A function defined on a finite interval, which is improperly Riemann integrable but not absolutely integrable and does not satisfy the conclusion of the Riemann Lebesgue Theorem.

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The function is modified from a Lebesgue function.

For each integer $k \geq 0$, let $a_k = 3^{k^4}$ and for each integer $r \geq 1$, let $c_r = \dfrac{1}{r^2}$. Subdivide the interval (0, π] by the interval $I_k = \left[\dfrac{\pi}{a_k}, \dfrac{\pi}{a_{k-1}}\right]$, $k = 1, 2, \cdots$. Define the function *f* in $[0, \pi]$ by

$$f(x) = c_k \sin(a_k x) \text{ for } x \in I_k = \left[\frac{\pi}{a_k}, \frac{\pi}{a_{k-1}}\right]$$

and $f(0) = 0$.

Note that the sequence (c_r) decreases to 0 and the sequence (a_k) is increasing and tends to ∞ .

Then we have:

(1) The function *f* is continuous in $[0, \pi]$.

(2) The Fourier cosine series of f diverges at $x = 0$. That is to say, if we extend f to an even function $F(x)$ in $[-\pi, \pi]$, so that $F(x) = f(x)$ and $F(-x) = f(x)$ for $x \geq 0$, then the Fourier series of F diverges at $x = 0$.

(3) The function g(x) defined by $g(x) = \begin{cases} \dfrac{f(x)}{x}, & 0 < x \leq \pi, \\ 0, & x = 0 \end{cases}$ is improperly Riemann integrable on $[0, \pi]$.

(4) g is not absolutely Riemann integrable on $[0, \pi]$ and so it is not Lebesgue integrable on $[0, \pi]$.

(5) The sequence $\left(\int_0^{\pi} g(t)\sin(nt)dt\right) = \left(\int_0^{\pi} f(t)\dfrac{\sin(nt)}{t}dt\right)$ diverges. Hence $\int_0^{\pi} g(t)\sin(nt)dt$ does not tend to 0 as *n* tends to infinity. That is to say, *g* does not satisfy the conclusion of the Riemann Lebesgue Theorem.

Proof of (1).

By definition, *f* is plainly continuous in (0, π]. Note that $\lim_{x\to 0^+} f(x) = 0$. This is because given $\varepsilon > 0$ there exists integer *N* such that for all $n \geq N$, $0 < c_n < \varepsilon$. Take $\delta > 0$ such that $0 < \delta < \dfrac{\pi}{a_N}$. Then $0 < x < \delta$ implies that $x \in I_k$, $k > N$ and so $|f(x)| \leq c_k < c_N < \varepsilon$.

Proof of (2).

In view of Theorem 19 in *Convergence of Fourier series*, this is equivalent to that the sequence $\left(\int_0^{\pi} f(t)\dfrac{\sin(nt)}{t}dt\right)$ is divergent. We shall show this later.

Proof of (3).

Observe that:

$$\left|\int_{\pi/a_k}^{\pi/a_{k-1}} g(x)dx\right| = c_k\left|\int_{\pi/a_k}^{\pi/a_{k-1}} \frac{\sin(a_k x)}{x}dx\right| = c_k\left|\int_{\pi}^{\pi a_k/a_{k-1}} \frac{\sin(u)}{u}du\right| < c_k K \text{ for some}$$

constant *K* since the integral $G(x) = \int_0^{x} \dfrac{\sin(t)}{t}dt$ is bounded in [0, ∞). Therefore,

$$\sum_{k=1}^{\infty}\left|\int_{\pi/a_k}^{\pi/a_{k-1}} g(x)dx\right| \leq K\sum_{k=1}^{\infty} c_k = K\frac{\pi^2}{6} < \infty \quad \text{-----------------------} \quad (1)$$

Hence $\sum_{k=1}^{\infty}\int_{\pi/a_k}^{\pi/a_{k-1}} g(x)dx < \infty$.

