Fourier Cosine and Sine Series

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Consider the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \qquad \text{-------------} \quad (C)$$

and the series

$$\sum_{n=1}^{\infty} a_n \sin(nx) \qquad \text{-------------} \quad (S)$$

for the case that the sequence (a_n) is a non-negative sequence converging to 0. We investigate the convergence of the above series and when they do converge whether the series is the series of a Lebesgue integrable function. When they do converge to a Lebesgue integrable function, we investigate sufficient condition so that the series is also convergent in the L^1 norm.

We recall the following definitions. Suppose f is a function Lebesgue integrable on $(-\pi, \pi)$. We assume that the function is periodic with period 2π, that is, $f(x) = f(x+2\pi)$ whenever anyone of $f(x)$ or $f(x+2\pi)$ is defined and that $f(-\pi) = f(\pi)$. Note that $f(\pi)$ or $f(-\pi)$ need not necessarily be defined and the restriction of f to the interval $[-\pi, \pi]$ need not necessarily be continuous at the end points. It is convenient to assume that f is defined for all values of x in $[-\pi, \pi]$ and by periodicity to all of $\mathbb{R}$. We may need to define values of f appropriately where it is not defined in $[-\pi, \pi]$ and extend to $\mathbb{R}$ by periodicity.

Then we have the following formula for the definition of the coefficients

of a Fourier series of f:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad n = 0, 1, 2, \ldots \qquad \text{---------} \quad (1)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \quad n = 1, 2, \ldots \qquad \text{---------} \quad (2)$$

Consider the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right) \text{ . ------------------ (A)}$$

When a_n and b_n are given by (1) and (2), (A) is called a Fourier series of the function f. When f is even, $b_n = 0$ for $n \geq 1$ and the series (A) is just (C). When f is odd, $a_n = 0$ for $n \geq 0$ and the series (A) is just (S).

Note that we assume that f is integrable in $[-\pi, \pi]$ so that (1) and (2) are meaningfully defined. Thus, (A) or (C) or (S) is a Fourier series if it is the Fourier series of some integrable function f. (However, for (2) to be defined it is sufficient to have the integrability of $f(x)\sin(x)$ over $[0, \pi]$ and we call (S) the generalized Fourier sine series.)

Note that (A) may or may not converge and may not be the Fourier series of its limiting function. And when (A) is a Fourier series, it may or may not converge at all points. Indeed, there exists a Lebesgue integrable function f whose Fourier series diverges at every point.

If we assume *nice* convergence, we do have some positive result. This is Theorem S below.

Theorem S. If the series (A) converges uniformly to a function f, then it is the Fourier series of its sum function f. More is true, if (A) converges almost everywhere to a function f and the n-th partial sums of (A) are absolutely dominated by a Lebesgue integrable function, then (A) is the Fourier series of f. More precisely the n-th partial sum converges to f in the L^1 norm.

We note that in all two cases of Theorem S, f is Lebesgue integrable and the series (A), by using either the consequence of uniform convergence or the Lebesgue Dominated Convergence Theorem, can be shown to be the Fourier series of f by the method of Theorem 11. The convergence to f in the L^1 norm is a consequence of uniform convergence for the first case and in the other of being absolutely dominated by a Lebesgue integrable function. We shall not prove this result but only for the special case that this note is concerned with.

This note is concerned mainly with the special case of the two series (C) and (S) when the coefficients (a_n) is a sequence of non-negative terms and $a_n \rightarrow 0$.

1. The Main Results.

For the sine series (S) we have the following result giving a necessary and sufficient condition for (S) to be a Fourier series.

Theorem 1. Suppose (a_n) is a sequence of nonnegative terms, $\Delta a_n = a_n - a_{n+1} \geq 0$ and $a_n \to 0$. Then the limit function or sum function of (S), g, is Lebesgue integrable if, and only if, $\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$. If $\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$, then (S) is the Fourier series of g and $\int_{-\pi}^{\pi} |s_n(x) - g(x)| dx \to 0$, where $s_n(x)$ is the n-th partial sum of the series (S), that is, $s_n(x)$ converges to g in the L^1 norm.

The situation with the cosine series is somewhat different. We state the result as follows.

Theorem 2. Suppose the sequence (a_0, a_1, …) is *convex* and $a_n \to 0$. Then

(1) The cosine series (C) converges, except possibly at $x = 0$, to a non-negative Lebesgue integrable function f.

(2) The series (C) is the Fourier series of f.

(3) $\int_{-\pi}^{\pi} |\sigma_n(x) - f(x)| dx \to 0$, where $\sigma_n(x)$ is the *Cesaro 1* or (C, 1) means of the series (C).

(4) $\int_{-\pi}^{\pi} |t_n(x) - f(x)| dx \to 0$ if, and only if, $a_n = o(\frac{1}{\ln(n)})$ or, equivalently, $a_n \ln(n) \to 0$. Here, $t_n(x)$ is the n-th partial sum of the series (C).

We now elaborate on the terms in italic in Theorem 2.

Suppose (a_n) is a sequence and $\Delta a_n = a_n - a_{n+1}$. Then (Δa_n) is also a sequence. The sequence (a_n) = (a_0 , a_1, …) is said to be *convex* if $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1} \geq 0$ for all $n \geq 0$. The *Cesaro 1* or (C,1) means of the sequence is defined to be

$$\sigma_{n+1} = \frac{1}{n+1}\left(s_0 + s_1 + \cdots + s_n\right),$$

where $s_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$. The (C,1) means of the cosine series is then given by

$$\sigma_{n+1}(x) = \frac{1}{n+1}\left(t_0(x) + t_1(x) + \cdots + t_n(x)\right),$$

where $t_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos(kx)$ for $n \geq 1$ and $t_0(x) = \frac{1}{2}a_0$.

If the series $(a_n) = (a_0, a_1, \ldots)$ is only decreasing and $a_n \to 0$, then we may not always have Lebesgue integrability of the sum function for the series (C) but the sum function does have improper Riemann integrability.

Theorem 3. Suppose the sequence $(a_0, a_1, \ldots)$ is decreasing and $a_n \to 0$. Then the cosine series (C) converges except possibly at $x = 0$ to a function f on $[-\pi, \pi]$, which is continuous at x for $x \neq 0$ in $[-\pi, \pi]$. The sum function f is, in general, improperly Riemann integrable. Thus, if we use improper integral in the formula for the Fourier coefficients a_n, then (C) is the Fourier Riemann series of f.

In the next section, we collect together the useful technical results such as summation techniques and properties of special sums for the proofs of these three theorems.

2. Technical and Useful Results.

We recall first the Riemann Lebesgue Theorem:

Theorem R L. Suppose f is Lebesgue integrable on $[-\pi, \pi]$. Then $\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0$ and $\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = 0$.

In view of Theorem RL, the condition that the sequence (a_n) be a null sequence, that is, $a_n \to 0$, is a necessary condition in Theorem 1 and 2.

2.1 Summation formula

Summation technique features prominently in the proof. We use predominantly Abel's summation formula, which we describe below.

Abel's Summation Formula.

Suppose (a_n) and (b_n) are two sequences. Let $s_n = \sum_{k=1}^{n} b_k$. Then we have the following summation formula:

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) s_k + a_n s_n$$

$$= \sum_{k=1}^{n} (a_k - a_{k+1}) s_k + a_{n+1} s_n \ . \quad \text{-----------------------------} \quad (3)$$

For the truncated sum we have:

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q-1} (a_k - a_{k+1}) s'_k + a_q s'_q \ , \qquad \text{----------------} \quad (4)$$

where $s'_k = \sum_{j=p}^{k} b_j$, $k \geq p$.

We have similar formula when the summation starts from 0 instead of 1. We interpret formula (3) and (4) accordingly. Formula (3) is sometimes called summation by parts.

Formula (3) or (4) is used to give an alternative useful way to sum the series (C) or (S). We have the following estimate of the sum, particularly useful in showing convergence of Fourier series.

Lemma 4. Suppose (a_n) is a decreasing sequence and $a_n \geq 0$ for all n. Then

$$\left| \sum_{k=1}^{n} a_k b_k \right| \leq a_1 \max_{1 \leq k \leq n} | s_k | \quad \text{-------------------} \quad (5)$$

and

$$\left| \sum_{k=p}^{q} a_k b_k \right| \leq a_p \max_{p \leq k \leq q} | s'_k | . \quad \text{------------------} \quad (6)$$

Proof.

By the summation formula (3), we have

$$\left| \sum_{k=1}^{n} a_k b_k \right| \leq \sum_{k=1}^{n-1} |a_k - a_{k+1}| |s_k| + |a_n| |s_n|$$

$$\leq \sum_{k=1}^{n-1}(a_k - a_{k+1}) \max_{1\leq j\leq n}|s_j| + a_n \max_{1\leq j\leq n}|s_j|,$$

since (a_n) is a decreasing sequence and $a_n \geq 0$ for all *n,*

$$= a_1 \max_{1\leq j\leq n} |s_j| \;.$$

Inequality (6) is derived from (4) in exactly the same way.

2.2 Properties of Convex Sequence

Recall that a sequence (a_n) = (a_0 ,a_1, …) is said to be *convex* if $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1} \geq 0$ for all $n \geq 0$. For most of the time, the sequence that we deal with is usually convergent or a null sequence. Hence, it is always bounded. For convex sequence that is bounded we have:

Lemma 5. If (a_n) = (a_0 ,a_1, …) is convex and bounded, then it is decreasing, i.e., $\Delta a_n = a_n - a_{n+1} \geq 0$ for all $n \geq 0$.

Proof. By hypothesis, the sequence (Δa_n) is decreasing. Then we claim that $\Delta a_n \geq 0$ for all $n \geq 0$. We show this by contradiction.

Suppose there is an integer $N \geq 0$, such that $\Delta a_N = a_N - a_{N+1} < 0$. Then since

(Δa_n) is decreasing, for all $n \geq N$, $\Delta a_n = a_n - a_{n+1} \leq \Delta a_N < 0$. Thus

$a_{N+1} = a_N - \Delta a_N$, $a_{N+2} = a_{N+1} - \Delta a_{N+1} \geq a_{N+1} - \Delta a_N \geq a_N - 2\Delta a_N$, …,

$a_{N+p} \geq a_N - p\Delta a_N$. Since $-\Delta a_N > 0$, ($a_N - p\Delta a_N$) is unbounded and so (a_n) is unbounded. This contradicts that (a_n) is bounded. Hence $\Delta a_n \geq 0$ for all $n \geq 0$.

More is true:

Lemma 6. If (a_n) = (a_0 ,a_1, …) is convex and bounded, then $n\Delta a_n \to 0$ and the series $\sum_{n=0}^{\infty}(n+1)\Delta^2 a_n$ converges to $a_0 - \lim_{n\to\infty} a_n$.

Proof. By Lemma 5, (a_n) is decreasing and bounded, and so by the Monotone Convergence Theorem, (a_n) is convergent.

Observe that $\sum_{k=0}^{n} \Delta a_k = a_0 - a_{n+1}$ so that $\sum_{k=0}^{\infty} \Delta a_k$ is convergent and $\sum_{k=0}^{\infty} \Delta a_k = a_0 - \lim_{n\to\infty} a_n$.

That is, $\sum_{k=0}^{\infty} \Delta a_k$ is a Cauchy series. Therefore, given any $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$,

$$\left| \sum_{k=n+1}^{2n} \Delta a_k \right| < \varepsilon .$$

Since $\Delta a_k \geq 0$ and $\Delta^2 a_k \geq 0$ for all $k \geq 0$, for all $n \geq N$,

$$n\Delta a_{2n} = \Delta a_{2n} + \cdots + \Delta a_{2n} \leq \Delta a_{n+1} + \cdots + \Delta a_{2n} = \left| \sum_{k=n+1}^{2n} \Delta a_k \right| < \varepsilon .$$

Hence $n \Delta a_{2n} \to 0$. It follows that $2n \Delta a_{2n} \to 0$.

Now $(2n+1)\Delta a_{2n+1} \leq (2n+1)\Delta a_{2n} \leq 3n\Delta a_{2n}$ for $n > 0$ and since $n \Delta a_{2n} \to 0$, by the Comparison Test, $(2n+1) \Delta a_{2n+1} \to 0$. Therefore, $n\Delta a_n \to 0$.

Let $s_n = \sum_{k=0}^{n} \Delta a_k$ for $n \geq 0$. Then by Abel's summation formula (3)

$$s_n = \sum_{k=0}^{n} \Delta^2 a_k (k+1) + \Delta a_{n+1}(n+1) .$$

Since $(n+1) \Delta a_{n+1} \to 0$ and (s_n) is convergent, $\sum_{n=0}^{\infty} (n+1)\Delta^2 a_n$ is convergent and

$$\sum_{n=0}^{\infty} (n+1)\Delta^2 a_n = \lim_{n\to\infty} s_n = a_0 - \lim_{n\to\infty} a_n .$$

2.3 Summing the sine and cosine series.

Consider the $(n+1)$-th partial sum of the cosine series (C),

$$t_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos(kx) \text{ . ----------------------- } \quad (7)$$

Applying the Abel summation formula (3), we have

$$t_n(x) = \sum_{k=0}^{n-1} D_k(x)\Delta a_k + a_n D_n(x)\,, \quad \text{---------------------} \quad (8)$$

where

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kx) \quad \text{------------------------------} \quad (9)$$

for $n > 0$ and $D_0(x) = ½$.

$D_n(x)$ is called the *Dirichlet kernel*. Note that $D_n(x)$ is defined and continuous for all x in $[-\pi, \pi]$. We shall use this form of the $(n+1)$-th partial sum of (C) to investigate convergence of (C).

Now consider the n-th partial sum of the sine series (S):

$$s_n(x) = \sum_{k=1}^{n} a_k \sin(kx)\,. \quad \text{--------------------------------} \quad (10)$$

Applying the Abel summation formula (3) to (10) gives

$$s_n(x) = \sum_{k=1}^{n-1} D_k(x)\Delta a_k + a_n D_n(x)\,, \quad \text{-------------------------} \quad (11)$$

where $$D_n(x) = \sum_{k=1}^{n} \sin(kx) \quad \text{--------------------------------} \quad (12)$$

for $n \geq 1$.

$D_n(x)$ is called the *conjugate Dirichlet kernel.* Observe that $D_n(x)$ is defined and continuous in $[-\pi, \pi]$. The name conjugate Dirichlet kernel has its origin in considering complex Fourier series as a power series on the unit circle so that (A) is the real part of $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n - ib_n)e^{inx}$ and if (C) is the series, then the sine series is the conjugate series appearing as the imaginary part of the power series.

Now we proceed to investigate the properties of the Dirichlet kernels. Before we do that we introduce a second summation formula involving the Dirichlet kernels.

If $a_n D_n(x) \to 0$, this will then bring us by taking limits of (8) to the series

$$\sum_{k=0}^{\infty} D_k(x)\Delta a_k$$

and the problem of the convergence of this series.

