) we get

$$\sum_{k=1}^{2n}(1-\frac{k}{2n+1})b_k\frac{k}{2n}\le K$$

and so

$$\sum_{k=1}^{2n}(1-\frac{k}{2n+1})b_k k\le 2nK\,. \quad \text{------------------------ (15)}$$

Now, for $1\le k\le n$, $\dfrac{k}{2n+1}\le\dfrac{1}{2}$ so that $1-\dfrac{k}{2n+1}\ge\dfrac{1}{2}$ and so

$$\sum_{k=1}^{n}\frac{1}{2}b_k k\le\sum_{k=1}^{n}(1-\frac{k}{2n+1})b_k k\le\sum_{k=1}^{2n}(1-\frac{k}{2n+1})b_k k\le 2nK$$

and hence

$$\sum_{k=1}^{n}b_k k\le 4nK\,. \quad \text{--- (16)}$$

Therefore, it follows from (16), that for all x in $[-\pi,\pi]$,

$$\left|\sum_{k=1}^{n}\frac{k}{n+1}b_k\sin(kx)\right|\le\sum_{k=1}^{n}\frac{k}{n+1}b_k\le\frac{4n}{n+1}K\le 4K\,. \quad \text{-------------------- (17)}$$

Hence, for all integer $n\ge 1$ and x in $[-\pi,\pi]$,

$$|s_n(x)|=\left|\sum_{k=1}^{n}b_k\sin(kx)\right|=\left|\sum_{k=1}^{n}\frac{k}{n+1}b_k\sin(kx)+\sum_{k=1}^{n}(1-\frac{k}{n+1})b_k\sin(kx)\right|$$

$$\le\left|\sum_{k=1}^{n}\frac{k}{n+1}b_k\sin(kx)\right|+\left|\sum_{k=1}^{n}(1-\frac{k}{n+1})b_k\sin(kx)\right|$$

$$\le 4K+K=5K\,, \quad \text{--- (18)}$$

by inequality (17) and (12).

This proves that the sequence of partial sums of the series (S) is uniformly bounded. This proves part (1)

Suppose now f is continuous in $[-\pi,\pi]$. Then by Theorem 34 part (iii) in *Convergence of Fourier Series*, the sequence of (C,1) means of the Fourier series of f, (S), converges uniformly and boundedly to $f(x)$ in $[-\pi,\pi]$. We shall make use of this fact to show that (S) is uniformly Cauchy.

Thus, given $\varepsilon > 0$, there exists integer N such that

$$n \geq N \Rightarrow \left|\sigma_n(x) - f(x)\right| < \varepsilon \text{ for all } x \text{ in } [-\pi, \pi]. \text{ -------------- } (19)$$

Let $g(x) = f(x) - \sigma_N(x)$. Then g is Lebesgue integrable, periodic of period 2π and is an odd function. In particular, for all x in $[-\pi, \pi]$.

$$\left|g(x)\right| = \left|f(x) - \sigma_N(x)\right| < \varepsilon \text{ .------------------------ } (20)$$

This means g is bounded in $[-\pi, \pi]$.

Note that $\sigma_N(x) = \sum_{k=1}^{N}(1 - \frac{k}{N+1})b_k \sin(kx)$ and so the Fourier coefficients of $g(x)$, (B_k) is given by

$$B_k = \begin{cases} \frac{k}{N+1} b_k, & 1 \leq k \leq N, \\ b_k, & k \geq N+1 \end{cases} . \qquad \text{------------------------ } (21)$$

Hence, $B_k \geq 0$ for all integer $k \geq 1$ since $b_k \geq 0$ for all integer $k \geq 1$. Note that the Fourier series of g is $\sum_{k=1}^{\infty} B_k \sin(kx)$. Let $s_n(x, g) = \sum_{k=1}^{n} B_k \sin(kx)$.

Therefore, by part (1) and (18), the sequence of partial sums of the Fourier series of g is uniformly bounded by 5ε. That is to say, for all $n \geq 1$ and for all x in $[-\pi, \pi]$,

$$\left|s_n(x, g)\right| = \left|\sum_{k=1}^{n} B_k \sin(kx)\right| \leq 5\varepsilon. \text{ ------------------------------ } (22)$$

If $m > n \geq N$, then for all x in $[-\pi, \pi]$,

$$\left|s_m(x) - s_n(x)\right| = \left|\sum_{k=n+1}^{m} b_k \sin(kx)\right| = \left|\sum_{k=n+1}^{m} B_k \sin(kx)\right|, \text{ by using (21),}$$

$$= \left|s_m(x, g) - s_n(x, g)\right| \leq \left|s_m(x, g)\right| + \left|s_n(x, g)\right| \leq 5\varepsilon + 5\varepsilon = 10\varepsilon,$$

by using (22).