For any $0 < \delta < \pi$, $\dfrac{\pi}{a_k} \leq \delta < \dfrac{\pi}{a_{k-1}}$ and

$$\int_{\delta}^{\pi} g(x)dx = c_k\int_{\delta}^{\pi/a_{k-1}} \frac{\sin(a_k x)}{x}dx + \sum_{n=1}^{k-1} c_n\int_{\pi/a_n}^{\pi/a_{n-1}} \frac{\sin(a_n x)}{x}dx$$

$$= c_k\int_{\pi/a_k}^{\pi/a_{k-1}} \frac{\sin(a_k x)}{x}dx - c_k\int_{\pi/a_k}^{\delta} \frac{\sin(a_k x)}{x}dx + \sum_{n=1}^{k-1} c_n\int_{\pi/a_n}^{\pi/a_{n-1}} \frac{\sin(a_n x)}{x}dx$$

$$= \sum_{n=1}^{k} c_n\int_{\pi}^{\pi a_n/a_{n-1}} \frac{\sin(u)}{u}dx - c_k\int_{\pi}^{a_k\delta} \frac{\sin(u)}{u}du . \qquad \text{----------------------} \quad (2)$$

Now given any $\varepsilon > 0$, there exists an integer N such that $k \geq N$ implies that $c_k < \dfrac{\varepsilon}{2K}$ --- (3)

Since $\displaystyle\sum_{k=1}^{\infty}\int_{\pi/a_k}^{\pi/a_{k-1}} g(x)dx < \infty$, there exists an integer M such that

$$k \geq M \Rightarrow \left|\sum_{k=M}^{\infty}\int_{\pi/a_k}^{\pi/a_{k-1}} g(x)dx\right| < \frac{\varepsilon}{2} \ . \quad \text{------------------------} \quad (4)$$

Let L = max (N, M). Then for $0 < \delta < \dfrac{\pi}{a_{L-1}}$, $\dfrac{\pi}{a_k} \leq \delta < \dfrac{\pi}{a_{k-1}}$ and $k \geq L$,

$$\left|\int_{\delta}^{\pi} g(x)dx - \sum_{k=1}^{\infty}\int_{\pi/a_k}^{\pi/a_{k-1}} g(x)dx\right|$$

$$= \left|\sum_{n=1}^{k}\int_{\pi/a_n}^{\pi/a_{n-1}} g(x)dx - \int_{\pi/a_k}^{\delta} g(x)dx - \sum_{k=1}^{\infty}\int_{\pi/a_k}^{\pi/a_{k-1}} g(x)dx\right|$$

$$= \left|-\int_{\pi/a_k}^{\delta} g(x)dx - \sum_{n=k+1}^{\infty}\int_{\pi/a_n}^{\pi/a_{n-1}} g(x)dx\right|$$

$$\leq \left|\int_{\pi/a_k}^{\delta} g(x)dx\right| + \left|\sum_{n=k+1}^{\infty}\int_{\pi/a_n}^{\pi/a_{n-1}} g(x)dx\right| \leq c_k K + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $\displaystyle\int_0^{\pi} g(x)dx = \sum_{k=1}^{\infty}\int_{\pi/a_k}^{\pi/a_{k-1}} g(x)dx$. This proves that g is improperly integrable in [0, π].

Proof of (4).

$$\int_{\pi/a_k}^{\pi/a_{k-1}} |g(x)|dx = c_k\int_{\pi/a_k}^{\pi/a_{k-1}} \frac{|\sin(a_k x)|}{x}dx = c_k\int_{\pi}^{\pi a_k/a_{k-1}} \frac{|\sin(u)|}{u}dx . \quad \text{-------------------} \quad (5)$$

Now $\dfrac{a_k}{a_{k-1}} = \dfrac{3^{k^4}}{3^{(k-1)^4}} = 3^{k^3+k^2(k-1)+k(k-1)^2+(k-1)^3}$ and so

$$\int_{\pi}^{\pi a_k/a_{k-1}} \frac{|\sin(u)|}{u}\,dx = \sum_{n=1}^{3^{k^3+k^2(k-1)+k(k-1)^2+(k-1)^3-1}} \int_{n\pi}^{(n+1)\pi} \frac{|\sin(u)|}{u}\,dx$$

$$\geq \sum_{n=1}^{3^{k^3+k^2(k-1)+k(k-1)^2+(k-1)^3-1}} \frac{1}{(n+1)\pi}\int_{n\pi}^{(n+1)\pi} |\sin(u)|\,dx$$

$$= \sum_{n=1}^{3^{k^3+k^2(k-1)+k(k-1)^2+(k-1)^3-1}} \frac{2}{(n+1)\pi} > \frac{2}{\pi}\left(\ln(3^{k^3+k^2(k-1)+k(k-1)^2+(k-1)^3}) - 1\right) > \frac{2}{\pi}k^3\ln(3) .$$

Therefore, $\int_{\pi/a_k}^{\pi/a_{k-1}} |g(x)|\,dx > c_k\dfrac{2}{\pi}k^3\ln(3) = \dfrac{2}{\pi}k\ln(3)$. Hence $\int_0^{\pi} |g(x)|\,dx$ is divergent

Proof of Claim 5.