Applying the Abel summation formula (3) to the (n+1)-th partial sum $F_n(x)$ of this series, we get

$$F_n(x) = \sum_{k=0}^{n} D_k(x)\Delta a_k = \sum_{k=0}^{n-1} E_k(x)\Delta^2 a_k + E_n(x)\Delta a_n \text{ ,} \quad \text{-----------------------} \quad (13)$$

where $E_k(x) = \sum_{j=0}^{k} D_k(x) = (k+1)K_k(x)$

and $$K_k(x) = \frac{1}{k+1}\sum_{j=0}^{k} D_j(x) \quad \text{-----------------------} \quad (14)$$

is called the *Fejér kernel* and is actually the mean of the Dirichlet kernel. It is also the (C,1) mean of the sequence $\left(\frac{1}{2}, \cos(x), \cos(2x), \cos(3x), \cdots\right)$.

In view of (13), we then have

$$t_n(x) = \sum_{k=0}^{n-2} (k+1)K_k(x)\Delta^2 a_k + nK_{n-1}(x)\Delta a_{n-1} + a_n D_n(x) \text{ ,} \quad \text{----------} \quad (15)$$

the result of applying the summation formula twice to the (n+1)-th partial sum of the series (C).

We shall use formula (8), (11) and (15) to study the convergence of (C) and (S).

Thus we derive below some properties of the Dirichlet and Fejér kernels.

2.4 Dirichlet, Fejér and Conjugate Kernels

$$2\sin(\tfrac{1}{2}x)D_n(x) = \sin(\tfrac{1}{2}x) + \sum_{k=1}^{n} 2\sin(\tfrac{1}{2}x)\cos(kx)$$

$$= \sin(\tfrac{1}{2}x) + \sum_{k=1}^{n}\left(\sin(kx + \tfrac{1}{2}x) - \sin((k-1)x + \tfrac{1}{2}x)\right)$$

$$= \sin(\tfrac{1}{2}x) + \sin((n+\tfrac{1}{2})x) - \sin(\tfrac{1}{2}x) = \sin((n+\tfrac{1}{2})x) \text{ .}$$

Thus, for $x \neq 0$ and x in $[-\pi, \pi]$, or $0 < x < 2\pi$,

$$D_n(x) = \frac{\sin((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)}. \qquad \text{-----------------------------} \quad (16)$$

Observe that $\lim_{x\to 0} \frac{\sin((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)} = \lim_{x\to 0} \frac{(n+\frac{1}{2})\cos((n+\frac{1}{2})x)}{\cos(\frac{1}{2}x)} = n + \frac{1}{2} = D_n(0)$

and the Dirichlet kernel in its functional form (16) is continuous at 0.

For the estimate of the Dirichlet kernel it is useful to consider the modified Dirichlet kernel defined by

$$D_n^*(x) = D_n(x) - \tfrac{1}{2}\cos(nx)$$

$$= \frac{\sin((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)} - \tfrac{1}{2}\cos(nx) = \frac{\sin((n+\frac{1}{2})x) - \cos(nx)\sin(\frac{1}{2}x)}{2\sin(\frac{1}{2}x)}$$

$$= \frac{\sin(nx)\cos(\frac{1}{2}x)}{2\sin(\frac{1}{2}x)}$$

$$= \frac{\sin(nx)}{2\tan(\frac{1}{2}x)} . \qquad \text{-------------------------------} \quad (17)$$

Note that the modified Dirichlet kernel is continuous on $[-\pi, \pi]$ and

$$D_n^*(0) = n \quad \text{and} \quad D_n^*(\pi) = 0 \qquad \text{------------------------} \quad (18)$$

The Fejér kernel has too a useful functional form. Using (16),

$$K_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(x) = \frac{1}{n+1}\sum_{k=0}^{n} \frac{\sin((k+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)}$$

$$= \frac{1}{n+1}\frac{1}{2\sin^2(\frac{1}{2}x)}\sum_{k=0}^{n} \sin((k+\tfrac{1}{2})x)\sin(\tfrac{1}{2}x)$$

$$= \frac{1}{n+1}\frac{1}{2\sin^2(\frac{1}{2}x)}\sum_{k=0}^{n} \frac{\cos(kx) - \cos((k+1)x)}{2}$$

$$= \frac{1}{n+1}\frac{1 - \cos((n+1)x)}{4\sin^2(\frac{1}{2}x)}$$

$$= \frac{1}{n+1} \frac{2\sin^2(\frac{1}{2}(n+1)x)}{4\sin^2(\frac{1}{2}x)}$$

$$= \frac{2}{n+1}\left\{\frac{\sin(\frac{1}{2}(n+1)x)}{2\sin(\frac{1}{2}x)}\right\}^2 . \quad \text{--------------------------------} \quad (19)$$

Observe that the Fejér kernel in its functional form (19) is continuous in [−π, π].

Since $D_k(0) = k + \frac{1}{2}$, $K_n(0) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(0) = \frac{1}{n+1}\sum_{k=0}^{n}(k+\frac{1}{2}) = \frac{1}{2} + \frac{n}{2}$. -------- (20)

Note that from (9)

$$\int_{-\pi}^{\pi} D_n(x)dx = \int_{-\pi}^{\pi} \frac{1}{2}dx + \sum_{k=1}^{n}\int_{-\pi}^{\pi}\cos(kx)dx = \pi \qquad \text{-------------------} \quad (21)$$

and so $\int_{-\pi}^{\pi} K_n(x)dx = \frac{1}{n+1}\sum_{k=0}^{n}\int_{-\pi}^{\pi} D_k(x) = \frac{1}{n+1}\sum_{k=0}^{n}\pi = \pi$

and $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1$. ---------------- (22)

We have similar derivations for the conjugate kernels.

Now $2\sin(\frac{1}{2}x)\tilde{D}_n(x) = \sum_{k=1}^{n} 2\sin(\frac{1}{2}x)\sin(kx)$

$$= \sum_{k=1}^{n}\left(\cos((k-\tfrac{1}{2})x) - \cos((k+\tfrac{1}{2})x)\right)$$

$$= \cos(\tfrac{1}{2}x) - \cos((n+\tfrac{1}{2})x) ,$$

so that for $x \neq 0$ and in [−π, π],

$$\tilde{D}_n(x) = \frac{\cos(\frac{1}{2}x) - \cos((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)} . \qquad \text{---------------------} \quad (23)$$

Observe that $\lim_{x\to 0}\frac{\cos(\frac{1}{2}x) - \cos((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)} = 0 = \tilde{D}_n(0)$ so that in its functional form (23), the conjugate Dirichlet kernel is continuous in [−π, π]. We shall also use the modified conjugate Dirichlet kernel, particularly because it is nonnegative.

The modified conjugate Dirichlet kernel is defined by

$$D_n^*(x) = D_n(x) - \frac{1}{2}\sin(nx) \quad . \qquad \text{-----------------------} \quad (24)$$

Using (23) we have for $0 < x < 2\pi$ or x in $[-\pi, \pi] - \{0\}$,

$$D_n^*(x) = \frac{\cos(\frac{1}{2}x) - \cos((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)} - \frac{1}{2}\sin(nx)$$

$$= \frac{\cos(\frac{1}{2}x) - \cos((n+\frac{1}{2})x) - \sin(nx)\sin(\frac{1}{2}x)}{2\sin(\frac{1}{2}x)}$$

$$= \frac{\cos(\frac{1}{2}x) - \cos(\frac{1}{2}x)\cos(nx)}{2\sin(\frac{1}{2}x)}$$

$$= \frac{1-\cos(nx)}{2\tan(\frac{1}{2}x)} \, . \qquad \text{--} \quad (25)$$

Note that as $\lim_{x\to 0}\frac{1-\cos(nx)}{2\tan(\frac{1}{2}x)} = 0 = D_n^*(0)$ and $\lim_{x\to \pi}\frac{1-\cos(nx)}{2\tan(\frac{1}{2}x)} = 0 = D_n^*(\pi)$, the conjugate Dirichlet kernel in its functional form (25) is continuous in $[-\pi, \pi]$.

The Diriichlet and Fejér kernels involved trigonometric functions. We now state the useful inequalities that we shall use.

(1) For all x, $|\sin(x)| \leq |x|$; $|\sin(x)| < x$ for $x > 0$. -------------------- (26)

(2) For $0 \leq x \leq \frac{\pi}{2}$, $\sin(x) \geq \frac{2}{\pi}x$. -------------------------- (27)

(3) For $0 \leq x \leq \pi$, $1-\cos(x) \geq 2\frac{x^2}{\pi^2}$. ----------------------------- (28)

(4) For all x, $1-\cos(x) \leq \frac{1}{2}x^2$. -------------------------- (29)

Inequality (1) is easy.

Inequality (2) is a consequence of the fact that cos(x) is decreasing on $[0, \frac{\pi}{2}]$ or that sin(x) is concave downward on $\left(0, \frac{\pi}{2}\right)$. By the Mean Value Theorem, for $0 < x < \frac{\pi}{2}, \frac{\sin(x)}{x} = \cos(\eta)$ for some η with $0 < \eta < x$. Also by the Mean Value Theorem,

$\frac{1-\sin(x)}{\frac{\pi}{2}-x}=\cos(\varphi)$ for some φ with $x<\varphi<\frac{\pi}{2}$. Since $\eta<\varphi$, $\cos(\eta)>\cos(\varphi)$ and so $\frac{\sin(x)}{x}>\frac{1-\sin(x)}{\frac{\pi}{2}-x}$. It follows that $\sin(x)>\frac{2}{\pi}x$ for $0<x<\frac{\pi}{2}$. Therefore, including the end points 0 and $\frac{\pi}{2}$, we have $\sin(x)\geq\frac{2}{\pi}x$.

For $0\leq x\leq\pi$, $1-\cos(x)=2\sin^2(\frac{x}{2})\geq 2\frac{x^2}{\pi^2}$ by inequality (2). Inequality (4) follows from inequality (1).

2.5 Lebesgue Constants.

To investigate convergence in the L^1 norm, we need some estimates of the integral of the modulus of the Dirichlet kernels.

The Lebesgue constant L_n is defined by $L_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\left|D_n(t)\right|dt=\frac{2}{\pi}\int_0^{\pi}\left|D_n(t)\right|dt$. The conjugate Lebesgue constant is similarly defined by

$$L_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\left|D_n(t)\right|dt=\frac{2}{\pi}\int_0^{\pi}\left|D_n(t)\right|dt\ .$$

It is useful to use the modified Dirichlet kernel. Since $\tan(x)\geq x$ for $0\leq x\leq\pi/2$, for $0<x\leq\pi$,

$$\left|D_n^*(x)\right|=\left|\frac{\sin(nx)}{2\tan(\frac{1}{2}x)}\right|\leq\frac{1}{x}.\qquad\text{------------------ (30)}$$

Obviously, from the definitions of modified Dirichlet kernel and conjugate kernel,

$$\left|D_n^*(x)\right|\leq n\quad\text{-----------------------------------(31)}$$

and

$$\left|\overset{*}{D}_n(x)\right|<n\ .\ \text{-----------------------------------(32)}$$

Also, we have

$$\left|\overset{*}{D}_n(x)\right| = \overset{*}{D}_n(x) = \frac{1-\cos(nx)}{2\tan(\frac{1}{2}x)} \le \frac{2}{x} \quad \text{-------------------------- (33)}$$

for $0 < x \le \pi$.

For the conjugate Dirichlet kernel, from (25), for $0 < x \le \pi$,

$$\left|\tilde{D}_n(x)\right| = \left|\frac{\cos(\frac{1}{2}x) - \cos((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)}\right| \le \left|\frac{1}{\sin(\frac{1}{2}x)}\right|$$

$$\le \frac{\pi}{x}, \quad \text{-- (34)}$$

by (27).

Similarly,

$$\left|D_n(x)\right| = \left|\frac{\sin((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)}\right| \le \left|\frac{1}{2\sin(\frac{1}{2}x)}\right| \le \frac{\pi}{2x}, \quad \text{-------------------------- (35)}$$

for $0 < x \le \pi$.

We have the following estimates for the Lebesgue constants.

Theorem 7.

(1) $L_n = \frac{4}{\pi^2}\ln(n) + O(1)$ **;** $L_n \simeq \frac{4}{\pi^2}\ln(n)$ as $n \to \infty$.

(2) $\tilde{L}_n \simeq \frac{2}{\pi}\ln(n)$**;** $\int_0^{\pi} \overset{*}{D}_n(t)dt \simeq \ln(n)$ as $n \to \infty$.

Before embarking on the proof of Theorem 7, we deduce the following estimate of the function $\frac{1}{2\tan(\frac{x}{2})}$.

Lemma 8. Let $h(x) = \frac{1}{x} - \frac{1}{2\tan(\frac{x}{2})}$. Then $h(x)$ is continuous, bounded and increasing on $(0, \pi)$, $\lim_{x\to 0^+} h(x) = 0$, $\lim_{x\to\pi^-} h(x) = \frac{1}{\pi}$, so that $0 < h(x) < 1/\pi$ and $\sup_{0<x<\pi} h(x) = \frac{1}{\pi}$. In particular, $\frac{1}{2\tan(\frac{x}{2})} = \frac{1}{x} + O(1)$ in $(0, \pi)$.

Proof. Observe that $h'(x) = -\frac{1}{x^2} + \frac{\csc^2(\frac{x}{2})}{4} = \frac{\left(\frac{x}{2}\right)^2 - \sin^2(\frac{x}{2})}{x^2\sin^2(\frac{x}{2})} > 0$ for $0 < x < \pi$, since $\frac{x}{2} > |\sin(\frac{x}{2})|$ for $x > 0$ (see (26)). Therefore, h is strictly increasing on $(0, \pi)$. Now

$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} \frac{2\tan(\frac{x}{2}) - x}{2x\tan(\frac{x}{2})}$$

$$= \lim_{x\to 0^+} \frac{\sec^2(\frac{x}{2}) - 1}{2\tan(\frac{x}{2}) + x\sec^2(\frac{x}{2})} = \lim_{x\to 0^+} \frac{\sec^2(\frac{x}{2})\tan(\frac{x}{2})}{2\sec^2(\frac{x}{2}) + x\sec^2(\frac{x}{2})\tan(\frac{x}{2})} = 0 \ ,$$

by applying L' Hôpital's Rule twice.

Observe that $\lim_{x\to\pi^-} h(x) = \lim_{x\to\pi^-} \frac{1}{x} - \lim_{x\to\pi^-} \frac{1}{2\tan(\frac{x}{2})} = \frac{1}{\pi} - 0 = \frac{1}{\pi}$. Hence, $\sup_{0<x<\pi} h(x) = \frac{1}{\pi}$ and $\inf_{0<x<\pi} h(x) = 0$. Since h is strictly increasing on $(0, \pi)$, it follows that $0 < h(x) < 1/\pi$. Therefore, for all x in $(0, \pi)$,

$$0 < \frac{1}{x} - \frac{1}{\pi} < \frac{1}{2\tan(\frac{x}{2})} < \frac{1}{x}. \qquad \text{--------------------------} \quad (36)$$

This means $\frac{1}{2\tan(\frac{x}{2})} = \frac{1}{x} + O(1)$ in $(0, \pi)$.

Proof of Theorem 7 Part (1).