Hence, $s_k(x)$ is uniformly Cauchy and so it converges uniformly to a continuous function $h(x)$ in $[-\pi, \pi]$. Therefore, by the regularity of (C,1) convergence, $\sigma_n(x)$ converges to $h(x)$ in $[-\pi, \pi]$. Since $\sigma_n(x)$ converges to $f(x)$ in $[-\pi, \pi]$, $h(x) = f(x)$ in $[-\pi, \pi]$. This completes the proof of part (2).

Situation when Theorem 1 may be used is when the periodic function is convex in $(0, 2\pi)$.

Definition 3. A function f is said to be convex in $[a, b]$ if for any $\alpha < \beta$ in $[a, b]$,

$$f(x) \leq \frac{(\beta - x)f(\alpha) + (x-\alpha)f(\beta)}{\beta - \alpha} \quad \text{----------------------} \quad (23)$$

for all x in $[\alpha, \beta]$.

We list below the properties of convex function.

Properties of convex function

Suppose g is convex in $[a, b]$.

(1) g is continuous in (a, b).

(2) g is mid-point convex in (a, b), i.e., for all x, y in (a, b).

$$x < y \Rightarrow f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}.$$

(3) For all x in (a, b), for all h such that $x + h$ and $x - h$ are in (a, b),

$$g(x+h) + g(x-h) - 2g(x) \geq 0. \quad \text{---------------} \; (24)$$

(4) The upper symmetric second derivative of g, $\overline{D_2}g(x) \geq 0$ for all x in (a, b).

(5) The right hand derivative exists for all x in (a, b) and is increasing in (a, b).

(6) The left hand derivative exists for all x in (a, b) and is increasing in (a, b).

(7) $g'(x)$ exists except for at most a countable number of points in (a, b).

Moreover, $-\infty < g'_-(x) \le g'_+(x) < \infty$ for all x in (a, b).

Property (2) is also a sufficient condition for a continuous g to be convex in $[a, b]$.

For a continuous function g in $[a, b]$, (3) and (4) are also sufficient conditions for convexity in $[a, b]$. Of course, (3) and (4) are sufficient conditions for convexity of continuous g in (a, b).

We shall need the following observation for a function g convex in $[a, b]$.

If x and y are in $[a, b)$ and $x > y$, then either $y + h \le x$ or $y + h > x$.

Suppose both $x + h$ and $y + h$ are in $[a, b]$.

If $y + h \le x$, then

$$\frac{g(y+h)-g(y)}{h} \le \frac{g(x)-g(y+h)}{y-x+h} \le \frac{g(x+h)-g(x)}{h},$$

and so

$$g(y+h)-g(y) \le g(x+h)-g(x).$$

If $y + h > x$,

$$\frac{g(y+h)-g(y)}{h} \le \frac{g(y+h)-g(x)}{y+h-x} \le \frac{g(x+h)-g(x)}{h}$$

and we have again $g(y+h)-g(y) \le g(x+h)-g(x)$.

Thus, we have:

(8) The function $g(x+h)-g(x)$ for positive h is increasing in x whenever it is defined.

Theorem 4. Suppose f is a Lebesgue integrable and periodic function of period 2π. If f is convex in $(0, 2\pi)$, then for integer $n \ge 1$,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx \ge 0.$$

Proof. For integer $n \geq 1$, partition $[0, 2\pi]$ into $4n$ equal intervals and sum the integral of f over each interval.