We want to show that $\int_0^{\pi} g(t)\sin(nt)\,dt = \int_0^{\pi} f(t)\dfrac{\sin(nt)}{t}\,dt$ is divergent. It is sufficient to show that $J_k = \int_0^{\pi} f(t)\dfrac{\sin(a_k t)}{t}\,dt \to \infty$ as $k \to \infty$.

$$J_k = \int_0^{\pi/a_k} f(t)\frac{\sin(a_k t)}{t}\,dt + \int_{\pi/a_k}^{\pi/a_{k-1}} f(t)\frac{\sin(a_k t)}{t}\,dt + \int_{\pi/a_{k-1}}^{\pi} f(t)\frac{\sin(a_k t)}{t}\,dt .$$

Let $J'_k = \int_0^{\pi/a_k} f(t)\dfrac{\sin(a_k t)}{t}\,dt$, $J''_k = \int_{\pi/a_k}^{\pi/a_{k-1}} f(t)\dfrac{\sin(a_k t)}{t}\,dt$ and

$$J'''_k = \int_{\pi/a_{k-1}}^{\pi} f(t)\frac{\sin(a_k t)}{t}\,dt .$$

Now, $|J'_k| \leq \int_0^{\pi/a_k} |f(t)|\left|\dfrac{\sin(a_k t)}{t}\right| dt \leq a_k \max\limits_{t\in[0,\,\pi/a_k]} |f(t)|\dfrac{\pi}{a_k} = c_{k+1}\pi = \dfrac{\pi}{(k+1)^2} < 1$. ----(6)

$$J''_k = \int_{\pi/a_k}^{\pi/a_{k-1}} f(t)\frac{\sin(a_k t)}{t}\,dt = c_k\int_{\pi/a_k}^{\pi/a_{k-1}} \frac{\sin^2(a_k t)}{t}\,dt = \frac{c_k}{2}\int_{\pi/a_k}^{\pi/a_{k-1}} \frac{1-\cos(2a_k t)}{t}\,dt$$

$$= \frac{c_k}{2}\int_{\pi/a_k}^{\pi/a_{k-1}} \frac{1}{t}\,dt - \frac{c_k}{2}\int_{\pi/a_k}^{\pi/a_{k-1}} \frac{\cos(2a_k t)}{t}\,dt$$

$$=\frac{c_k}{2}\ln\left(3^{k^3+k^2(k-1)+k(k-1)^2+(k-1)^3}\right)-\frac{c_k}{2}\int_{\pi/a_k}^{\pi/a_{k-1}}\frac{\cos(2a_kt)}{t}dt$$

$$=\frac{c_k}{2}\ln\left(3^{k^3+k^2(k-1)+k(k-1)^2+(k-1)^3}\right)-\frac{c_k}{2}\int_{\pi/a_k}^{\pi/a_{k-1}}\frac{\cos(2a_kt)}{t}dt$$

$$=\frac{c_k}{2}(k^3+k^2(k-1)+k(k-1)^2+(k-1)^3)\ln(3)-\frac{c_k}{2}\int_{\pi/a_k}^{\pi/a_{k-1}}\frac{\cos(2a_kt)}{t}dt$$

$$=\frac{\ln(3)}{2k^2}(k^3+k^2(k-1)+k(k-1)^2+(k-1)^3)-\frac{c_k}{2}\int_{\pi/a_k}^{\pi/a_{k-1}}\frac{\cos(2a_kt)}{t}dt\,. \text{ -------- (7)}$$

By the Second Mean Value Theorem,

$$\int_{\pi/a_k}^{\pi/a_{k-1}}\frac{\cos(2a_kt)}{t}dt=\frac{a_k}{\pi}\int_{\pi/a_k}^{C}\cos(2a_kt)dt=\frac{1}{2\pi}\left[\sin(2a_kt)\right]_{\pi/a_k}^{C}$$

for some C such that $\frac{\pi}{a_k}<C<\frac{\pi}{a_{k-1}}$. Therefore,

$$\left|\frac{c_k}{2}\int_{\pi/a_k}^{\pi/a_{k-1}}\frac{\cos(2a_kt)}{t}dt\right|\le\frac{c_k}{4\pi}\left|\left[\sin(2a_kt)\right]_{\pi/a_k}^{C}\right|\le\frac{c_k}{4\pi}=\frac{1}{4k^2\pi}<1. \text{ ------------ (8)}$$

