Convergence of Fourier Series

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This article is about convergence of Fourier series. Pointwise convergence, uniform convergence, (C,1) summability and convergence almost everywhere will be discussed. We divide this note into three sections. In Section A we set up the notations and describe the concepts involved while in section B we describe some results in Lebesgue theory. In Section C, we give necessary and sufficient conditions for convergence, uniform convergence and (C,1) summability. Some tests and sufficient conditions for uniform convergence for function of bounded variation are derived. Application to sectionally continuous functions is given. We have included Féjer Lebesgue Theorem which gives a sufficient condition for the (C,1) summability of Fourier series and some theorems concerning (C,1) summability and convergence almost everywhere.

Section A. Definitions, Notations and Preliminaries

Definition 1.

Let f be a Lebesgue integrable periodic function of period 2π.

It is convenient to assume that f is defined for all values of x in $[0, 2\pi]$ and by periodicity to all of $\mathbb{R}$. We may need to define values of f appropriately where it is not defined in $[0, 2\pi]$ and extend to $\mathbb{R}$ by periodicity.

Then we have the following formula for the definition of the coefficients

of a Fourier series of f:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx \text{ , } n = 0, 1, 2, \ldots \text{ . } \quad \text{---------} \quad (1)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx \text{ , } n = 1, 2, \ldots \text{ . } \quad \text{---------} \quad (2)$$

Consider the series

$$T(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(x) \text{ . } \quad \text{---------} \quad (A)$$

If a_n and b_n are given by (1) and (2), then $T(x)$ is called the *Fourier series of the function f.*

Note that we assume the function f is integrable in $[-\pi, \pi]$ so that (1) and (2) are meaningfully defined. Thus (A) is a Fourier series if it is the Fourier series of some integrable function f, otherwise it is called a *trigonometric series*, i.e., when the coefficients, a_n, b_n of (A) are not known to be given by (1) and (2).

The trigonometric series $T(x)$ may or may not converge and may not be the Fourier series of its limiting function. When $T(x)$ is a Fourier series it may not converge at all points. Kolmogorov showed that there exists a Lebesgue integrable function f whose Fourier series diverges at every point.

If we assume *nice* convergence for the trigonometric series, we do have some positive result. This is Theorem S below.

Theorem S. If the trigonometric series $T(x)$ converges uniformly to a function f, then it is the Fourier series of its sum function f. More is true, if $T(x)$ converges almost everywhere to a function f and the n-th partial sums of $T(x)$ are absolutely dominated by a Lebesgue integrable function, then $T(x)$ is the Fourier series of f. More precisely the n-th partial sum converges to f in the L^1 norm.

We note that in all two cases of Theorem S, the limiting function f is Lebesgue integrable and the trigonometric series $T(x)$, by using either the consequence of uniform convergence or the Lebesgue Dominated Convergence Theorem, can be shown to be the Fourier series of f. The convergence to f in the L^1 norm is a consequence of uniform convergence for the first case and in the other of being absolutely dominated by a Lebesgue integrable function.

In this note we are concerned mainly with Fourier series. Where trigonometric series is meant, it will always be specified.

Definition 2.

Suppose (a_n) is a sequence. The *Cesaro 1* or (C,1) means of the sequence is defined to be

$$\sigma_{n+1} = \frac{1}{n+1}\left(s_0 + s_1 + \cdots + s_n\right),$$

where $s_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$.

For $n \geq 1$, let

$$t_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} A_k(x) \quad \text{-----------------------} \quad (1)$$

be the sum of the first n +1 terms of the Fourier series $T(x)$ and $t_0(x) = \frac{1}{2}a_0$.

The $(C,1)$ means of the Fourier series is then given by

$$\sigma_{n+1}(x) = \frac{1}{n+1}\left(t_0(x) + t_1(x) + \cdots + t_n(x)\right). \quad \text{----------------} \quad (2)$$

If $\sigma_n(\theta) \to s$, then we say the Fourier series $T(x)$ is (C, *1*)-summable to s at θ.

We are concerned when $T(x)$ is a Fourier series of f, if $t_n(x)$ is pointwise convergent, uniform convergent or boundedly convergent and if it converges to the function f. We are also concerned with the pointwise convergence, uniform convergence and almost everywhere convergence of the (C,1) sums $\sigma_n(x)$ of $T(x)$.

Summation formula

We use Abel's summation technique. For this we reproduce the following material from my article *Fourier Cosine and sine series.*

Abel's Summation Formula.

Suppose (a_n) and (b_n) are two sequences. Let $s_n = \sum_{k=1}^{n} b_k$. Then we have the following summation formula:

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) s_k + a_n s_n$$

$$= \sum_{k=1}^{n} (a_k - a_{k+1}) s_k + a_{n+1} s_n . \quad \text{-----------------------------} \quad (3)$$

For the truncated sum we have:

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q-1} (a_k - a_{k+1}) s'_k + a_q s'_q , \quad \text{-----------------------} \quad (4)$$

where $s'_k = \sum_{j=p}^{k} b_j$, $k \ge p$.