$$\int_0^{2\pi} f(x)\cos(nx)dx = \sum_{k=1}^{4n}\int_{\frac{\pi}{2n}(k-1)}^{\frac{\pi}{2n}k} f(x)\cos(nx)dx$$

$$= \sum_{j=0}^{n-1}\int_{\frac{\pi}{2n}(4j+2)}^{\frac{\pi}{2n}(4j+3)} f(x)\cos(nx)dx + \sum_{j=0}^{n-1}\int_{\frac{\pi}{2n}(4j)}^{\frac{\pi}{2n}(4j+1)} f(x)\cos(nx)dx$$

$$+\sum_{j=0}^{n-1}\int_{\frac{\pi}{2n}(4j+1)}^{\frac{\pi}{2n}(4j+2)} f(x)\cos(nx)dx + \sum_{j=0}^{n-1}\int_{\frac{\pi}{2n}(4j+3)}^{\frac{\pi}{2n}(4j+4)} f(x)\cos(nx)dx$$

$$= \sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f(t+\tfrac{\pi}{n}(2j+1))\cos(nt+(2j+1)\pi)dt + \sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f(t+\tfrac{2\pi}{n}j)\cos(nt+2\pi j)dt$$

$$+\sum_{j=0}^{n-1}\int_{-\frac{\pi}{2n}(4j+2)}^{-\frac{\pi}{2n}(4j+1)} f(-x)\cos(nx)dx + \sum_{j=0}^{n-1}\int_{-\frac{\pi}{2n}(4j+4)}^{\frac{\pi}{2n}(4j+3)} f(-x)\cos(nx)dx$$

$$= -\sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f(t+\tfrac{\pi}{n}(2j+1))\cos(nt)dt + \sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f(t+\tfrac{2\pi}{n}j)\cos(nt)dt$$

$$+\sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f(-t+\tfrac{\pi}{2n}(4j+2))\cos(nt-(2j+1)\pi)dt + \sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f(-t+\tfrac{\pi}{n}(2j+2))\cos(nt-2\pi(j+1))dt$$

$$= \sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f(t+\tfrac{2\pi}{n}j)\cos(nt)dt - \sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f((t+\tfrac{2\pi}{n}j)+\tfrac{\pi}{n})\cos(nt)dt$$

$$-\sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f(-t+\tfrac{\pi}{2n}(4j+2))\cos(nt)dt + \sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f(-t+\tfrac{\pi}{n}(2j+2))\cos(nt)dt$$

$$= \sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f(t+\tfrac{2\pi}{n}j)\cos(nt)dt - \sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f((t+\tfrac{2\pi}{n}j)+\tfrac{\pi}{n})\cos(nt)dt$$

$$-\sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f(-t+\tfrac{\pi}{n}(2j+1))\cos(nt)dt + \sum_{j=0}^{n-1}\int_0^{\frac{\pi}{2n}} f(-t+\tfrac{2\pi}{n}(j+1))\cos(nt)dt\,. \quad \text{-------------------(25)}$$

Now, for $0 \leq t \leq \dfrac{\pi}{2n}$, $0 \leq 2t \leq \dfrac{\pi}{n}$ and so $\dfrac{\pi}{n} - 2t \geq 0$. So, for $0 \leq t \leq \dfrac{\pi}{2n}$, we can write

$$f(-t+\tfrac{\pi}{n}(2j+2))\cos(nt) - f(t+\tfrac{\pi}{n}(2j+1))\cos(nt)$$

$$= f\left((t+\tfrac{\pi}{n}(2j+1))+(\tfrac{\pi}{n}-2t)\right)\cos(nt) - f\left(t+\tfrac{\pi}{n}(2j+1)\right)\cos(nt) \quad \text{------------------ (26)}$$

and

$f(-t+\frac{\pi}{n}(2j+1))\cos(nt)-f(t+\frac{2\pi}{n}j)\cos(nt)$

$=f\left((t+\frac{2\pi}{n}j)+(\frac{\pi}{n}-2t)\right)\cos(nt)-f(t+\frac{2\pi}{n}j)\cos(nt)$. -------------------- (27)

For $0\le t\le\dfrac{\pi}{2n}$, $h=\dfrac{\pi}{n}-2t\ge 0$, let $x=t+\frac{\pi}{n}(2j+1)$ and $y=t+\frac{2\pi}{n}j$ and so $x>y$.

By property (8),

$$f(x+h)-f(x)\ge f(y+h)-f(y). \quad \text{-------------------------- (28)}$$

This means

$f(t+\frac{2\pi}{n}j)-f((t+\frac{2\pi}{n}j)+\frac{\pi}{n})-f(-t+\frac{\pi}{n}(2j+1))+f(-t+\frac{2\pi}{n}(j+1))$

$=f(-t+\frac{2\pi}{n}(j+1))-f(t+\frac{\pi}{n}(2j+1))-\left\{f(-t+\frac{\pi}{n}(2j+1))-f(t+\frac{2\pi}{n}j)\right\}\ge 0$, by (28).