We shall use the modified conjugate Dirichlet kernel because $\left|D_n(x) - D_n^*(x)\right| \le \frac{1}{2}$ for x in $[-\pi, \pi]$. We have $\left|D_n^*(x)\right| - \frac{1}{2} \le \left|D_n(x)\right| \le \left|D_n^*(x)\right| + \frac{1}{2}$ for x in $[0, \pi]$ and so

$$\frac{2}{\pi}\int_0^\pi \left|D_n^*(x)\right| dx - 1 \le \frac{2}{\pi}\int_0^\pi \left|D_n(x)\right| dx = L_n \le \frac{2}{\pi}\int_0^\pi \left|D_n^*(x)\right| dx + 1. \quad \text{------------} \ (37)$$

This means $$L_n = \frac{2}{\pi}\int_0^\pi \left|D_n^*(x)\right| dx + O(1). \qquad \text{-----------------------} \quad (38)$$

By (17) and (36), for $0 < x < \pi$,

$$0 \le \frac{|\sin(nx)|}{x} - \frac{|\sin(nx)|}{\pi} \le \left|\frac{\sin(nx)}{2\tan(\frac{x}{2})}\right| = \left|D_n^*(x)\right| \le \frac{|\sin(nx)|}{x}. \quad \text{-----------} \quad (39)$$

Observe that all the three functions in the above inequality are bounded in the closed and bounded interval [0, π]. Thus taking the integrals we have

$$\int_0^{\pi}\frac{|\sin(nx)|}{x}dx-\int_0^{\pi}\frac{|\sin(nx)|}{\pi}dx\le\int_0^{\pi}\left|D_n^*(x)\right|dx\le\int_0^{\pi}\frac{|\sin(nx)|}{x}dx\,.$$

Therefore,

$$\frac{2}{\pi}\int_0^{\pi}\frac{|\sin(nx)|}{x}dx-\frac{2}{\pi}\int_0^{\pi}\frac{|\sin(nx)|}{\pi}dx\le\frac{2}{\pi}\int_0^{\pi}\left|D_n^*(x)\right|dx\le\frac{2}{\pi}\int_0^{\pi}\frac{|\sin(nx)|}{x}dx\,.$$

Consequently,

$$\frac{2}{\pi}\int_0^{\pi}\frac{|\sin(nx)|}{x}dx-\frac{2}{\pi}\le\frac{2}{\pi}\int_0^{\pi}\left|D_n^*(x)\right|dx\le\frac{2}{\pi}\int_0^{\pi}\frac{|\sin(nx)|}{x}dx\ . \qquad \text{----------- (40)}$$

Thus, $$\frac{2}{\pi}\int_0^{\pi}\left|D_n^*(x)\right|dx=\frac{2}{\pi}\int_0^{\pi}\frac{|\sin(nx)|}{x}dx+O(1)\,. \qquad \text{------------ (41)}$$

We now estimate the integral $\int_0^{\pi}\frac{|\sin(nx)|}{x}dx$.

Divide [0, π] into n equal subintervals so that

$$\int_0^{\pi}\frac{|\sin(nx)|}{x}dx=\sum_{k=0}^{n-1}\int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}}\frac{|\sin(nx)|}{x}dx=\sum_{k=0}^{n-1}\int_0^{\frac{\pi}{n}}\frac{|\sin(nt+k\pi)|}{t+k\frac{\pi}{n}}dt=\sum_{k=0}^{n-1}\int_0^{\frac{\pi}{n}}\frac{|\sin(nt)|}{t+k\frac{\pi}{n}}dt$$

$$=\int_0^{\frac{\pi}{n}}\frac{|\sin(nt)|}{t}dt+\sum_{k=1}^{n-1}\int_0^{\frac{\pi}{n}}\frac{|\sin(nt)|}{t+k\frac{\pi}{n}}dt$$

$$=\int_0^{\frac{\pi}{n}}\frac{\sin(nt)}{t}dt+\int_0^{\frac{\pi}{n}}\sin(nt)\left(\sum_{k=1}^{n-1}\frac{1}{t+k\frac{\pi}{n}}\right)dt\,, \qquad \text{---------------------- (42)}$$

by using change of variable, $x=t+k\frac{\pi}{n}$.

Now for $k\ge 1$ and $0\le\ t\ \le\frac{\pi}{n}$, $k\frac{\pi}{n}\le t+k\frac{\pi}{n}\le(k+1)\frac{\pi}{n}$ so that

$$\frac{1}{k\frac{\pi}{n}}\ge\frac{1}{t+k\frac{\pi}{n}}\ge\frac{1}{(k+1)\frac{\pi}{n}}\,. \qquad \text{----------------------- (43)}$$

Observe that $\int_0^{\frac{\pi}{n}}\frac{|\sin(nt)|}{t}dt\le\int_0^{\frac{\pi}{n}}\frac{nt}{t}dt=\pi$. It then follows from (42) and (43) that

$$\int_0^{\pi}\frac{|\sin(nx)|}{x}dx\le\pi+\int_0^{\frac{\pi}{n}}\sin(nt)dt\sum_{k=1}^{n-1}\frac{1}{k\frac{\pi}{n}}=\pi+\frac{2}{n}\sum_{k=1}^{n-1}\frac{1}{k\frac{\pi}{n}}=\pi+\frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{k}\,. \qquad \text{----- (44)}$$

Let $d_n = \sum_{k=1}^{n} \frac{1}{k} - \ln(n)$. Then (d_n) is a non-negative decreasing sequence converging to the Euler constant $\gamma < 1$. Now $d_1 = 1$. Therefore, for all $n \geq 1$,

$$\gamma \leq \sum_{k=1}^{n} \frac{1}{k} - \ln(n) \leq d_1 = 1 \quad \text{or}$$

$$\gamma + \ln(n) \leq \sum_{k=1}^{n} \frac{1}{k} \leq \ln(n) + 1 . \qquad \text{------------------} \quad (45)$$

Hence it follows from (44) and (45) that for $n \geq 1$,

$$\int_0^{\pi} \frac{|\sin(nx)|}{x} dx \leq \pi + \frac{2}{\pi}(\ln(n)+1) = \frac{2}{\pi}\ln(n) + \left(\pi + \frac{2}{\pi}\right) \text{ ----------------} \quad (46)$$

and so

$$\frac{2}{\pi}\int_0^{\pi} \frac{|\sin(nx)|}{x} dx \leq \frac{4}{\pi^2}\ln(n) + \left(2 + \frac{4}{\pi^2}\right) . \qquad \text{---------------} \quad (47)$$

From (42) and (43) we obtain,

$$\int_0^{\pi} \frac{|\sin(nx)|}{x} dx \geq \int_0^{\frac{\pi}{n}} \frac{\sin(nt)}{t} dt + \int_0^{\frac{\pi}{n}} \sin(nt) dt \sum_{k=1}^{n-1} \frac{1}{(k+1)\frac{\pi}{n}} = \int_0^{\pi} \frac{\sin(x)}{x} dx + \frac{2}{n}\sum_{k=1}^{n-1} \frac{1}{(k+1)\frac{\pi}{n}}$$

$$\geq \int_0^{\pi/2} \frac{\sin(x)}{x} dx + \frac{2}{\pi}\sum_{k=1}^{n-1} \frac{1}{(k+1)}$$

$$\geq 1 + \frac{2}{\pi}\sum_{k=1}^{n-1} \frac{1}{(k+1)} \geq \frac{2}{\pi}\sum_{k=1}^{n} \frac{1}{k} \text{ by inequality (27).} \quad \text{-------------} \quad (48)$$

So, for $n \geq 1$,

$$\int_0^{\pi} \frac{|\sin(nx)|}{x} dx \geq \frac{2}{\pi}(\gamma + \ln n) = \frac{2}{\pi}\ln n + \frac{2}{\pi}\gamma ,$$

by using (45).

Hence, for $n \geq 1$,

$$\frac{2}{\pi}\int_0^{\pi} \frac{|\sin(nx)|}{x} dx \geq \frac{4}{\pi^2}\ln(n) + \frac{4}{\pi^2}\gamma . \text{ -------------------} \quad (49)$$

That is, $$\frac{2}{\pi}\int_0^{\pi} \frac{|\sin(nx)|}{x} dx = \frac{4}{\pi^2}\ln(n) + O(1) . \text{ --------------------} \quad (50)$$

It follows then from (41) and (50) that

$$\frac{2}{\pi}\int_0^{\pi}\left|D_n^*(x)\right|dx = \frac{4}{\pi^2}\ln(n)+O(1). \qquad \text{---------------------- (51)}$$

Therefore, from (38) and (51), $L_n = \frac{4}{\pi^2}\ln(n)+O(1)$.

Proof of Theorem 7 Part (2).

We now estimate the conjugate Lebesgue constant. As above we shall use the modified conjugate Dirichlet kernel since $\left|\overset{*}{D}_n(x)-\overset{*}{D}_n(x)\right|\leq\frac{1}{2}$ for x in $[0, \pi]$. It is useful to note that $\overset{*}{D}_n(x)\geq 0$. As for L_n, we deduce that

$$L_n = \frac{2}{\pi}\int_0^{\pi}\overset{*}{D}_n(x)dx+O(1). \text{ --------------------- (52)}$$

Recall that $\overset{*}{D}_n(x)=\frac{1-\cos(nx)}{2\tan(\frac{1}{2}x)}$ for $0 < x < \pi$ and extend the definition at 0 and π by taking appropriate limits. Hence, we obtain, by using Lemma 8, (see (36))

$$0\leq\frac{1-\cos(nx)}{x}-\frac{1-\cos(nx)}{\pi}\leq\left|\frac{1-\cos(nx)}{2\tan(\frac{x}{2})}\right| = \overset{*}{D}_n(x)\leq\frac{1-\cos(nx)\,|}{x},$$

for $0 < x < \pi$. Note that all the functions in the above inequality are bounded.

Hence, taking integral we get,

$$0\leq\int_0^{\pi}\frac{1-\cos(nx)}{x}dx-\int_0^{\pi}\frac{1-\cos(nx)}{\pi}\leq\int_0^{\pi}\overset{*}{D}_n(x)dx\leq\int_0^{\pi}\frac{1-\cos(nx)}{x}dx,$$

that is,

$$0\leq\int_0^{\pi}\frac{1-\cos(nx)}{x}dx-1\leq\int_0^{\pi}\overset{*}{D}_n(x)dx\leq\int_0^{\pi}\frac{1-\cos(nx)}{x}dx. \text{ ---------------- (53)}$$

It follows that

$$L_n = \frac{2}{\pi}\int_0^{\pi}\overset{*}{D}_n(x)dx = \frac{2}{\pi}\int_0^{\pi}\frac{1-\cos(nx)}{x}dx+O(1). \qquad \text{--------------- (54)}$$

As in the case for the Lebesgue constant, we divide the interval $[0, \pi]$ into n equal subintervals and spread the integral over these n intervals.

$$\int_0^{\pi}\frac{1-\cos(nx)}{x}dx=\sum_{k=0}^{n-1}\int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}}\frac{1-\cos(nx)}{x}dx=\sum_{k=0}^{n-1}\int_0^{\frac{\pi}{n}}\frac{1-\cos(nt+k\pi)}{t+k\frac{\pi}{n}}dt$$

$$=\int_0^{\frac{\pi}{n}}\frac{1-\cos(nt)}{t}dt+\sum_{k=1}^{n-1}\int_0^{\frac{\pi}{n}}\frac{1-\cos(nt+k\pi)}{t+k\frac{\pi}{n}}dt\,. \qquad \text{-------------}\quad (55)$$

Observe that

$$\int_0^{\frac{\pi}{n}}\frac{1-\cos(nt)}{t}dt=\int_0^{\pi}\frac{1-\cos(x)}{x}dx\le\int_0^{\pi}\frac{2\sin^2(\frac{x}{2})}{x}dx\le\int_0^{\pi}\frac{1}{2}xdx=\frac{\pi^2}{4} \qquad \text{----------}\quad (56)$$

and for $k\ge 1$,

$$\int_0^{\frac{\pi}{n}}\frac{1-\cos(nt+k\pi)}{t+k\frac{\pi}{n}}dt\le\frac{1}{k\frac{\pi}{n}}\int_0^{\frac{\pi}{n}}(1-\cos(nt+k\pi))dt=\frac{1}{k}. \qquad \text{----------------}\quad (57)$$

Thus, combining (55), (56) and (57), we have for $n\ge 1$,

$$\int_0^{\pi}\frac{1-\cos(nx)}{x}dx\le\frac{\pi^2}{4}+\sum_{k=1}^{n-1}\frac{1}{k}\le\frac{\pi^2}{4}+\sum_{k=1}^{n}\frac{1}{k}$$

$$\le\ln(n)+1+\frac{\pi^2}{4}, \qquad \text{----------------------}\quad (58)$$

by (45).

Using inequality (28),

$$\int_0^{\frac{\pi}{n}}\frac{1-\cos(nt)}{t}dt=\int_0^{\pi}\frac{1-\cos(x)}{x}dx\ge\int_0^{\pi}\frac{2x^2}{\pi^2}dx=\frac{2\pi}{3}, \qquad \text{----------------}\quad (59)$$

and for $k\ge 1$,

$$\int_0^{\frac{\pi}{n}}\frac{1-\cos(nt+k\pi)}{t+k\frac{\pi}{n}}dt\ge\frac{1}{(k+1)\frac{\pi}{n}}\int_0^{\frac{\pi}{n}}(1-\cos(nt+k\pi))dt=\frac{1}{k+1}. \qquad \text{--------}\quad (60)$$

Using (55), (59) and (60) we have for $n\ge 1$,

$$\int_0^{\pi}\frac{1-\cos(nx)}{x}dx\ge 1+\sum_{k=1}^{n-1}\frac{1}{k+1}=\sum_{k=1}^{n}\frac{1}{k}$$

$$\ge\gamma+\ln(n)\,, \qquad \text{-----------------------------}\quad (61)$$

by inequality (45).

Thus (58) and (61) says that

$$\int_0^{\pi} \frac{1-\cos(nx)}{x}dx = \ln(n) + O(1) . \quad \text{---------------------------} \quad (62)$$

It follows from (53) and (62) that $\int_0^{\pi} D_n^*(x)dx = \ln(n) + O(1)$ and so by (54),

$$L_n = \frac{2}{\pi}\int_0^{\pi} D_n^*(x)dx = \frac{2}{\pi}\ln(n) + O(1) .$$

This proves part (2).

2.6 Convergence of (S) and (C)

In this section we investigate the convergence of the series (S) and (C) when the coefficients are nonnegative and converge to 0. We deduce when the convergence is uniform and when the sum function is continuous. The technique is usually known as Dirichlet test.

We shall begin with the cosine series (C):

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

Let $t_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos(kx)$ be the $(n+1)th$ partial sum of (C). Suppose $\Delta a_n \geq 0$ for $n \geq 0$ and $a_n \to 0$.