Estimates of the sum are expressed in the following technical lemma.

Lemma 3. Suppose (a_n) is a decreasing sequence and $a_n \geq 0$ for all n. Then

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq a_1 \max_{1\leq k\leq n} |s_k| \quad \text{----------------------} \quad (5)$$

and

$$\left|\sum_{k=p}^{q} a_k b_k\right| \leq a_p \max_{p\leq k\leq q} |s'_k|. \quad \text{--------------------} \quad (6)$$

The proof can be found in *Fourier Cosine and sine series.*

Summing the Fourier Series

Dirichlet and Fejer Kernels

Definition 4.

Consider the (n+1)-th partial sum of the Fourier series $T(x)$,

$$t_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} A_k(kx) \ . \quad \text{---------------------- -----------} \quad (7)$$

$$t_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}\left(a_k \cos(kx) + b\sin(kx)\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left\{\frac{1}{2} + \sum_{k=1}^{n}\cos(k(t-x))\right\}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(t-x)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)D_n(u)du , \quad \text{------- ----} \ (8)$$

by Change of Variable and periodicity,

where

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n}\cos(kx) \quad \text{-----------------------------------} \quad (9)$$

for $n > 0$ and $D_0(x) = ½$.

$D_n(x)$ is called the *Dirichlet kernel*. Note that $D_n(x)$ is defined and continuous for all x in $[-\pi, \pi]$. We shall use this form of the (n+1)-th partial sum of $T(x)$ to investigate convergence of $T(x)$.

The (C,1) mean of $T(x)$,

$$\sigma_{n+1}(x) = \frac{1}{n+1}\left(t_0(x) + t_1(x) + \cdots + t_n(x)\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\frac{1}{n+1}\sum_{k=0}^{n} D_k(u)du$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)K_n(u)du\,, \quad \text{------------------------------------} \quad (10)$$

where

$$K_n(u) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(u)\,, \quad \text{---------------------------------} \quad (11)$$

is called the *Fejér kernel*.

We now describe some properties of the Dirichlet and Fejér kernels.

Now, $2\sin(\frac{1}{2}x)D_n(x) = \sin(\frac{1}{2}x) + \sum_{k=1}^{n} 2\sin(\frac{1}{2}x)\cos(kx)$

$$= \sin(\tfrac{1}{2}x) + \sum_{k=1}^{n}\left(\sin(kx + \tfrac{1}{2}x) - \sin((k-1)x + \tfrac{1}{2}x)\right)$$

$$= \sin(\tfrac{1}{2}x) + \sin((n+\tfrac{1}{2})x) - \sin(\tfrac{1}{2}x) = \sin((n+\tfrac{1}{2})x)\,.$$

Thus, for $x \neq 0$ and x in $[-\pi, \pi]$, or $0 < x < 2\pi$,

$$D_n(x) = \frac{\sin((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)}\,. \quad \text{---------------------------} \quad (12)$$

Observe that $\lim_{x\to 0}\frac{\sin((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)} = \lim_{x\to 0}\frac{(n+\frac{1}{2})\cos((n+\frac{1}{2})x)}{\cos(\frac{1}{2}x)} = n + \frac{1}{2} = D_n(0)$

and the Dirichlet kernel in its functional form (16) is continuous at 0.

For the estimate of the Dirichlet kernel it is useful to consider the modified Dirichlet kernel defined by

$$D_n^*(x) = D_n(x) - \tfrac{1}{2}\cos(nx)$$

$$= \frac{\sin((n+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)} - \tfrac{1}{2}\cos(nx) = \frac{\sin((n+\frac{1}{2})x) - \cos(nx)\sin(\frac{1}{2}x)}{2\sin(\frac{1}{2}x)}$$

$$= \frac{\sin(nx)\cos(\frac{1}{2}x)}{2\sin(\frac{1}{2}x)}$$

$$= \frac{\sin(nx)}{2\tan(\frac{1}{2}x)} \quad . \qquad \text{-------------------------------} \quad (13)$$

Note that the modified Dirichlet kernel is continuous in [−π, π] and

$$D_n^*(0) = n \quad \text{and} \quad D_n^*(\pi) = 0 \,. \qquad \text{------------------------} \quad (14)$$