Thus, it follows from this and the fact that for $0\le t\le\dfrac{\pi}{2n}$, $\cos(nt)\ge 0$, that

$\int_0^{\frac{\pi}{2n}}\left(f(-t+\frac{2\pi}{n}(j+1))-f(t+\frac{\pi}{n}(2j+1))-\left\{f(-t+\frac{\pi}{n}(2j+1))-f(t+\frac{2\pi}{n}j)\right\}\right)\cos(nt)dt\ge 0$

and from (25), $\int_0^{2\pi}f(x)\cos(nx)dx\ge 0$.

Consequently, $a_n=\dfrac{1}{\pi}\int_0^{2\pi}f(x)\cos(nx)dx\ge 0$ for all integer $n\ge 1$. This completes the proof.

Suppose f is periodic of period 2π, Lebesgue integrable and is convex in (0, 2π). Suppose

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right) \quad \text{----------------------- (29)}$$

is the Fourier series of f. By Theorem 4, $a_n\ge 0$ for all integer $n\ge 1$.

If f is also bounded in (0, 2π), then since f is continuous in (0, 2π) by Property (1), f is Riemann integrable and hence Lebesgue integrable. Moreover, by Property (5) and (6), the right and left derivatives of f at every point in (0, 2π) exist. Therefore, by Corollary 28 of *Convergence of Fourier Series page 38*, the Fourier series of f converges to $f(x)$ for x in (0, 2π).

For even function f convex and bounded in (0, 2π), its Fourier series reduces to

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx),$$

where $a_n \geq 0$ for all integer $n \geq 1$. Theorem 1 then says that

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k < \infty$$

and the Fourier series of f converges uniformly to f in (0, 2π). It converges to the right limit of f at 0 and the left limit of f at 2π and these two values are the same. That is to say, the limit of f at all multiples of 2π exists.

In summary:

Theorem 5. Suppose f is a Lebesgue integrable and periodic function of period 2π.

(1) If f is convex and bounded in (0, 2π), then the Fourier series of f converges to $f(x)$ in (0, 2π). Moreover, its Fourier series

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty}\left(a_k \cos(kx) + b_k \sin(kx)\right)$$

satisfies $a_n \geq 0$ for all integer $n \geq 1$.

(2) If f is convex, bounded and even in (0, 2π), then the Fourier series of f converges uniformly in $[0, 2\pi]$, it converges to $f(x)$ in (0, 2π) and to the limit of f at $x = 0$ and 2π. Moreover its Fourier series is

$$\frac{1}{2}a_0+\sum_{k=1}^{\infty}a_k\cos(kx)$$

and satisfies $\frac{1}{2}a_0+\sum_{k=1}^{\infty}a_k<\infty$.

Now we turn to the case of odd function. To use Theorem 2, we need to have a function satisfying that the Fourier coefficients are non-negative.

The following give a sufficient condition for the Fourier coefficients b_n to be non-negative.

Theorem 6. Suppose f is a Lebesgue integrable and periodic function of period 2π. If f is decreasing in $(0, 2\pi)$, then for integer $n \geq 1$,

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin(nx)dx\geq 0.$$

Proof.

For integer $n \geq 1$, partition $[0, 2\pi]$ into n equal intervals and sum the integral of f over each interval, we obtain:

$$\int_0^{2\pi}f(x)\sin(nx)dx=\sum_{k=0}^{n-1}\int_{\frac{2\pi}{n}k}^{\frac{2\pi}{n}(k+1)}f(x)\sin(nx)dx$$

$$=\sum_{k=0}^{n-1}\int_0^{\frac{2\pi}{n}}f(t+\tfrac{2\pi}{n}k)\sin(nt+2k\pi)dt=\sum_{k=0}^{n-1}\int_0^{\frac{2\pi}{n}}f(t+\tfrac{2\pi}{n}k)\sin(nt)dt$$

$$=\sum_{k=0}^{n-1}\int_0^{\frac{\pi}{n}}f(t+\tfrac{2\pi}{n}k)\sin(nt)dt+\sum_{k=0}^{n-1}\int_{\frac{\pi}{n}}^{\frac{2\pi}{n}}f(t+\tfrac{2\pi}{n}k)\sin(nt)dt$$

$$=\sum_{k=0}^{n-1}\int_0^{\frac{\pi}{n}}f(t+\tfrac{2\pi}{n}k)\sin(nt)dt+\sum_{k=0}^{n-1}\int_0^{\frac{\pi}{n}}f(t+\tfrac{2\pi}{n}k+\tfrac{\pi}{n})\sin(nt+\pi)dt$$