For $m > n$, by Abel's summation formula (3) or (8),

$$t_m(x) - t_n(x) = \sum_{k=n}^{m} D_k(x)\Delta a_k + a_{m+1}D_m(x) - a_n D_n(x) . \quad \text{-----------------} \quad (63)$$

Then by triangle inequality,

$$\left|t_m(x) - t_n(x)\right| \leq \left(\sum_{k=n}^{m} \Delta a_k\right) \max_{n\leq k\leq m}\left|D_k(x)\right| + (a_{m+1} + a_n)\max_{n\leq k\leq m}\left|D_k(x)\right| = 2a_n \max_{n\leq k\leq m}\left|D_k(x)\right| . \quad \text{----} \quad (64)$$

Now restrict the domain to the interval $[\delta, 2\pi-\delta]$, $0 < \delta < \pi$. Then for all x in $[\delta, 2\pi-\delta]$ and for all $n \geq 0$,

$$\left|D_n(x)\right| = \left|\frac{\sin((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)}\right| \leq \frac{1}{2\sin(\frac{\delta}{2})} . \quad \text{--------------------------------} \quad (65)$$

Thus, by (64) and (65), for all x in $[\delta, 2\pi-\delta]$ and for all $m > n \geq 0$,

$$\left|t_n(x) - t_m(x)\right| \leq \frac{a_n}{\sin(\frac{\delta}{2})} \; . \qquad \text{------------------------------} \; (66)$$

Since $a_n \to 0$, given any $\varepsilon > 0$, there exist integer N such that

$$n \geq N \Rightarrow |a_n| = a_n < \varepsilon \sin(\tfrac{\delta}{2})) \, .$$

Thus, for any n, m with $m > n \geq N$ and for all x in $[\delta, 2\pi-\delta]$,

$$\left|t_n(x) - t_m(x)\right| < \varepsilon \, .$$

It follows that the sequence $(t_n(x))$ is uniformly Cauchy on $[\delta, 2\pi-\delta]$. Therefore, (C) converges uniformly to a continuous function on $[\delta, 2\pi-\delta]$. It follows that (C) converges pointwise to a continuous function on $(0, 2\pi)$. Hence (C) converges pointwise to a continuous function f on $[-\pi, \pi] - \{0\}$. More precisely (C) converges pointwise for all x not a multiple of 2π. The sum function is continuous at every point not a multiple of 2π. The series (C) may or may not be convergent at 0 and when it does, the sum function may or may not be continuous at 0.

Now we consider the sine series (S):

$$\sum_{n=1}^{\infty} a_n \sin(nx)$$

Let the n-th partial sum of (S) be $s_n(x) = \sum_{k=1}^{n} a_k \sin(kx)$. Suppose $\Delta a_n \geq 0$ for $n \geq 1$ and $a_n \to 0$. For $m > n$, as above by Abel's summation formula (3) or (11),

$$s_m(x) - s_n(x) = \sum_{k=n}^{m} D_k(x) \Delta a_k + a_{m+1} D_m(x) - a_n D_n(x) \, . \quad \text{----------------} \quad (67)$$

And we have by triangle inequality, for $m > n > 0$,

$$\left|s_m(x) - s_n(x)\right| \leq 2 a_n \max_{n \leq k \leq m} \left|D_k(x)\right| . \quad \text{------------------------------} \; (68)$$

Now for all x in $[\delta, 2\pi-\delta]$, $0 < \delta < \pi$ and for all $n \geq 0$,

$$\left|D_n(x)\right| = \left|\frac{\cos(\frac{1}{2}x) - \cos((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)}\right| \leq \frac{1}{\sin(\frac{\delta}{2})} \; . \quad \text{----------------------} \quad (69)$$

Therefore, it follows from (68) and (69) that for all x in $[\delta, 2\pi-\delta]$, $0 < \delta < \pi$ and for all $m > n > 0$,

$$|s_m(x) - s_n(x)| \le \frac{2a_n}{\sin(\frac{\delta}{2})}. \quad \text{-------------------------------- (70)}$$

Since $a_n \to 0$, we deduce as for the cosine series that $(s_n(x))$ is uniformly Cauchy on $[\delta, 2\pi-\delta]$ and so (S) converges uniformly on $[\delta, 2\pi-\delta]$. Therefore, (S) converges uniformly to a continuous sum function on $[\delta, 2\pi-\delta]$. It follows that (S) converges pointwise to a continuous function on $(0, 2\pi)$. Hence, by periodicity it converges to a sum function continuous at every point not a multiple of 2π. Since (S) converges at 0, (S) is convergent on the whole of **R**. The series (S) converges to a sum function g on $[-\pi, \pi]$ continuous at $x \neq 0$. The function g may or may not be continuous at 0.

We have thus proved the following theorem.

Theorem 9. Suppose $\Delta a_n \ge 0$ and $a_n \to 0$ for the series (C) and (S).

Then the series (C) converges pointwise except possibly at $x = 0$ to a function f continuous at x for all x in $[-\pi, \pi]-\{0\}$. The series (S) converges pointwise to a function g on $[-\pi, \pi]$ and g is continuous at x for all $x \neq 0$ in $[-\pi, \pi]$. Both series converge uniformly on $[\delta, 2\pi-\delta]$ for any $0 < \delta < \pi$.

For the sine series (S) to converge uniformly on the whole of **R**, we have the following result.

Theorem 10. Suppose $\Delta a_n \ge 0$ for $n \ge 1$ and $a_n \to 0$. Then the series (S) converges uniformly on **R** if and only if $na_n \to 0$.

Proof.

Suppose (S) converges uniformly on **R**. Then (S) is uniformly Cauchy.

Hence, given $\varepsilon > 0$, there exists an integer N such that for all $n \ge N$ and for all $m \ge n$ and for all x in **R**,

$$\left|\sum_{k=n}^{m} a_k \sin(kx)\right| < \frac{\varepsilon}{\sqrt{2}}. \quad \text{---------------------- (71)}$$

Take any $n \ge N$. Let $y = \frac{\pi}{4n}$. Since $a_n \to 0$ and (a_n) is decreasing, $a_n \ge 0$ for all $n \ge 1$. Therefore,

$$\sum_{k=n+1}^{2n} a_k \sin(ky) \ge a_{2n} \sum_{k=n+1}^{2n} \sin(ky) \ge a_{2n} \sum_{k=n+1}^{2n} \sin(ny) \ge na_{2n} \sin(\frac{\pi}{4}) \ge 0 .$$

It then follows from (71) that for any $n \ge N$,

$$na_{2n} \sin\left(\frac{\pi}{4}\right) \le \sum_{k=n+1}^{2n} a_k \sin(ky) = \left|\sum_{k=n+1}^{2n} a_k \sin(ky)\right| < \frac{\varepsilon}{\sqrt{2}} ,$$

that is, $na_{2n} < \varepsilon$. This means $na_{2n} \to 0$ and so $2na_{2n} \to 0$. Since $a_{2n-2} \ge a_{2n-1}$, by the Comparison Test, $(2n-2)a_{2n-1} \to 0$. Thus, $(2n-1)a_{2n-1} = (2n-2)a_{2n-1} + a_{2n-1} \to 0$. It follows that $na_n \to 0$.

Conversely, suppose $na_n \to 0$. Then $\limsup_{n\to\infty} na_n = 0$. Let $\beta_k = \sup_{j\ge k} ja_j$. Then $\beta_n \to 0$. We shall estimate the tail end of the series and show that the estimate is independent of x and depends only on β_n .

Take any x in $(0, \pi]$. Let $N_x = \left[\frac{\pi}{x}\right]$, the integer part of π/x. Then $1 \le N_x \le \frac{\pi}{x} < N_x + 1$. By Theorem 9, (S) is convergent on **R**. It follows that the truncated sum

$$T_k(x) = \sum_{n=k}^{\infty} a_n \sin(nx)$$

is convergent for all x. For any x in $(0, \pi]$ we split $T_k(x)$ into two summations according to x using N_x . For convenience we drop the subfix and let $N = N_x$ and note that it depends on x.

Let $T_k'(x) = \sum_{n=k}^{k+N-1} a_n \sin(nx)$ and $T_k''(x) = \sum_{n=k+N}^{\infty} a_n \sin(nx)$. For the first summation we have

$$|T_k'(x)| \le \sum_{n=k}^{k+N-1} a_n |\sin(nx)| \le \sum_{n=k}^{k+N-1} a_n nx = x \sum_{n=k}^{k+N-1} na_n \le x \sum_{n=k}^{k+N-1} \beta_k = xN\beta_k$$

$$\le \pi\beta_k . \qquad \text{----------------} \quad (72)$$

From (67) we have

$$T_k''(x) = \lim_{m\to\infty}\left(s_m(x) - s_{k+N-1}(x)\right)$$

$$= \lim_{m\to\infty}\left(\sum_{n=k+N-1}^{m-1} D_n(x)\Delta a_n + a_m D_m(x) - a_{k+N-1}D_{k+N-1}(x)\right) \text{ by using (67)}$$

$$= \lim_{m\to\infty}\left(\sum_{n=k+N}^{m-1} D_n(x)\Delta a_n + a_m D_m(x) - a_{k+N}D_{k+N-1}(x)\right).$$

Therefore, since the above limit exists, $a_n \to 0$ and $\left|D_n(x)\right| \le \frac{1}{\sin(\frac{\delta}{2})}$ for some δ with $0 < \delta < \pi$, we deduce that

$$T_k''(x) = \sum_{n=k+N}^{\infty} D_n(x)\Delta a_n - a_{k+N}D_{k+N-1}(x) \;. \qquad \text{-------------- (73)}$$

Hence, $\left|T_k''(x)\right| \le \sum_{n=k+N}^{\infty}\left|D_n(x)\right|\Delta a_n + a_{k+N}\left|D_{k+N-1}(x)\right|$

$$\le \sum_{n=k+N}^{\infty}\frac{\pi}{x}\Delta a_n + a_{k+N}\frac{\pi}{x} = 2a_{k+N}\frac{\pi}{x}, \text{ by using inequality (34),}$$

$$\le 2a_{k+N}(N+1), \qquad \text{since } \frac{\pi}{x} < N+1,$$

$$\le 2a_{k+N}(N+k) \le 2\beta_k\,. \qquad \text{------------------------------ (74)}$$

Therefore, combining (72) and (74) we have, for any x in $(0, \pi]$,

$$\left|T_k(x)\right| \le (2+\pi)\beta_k\,. \qquad \text{------------------------------ (75)}$$

Inequality (75) is obviously true for $x = 0$. Since $\beta_n \to 0$, $\left|T_k(x)\right| \to 0$ uniformly on $[0, \pi]$. Hence the series (S) converges uniformly on $[0, \pi]$ and since the sum function is odd, (S) also converges uniformly on $[-\pi, 0]$ and hence on $[-\pi, \pi]$. It then follows by periodicity that (S) converges uniformly on the whole of **R**.

Under the hypothesis that $\Delta a_n \ge 0$ for $n \ge 1$ and $a_n \to 0$, if the series (C) or (S) converges to a Lebesgue integrable function, then (C) or (S) is the Fourier series of their respective sum function. This is a special case of a more general result namely, that if a trigonometric series converges except for a denumerable subset to a finite and integrable function, then it is the Fourier series of this function. There are other generalizations of this result. The proofs of these general results are much more difficult. We present the proof for this special case.

Theorem 11. Suppose that $\Delta a_n \geq 0$ for $n \geq 0$ and $a_n \to 0$. Suppose the series (C) converges to a Lebesgue integrable function *f* and the series (S) converges to a Lebesgue integrable function *g*. Then (C) is the Fourier series of *f* and (S) is the Fourier series of *g*.

Proof. Observe that g(x)sin(mx) is the limit of the series $\sum_{k=1}^{\infty} \sin(mx) a_k \sin(kx)$.

That is,

$$g(x)\sin(mx) = \sum_{k=1}^{\infty} \sin(mx) a_k \sin(kx). \qquad \text{---------------------} \quad (76)$$

We claim that this series is uniformly convergent on **R**.

For $d > n$, the truncated series

$$\sum_{k=n}^{d} \sin(mx) a_k \sin(kx) = \sin(mx) \sum_{k=n}^{d} a_k \sin(kx)$$

and so, for $0 < x \leq \pi$,

$$\left|\sum_{k=n}^{d} \sin(mx) a_k \sin(kx)\right| \leq mx \left|\sum_{k=n}^{d} a_k \sin(kx)\right|$$

$$\leq mx a_n \max_{n \leq j \leq d} \left|\sum_{k=n}^{j} \sin(kx)\right|, \text{ by Lemma 4,}$$

$$\leq mx a_n \tfrac{1}{\sin(\frac{x}{2})}, \text{ by a similar formula to (23),}$$

$$\leq m a_n \pi, \text{ by inequality (27).}$$

This inequality is obviously true for $x = 0$. Hence for $0 \leq x \leq \pi$,

$$\left|\sum_{k=n}^{d} \sin(mx) a_k \sin(kx)\right| \leq m a_n \pi$$

and so

$$\left|\sum_{k=n}^{\infty} \sin(mx) a_k \sin(kx)\right| \leq m a_n \pi. \qquad \text{----------------} \quad (76)$$

Since $a_n \to 0$, (76) implies that $\sum_{k=1}^{\infty} \sin(mx) a_k \sin(kx)$ is uniformly Cauchy on $[0, \pi]$ and so converges uniformly on $[0, \pi]$ to $g(x)\sin(mx)$.

Therefore,

$$\frac{2}{\pi}\int_0^{\pi} g(x)\sin(mx)dx = \frac{2}{\pi}\sum_{k=1}^{\infty} a_k \int_0^{\pi} \sin(mx)\sin(kx)dx$$

$$= \frac{2}{\pi}\sum_{k=1}^{\infty} a_k \int_0^{\pi} \frac{\cos((m-k)x) - \cos((m+k)x)}{2} dx = a_m \,.$$

This means (a_n) are the Fourier coefficients of $g(x)$. Thus (S) is the Fourier series of g.

For the cosine series (C) we use the following device.

Consider $(1 - \cos(mx)) f(x)$.

The series $\sum_{k=1}^{\infty} a_k (1-\cos(mx))\cos(kx)$ converges to $(1-\cos(mx)) f(x)$. We show that the convergence is uniform on $[0, \pi]$.

Now for any $d \geq n$ and x in $(0, \pi]$,

$$\left|\sum_{k=n}^{d} a_k (1-\cos(mx))\cos(kx)\right| = (1-\cos(mx))\left|\sum_{k=n}^{d} a_k \cos(kx)\right|$$

$$\leq \frac{1}{2} m^2 x^2 \left|\sum_{k=n}^{d} a_k \cos(kx)\right|$$

$$\leq \frac{1}{2} m^2 x^2 a_n \max_{n \leq j \leq d} \left|\sum_{k=n}^{j} \cos(kx)\right| \text{, by Lemma 4,}$$

$$\leq \frac{1}{2} m^2 a_n x^2 \frac{1}{\sin(\frac{x}{2})} \text{, by using the summation method of (16),}$$

$$\leq \frac{1}{2} m^2 a_n x\pi \leq \frac{1}{2} m^2 \pi^2 a_n \text{, by (27).}$$

This inequality is obviously true for $x = 0$. Hence, for x in $[0, \pi]$,

$$\left|\sum_{k=n}^{\infty} a_k (1-\cos(mx))\cos(kx)\right| \leq \frac{1}{2} m^2 \pi^2 a_n \,. \quad \text{----------------- (77)}$$

It then follows that the series converges to $(1 - \cos(mx))\, f(x)$ uniformly on $[0, \pi]$. Therefore,

$$\frac{2}{\pi}\int_0^{\pi}(1-\cos(mx))f(x)dx = \frac{a_0}{\pi}\int_0^{\pi}(1-\cos(mx))dx + \sum_{k=1}^{\infty} a_k \frac{2}{\pi}\int_0^{\pi}(1-\cos(mx))\cos(kx)dx$$

$$= a_0 - \sum_{k=1}^{\infty} a_k \frac{2}{\pi}\int_0^{\pi}\cos(mx)\cos(kx)dx$$

$$= a_0 - \sum_{k=1}^{\infty} a_k \frac{2}{\pi}\int_0^{\pi}\frac{\cos(mx+kx)+\cos(mx-kx)}{2}dx$$

$$= a_0 - a_m . \qquad \text{-----------------------------------} \quad (78)$$

Taking limit as m tends to infinity we have, by the Riemann-Lebesgue Theorem,

$$\frac{2}{\pi}\int_0^{\pi} f(x)dx = a_0 \qquad \text{-----------------------------------} \quad (79)$$

It follows now from (78) that

$$\frac{2}{\pi}\int_0^{\pi}\cos(mx)f(x)dx = a_m .$$

Hence the series (C) is the Fourier series for f.