The Fejér kernel has too a useful functional form. Using (12),

$$K_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(x) = \frac{1}{n+1}\sum_{k=0}^{n}\frac{\sin((k+\frac{1}{2})x)}{2\sin(\frac{1}{2}x)}$$

$$= \frac{1}{n+1}\frac{1}{2\sin^2(\frac{1}{2}x)}\sum_{k=0}^{n}\sin((k+\tfrac{1}{2})x)\sin(\tfrac{1}{2}x)$$

$$= \frac{1}{n+1}\frac{1}{2\sin^2(\frac{1}{2}x)}\sum_{k=0}^{n}\frac{\cos(kx)-\cos((k+1)x)}{2}$$

$$= \frac{1}{n+1}\frac{1-\cos((n+1)x)}{4\sin^2(\frac{1}{2}x)}$$

$$= \frac{1}{n+1}\frac{2\sin^2(\frac{1}{2}(n+1)x)}{4\sin^2(\frac{1}{2}x)}$$

$$= \frac{2}{n+1}\left\{\frac{\sin(\frac{1}{2}(n+1)x)}{2\sin(\frac{1}{2}x)}\right\}^2 . \qquad \text{--------------------------------} \quad (15)$$

The Fejér kernel in its functional form (15) is continuous in [−π, π].

Since $D_k(0) = k + \frac{1}{2}$, $K_n(0) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(0) = \frac{1}{n+1}\sum_{k=0}^{n}(k+\frac{1}{2}) = \frac{1}{2}+\frac{n}{2}$. -------- (16)

Note that from (9),

$$\int_{-\pi}^{\pi} D_n(x)dx = \int_{-\pi}^{\pi}\frac{1}{2}dx + \sum_{k=1}^{n}\int_{-\pi}^{\pi}\cos(kx)dx = \pi \qquad \text{-------------------} \quad (17)$$

and so $\quad \int_{-\pi}^{\pi} K_n(x)dx = \frac{1}{n+1}\sum_{k=0}^{n}\int_{-\pi}^{\pi} D_k(x) = \frac{1}{n+1}\sum_{k=0}^{n}\pi = \pi$

and $\quad \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1 \,.$ ---------------- (18)

Useful properties of trigonometric functions

Lemma 5.

(1) For all x, $|\sin(x)| \le |x|$; $|\sin(x)| < x$ for $x > 0$. -------------------- (19)

(2) For $0 \le x \le \frac{\pi}{2}$, $\sin(x) \ge \frac{2}{\pi} x$. -------------------------- (20)

(3) For $0 \le x \le \pi$, $1 - \cos(x) \ge 2\frac{x^2}{\pi^2}$. ---------------------------- (21)

(4) For all x, $1 - \cos(x) \le \frac{1}{2} x^2$. ------------------------- (22)

(5) Let $h(x) = \frac{1}{x} - \frac{1}{2\tan(\frac{x}{2})}$. Then $h(x)$ is continuous, bounded and

increasing on $(0, \pi)$, $\lim_{x \to 0^+} h(x) = 0$, $\lim_{x \to \pi^-} h(x) = \frac{1}{\pi}$, so that $0 < h(x) < 1/\pi$ and

$\sup_{0<x<\pi} h(x) = \frac{1}{\pi}$. In particular, $\frac{1}{2\tan(\frac{x}{2})} = \frac{1}{x} + O(1)$ in $(0, \pi)$.

Consider the problem of the Fourier series $T(x)$ converging to c. Our aim is to examine when the difference $t_n(x) - c$ tends to 0 and formulate conditions for pointwise, uniform or almost everywhere convergence. The difference has a nice integral form in terms of the Dirichlet kernel function.

$$t_n(x) - c = \frac{1}{\pi}\int_{-\pi}^{\pi} \big(f(x+u) - c\big) D_n(u) du \text{ , --------------- (23)}$$

since $\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x) dx = 1$.

We shall do the same for the problem of (C,1) summability of the Fourier series $T(x)$ to c. The difference $\sigma_{n+1}(x) - c$ too has a nice integral form in terms of the Fejér kernel function.

$$\sigma_{n+1}(x) - c = \frac{1}{\pi}\int_{-\pi}^{\pi} \big(f(x+u) - c\big) K_n(u) du \text{ , --------------- (24)}$$

since $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x) dx = 1$.

Section B Lebesgue Theory and Mean Value Theorem

We shall need a generalized form of the Lebesgue Riemann Theorem. This is used to show uniform convergence and to show that the problem of convergence, be it pointwise, uniform or boundedly, only depends on the local behaviour of the function. Results from Lebesgue integration theory will be used.

We shall state some of the well-known results without proof.

Theorem 6. (Uniform continuity of the Lebesgue integral)

If g is Lebesgue integrable over the measurable set E, then given any $\varepsilon > 0$, there exists $\delta > 0$ such that for any measurable subset U in E,

$$m(U) < \delta \Rightarrow \int_U g < \varepsilon \ .$$

Theorem 7.