$$=\sum_{k=0}^{n-1}\int_0^{\frac{\pi}{n}}f(t+\tfrac{2\pi}{n}k)\sin(nt)dt-\sum_{k=0}^{n-1}\int_0^{\frac{\pi}{n}}f(t+\tfrac{2\pi}{n}k+\tfrac{\pi}{n})\sin(nt)dt$$

$$=\sum_{k=0}^{n-1}\int_0^{\frac{\pi}{n}}\left\{f(t+\tfrac{2\pi}{n}k)-f(t+\tfrac{2\pi}{n}k+\tfrac{\pi}{n})\right\}\sin(nt)dt. \qquad \text{----------------- (30)}$$

Since f is decreasing and $\sin(nt) \geq 0$ for $0 \leq t \leq \frac{\pi}{n}$,

$$\sum_{k=0}^{n-1} \int_0^{\frac{\pi}{n}} \{f(t + \tfrac{2\pi}{n}k) - f(t + \tfrac{2\pi}{n}k + \tfrac{\pi}{n})\} \sin(nt)dt \geq 0 .$$

It follows then from (30) that $b_n = \dfrac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx \geq 0$ for all integer $n \geq 1$. This completes the proof of Theorem 6.

If f is bounded and decreasing in (0, 2π), then we may assume without loss of generality that f is decreasing in [0, 2π]. Then f is of bounded variation in [0, 2π]. Thus, by Theorem 24 of *Convergence of Fourier Series page 33*, the Fourier series of f converges boundedly to $\frac{1}{2}\{f(x+) + f(x-)\}$ in [0, 2π]. This means that the sequence of n-th partial sums of the Fourier series is uniformly bounded in [0, 2π]. Moreover, if in addition, f is continuous in [0, 2π], then the Fourier series of f converges uniformly to $f(x)$ in [0, 2π].

For an odd function f which is decreasing and bounded in [0, 2π], by virtue of Theorem 6, we may use Theorem 2 to deduce that its Fourier series is uniformly bounded in [0, 2π]. However, it does not say that it is convergent. It gives less information than Theorem 24 of *Convergence of Fourier Series.* Unlike the case of even function, we may not be able to redefine the function f to a continuous periodic function in [0, 2π] by changing its value at 0. Take the well-known function $J(x)$ defined by,

$$J(x) = \begin{cases} \dfrac{1}{2}(\pi - x), & 0 < x < 2\pi, \\ 0, & x = 0 \end{cases}$$

and extended to whole of $\mathbb{R}$ by periodicity.

$J(x)$ is odd and decreasing in (0, 2π) and its Fourier series is

$$S(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} .$$

Note that $S(x)$ converges uniformly in any closed interval free from multiples of 2π and converges boundedly to the function $J(x)$ in [0, 2π]. Obviously $J(x)$ is continuous in $(0, 2\pi)$ and has a jump discontinuity at 0 and indeed at all multiples of 2π. Plainly we cannot redefine its value at 0 to give a continuous periodic function.

In summary we have:

Theorem 7. Suppose *f* is a Lebesgue integrable and periodic function of period 2π.

(1) If *f* is decreasing and bounded in $(0, 2\pi)$, then the Fourier series of *f*

$$\frac{1}{2}a_0+\sum_{k=1}^{\infty}\left(a_k\cos(kx)+b_k\sin(kx)\right)$$

converges boundedly to $\frac{1}{2}\{f(x+)+f(x-)\}$ in $[0, 2\pi]$. Moreover, its Fourier coefficients (b_n) satisfy $b_n \geq 0$ for all integer $n \geq 1$. If, in addition, *f* is continuous in $[0, 2\pi]$, then its Fourier series converges uniformly to $f(x)$ in $[0, 2\pi]$.

(2) If *f* is odd, bounded and decreasing in $(0, 2\pi)$, then the Fourier series of *f*

$$\sum_{k=1}^{\infty} b_k \sin(kx)$$

converges boundedly to $\frac{1}{2}\{f(x+)+f(x-)\}$ in $[0, 2\pi]$. Moreover, its Fourier coefficients (b_n) satisfy $b_n \geq 0$ for all integer $n \geq 1$. If, in addition, *f* is continuous in $[0, 2\pi]$, then its Fourier series converges uniformly to $f(x)$ in $[0, 2\pi]$.