3. Proof of The Main Results

3.1 Proof of Theorem 1.

By hypothesis $\Delta a_n \geq 0$ for all $n \geq 1$ and $a_n \to 0$. By Theorem 9, the series (S) converges pointwise on **R** and uniformly on $[\delta, 2\pi-\delta]$. We shall show that the sum function g is Lebesgue integrable if, and only if, $\sum_{n=1}^{\infty}\frac{a_n}{n} < \infty$.

Recall from (11) that the n-th partial sum of (S) is:

$$s_n(x) = \sum_{k=1}^{n-1} D_k(x)\Delta a_k + a_n D_n(x) = \sum_{k=1}^{n} D_k(x)\Delta a_k + a_{n+1} D_n(x) .$$

Since $s_n(x) \to g(x)$, $a_{n+1} \to 0$ and $\left|D_n(x)\right| \le \frac{\pi}{x}$, by (34),

$$\sum_{k=1}^{\infty} D_k(x)\Delta a_k \to g(x) \qquad \text{--------------------------- (80)}$$

pointwise on [−π, π].

We now consider the use of the modified conjugate Dirichlet kernel. Take the series $\sum_{k=1}^{\infty} \overset{*}{D}_k(x)\Delta a_k$. It converges to a function g^* on [−π, π], because

$\overset{*}{D}_k(x) = D_k(x) - \frac{1}{2}\sin(kx)$ and $\frac{1}{2}\sum_{k=1}^{\infty}\sin(kx)\Delta a_k$ converges uniformly and absolutely to a continuous function h on **R** by application of the Weierstrass M test. That is, we have

$$\sum_{k=1}^{\infty} \overset{*}{D}_k(x)\Delta a_k \to g^*(x) \qquad \text{------------------------ (81)}$$

on [−π, π] and $g(x) = g^*(x) + h(x)$.

Note that $\overset{*}{D}_k(x)\Delta a_k \ge 0$ and so by the Lebesgue Monotone Convergence Theorem,

$$\int_0^{\pi} g^*(x)dx = \sum_{k=1}^{\infty}\left(\int_0^{\pi} \overset{*}{D}_k(x)dx\right)\Delta a_k \qquad \text{--------------------- (82)}$$

and g^* is Lebesgue integrable if, and only if, $\sum_{k=1}^{\infty}\left(\int_0^{\pi} \overset{*}{D}_k(x)dx\right)\Delta a_k < \infty$.

Since $g(x) = g^*(x) + h(x)$ and h is continuous, g is Lebesgue integrable if, and only if, g^* is Lebesgue integrable.

Now, by Theorem 7 Part (2), $\int_0^{\pi} \overset{*}{D}_k(x)dx = \ln(n) + O(1)$ and since $\sum_{k=1}^{\infty} K\Delta a_k < \infty$ for any constant K,

$$\sum_{k=1}^{\infty}\left(\int_0^{\pi} \overset{*}{D}_k(x)dx\right)\Delta a_k < \infty \Leftrightarrow \sum_{k=1}^{\infty}\ln(k)\Delta a_k < \infty . \qquad \text{------------------ (83)}$$

It remains to show that

$$\sum_{k=1}^{\infty} \ln(k)\Delta a_k < \infty \Leftrightarrow \sum_{k=1}^{\infty} \frac{a_k}{k} < \infty \ . \qquad \text{-----------------} \ (84)$$

Now, let $t_n = \sum_{k=1}^{n} \frac{a_k}{k}$ be the n-th partial sum of $\sum_{k=1}^{\infty} \frac{a_k}{k}$. Then by Abel summation formula (3),

$$t_n = \sum_{k=1}^{n} s_k \Delta a_k + a_{n+1} s_n = \sum_{k=1}^{n-1} s_k \Delta a_k + a_n s_n \ , \qquad \text{------------} \ (85)$$

where $s_n = \sum_{k=1}^{n} \frac{1}{k}$.

Suppose now that $\sum_{k=1}^{\infty} \frac{a_k}{k} < \infty$, that is (t_n) is convergent. Therefore, (t_n) is bounded above. Since all the terms are nonnegative, $0 \le \sum_{k=1}^{n} s_k \Delta a_k$ is bounded above and so the series $\sum_{k=1}^{\infty} s_k \Delta a_k$ is convergent. Now by (45), $s_k = \ln(k) + O(1)$ and so it follows that $\sum_{k=1}^{\infty} \ln(k)\Delta a_k$ is convergent.

Conversely, suppose $\sum_{k=1}^{\infty} \ln(k)\Delta a_k$ is convergent. Then $\sum_{k=1}^{n} s_k \Delta a_k$ is convergent.

Observe that

$$\ln(n) a_n = \ln(n) \sum_{k=n}^{\infty} \Delta a_k \le \sum_{k=n}^{\infty} \ln(k)\Delta a_k \ .$$

Since $\sum_{k=n}^{\infty} \ln(k)\Delta a_k \to 0$, by the Comparison test

$$\ln(n) a_n \to 0 \ . \qquad \text{----------------------------------} \ (86)$$

Therefore, since $s_n = \ln(n) + O(1)$ and $a_n \to 0$, $a_n s_n \to 0$. It follows then form (85) that (t_n) is convergent, i.e., $\sum_{k=1}^{\infty} \frac{a_k}{k} < \infty$.

Next, we show that if $\sum_{k=1}^{\infty} \frac{a_k}{k} < \infty$, then (S) converges to g in the L^1 norm.

By the Lebesgue Monotone Convergence Theorem,

$$\int_0^{\pi}\left|g^*(x)-\sum_{k=1}^{n}\overset{*}{D}_k(x)\Delta a_k\right|dx\to 0 \quad as\ n\to\infty\ . \qquad \text{------------} \quad (87)$$

Therefore,

$$\int_0^{\pi}\left|g(x)-\sum_{k=1}^{n-1}D_k(x)\Delta a_k-a_nD_n(x)\right|dx$$

$$=\int_0^{\pi}\left|g^*(x)+h(x)-\sum_{k=1}^{n-1}\overset{*}{D}_k(x)\Delta a_k-\frac{1}{2}\sum_{k=1}^{n-1}\sin(kx)\Delta a_k-a_nD_n(x)\right|dx$$

$$\le\int_0^{\pi}\left|g^*(x)-\sum_{k=1}^{n-1}\overset{*}{D}_k(x)\Delta a_k\right|dx+\int_0^{\pi}\left|h(x)-\frac{1}{2}\sum_{k=1}^{n-1}\sin(kx)\Delta a_k\right|dx+a_n\int_0^{\pi}\left|D_n(x)\right|dx\ . \text{-----} \quad (88)$$

Since $\frac{1}{2}\sum_{k=1}^{\infty}\sin(kx)\Delta a_k$ converges uniformly to h on $[0, \pi]$,

$$\int_0^{\pi}\left|h(x)-\frac{1}{2}\sum_{k=1}^{n-1}\sin(kx)\Delta a_k\right|dx\to 0 \text{ as } n\to\infty\ . \qquad \text{------------------} \quad (89)$$

Since $\int_0^{\pi}\left|D_n(x)\right|dx=\ln(n)+O(1)$, $a_n\to 0$ and $\ln(n)a_n\to 0$ (See (86)),

$$a_n\int_0^{\pi}\left|D_n(x)\right|dx\to 0\ . \qquad \text{--------------------------------} \quad (90)$$

Therefore, by the Comparison Test, using (88), (87), (89) and (90), we have

$$\int_0^{\pi}\left|g(x)-s_n(x)\right|dx=\int_0^{\pi}\left|g(x)-\sum_{k=1}^{n-1}D_k(x)\Delta a_k-a_nD_n(x)\right|dx\to 0\ .$$

Thus (S) converges to g in the L^1 norm. This completes the proof of Theorem 1.

3.2 Proof of Theorem 2.

If $(a_0, a_1, \ldots)$ is *convex* and $a_n\to 0$, then by Lemma 5, $\Delta a_n \ge 0$ for all $n\ge 0$. Part (2) is a consequence of Part (1) by Theorem 11. By Theorem 9, the cosine series (C) converges pointwise at x except possibly for $x = 0$ in $[-\pi, \pi]$. The limiting function or sum function f is continuous at every $x \ne 0$ in $[-\pi, \pi]$.

We shall show that f is a non-negative Lebesgue integrable function.

Let $t_n(x)=\frac{1}{2}a_0+\sum_{k=1}^{n}a_k\cos(kx)$ be the (n+1)-th partial sum of (C). Then we have, by Abel's summation formula (3) (see (8)),

$$t_n(x)=\sum_{k=0}^{n-1}D_k(x)\Delta a_k+a_nD_n(x).$$

By using Abel's summation formula on the summand $\sum_{k=0}^{n-1}D_k(x)\Delta a_k$, we get

$$t_n(x)=\sum_{k=0}^{n-2}(k+1)K_k(x)\Delta^2 a_k+nK_{n-1}(x)\Delta a_{n-1}+a_nD_n(x)$$

(see 15). Here, $K_n(x)$ is the Fejér kernel. Note that for $x\neq 0$ and x in $[-\pi, \pi]$,

$D_n(x)\leq\frac{\pi}{2|x|}$ (see (35)). It follows that $a_nD_n(x)\to 0$. Observe that $K_n(x)\geq 0$ for all x in $[-\pi, \pi]$ and for x in $[\delta, \pi]$, $0<\delta<\pi$, from (19) we have

$$K_n(x)=\frac{1}{n+1}\frac{1-\cos((n+1)x)}{4\sin^2(\frac{1}{2}x)}\leq\frac{1}{2(n+1)\sin^2(\frac{1}{2}x)}\leq\frac{1}{2(n+1)\sin^2(\frac{1}{2}\delta)}$$

or $$\max_{\delta\leq x\leq\pi}K_n(x)\leq\frac{1}{2(n+1)\sin^2(\frac{1}{2}\delta)}. \quad \text{------------------} \quad (91)$$

Therefore, $nK_{n-1}(x)\Delta a_{n-1}\leq\frac{n}{2n\sin^2(\frac{1}{2}\delta)}\Delta a_{n-1}=\frac{1}{2\sin^2(\frac{1}{2}\delta)}\Delta a_{n-1}$. Since $\Delta a_n\to 0$, $nK_{n-1}(x)\Delta a_{n-1}\to 0$. It follows that

$$t_n(x)\to\sum_{k=0}^{\infty}(k+1)K_k(x)\Delta^2 a_k$$

pointwise on $[-\pi, \pi]-\{0\}$. Hence for x in $[-\pi, \pi]-\{0\}$,

$$f(x)=\sum_{k=0}^{\infty}(k+1)K_k(x)\Delta^2 a_k\geq 0.$$

Because $(k+1)K_k(x)\Delta^2 a_k\geq 0$ for all $k\geq 0$, by the Lebesgue Monotone Convergence Theorem,

$$\int_{-\pi}^{\pi}f(x)dx=\sum_{k=0}^{\infty}\left(\int_{-\pi}^{\pi}K_k(x)dx\right)(k+1)\Delta^2 a_k$$

$$= \sum_{k=0}^{\infty} \pi (k+1)\Delta^2 a_k \text{ , by (22)}$$

$$< \infty \text{ , by Lemma 6.}$$

It follows that *f* is Lebesgue integrable. This proves Part (1) and hence Part (2).

Now, we examine the convergent series

$$\sum_{k=0}^{\infty} (k+1) K_k(x) \Delta^2 a_k \ .$$

Let the (n+1)-partial sum of this series be $G_n(x) = \sum_{k=0}^{n} (k+1) K_k(x) \Delta^2 a_k$.

By the Lebesgue Monotone Convergence Theorem,

$$\int_{-\pi}^{\pi} \left| f(x) - G_n(x) \right| dx \to 0. \qquad \text{-------------------} \quad (92)$$

More precisely,

$$\int_{-\pi}^{\pi} G_n(x) dx = \sum_{k=0}^{n} \left(\int_{-\pi}^{\pi} K_k(x) dx \right) (k+1) \Delta^2 a_k \to \int_{-\pi}^{\pi} f(x) dx \, .$$

Now,

$$\left| \left| t_n(x) - f(x) \right| - \left| a_n D_n(x) \right| \right| \le \left| t_n(x) - f(x) - a_n D_n(x) \right|$$

$$= \left| \sum_{k=0}^{n-2} (k+1) K_k(x) \Delta^2 a_k - f(x) + n K_{n-1}(x) \Delta a_{n-1} \right|, \text{ by (15),}$$

$$\le \left| G_{n-2}(x) - f(x) \right| + n K_{n-1}(x) \Delta a_{n-1} \, .$$

Therefore,

$$a_n \left| D_n(x) \right| - \left| G_{n-2}(x) - f(x) \right| - n K_{n-1}(x) \Delta a_{n-1} \le \left| t_n(x) - f(x) \right|$$

$$\le a_n \left| D_n(x) \right| + \left| G_{n-2}(x) - f(x) \right| + n K_{n-1}(x) \Delta a_{n-1} \, .$$

$$\text{---------------------} \quad (93)$$

Hence,

$$a_n \int_{-\pi}^{\pi} \left| D_n(x) \right| dx - \int_{-\pi}^{\pi} \left| G_{n-2}(x) - f(x) \right| dx - n \Delta a_{n-1} \int_{-\pi}^{\pi} K_{n-1}(x) dx \le \int_{-\pi}^{\pi} \left| t_n(x) - f(x) \right| dx$$

$$\le a_n\int_{-\pi}^{\pi}\left|D_n(x)\right|dx+\int_{-\pi}^{\pi}\left|G_{n-2}(x)-f(x)\right|dx+n\Delta a_{n-1}\int_{-\pi}^{\pi}K_{n-1}(x)dx .$$

Thus, by (22) we get

$$a_n\int_{-\pi}^{\pi}\left|D_n(x)\right|dx-\int_{-\pi}^{\pi}\left|G_{n-2}(x)-f(x)\right|dx-n\Delta a_{n-1}\pi\le\int_{-\pi}^{\pi}\left|t_n(x)-f(x)\right|dx$$

$$\le a_n\int_{-\pi}^{\pi}\left|D_n(x)\right|dx+\int_{-\pi}^{\pi}\left|G_{n-2}(x)-f(x)\right|dx+n\Delta a_{n-1}\pi .$$

By Lemma 6, $n\Delta a_{n-1}\pi=(n-1)\Delta a_{n-1}\pi+\Delta a_{n-1}\pi\to 0$ and so it follows from the above inequality and (92) that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\left|t_n(x)-f(x)\right|dx=\lim_{n\to\infty}a_n\int_{-\pi}^{\pi}\left|D_n(x)\right|dx . \qquad \text{------------------ (94)}$$

Since $\int_{-\pi}^{\pi}\left|D_n(x)\right|dx=\frac{4}{\pi}\ln(n)+O(1)$ and $a_n\to 0$, $\lim_{n\to\infty}a_n\int_{-\pi}^{\pi}\left|D_n(x)\right|dx=0\Leftrightarrow a_n\ln(n)\to 0$.