Suppose g is Lebesgue integrable on an interval containing $[a, b]$ or g is Lebesgue integrable on $[a, b]$ and g is formally extended to an interval containing $[a, b]$ by defining g to be 0 outside $[a, b]$. Then

$$\lim_{h \to 0} \int_a^b |g(t+h) - g(t)| dt = 0.$$

Theorem 8. Suppose g is Lebesgue integrable on $[a, b]$. Let $G(x) = \int_a^x g(t)dt$.

Then G is absolutely continuous on $[a, b]$, differentiable almost everywhere and $G'(x) = g(x)$ almost everywhere on $[a, b]$.

Definition 9. Suppose g is Lebesgue integrable on $[a, b]$. A point x in $[a, b]$ is a *Lebesgue point of* g if $\int_0^h |g(x+t) - g(x)| dt = o(h)$ or equivalently,

$$\lim_{h \to 0} \frac{1}{h} \int_0^h |g(x+t) - g(x)| dt = 0.$$

The *Lebesgue set of* g: $[a, b] \to \mathbb{R}$ is the set of Lebesgue points of g in $[a, b]$.

Almost every point of $[a, b]$ is a Lebesgue point of g. This is a consequence of the following more general result.

Lemma 10. Suppose g: $[a, b] \to \mathbb{R}$ is Lebesgue integrable. Then for any c in $\mathbb{R}$, $\lim_{t\to 0}\frac{1}{t}\int_0^t |g(x+u)-c|\,du = |g(x)-c|$ for almost all x in $[a, b]$. More precisely, there exists a subset E of $[a, b]$ of measure 0 such that

$$\lim_{t\to 0}\frac{1}{t}\int_0^t |g(x+u)-c|\,du = |g(x)-c|$$

for all x in $[a, b] - E$ and for any c in $\mathbb{R}$.

Proof. For any rational number p, there exists a subset E_p of zero measure in $[a, b]$ such that for all x in $[a, b] - E_p$

$$\lim_{t\to 0}\frac{1}{t}\int_0^t |g(x+u)-p|\,du = |g(x)-p|.$$

This is deduced by noting that

$$\frac{1}{t}\int_0^t |g(x+u)-p|\,du = \frac{1}{t}\int_x^{x+t} |g(s)-p|\,ds = \frac{G(x+t)-G(x)}{t},$$

where $G(u)=\int_a^u |g(s)-p|\,ds$. By Theorem 8, G is differentiable almost everywhere in $[a, b]$. Therefore, there exists a set of measure 0, E_p, such that for all x in $[a, b] - E_p$, $G'(x)=\lim_{t\to 0}\frac{G(x+t)-G(x)}{t} = |g(x)-p|$.

Now let $E=\bigcup_{p\in\mathbb{Q}} E_p$. Since the set of rational numbers is countable, E is a countable union of sets of measure zero and so is also of zero measure. In particular for all x in $[a, b] - E$ and for all rational number p in $[a, b]$,

$$\lim_{t\to 0}\frac{1}{t}\int_0^t |g(x+u)-p|\,du = |g(x)-p|.$$

Suppose now x is in $[a, b] - E$ and β is irrational. We want to show that

$$\lim_{t\to 0}\frac{1}{t}\int_0^t |g(x+u)-\beta|\,du = |g(x)-\beta|\,.$$

Given any $\varepsilon > 0$, by the density of rational numbers we can choose a rational number p in $[a, b]$ such that

$$|\beta - p| < \frac{\varepsilon}{3}\ . \qquad \text{------------------------------ (25)}$$

Note that for $t \neq 0$,

$$\left|\frac{1}{t}\int_0^t |g(x+u)-\beta|\,du - \frac{1}{t}\int_0^t |g(x+u)-p|\,du\right|$$

$$\le \frac{1}{t}\int_0^t \big|\big(g(x+u)-\beta\big)-\big(g(x+u)-p\big)\big|\,du = \frac{1}{t}\int_0^t |\beta-p|\,du = |\beta-p| < \frac{\varepsilon}{3}\,. \text{ ---- (26)}$$

Thus, for $t \neq 0$,

$$\left|\frac{1}{t}\int_0^t |g(x+u)-\beta|\,du - |g(x)-\beta|\right| \le \left|\frac{1}{t}\int_0^t |g(x+u)-\beta|\,du - \frac{1}{t}\int_0^t |g(x+u)-p|\,du\right|$$

$$+\left|\frac{1}{t}\int_0^t |g(x+u)-p|\,du - |g(x)-p|\right| + \big||g(x)-p| - |g(x)-\beta|\big|$$

$$< \frac{\varepsilon}{3} + \left|\frac{1}{t}\int_0^t |g(x+u)-p|\,du - |g(x)-p|\right| + |\beta - p|$$

$$< \frac{2}{3}\varepsilon + \left|\frac{1}{t}\int_0^t |g(x+u)-p|\,du - |g(x)-p|\right|, \qquad \text{------------- (27)}$$

by using (26).