It follows then from (7) and (8) that

$$J_k''\to\infty \text{ as } k\to\infty. \text{ --------------------------------- (9).}$$

For $k\ge 2$, $$J_k'''=\int_{\pi/a_{k-1}}^{\pi}f(t)\frac{\sin(a_kt)}{t}dt=\sum_{n=1}^{k-1}c_n\int_{\pi/a_n}^{\pi/a_{n-1}}\sin(a_nt)\frac{\sin(a_kt)}{t}dt$$

$$=\sum_{n=1}^{k-1}\frac{c_n}{2}\int_{\pi/a_n}^{\pi/a_{n-1}}\frac{\cos((a_k-a_n)t)-\cos((a_k+a_n)t)}{t}dt\,. \text{ --------- (10)}$$

Now,

$$\int_{\pi/a_n}^{\pi/a_{n-1}}\frac{\cos((a_k-a_n)t)}{t}dt=\int_{\pi/a_n}^{1/(a_k-a_n)}\frac{\cos((a_k-a_n)t)}{t}dt+\int_{1/(a_k-a_n)}^{\pi/a_{n-1}}\frac{\cos((a_k-a_n)t)}{t}dt$$

$$=\int_{\pi(a_k-a_n)/a_n}^{1}\frac{\cos(u)}{u}du+\int_{1}^{\pi(a_k-a_n)/a_{n-1}}\frac{\cos(u)}{u}du$$

$$= \int_1^{\pi(a_k - a_n)/a_{n-1}} \frac{\cos(u)}{u} du - \int_1^{\pi(a_k - a_n)/a_n} \frac{\cos(u)}{u} du .$$

Similarly,

$$\int_{\pi/a_n}^{\pi/a_{n-1}} \frac{\cos((a_k + a_n)t)}{t} dt = \int_{\pi/a_n}^{1/(a_k + a_n)} \frac{\cos((a_k + a_n)t)}{t} dt + \int_{1/(a_k + a_n)}^{\pi/a_{n-1}} \frac{\cos((a_k + a_n)t)}{t} dt$$

$$= \int_1^{\pi(a_k + a_n)/a_{n-1}} \frac{\cos(u)}{u} du - \int_1^{\pi(a_k + a_n)/a_n} \frac{\cos(u)}{u} du .$$

Observe that for $k \geq 2$, and $n > k$, $\frac{a_k}{a_n - 1} > \frac{a_k}{a_n} \geq \frac{a_k}{a_{k-1}} = 3^{k^3 + k^2(k-1) + k(k-1)^2 + (k-1)^3} > 3$, and so $\frac{\pi(a_k + a_n)}{a_n}$, $\frac{\pi(a_k + a_n)}{a_{n-1}}$ and $\frac{\pi(a_k - a_n)}{a_n}$ are all greater than 1. Plainly,

$$\frac{\pi(a_k - a_n)}{a_{n-1}} > \frac{\pi(a_k - a_n)}{a_n} > 1.$$

Since the improper integral $\int_1^{\infty} \frac{\cos(u)}{u} du$ is convergent, the function $H(x) = \int_1^{x} \frac{\cos(u)}{u} du$ is bounded, say by U. It follows that, for $k > n$,

$$\left| \int_{\pi/a_n}^{\pi/a_{n-1}} \frac{\cos((a_k - a_n)t) - \cos((a_k + a_n)t)}{t} dt \right| \leq 4U .$$

Hence, $$\left| J_k''' \right| \leq \sum_{n=1}^{k-1} 2c_n U < 2U \sum_{n=1}^{\infty} c_n = U \frac{\pi^2}{6} < \infty .$$

Thus, since J_k' and J_k''' are uniformly bounded and $J_k'' \to \infty$ as $k \to \infty$, it follows that $J_k \to \infty$ as $k \to \infty$. Hence $\int_0^{\pi} g(t)\sin(nt)dt = \int_0^{\pi} f(t)\frac{\sin(nt)}{t} dt$ diverges.