Therefore, $\lim_{n\to\infty}\int_{-\pi}^{\pi}\left|t_n(x)-f(x)\right|dx=0\Leftrightarrow\lim_{n\to\infty}a_n\ln(n)=0$. This proves Part (4).

Now we examine the (C,1) mean of the Fourier series (C):

$$\sigma_{n+1}(x)=\frac{1}{n+1}\left(t_0(x)+t_1(x)+\cdots+t_n(x)\right)$$

For x in $[-\pi,\pi]-\{0\}$, $t_n(x)\to f(x)$. Therefore, by *the regularity of Cesaro summability,* $\sigma_{n+1}(x)\to f(x)$. [If a series converges, then its (C,1) mean also converges to the same value.] It remains to prove Part (3) that

$$\int_{-\pi}^{\pi}\left|\sigma_n(x)-f(x)\right|dx\to 0 .$$

Firstly, we show that $\int_{-\pi}^{\pi}\sigma_{n+1}(x)dx\to\int_{-\pi}^{\pi}f(x)dx$.

We shall use the following formula that for a (C,1) mean of a series with index starting from 0,

$$\sigma_{n+1}=\frac{1}{n+1}\left(s_0+s_1+\cdots+s_n\right)=\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)a_k\ , \qquad \text{------------------- (95)}$$

where $s_n=\sum_{k=0}^{n}a_k$.

Using (95), we have

$$\sigma_{n+1}(x) = \frac{1}{n+1}\left(t_0(x) + t_1(x) + \cdots + t_n(x)\right)$$

$$= \frac{1}{2}a_0 + \sum_{k=1}^{n}\left(1 - \frac{k}{n+1}\right)a_k \cos(kx)$$

$$= \sum_{k=0}^{n-1}\left\{\left(1 - \frac{k}{n+1}\right)a_k - \left(1 - \frac{k+1}{n+1}\right)a_{k+1}\right\}D_k(x) + \left(1 - \frac{n}{1+n}\right)a_n D_n(x),$$

by Abel's summation formula (3),

$$= \sum_{k=0}^{n-1}\Delta a_k D_k(x) + \frac{1}{n+1}\sum_{k=0}^{n-1}\left\{(-k)a_k + (k+1)a_{k+1}\right\}D_k(x) + \left(1 - \frac{n}{1+n}\right)a_n D_n(x)$$

$$= \sum_{k=0}^{n-1}\Delta a_k D_k(x) + \frac{1}{n+1}\sum_{k=0}^{n-1}\left\{a_k - (k+1)\Delta a_k\right\}D_k(x) + \left(1 - \frac{n}{1+n}\right)a_n D_n(x)$$

$$= \sum_{k=0}^{n-1}\Delta a_k D_k(x) + \frac{1}{n+1}\sum_{k=0}^{n-1}a_k D_k(x) - \frac{1}{n+1}\sum_{k=0}^{n-1}(k+1)\Delta a_k D_k(x) + \left(1 - \frac{n}{1+n}\right)a_n D_n(x)$$

$$= \sum_{k=0}^{n-1}\Delta a_k D_k(x) + \frac{1}{n+1}\sum_{k=0}^{n}a_k D_k(x) - \frac{1}{n+1}\sum_{k=0}^{n-1}(k+1)\Delta a_k D_k(x). \text{ ----------- } (96)$$

Now $\sum_{k=0}^{n-1}\Delta a_k D_k(x) = \sum_{k=0}^{n-2}\Delta^2 a_k (k+1)K_k(x) + \Delta a_{n-1} n K_{n-1}(x)$,

$\frac{1}{n+1}\sum_{k=0}^{n}a_k D_k(x) = \frac{1}{n+1}\sum_{k=0}^{n-1}\Delta a_k (k+1)K_k(x) + a_n K_n$ and

$$\frac{1}{n+1}\sum_{k=0}^{n-1}(k+1)\Delta a_k D_k(x) = \frac{1}{1+n}\sum_{k=0}^{n-2}\left\{(k+1)\Delta a_k - (k+2)\Delta a_{k+1}\right\}(k+1)K_k(x) + \frac{n^2}{1+n}\Delta a_{n-1}K_{n-1},$$

by using Abel's summation (3).

Therefore, it follows from (96),

$$\sigma_{n+1}(x) = \sum_{k=0}^{n-2}\Delta^2 a_k (k+1)K_k(x) + \Delta a_{n-1} n K_{n-1}(x)$$

$$+ \frac{1}{n+1}\sum_{k=0}^{n-1}\Delta a_k (k+1)K_k(x) + a_n K_n$$

$$- \frac{1}{1+n}\sum_{k=0}^{n-2}\left\{(k+1)\Delta a_k - (k+2)\Delta a_{k+1}\right\}(k+1)K_k(x) - \frac{n^2}{1+n}\Delta a_{n-1}K_{n-1}$$

$$= \sum_{k=0}^{n-2} \Delta^2 a_k (k+1) K_k(x) + \Delta a_{n-1} n K_{n-1}(x) + \frac{1}{n+1} \Delta a_{n-1} n K_{n-1}(x) + a_n K_n$$

$$- \frac{1}{1+n} \sum_{k=0}^{n-2} \left\{ k \Delta a_k - (k+2) \Delta a_{k+1} \right\} (k+1) K_k(x) - \frac{n^2}{1+n} \Delta a_{n-1} K_{n-1}$$

$$= \sum_{k=0}^{n-2} \Delta^2 a_k (k+1) K_k(x) + \Delta a_{n-1} n K_{n-1}(x) + \frac{1}{n+1} \Delta a_{n-1} n K_{n-1}(x) + a_n K_n$$

$$- \frac{1}{1+n} \sum_{k=0}^{n-2} \left\{ k \Delta^2 a_k - 2\Delta a_{k+1} \right\} (k+1) K_k(x) - \frac{n^2}{1+n} \Delta a_{n-1} K_{n-1}. \quad \text{--------- (97)}$$

Thus,

$$\sigma_{n+1}(x) - G_{n-2}(x) = \Delta a_{n-1} n K_{n-1}(x) + \frac{1}{n+1} \Delta a_{n-1} n K_{n-1}(x) + a_n K_n$$

$$- \frac{1}{1+n} \sum_{k=0}^{n-2} \left\{ k \Delta^2 a_k - 2\Delta a_{k+1} \right\} (k+1) K_k(x) - \frac{n^2}{1+n} \Delta a_{n-1} K_{n-1}.$$

---------------- (98)

Now,

$$\int_{-\pi}^{\pi} \left(\sum_{k=0}^{n-2} \left\{ k \Delta^2 a_k - 2\Delta a_{k+1} \right\} (k+1) K_k(x) \right) = \pi \left(\sum_{k=0}^{n-2} \left\{ k \Delta^2 a_k - 2\Delta a_{k+1} \right\} (k+1) \right)$$

$$= \pi \sum_{k=0}^{n-2} k \Delta^2 a_k (k+1) - 2\pi \sum_{k=0}^{n-2} \Delta a_{k+1} (k+1)$$

$$= \pi \sum_{k=0}^{n-2} k \Delta^2 a_k (k+1) - 2\pi \sum_{k=0}^{n-3} \Delta^2 a_{k+1} \left(\sum_{j=0}^{k} (j+1) \right) - 2\pi \Delta a_{n-1} \left(\sum_{j=0}^{n-2} (j+1) \right),$$

by Abel's summation formula (3)

$$= \pi \sum_{k=0}^{n-2} k \Delta^2 a_k (k+1) - \pi \sum_{k=0}^{n-3} \Delta^2 a_{k+1} (k+1)(k+2) - \pi \Delta a_{n-1} n(n-1)$$

$$= \pi \sum_{k=0}^{n-2} k \Delta^2 a_k (k+1) - \pi \sum_{k=1}^{n-2} \Delta^2 a_k k(k+1) - \pi \Delta a_{n-1} n(n-1)$$

$$= -\pi \Delta a_{n-1} n(n-1). \quad \text{------------------------ (99)}$$

Therefore, it follows from (98) and (99) that

$$\int_{-\pi}^{\pi}\left(\sigma_{n+1}(x)-G_{n-2}(x)\right)dx=\pi\Delta a_{n-1}n+\frac{1}{n+1}\pi\Delta a_{n-1}n+\pi a_n+\pi\frac{1}{n+1}\Delta a_{n-1}n(n-1)-\pi\frac{n^2}{1+n}\Delta a_{n-1}$$

$$=\pi\Delta a_{n-1}n+\pi a_n\to 0 \text{ as } n\to\infty, \text{ by Lemma 6.}$$

Therefore,

$$\int_{-\pi}^{\pi}\left(\sigma_{n+1}(x)-f(x)\right)dx=\int_{-\pi}^{\pi}\left(\sigma_{n+1}(x)-G_{n-2}(x)\right)dx+\int_{-\pi}^{\pi}\left(G_{n-2}(x)-f(x)\right)dx\to 0 \text{ as } n\to\infty .$$

That is, $\lim_{n\to\infty}\int_{-\pi}^{\pi}\sigma_{n+1}(x)dx=\int_{-\pi}^{\pi}f(x)dx$.

Convergence in the L^1 norm is more difficult. We shall need some technical result concerning the Fejér kernels and the (C, 1) mean of a Fourier series.

We shall need to use more general result to do this.

3.3 Proof of Theorem 2 part (3).

We write the (n+1)-th partial sum t_n as an integral:

$$t_n(x)=\frac{1}{2}a_0+\sum_{k=1}^{n}a_k\cos(kx)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left\{\frac{1}{2}+\sum_{k=1}^{n}\cos(k(t-x))\right\}dt,$$

since the limiting function f is Lebesgue integrable,

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)D_n(t-x)dt=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+u)D_n(u)du, \qquad \text{-------------- (100)}$$

by Change of Variable and periodicity.

Note that (100) is also true for a general Lebesgue integrable function f not necessarily an even function.

Then the (C,1) mean,

$$\sigma_{n+1}(x)=\frac{1}{n+1}\left(t_0(x)+t_1(x)+\cdots+t_n(x)\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+u)\frac{1}{n+1}\sum_{k=0}^{n}D_k(u)du$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+u)K_n(u)du. \qquad \text{----------------- (101)}$$

Therefore,

$$\sigma_{n+1}(x)-f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x+u)-f(x)\right)K_n(u)du\,, \quad \text{----------------} \quad (102)$$

since $\frac{1}{\pi}\int_{-\pi}^{\pi}K_n(u)du=1$.

Before we proceed further, we state a result of Fejér:

Theorem 12. Suppose f is a Lebesgue integrable periodic function of period 2π. Then f has a Fourier series (A).

(1) If f is continuous at x, then the (C,1) mean of the Fourier series (A) converges to $f(x)$;
(2) If f is continuous on $[-\pi, \pi]$, then the (C,1) mean of the Fourier series (A) converges uniformly to f;
(3) If f has a jump discontinuity at x, that is, $\lim_{t\to x^-}f(t)=f(x_-)$ and $\lim_{t\to x^+}f(t)=f(x_+)$ exist, finite and not equal, then the (C,1) mean of the Fourier series at x converges to $\frac{1}{2}\left(f(x_-)+f(x_+)\right)$.

Proof. Using (102), we have

$$\sigma_{n+1}(x)-f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x+u)-f(x)\right)K_n(u)du$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\left(f(x+u)-f(x)\right)K_n(u)du+\frac{1}{\pi}\int_{-\pi}^{0}\left(f(x+u)-f(x)\right)K_n(u)du$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\left(f(x+u)-f(x)\right)K_n(u)du+\frac{1}{\pi}\int_{0}^{\pi}\left(f(x-u)-f(x)\right)K_n(u)du\,,$$

by Change of Variable and that $K_n(-u) = K_n(u)$,

$$=\frac{2}{\pi}\int_{0}^{\pi}\left(\frac{f(x+u)+f(x-u)}{2}-f(x)\right)K_n(u)du$$

$$=\frac{2}{\pi}\int_{0}^{\pi}\phi(x,u)K_n(u)du\,, \quad \text{---------------------} \quad (103)$$

where $\phi(x,u)=\dfrac{f(x+u)+f(x-u)}{2}-f(x)$.

If f is continuous at x, then given $\varepsilon > 0$, there exists $\delta > 0$ depending on x so that

$$|u| \leq \delta \Rightarrow |f(x+u) - f(x)| < \varepsilon . \quad \text{----------------------- (104)}$$

Therefore,

$$|u| \leq \delta \Rightarrow |\phi(x,u)| < \varepsilon \quad \text{----------------------- (105)}$$

and it follows that

$$\int_0^\delta |\phi(x,u) K_n(u)| du \leq \varepsilon \int_0^\delta K_n(u) du \leq \varepsilon \int_0^1 K_n(u) du = \varepsilon \frac{\pi}{2} . \quad \text{------------- (106)}$$

Now,

$$\int_\delta^\pi |\phi(x,u) K_n(u)| du \leq \mu_n(\delta) \int_\delta^\pi |\phi(x,u)| du , \quad \text{--------------- (107)}$$

where

$$\mu_n(\delta) = \max_{\delta \leq x \leq \pi} K_n(x) \leq \frac{1}{2(n+1)\sin^2(\frac{1}{2}\delta)} . \quad \text{------------------ (108)}$$

Thus, by (103), (106) and (107),

$$|\sigma_{n+1}(x) - f(x)| = \frac{2}{\pi} \int_0^\pi |\phi(x,u)| K_n(u) du \leq \frac{2}{\pi} \left(\varepsilon \frac{\pi}{2} + \mu_n(\delta) \int_\delta^\pi |\phi(x,u)| du \right), \text{ i.e.,}$$

$$|\sigma_{n+1}(x) - f(x)| \leq \varepsilon + \frac{2}{\pi} \mu_n(\delta) \int_\delta^\pi |\phi(x,u)| du . \quad \text{------------ (109)}$$

Since the inequality (108) implies that $\mu_n(\delta) \to 0$, it follows from (109) that $|\sigma_{n+1}(x) - f(x)| \to 0$. That is to say, $\sigma_{n+1}(x) \to f(x)$. This proves part (1).

If f is continuous on $[-\pi, \pi]$, then f is uniformly continuous on $[-\pi, \pi]$ and so (104) is valid for any x as $\delta > 0$ can be chosen for any x such that (104) holds true.