Since $\lim_{t\to 0}\frac{1}{t}\int_0^t |g(x+u)-p|\,du = |g(x)-p|$, there exists $\delta > 0$ such that

$$0 < |t| < \delta \Rightarrow \left|\frac{1}{t}\int_0^t |g(x+u)-p|\,du - |g(x)-p|\right| < \frac{\varepsilon}{3}. \quad \text{---------- (28)}$$

It follows then from (27) and (28) that there exists $\delta > 0$ such that

$$0<|t|<\delta \Rightarrow \left|\frac{1}{t}\int_0^t |g(x+u)-\beta|du - |g(x)-\beta|\right| \le \frac{2}{3}\varepsilon + \frac{\varepsilon}{3} = \varepsilon .$$

Hence, $\lim_{t\to 0}\frac{1}{t}\int_0^t |g(x+u)-\beta|du = |g(x)-\beta|$.

The next technical result, which proves to be very useful, is the Second Mean Value Theorem for integral. We state the generalised version of the theorem here without proof. A good reference is Hobson's classic, *The theory of Functions of a Real Variable and the Theory of Fourier's Series* and the improvement of Dixon in *The Second Mean Value Theorem in the Integral Calculus in Mathematical Proceedings of the Cambridge Philosophical society, 25, 1929, 282-284.*

Theorem 11 (Generalised Second Mean Value Theorem).

Suppose f is Lebesgue integrable on $[a, b]$ and g: $[a, b] \to \mathbb{R}$ is monotone.

Then

(i) $$\int_a^b f(x)g(x)dx = g(a)\int_a^C f(x)dx + g(b)\int_C^b f(x)dx \text{ ----------- (M)}$$

for some C with $a \le C \le b$;

(ii) (M) holds with $a < C < b$ *except* in some trivial cases, where $g(x)$ is constant in the open interval (a, b);

(iii) (M) holds with $g(a)$ and $g(b)$ replaced by A and B respectively so that

the function $h(x) = \begin{cases} A, & x = a, \\ g(x), & a < x < b, \\ B, & x = b \end{cases}$ is monotone; i.e.

$$\int_a^b f(x)g(x)dx = A\int_a^C f(x)dx + B\int_C^b f(x)dx$$

for some C with $a < C < b$, *except* in some trivial cases where $g(x)$ is constant in the open interval (a, b); $A \le \lim_{x\to a^+} g(x), B \ge \lim_{x\to b^-} g(x)$ if g is non-decreasing and $A \ge \lim_{x\to a^+} g(x), B \le \lim_{x\to b^-} g(x)$ if g is non-increasing.

We use Theorem 11 in the following special case which we state below.

Corollary 12. Suppose f is Lebesgue integrable on $[a, b]$ and g: $[a, b] \to \mathbb{R}$ is monotone. If g is non-negative, non-increasing and greater than or equal to 0, we can take $B = 0$ and $A = g(a)$; if g is non-negative, non-decreasing and greater than or equal to 0, we can take $A = 0$ and $B = g(b)$. This is sometimes called the Bonnet's Mean Value Theorem.

Theorem 13. (Riemann Lebesgue Theorem)

Suppose f and g are function of period 2π and that f is Lebesgue integrable and g is of bounded variation. Then for any a, b with $-\pi \leq a \leq b \leq \pi$ and any θ,

$$\int_a^b f(\theta+t)g(t)\cos(nt)dt \quad \text{and} \quad \int_a^b f(\theta+t)g(t)\sin(nt)dt$$

both tend to 0 uniformly in a, b, and θ as n tends to ∞. Moreover, both sequences are uniformly bounded in a, b, θ and n.

Proof. We shall prove only the conclusion for $\int_a^b f(\theta+t)g(t)\cos(nt)dt$. The proof for the other sequence is analogous.

Suppose first that g is the constant function $g(t) = 1$ for all t. Then

$$\int_a^b f(\theta+t)\cos(nt)dt = \int_{a-\frac{\pi}{n}}^{b-\frac{\pi}{n}} f(\theta+s+\tfrac{\pi}{n})\cos(ns+\pi)ds$$

$$= -\int_{a-\frac{\pi}{n}}^{b-\frac{\pi}{n}} f(\theta+s+\tfrac{\pi}{n})\cos(ns)ds$$

$$= -\int_a^b f(\theta+s+\tfrac{\pi}{n})\cos(ns)ds + \int_{b-\frac{\pi}{n}}^b f(\theta+s+\tfrac{\pi}{n})\cos(ns)ds$$

$$-\int_{a-\frac{\pi}{n}}^a f(\theta+s+\tfrac{\pi}{n})\cos(ns)ds.$$

Hence,

$$2\int_a^b f(\theta+t)\cos(nt)dt = \int_a^b \left(f(\theta+s) - f(\theta+s+\tfrac{\pi}{n})\right)\cos(ns)ds$$

$$+\int_{b-\frac{\pi}{n}}^{b} f(\theta+s+\tfrac{\pi}{n})\cos(ns)ds - \int_{a-\frac{\pi}{n}}^{a} f(\theta+s+\tfrac{\pi}{n})\cos(ns)ds\,.$$

It follows that

$$\int_a^b f(\theta+t)\cos(nt)dt = \frac{1}{2}\int_a^b \left(f(\theta+s) - f(\theta+s+\tfrac{\pi}{n})\right)\cos(ns)ds$$

$$+\frac{1}{2}\left\{\int_{b-\frac{\pi}{n}}^{b} f(\theta+s+\tfrac{\pi}{n})\cos(ns)ds - \int_{a-\frac{\pi}{n}}^{a} f(\theta+s+\tfrac{\pi}{n})\cos(ns)ds\right\}.$$

------------- (29)

Therefore,

$$\left|\int_a^b f(\theta+t)\cos(nt)dt\right| \leq \frac{1}{2}\int_a^b \left|f(\theta+s) - f(\theta+s+\tfrac{\pi}{n})\right|ds$$

$$+\frac{1}{2}\left\{\int_{b-\frac{\pi}{n}}^{b} \left|f(\theta+s+\tfrac{\pi}{n})\right|ds + \int_{a-\frac{\pi}{n}}^{a} \left|f(\theta+s+\tfrac{\pi}{n})\right|ds\right\}$$

$$= \frac{1}{2}\int_{a+\theta}^{b+\theta} \left|f(t) - f(t+\tfrac{\pi}{n})\right|dt + +\frac{1}{2}\left\{\int_{b+\theta}^{b+\theta+\frac{\pi}{n}} \left|f(t)\right|dt + \int_{a+\theta}^{a+\theta+\frac{\pi}{n}} \left|f(t)\right|dt\right\}$$

$$\leq \frac{1}{2}\int_{-\pi}^{\pi} \left|f(t) - f(t+\tfrac{\pi}{n})\right|dt + +\frac{1}{2}\left\{\int_{b+\theta}^{b+\theta+\frac{\pi}{n}} \left|f(t)\right|dt + \int_{a+\theta}^{a+\theta+\frac{\pi}{n}} \left|f(t)\right|dt\right\}.$$

---------------- (30)

By Theorem 7, there exists N such that

$$n \geq N \Rightarrow \int_{-\pi}^{\pi} \left|f(t) - f(t+\tfrac{\pi}{n})\right|dt < \varepsilon \quad \text{---------------- (31)}$$

and by Theorem 6, there exists M such that for all $n \geq M$ and for any a, b or θ

$$\int_{b+\theta}^{b+\theta+\frac{\pi}{n}} \left|f(t)\right|dt + \int_{a+\theta}^{a+\theta+\frac{\pi}{n}} \left|f(t)\right|dt < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,. \quad \text{------------- (32)}$$

Let $L = \max\{M, N\}$. It follows then from (30), (31) and (32) that for any a, b or θ,

$$n \geq L \Rightarrow \left|\int_a^b f(\theta+t)\cos(nt)dt\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

Hence, $\int_a^b f(\theta+t)\cos(nt)dt \to 0$ uniformly in a, b, and θ.

Observe that
$$\left|\int_a^b f(\theta+t)\cos(nt)dt\right| \le \int_a^b |f(\theta+t)|dt \le \int_{-\pi}^{\pi} |f(\theta+t)|dt = \int_{-\pi}^{\pi} |f(t)|dt < \infty .$$

Therefore, $\int_a^b f(\theta+t)\cos(nt)dt \to 0$ uniformly and boundedly in a, b, and θ.

Now suppose g is of bounded variation on $[-\pi, \pi]$. Then g is the difference of two decreasing functions. Thus, we may write

$$g = g_1 - g_2 ,$$

where g_1 and g_2 are bounded and decreasing. We may choose g_1 and g_2 to be both non-negative on $[-\pi, \pi]$. Then

$$\int_a^b f(\theta+t)g(t)\cos(nt)dt = \int_a^b f(\theta+t)g_1(t)\cos(nt)dt - \int_a^b f(\theta+t)g_2(t)\cos(nt)dt .$$

------------- (33)

By the Generalized Second Mean Value Theorem or the Bonnet Mean Value Theorem,

$$\int_a^b f(\theta+t)g_1(t)\cos(nt)dt = g_1(a)\int_a^{c_n} f(\theta+t)\cos(nt)dt , \quad \text{--------- (34)}$$

for some c_n with $a < c_n < b$ and

$$\int_a^b f(\theta+t)g_1(t)\cos(nt)dt = g_2(a)\int_a^{d_n} f(\theta+t)\cos(nt)dt , \quad \text{--------- (35)}$$

for some d_n with $a < d_n < b$.