Note that if $M = \max_{-\pi \leq x \leq \pi} |f(x)|$, then $|\phi(x,u)| \leq 2M$. It follows from (109) that for all x,

$$|\sigma_{n+1}(x) - f(x)| \leq \varepsilon + \frac{2}{\pi} \mu_n(\delta) \int_\delta^\pi 2M du \leq \varepsilon + 4M \mu_n(\delta) . \quad \text{-------------- (110)}$$

This implies that $\sigma_{n+1}(x) \to f(x)$ uniformly on [−π, π]. This completes the proof for part (2).

Suppose now *f* has a jump discontinuity at *x*. We may redefine the value of *f* at *x* to be $\frac{1}{2}\left(f(x_-)+f(x_+)\right)$. Then by the definition of the one-sided limit at *x*, there exists $\delta > 0$ so that $|u| \le \delta \Rightarrow |\phi(x,u)| < \varepsilon$. It follows in exactly the same manner, using (106) and (107), that $\sigma_{n+1}(x) \to f(x)$. This proves part (3).

Completion of the proof of Theorem 2 part (3)

By (102), $|\sigma_{n+1}(x) - f(x)| \le \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+u) - f(x)| K_n(u)du$.

Therefore,

$$\int_{-\pi}^{\pi} |\sigma_{n+1}(x) - f(x)| dx \le \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} |f(x+u) - f(x)| K_n(u)du\right)dx .$$

But

$$\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} |f(x+u) - f(x)| K_n(u)du\right)dx = \int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} |f(x+u) - f(x)| K_n(u)dx\right)du ,$$

by Fubini Theorem for non-negative function,

$$= \int_{-\pi}^{\pi} \eta(u) K_n(u)du , \quad \text{------------------------ (111)}$$

where $\eta(u) = \int_{-\pi}^{\pi} |f(x+u) - f(x)| dx$.

Note that $\eta(u)$ is a periodic, nonnegative continuous function. It is also an even function but we do not require this fact. That it is a continuous function can be deduced by the fact that *f* can be approximated by a continuous function since it is integrable (See the next theorem.) Hence $\eta(u)K_n(u)$ is integrable. Note that *f* is measurable since it is integrable and so there is an integrable Borel measurable function *g* such that *g* = *f* almost everywhere on [−2π, 2π]. We may replace *f* by *g* and the integral $\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+u) - f(x)| K_n(u)du$ as well as the integrals on both sides of (111) remain unchanged. Since *g* is Borel, $g(x + y)$ is measurable with respect to the product measure on $\mathbb{R} \times \mathbb{R}$ and so

$|g(x+u)-g(x)|K_n(u)$ is measurable and we may apply Fubini Theorem to conclude that

$$\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}|g(x+u)-g(x)|K_n(u)du\right)dx=\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}|g(x+u)-g(x)|K_n(u)dx\right)du$$

and so (111) follows since

$$\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}|g(x+u)-g(x)|K_n(u)dx\right)du=\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}|f(x+u)-f(x)|K_n(u)dx\right)du .$$

By (101), $\frac{1}{\pi}\int_{-\pi}^{\pi}\eta(u)K_n(u)du$ is the (C,1) mean of the Fourier series for η at 0.

Observe that $\eta(0) = 0$. We shall prove that $\eta(u)$ is continuous at 0. Indeed $\eta(u)$ is continuous on $[-\pi, \pi]$. The proof for any u in $[-\pi, \pi]$ is similar. We require the following approximation theorem:

Theorem 13. Given any $\varepsilon > 0$, any integrable function g on $\mathbb{R}$ may be approximated by a continuous function ϕ with compact support so that $\int_{\mathbb{R}}|\phi(x)-g(x)|<\varepsilon$.

To use this result, extend the domain of f beyond $[-2\pi, 2\pi]$ by defining it to take the value 0 outside $[-2\pi, 2\pi]$. Then there exists a continuous function ϕ with compact support so that $\int_{\mathbb{R}}|\phi(x)-f(x)|dx<\frac{\varepsilon}{3}$. Therefore, $\int_{-\pi}^{\pi}|\phi(x)-f(x)|dx<\frac{\varepsilon}{3}$ and $\int_{-\pi}^{\pi}|f(x+u)-\phi(x+u)|dx<\frac{\varepsilon}{3}$.

Thus, $\int_{-\pi}^{\pi}|f(x+u)-f(x)|dx$

$$\leq\int_{-\pi}^{\pi}|f(x+u)-\phi(x+u)|dx+\int_{-\pi}^{\pi}|f(x)-\phi(x)|dx+\int_{-\pi}^{\pi}|\phi(x+u)-\phi(x)|dx$$

$$\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\int_{-\pi}^{\pi}|\phi(x+u)-\phi(x)|dx .$$

The function ϕ is continuous on $[-2\pi, 2\pi]$ and so it is uniformly continuous on $[-2\pi, 2\pi]$. Hence, by uniform continuity, there exists $\pi > \delta > 0$ so that

$|\phi(x+u)-\phi(x)|<\frac{\varepsilon}{6\pi}$ for all x in $[-\pi, \pi]$ and for any $|u|<\delta$.

Hence, $\int_{-\pi}^{\pi}|\phi(x+u)-\phi(x)|dx \le \int_{-\pi}^{\pi}\frac{\varepsilon}{6\pi}dx = \frac{\varepsilon}{3}$ for $|u|<\delta$ and so

$$\int_{-\pi}^{\pi}|f(x+u)-f(x)|dx < \varepsilon \ .$$

It follows that $\int_{-\pi}^{\pi}|f(x+u)-f(x)|dx \to 0$ as $u \to 0$. This means η is continuous at 0.

By (111),

$$\int_{-\pi}^{\pi}|\sigma_{n+1}(x)-f(x)|dx \le \frac{1}{\pi}\int_{-\pi}^{\pi}\eta(u)K_n du \ .$$

Therefore, since the right-hand side of the above expression is the (C,1) mean of the Fourier series of η at 0 and $\eta(0) = 0$, by Theorem 12 Part (1),

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\eta(u)K_n du \to \eta(0) = 0 \, .$$

Hence, by the Comparison Test, we have

$$\int_{-\pi}^{\pi}|\sigma_{n+1}(x)-f(x)|dx \to 0$$

and this completes the proof of Theorem 2 part (3).

We have actually proved the following

Theorem 12*. Suppose *f* is a Lebesgue integrable function of period 2π. Then the sequence of (C,1) means of the Fourier series of *f* converges to *f* in the L^1 norm. More precisely, $\int_{-\pi}^{\pi}|\sigma_{n+1}(x)-f(x)|dx \to 0$.

3.4 Proof of Theorem 3

Suppose $a_n \to 0$ and $(a_n) = (a_0, a_1, \ldots)$ is decreasing. By Theorem 9, the cosine series (C),

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \, ,$$

converges pointwise to a function $f(x)$ in $[-\pi, \pi]$ except possibly at $x = 0$. The function f is continuous at $x \neq 0$.

We assume that $a_0 = 0$ and all partial sums involved begin with a_1. The series obtained by integrating (C) term-wise is

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \sin(nx) . \qquad \text{----------------------------- (SC)}$$

Since $a_n \to 0$ and so $n\left(\frac{a_n}{n}\right) \to 0$ and with this condition, by Theorem 10, (SC) converges uniformly to a continuous function $F(x)$ on $\mathbb{R}$. Note that the series (C) converges uniformly in $[\delta, 2\pi-\delta]$ for any $0 < \delta < \pi$. This implies that F is differentiable in $[-\pi, \pi]-\{0\}$ and $F'(x) = f(x)$. Since F is continuous and so Lebesgue integrable, by Theorem 11, (SC) is the Fourier series of F. Hence

$$\frac{a_n}{n} = \frac{2}{\pi} \int_0^{\pi} F(x) \sin(nx) dx . \qquad \text{--------------------- (112)}$$

Now, for $0 < \delta < \pi$,

$$\int_{\delta}^{\pi} F(x) \sin(nx) dx = \left[-\frac{1}{n} \cos(nx) F(x) \right]_{\delta}^{\pi} + \int_{\delta}^{\pi} \frac{1}{n} \cos(nx) f(x) dx ,$$

by integration by parts,

$$= \frac{1}{n} \cos(n\delta) F(\delta) - \frac{1}{n} \cos(n\pi) F(\pi) + \int_{\delta}^{\pi} \frac{1}{n} \cos(nx) f(x) dx \ . \quad \text{-------- (113)}$$

Note that $F(\pi) = 0$ and $\lim_{\delta \to 0} F(\delta) = F(0) = 0$. It then follows from (113) that

$$\lim_{\delta \to 0} \int_{\delta}^{\pi} F(x) \sin(nx) dx = \lim_{\delta \to 0} \frac{1}{n} \cos(n\delta) F(\delta) - \frac{1}{n} \cos(n\pi) F(\pi) + \frac{1}{n} \lim_{\delta \to 0} \int_{\delta}^{\pi} \cos(nx) f(x) dx$$

$$= \frac{1}{n} \cos(0) F(0) - \frac{1}{n} \cos(n\pi) \bullet 0 + \frac{1}{n} \lim_{\delta \to 0} \int_{\delta}^{\pi} \cos(nx) f(x) dx = \frac{1}{n} \lim_{\delta \to 0} \int_{\delta}^{\pi} \cos(nx) f(x) dx .$$

Hence, from (112),

$$\frac{a_n \pi}{2n} = \int_0^{\pi} F(x) \sin(nx) dx = \frac{1}{n} \lim_{\delta \to 0} \int_{\delta}^{\pi} \cos(nx) f(x) dx = \frac{1}{n} \int_0^{\pi} \cos(nx) f(x) dx \ , \quad \text{----- (114)}$$

where the right-hand side is an improper Riemann integral, and so

$$a_n = \frac{2}{\pi}\lim_{\delta\to 0}\int_\delta^\pi \cos(nx)f(x)dx = \frac{2}{\pi}\int_0^\pi \cos(nx)f(x)dx\,.$$

Thus, once we show that *f* has an improper integral that is 0, then (C) is the Riemann Fourier series of *f*.

For $0 < \delta < \pi$, $\int_\delta^\pi f(x)dx = \int_\delta^\pi F'(x)dx = F(\pi) - F(\delta) = -F(\delta)$. Therefore,

$$\int_0^\pi f(x)dx = -\lim_{\delta\to 0} F(\delta) = -F(0) = 0\,.$$

Thus, (C) is the Riemann Fourier series of *f*.

Suppose now that $a_0 \neq 0$. Then if (C) converges to *f*, $\sum_{n=1}^{\infty} a_n\cos(nx)$ converges to $f(x) - \frac{a_0}{2}$ and by the above argument $\int_0^\pi \left(f(x) - \frac{a_0}{2}\right)dx = 0$ so that

$$\frac{2}{\pi}\int_0^\pi f(x)dx = a_0\,.$$

From (114), we obtain, for $n \geq 1$.

$$\frac{a_n\pi}{2n} = \int_0^\pi F(x)\sin(nx)dx = \frac{1}{n}\lim_{\delta\to 0}\int_\delta^\pi \cos(nx)F'(x)dx$$

$$= \frac{1}{n}\int_0^\pi \cos(nx)\left(f(x) - \frac{a_0}{2}\right)dx = \frac{1}{n}\int_0^\pi \cos(nx)f(x)dx$$

and we have, as before, $a_n = \frac{2}{\pi}\int_0^\pi \cos(nx)f(x)dx$ for $n \geq 1$. Note that in interpreting (114) in the context that $a_0 \neq 0$, $F'(x) = f(x) - \frac{a_0}{2}$.

That is to say, (C) is the Riemann Fourier series of its sum function *f*. This completes the proof of Theorem 3.

4. Examples

(1) Because the sequence $\left(\frac{1}{\ln(n)}\right)$ is convex, by Theorem 2 part (4), translated appropriately with the series starting from n= 2, the series

$$\sum_{n=2}^{\infty}\frac{1}{\ln(n)}\cos(nx)$$

converges to a Lebesgue integrable function *f* and is the Fourier series of its sum function *f*. It does not converge to *f* in the L^1 norm. Indeed by (94) in the proof of Theorem 2 and Theorem 7,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}|t_n(x)-f(x)|dx=\lim_{n\to\infty}\frac{1}{\ln(n)}\int_{-\pi}^{\pi}|D_n(x)|dx=\lim_{n\to\infty}\frac{1}{\ln(n)}\frac{4}{\pi}\ln(n)=\frac{4}{\pi}\neq 0\,.$$

However, its (C,1) mean converges to *f* in the L^1 norm.

The conjugate series

$$\sum_{n=2}^{\infty}\frac{1}{\ln(n)}\sin(nx)$$

by Theorem 9, converges to a function *g* but it is not the Fourier series of *g* by Theorem 1 since $\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$ is divergent.

(2) The series

$$\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}\cos(nx)$$

converges to a Lebesgue integrable function *f* in the L^1 norm by Theorem 2 Part (4).

(3) The series

$$\sum_{n=2}^{\infty}\frac{1}{\sqrt{\ln(n)}}\cos(nx)$$

converges by Theorem 2, to a non-negative Lebesgue integrable function *f* because the sequence $\left(\frac{1}{\sqrt{\ln(n)}}\right)$ is convex. But it does not converge to *f* in the L^1 norm. Indeed, the integral of the modulus of its *n*-th partial sum $t_n(x)$ tends to infinity. We deduce this as follows. From (94),

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}|t_n(x)-f(x)|dx=\lim_{n\to\infty}a_n\int_{-\pi}^{\pi}|D_n(x)|dx$$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{\ln(n)}}\frac{4}{\pi}\ln(n)\quad\text{by Theorem 7}$$

$$= \lim_{n\to\infty} \frac{4}{\pi} \sqrt{\ln(n)} = \infty$$

and so $\lim_{n\to\infty} \int_{-\pi}^{\pi} |t_n(x)| dx = \infty$.

However, its (C,1) mean tends to f in the L^1 norm.

5. Related Results to Theorem 10 and Theorem 1.

There are two results that can be proved or deduced by the methods of Theorem 10. One of them concerns bounded convergence and the other continuity.

Theorem 14. Suppose $\Delta a_n \geq 0$ for $n \geq 1$ and $a_n \to 0$. Then the series (S) converges boundedly on $\mathbb{R}$ if, and only if, $a_n = O(\frac{1}{n})$ or $na_n \leq K$ for all $n \geq 1$ and for some $K > 0$.

Proof.