Therefore,

$$\left|\int_a^b f(\theta+t)g(t)\cos(nt)dt\right| \le g_1(a)\left|\int_a^{c_n} f(\theta+t)\cos(nt)dt\right|$$

$$+g_2(a)\left|\int_a^{d_n} f(\theta+t)\cos(nt)dt\right| .$$

Now g_1 and g_2 are bounded above and so both are bounded above by some constant K. We may take $K = \max\{g_1(-\pi), g_2(-\pi)\}$. Hence

$$\left|\int_a^b f(\theta+t)g(t)\cos(nt)dt\right| \le K\left|\int_a^{c_n} f(\theta+t)\cos(nt)dt\right| + K\left|\int_a^{d_n} f(\theta+t)\cos(nt)dt\right|.$$

By the first case that we have just proved, given $\varepsilon > 0$, there exists N such that for any a, b, and θ,

$$n \ge N \Rightarrow \left|\int_a^b f(\theta+t)\cos(nt)dt\right| < \frac{\varepsilon}{2K+1}.$$

It follows then that for any a, b, and θ,

$$n \ge N \Rightarrow \left|\int_a^b f(\theta+t)g(t)\cos(nt)dt\right| < \frac{2K\varepsilon}{2K+1} < \varepsilon.$$

This means $\int_a^b f(\theta+t)g(t)\cos(nt)dt \to 0$ uniformly in a, b, and θ.

Moreover, $\left|\int_a^b f(\theta+t)g(t)\cos(nt)dt\right| \le 2K\int_{-\pi}^{\pi}|f(t)|dt < \infty$. Therefore,

$\int_a^b f(\theta+t)g(t)\cos(nt)dt$ is uniformly bounded in a, b, θ and n and

$\int_a^b f(\theta+t)g(t)\cos(nt)dt \to 0$ uniformly and boundedly in a, b, and θ.

Uniform boundedness of the integral of the Dirichlet kernels, modified Dirichlet kernels and related integral.

Theorem 14. For ε in $[0, \pi]$, the integrals

$\int_0^\varepsilon D_n(t)dt = \int_0^\varepsilon \frac{\sin((n+\frac{1}{2})t)}{2\sin(\frac{1}{2}t)}dt$, $\int_0^\varepsilon D_n^*(t)dt = \int_0^\varepsilon \frac{\sin(nt)}{2\tan(\frac{1}{2}t)}dt$ and

$\int_0^\varepsilon \frac{\sin(nt)}{t}dt$ are uniformly bounded in n and ε.

$\int_0^{\varepsilon}\left(\frac{1}{t}-\frac{1}{2\tan(\frac{1}{2}t)}\right)\sin(nt)dt$ is uniformly bounded in n and ε and tends to 0 uniformly on $[0,\pi]$.

Proof. By Lemma 5, $h(t)=\frac{1}{t}-\frac{1}{2\tan(\frac{1}{2}t)}$ is continuous, bounded and increasing on (0, π). Note that $\lim_{t\to 0^+}h(t)=0$ and $\lim_{t\to\pi^-}h(t)=\frac{1}{\pi}$. Therefore, by the Second Mean Value Theorem (Theorem11), for any ε in (0, π],

$$\int_0^{\varepsilon}\left(\frac{1}{t}-\frac{1}{2\tan(\frac{1}{2}t)}\right)\sin(nt)dt = h(\varepsilon)\int_C^{\varepsilon}\sin(nt)dt\,,$$

for some C with $0 < C < \pi$,

$$= h(\varepsilon)\left[-\frac{1}{n}\cos(nt)\right]_C^{\varepsilon} = h(\varepsilon)\left(\frac{1}{n}\cos(nC)-\frac{1}{n}\cos(n\varepsilon)\right).$$

Therefore,

$$\left|\int_0^{\varepsilon}\left(\frac{1}{t}-\frac{1}{2\tan(\frac{1}{2}t)}\right)\sin(nt)dt\right| \le \frac{2}{n}\left|h(\varepsilon)\right| \le \frac{2}{n}\left|h(\pi)\right| = \frac{2}{n\pi} \le \frac{2}{\pi}\,,$$

for all n and for all ε in [0, π]. Since $\frac{2}{n\pi}\to 0$, $\int_0^{\varepsilon}\left(\frac{1}{t}-\frac{1}{2\tan(\frac{1}{2}t)}\right)\sin(nt)dt$ tends to 0 uniformly on [0, π].