Suppose (S) converges boundedly on $\mathbb{R}$. Then there exists a real number $M > 0$ such that for all $n \geq 1$ and for all x in $\mathbb{R}$,

$$\left| \sum_{k=1}^{n} a_k \sin(kx) \right| \leq M . \qquad \text{----------------------} \quad (115)$$

Take any $n \geq 1$. Let $y = \frac{\pi}{4n}$. Since $a_n \to 0$ and (a_n) is decreasing, $a_n \geq 0$ for all $n \geq 1$. Therefore,

$$\sum_{k=n+1}^{2n} a_k \sin(ky) \geq a_{2n} \sum_{k=n+1}^{2n} \sin(ky) \geq a_{2n} \sum_{k=n+1}^{2n} \sin(ny) \geq na_{2n} \sin(\frac{\pi}{4}) \geq 0 .$$

It then follows from (115) that for any $n \geq 1$,

$$na_{2n} \sin\left(\frac{\pi}{4}\right) \leq \sum_{k=n+1}^{2n} a_k \sin(ky) \leq \sum_{k=1}^{2n} a_k \sin(ky) = \left| \sum_{k=1}^{2n} a_k \sin(ky) \right| \leq M ,$$

that is, $na_{2n} \leq \sqrt{2}M$. Therefore $2na_{2n} \leq 2\sqrt{2}M$. Since $a_{2n-2} \geq a_{2n-1}$, $(2n-2)a_{2n-1} \leq (2n-2)a_{2n-2} \leq 2\sqrt{2}M$ for $n > 1$. Thus, for $n > 1$, $(2n-1)a_{2n-1} = (2n-2)a_{2n-1} + a_{2n-1} \leq 2\sqrt{2}M + a_{2n-1} \leq 2\sqrt{2}M + a_1$. If we let

$K = 2\sqrt{2}M + a_1$, then for all $n \geq 1$, $(2n-1)a_{2n-1} \leq K$. It follows that $na_n \leq K$ for all $n \geq 1$.

Conversely, suppose there exists $K > 0$ such that $na_n \leq K$ for all $n \geq 1$.

Take any x in (0, π]. Let $N_x = \left[\frac{\pi}{x}\right]$, the integer part of π/x. Then $1 \leq N_x \leq \frac{\pi}{x} < N_x + 1$. By Theorem 9, (S) is convergent on $\mathbb{R}$, i.e., $T(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ is convergent for all x. Let $T_k(x) = \sum_{n=1}^{k} a_n \sin(nx)$. We split $T_k(x)$ into two summations according to x using N_x . For convenience we drop the subfix and let $N = \min(k, N_x)$ and note that it depends on x.

Let $T'(x) = \sum_{n=1}^{N} a_n \sin(nx)$ and $T''(x) = \sum_{n=N+1}^{k} a_n \sin(nx)$ if $N < k$ and empty if $N \geq k$. For the first summation we have

$$\left|T'(x)\right| \leq \sum_{n=1}^{N} a_n \left|\sin(nx)\right| \leq \sum_{n=1}^{N} a_n nx \leq \sum_{n=1}^{N} Kx \leq KNx \leq K\pi . \quad \text{--------------} \quad (116)$$

From (67) we have

$$T''(x) = \left(s_k(x) - s_N(x)\right)$$

$$= \left(\sum_{n=N}^{k-1} D_n(x)\Delta a_n + a_k D_k(x) - a_N D_N(x)\right), \text{ by using (67),}$$

$$= \left(\sum_{n=1+N}^{k-1} D_n(x)\Delta a_n + a_k D_k(x) - a_{1+N} D_N(x)\right).$$

Therefore, when $N < k$,

$$\left|T''(x)\right| \leq \sum_{n=1+N}^{k-1} \left|D_n(x)\right| \Delta a_n + a_k \left|D_k(x)\right| + a_{1+N} \left|D_N(x)\right|$$

$$\leq \sum_{n=1+N}^{k-1} \frac{\pi}{x} \Delta a_n + a_k \frac{\pi}{x} + a_{1+N} \frac{\pi}{x} = 2a_{1+N} \frac{\pi}{x}, \text{ by using inequality (34),}$$

$$\leq 2a_{1+N}(N+1), \quad \text{since } \frac{\pi}{x} < N+1,$$

$$\leq 2K\,. \qquad \text{------------------------------} \quad (117)$$

Therefore, combining (116) and (117) we have for any x in $(0, \pi]$

$$\left|T_k(x)\right| \leq (2+\pi)K \qquad \text{------------------------------} \quad (118)$$

Inequality (118) is obviously true for $x = 0$. Therefore, $T_k(x)$ converges boundedly on $[0, \pi]$, i.e., (S) converges boundedly on $[0, \pi]$. Since the sum function is odd, (S) also converges boundedly on $[-\pi, 0]$ and hence on $[-\pi, \pi]$. It then follows by periodicity that (S) converges boundedly on the whole of $\mathbb{R}$.

The next result states that the uniform convergence of the series (S) is equivalent to the continuity of the limiting function g.

Theorem 15. Suppose $\Delta a_n \geq 0$ for $n \geq 1$ and $a_n \to 0$. Then the series (S) converges to a continuous function on $\mathbb{R}$ if, and only if, $na_n \to 0$.

Proof.

Suppose $na_n \to 0$. Then by Theorem 10 (S) converges uniformly to g on $\mathbb{R}$. Consequently, g is continuous.

Conversely suppose the limiting function g is continuous. Then g is Lebesgue integrable and so by Theorem 11, (S) is the Fourier series of g.

We assert that we may integrate g term by term. This is a special case that any Fourier series may be integrated term by term and the resulting series converges uniformly.

If we integrate (S) term by term we obtain the following series:

$$\text{(D)} \qquad \sum_{n=1}^{\infty}\frac{a_n}{n}(1-\cos(nx)) = \sum_{n=1}^{\infty}\frac{a_n}{n} - \sum_{n=1}^{\infty}\frac{a_n}{n}\cos(nx)\,.$$

Since (S) is the Fourier series of g,

$$\sum_{m=1}^{\infty}\frac{a_m}{m} = \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{1}{\pi}\int_0^{2\pi} g(t)\sin(mt)dx\right) = \frac{1}{\pi}\int_0^{2\pi} g(t)\sum_{m=1}^{\infty}\frac{\sin(mt)}{m}dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(t)(\pi - x)dx\,,$$

by the Lebesgue Dominated Convergence Theorem, since $\sum_{m=1}^{\infty}\frac{\sin(mt)}{m}$ converges boundedly to the function $h(x)=\begin{cases}\frac{1}{2}(\pi-x), & 0<x<2\pi \\ 0 & x=0,\quad 2\pi\end{cases}$.

This implies that $\sum_{n=1}^{\infty}\frac{a_n}{n}$ is convergent. It then follows that the series (D) converges uniformly and absolutely to a continuous function by the Weierstrass M-test. We now show that it converges to the integral of g, $G(x)=\int_0^x g(t)dt$.

Observe that $G(0) = G(2\pi) = 0$ and G is continuous of period 2π and is an even function. It follows that the Fourier series of $G(x)$ is a cosine series and its Fourier coefficients A_n is given by

$$A_0=\frac{1}{\pi}\int_0^{2\pi}G(t)dt=\frac{1}{\pi}\left(\left[G(t)t\right]_0^{2\pi}-\int_0^{2\pi}tg(t)dt\right)$$

$$=\frac{1}{\pi}\int_0^{2\pi}(\pi-t)g(t)dt \quad \text{since } G(0) = G(2\pi)=0,$$

$$=2\sum_{n=1}^{\infty}\frac{a_n}{n}$$

and for $n \geq 1$,

$$A_n=\frac{1}{\pi}\int_0^{2\pi}G(t)\cos(nt)dt=\frac{1}{\pi}\left(\left[\frac{\sin(nt)}{n}G(t)\right]_0^{2\pi}-\int_0^{2\pi}\frac{\sin(nt)}{n}g(t)dt\right),$$

by integration by parts,

$$=-\frac{1}{n}\frac{1}{\pi}\int_0^{2\pi}g(t)\sin(nt)dt=-\frac{a_n}{n}.$$

Thus the Fourier series of $G(x)$ is given by (D).

Since G is continuous on $[0, 2\pi]$ and its Fourier series (D) is convergent on $[0, 2\pi]$, by Theorem 12 Part (2), the (C,1) mean of (D) converges uniformly to G on $[0, 2\pi]$. Since (D) converges uniformly, its limiting function is the same as the limit of its (C,1) mean. Consequently (D) converges uniformly to G.

Let k be any positive integer,

$$G\left(\frac{\pi}{k}\right)=\int_0^{\pi/k} g(t)dt=\sum_{n=1}^{\infty}\frac{a_n}{n}\left(1-\cos\left(n\frac{\pi}{k}\right)\right) . \qquad \text{---------- (119)}$$

Since g is continuous at 0, $g(t) \to 0$ as $t \to 0$. Given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|t|<\delta \Rightarrow |g(t)|<\varepsilon$. Since $\frac{\pi}{k}\to 0$ because $k \to \infty$, there exists a positive integer N such that $k>N \Rightarrow \frac{\pi}{k}<\delta$. Therefore,

$$k>N \Rightarrow \left|G\left(\frac{\pi}{k}\right)\right|=\left|\int_0^{\pi/k} g(t)dt\right|\leq\int_0^{\pi/k}|g(t)|dt\leq\int_0^{\pi/k}\varepsilon dt=\varepsilon\frac{\pi}{k} . \qquad \text{-----------(120)}$$

This means

$$\lim_{k\to\infty} kG\left(\frac{\pi}{k}\right)=0 . \qquad \text{-------------------------- (121)}$$

Now,

$$\sum_{n=[k/2]+1}^{n=k}\frac{a_n}{n}\left(1-\cos\left(n\frac{\pi}{k}\right)\right)\geq\sum_{n=[k/2]+1}^{n=k}\frac{a_n}{n}2n^2\frac{\pi^2}{k^2}\frac{1}{\pi^2}=\sum_{n=[k/2]+1}^{n=k}2na_n\frac{1}{k^2} ,$$

by using inequality (28),

$$\geq\sum_{n=[k/2]+1}^{n=k}2na_k\frac{1}{k^2}=\frac{2}{k^2}a_k\sum_{n=[k/2]+1}^{n=k}n=\frac{2}{k^2}a_k\frac{k-[k/2]}{2}(k+[k/2]+1)$$

$$\geq\frac{2}{k^2}a_k\frac{k}{4}(k+[k/2]+1)\geq\frac{a_k}{2} . \qquad \text{----------------------- (122)}$$

Therefore, for $k > N$,

$$G\left(\frac{\pi}{k}\right)=\int_0^{\pi/k} g(t)dt=\sum_{n=1}^{\infty}\frac{a_n}{n}\left(1-\cos\left(n\frac{\pi}{k}\right)\right)\geq\frac{a_k}{2} . \qquad \text{----------------- (123)}$$

Hence, $ka_k \leq 2kG\left(\frac{\pi}{k}\right)$ for $k > N$. And so by the Squeeze Theorem and (121),

$\lim_{k\to\infty} ka_k = 0$.

This completes the proof.

The next result concerns the cosine series (C). It gives a sufficient condition for the Lebesgue integrability of the sum function of (C), whereas the same condition is a necessary condition for the sum function of (S) to be Lebesgue integrable.

The method of proof of Theorem 1 proves the following:

Theorem 16. Suppose (a_n) is a sequence of nonnegative terms, $\Delta a_n = a_n - a_{n+1} \geq 0$ and $a_n \to 0$. Then the limit function or sum function of (C), f, is Lebesgue integrable if $\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$. If $\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$, then (C) is the Fourier series of f and $\int_{-\pi}^{\pi} |t_n(x) - f(x)| dx \to 0$, where $t_n(x)$ is the $(n+1)$-th partial sum of the series (C), that is, $t_n(x)$ converges to f in the L^1 norm.

Proof.

Recall from (8) that the $(n+1)$-th partial sum of the series (C) is

$$t_n(x) = \sum_{k=0}^{n-1} D_k(x)\Delta a_k + a_n D_n(x).$$

As deduced in Section 2.6, $t_n(x)$ converges pointwise to a continuous function f on $[-\pi,\pi]-\{0\}$. It may or may not converge at 0. We want to show that f is Lebesgue integrable on $[-\pi,\pi]$.

Since for $x \neq 0$, $t_n(x) \to f(x)$, $a_n \to 0$ and $|D_n(x)| \leq \frac{\pi}{2x}$ by (35),

$$\sum_{k=0}^{\infty} D_k(x)\Delta a_k \to f(x) \qquad \text{--------------------------} \quad (124)$$

pointwise and absolutely on $[-\pi,\pi]-\{0\}$.

Recall from the proof of Theorem 1 (see (84)), that

$$\sum_{k=1}^{\infty} \ln(k)\Delta a_k < \infty \Leftrightarrow \sum_{k=1}^{\infty} \frac{a_k}{k} < \infty .$$

Note that $g(x)=\sum_{k=0}^{\infty}|D_k(x)|\Delta a_k$ is convergent on $[-\pi,\pi]-\{0\}$. Obviously, $\sum_{k=0}^{n}D_k(x)\Delta a_k$ is dominated by *g*. By Lemma 7 part (1) or (51), for $n\geq 1$,

$$\int_{-\pi}^{\pi}|D_n(x)|dx=\frac{4}{\pi}\ln(n)+O(1).$$

Therefore, $\sum_{k=0}^{\infty}\left(\int_{-\pi}^{\pi}|D_k(x)|dx\right)\Delta a_k<\infty$ as we are given that $\sum_{k=1}^{\infty}\frac{a_k}{k}<\infty$.

Therefore, by the Lebesgue Monotone Convergence Theorem, *g* is Lebesgue integrable on $[-\pi,\pi]$. It follows then by the Lebesgue Dominated Convergence Theorem and (124) that f is Lebesgue integrable on $[-\pi,\pi]$. Therefore, by Theorem 11, (C) is the Fourier series of *f*.

Next, we show that if $\sum_{k=1}^{\infty}\frac{a_k}{k}<\infty$, then (C) converges to *f* in the L^1 norm.

By the Lebesgue Domonated Convergence Theorem,

$$\int_{-\pi}^{\pi}\left|f(x)-\sum_{k=0}^{n}D_k(x)\Delta a_k\right|dx\to 0 \quad as\ n\to\infty. \qquad \text{---------} \quad (126)$$

Now,

$$\int_{-\pi}^{\pi}\left|f(x)-\sum_{k=0}^{n-1}D_k(x)\Delta a_k-a_nD_n(x)\right|dx$$

$$\leq\int_{-\pi}^{\pi}\left|f(x)-\sum_{k=0}^{n-1}D_k(x)\Delta a_k\right|dx+a_n\int_{-\pi}^{\pi}|D_n(x)|dx. \qquad \text{--------------------} \quad (127)$$

Since $\int_{-\pi}^{\pi}|D_n(x)|dx=\frac{4}{\pi}\ln(n)+O(1)$, $a_n\to 0$ and $\ln(n)a_n\to 0$ (See (86)),

$$a_n\int_{-\pi}^{\pi}|D_n(x)|dx\to 0. \qquad \text{---------------------------} \quad (128)$$

Therefore, by the Comparison Test, using (126), (127) and (128), we have

$$\int_{0}^{\pi}|f(x)-t_n(x)|dx=\int_{-\pi}^{\pi}\left|f(x)-\sum_{k=0}^{n-1}D_k(x)\Delta a_k-a_nD_n(x)\right|dx\to 0.$$

Thus, (C) converges to *f* in the L^1 norm. This completes the proof.

Remark. We have seen in Example 4 (1) that the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n)} \cos(nx)$$

converges to a Lebesgue integrable function *f* and is the Fourier series of *f*. As $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ is divergent, this shows that the converse of Theorem 16 is false.

Note that it does not converge to *f* in the L^1 norm.