Now, $\int_0^{\varepsilon}\frac{\sin(nt)}{t}dt = \int_0^{n\varepsilon}\frac{\sin(u)}{u}du$ by a change of variable. Let $G(x)=\int_0^{x}\frac{\sin(u)}{u}du$. It is well known that $G(x)\to\int_0^{\infty}\frac{\sin(u)}{u}du=\frac{\pi}{2}$ as $x\to\infty$. This means that $G(x)$ must be bounded on [0, ∞). It follows that $\int_0^{\varepsilon}\frac{\sin(nt)}{t}dt = G(n\varepsilon)$ is bounded for any ε in [0, ∞) and for all positive integer n. That is to say, $\int_0^{\varepsilon}\frac{\sin(nt)}{t}dt$ is bounded uniformly in ε in [0, ∞) and n.

Since

$$\int_0^{\varepsilon}\frac{\sin(nt)}{t}dt-\int_0^{\varepsilon}D_n^*(t)dt=\int_0^{\varepsilon}\frac{\sin(nt)}{t}dt-\int_0^{\varepsilon}\frac{\sin(nt)}{2\tan(\frac{1}{2}t)}dt$$

$$=\int_0^{\varepsilon}\left(\frac{1}{t}-\frac{1}{2\tan(\frac{1}{2}t)}\right)\sin(nt)dt$$

converges uniformly on $[0, \pi]$ to 0 as n tends to infinity, it follows that $\int_0^{\varepsilon}D_n^*(t)dt=\int_0^{\varepsilon}\frac{\sin(nt)}{2\tan(\frac{1}{2}t)}dt$ is uniformly bounded in n and ε.

Now $\int_0^{\varepsilon}D_n(t)dt-\int_0^{\varepsilon}D_n^*(t)dt=\int_0^{\varepsilon}\frac{1}{2}\cos(nt)dt=\frac{1}{2}\left[\frac{\sin(nt)}{n}\right]_0^{\varepsilon}=\frac{\sin(n\varepsilon)}{2n}$ and so

$\left|\int_0^{\varepsilon}D_n(t)dt-\int_0^{\varepsilon}D_n^*(t)dt\right|\le\frac{1}{2n}$. Hence $\int_0^{\varepsilon}D_n(t)dt-\int_0^{\varepsilon}D_n^*(t)dt\to 0$ uniformly on $[0, \pi]$ to 0 as n tends to infinity. It then follows that $\int_0^{\varepsilon}D_n(t)dt$ is uniformly bounded in n and ε.

This completes the proof.

Section C Convergence Conditions. Convergence Theorems.

We shall formulate the condition for convergence of Fourier series in terms of the Dirichlet kernel function and related much friendlier functions. To derive convergence condition for function which is of bounded variation, the following estimation result, in terms of the total variation function, is crucial in some of the proofs below.

Theorem 15. Suppose h is a function of bounded variation on $[0, 2\pi]$ such that $\lim_{x\to 0^+}h(x)=h(0+)=0$. Then for any $0\le\tau\le\ \delta\le 2\pi$, there is a constant K such that for all $\lambda > 0$,

$$\left|\int_{\tau}^{\delta}h(t)\frac{\sin(\lambda t)}{t}dt\right|\le KV_h(0,\delta)\ ,$$

where $V_h(0,x)$ is the total variation of h in the interval (0, x).

Proof. Let g: [0, 2π] → ℝ be defined by $g(u)=\begin{cases} h(u), & u>0 \\ \lim_{t\to 0^+} h(t)=0 \end{cases}$. Then g is continuous and is also of bounded variation. Note that the total variation function of h is the same as the total variation function of g. Let $P_g(x)$ denote the positive variation of g on [0, x] and $N_g(x)$ be the negative variation function of g on [0, x]. Then both $P_g(x)$ and $N_g(x)$ are non-negative increasing functions, $g(x) = P_g(x) - N_g(x)$ and the total variation function of g, $V_g[0, x]$ defined to be the total variation of g on [0, x], is equal to $P_g(x) + N_g(x)$.

Then for any $0 \le \tau \le \delta \le 2\pi$, by the Second Mean Value Theorem (Theorem 12) there exists η with $\tau < \eta < \delta$, such that

$$\int_\tau^\delta P_g(t)\frac{\sin(\lambda t)}{t}dt = P_g(\delta)\int_\eta^\delta \frac{\sin(\lambda t)}{t}dt .$$

Hence,

$$\left|\int_\tau^\delta P_g(t)\frac{\sin(\lambda t)}{t}dt\right| \le P_g(\delta)\left|\int_\eta^\delta \frac{\sin(\lambda t)}{t}dt\right| \le 2DP_g(\delta), \text{ ------- (36)}$$

where D is a bound for $G(x)=\int_0^x \frac{\sin(t)}{t}dt$ as shown in Theorem 14.

Similarly, we deduce that there exists η' with $\tau < \eta' < \delta$, such that

$$\left|\int_\tau^\delta N_g(t)\frac{\sin(\lambda t)}{t}dt\right| \le N_g(\delta)\left|\int_{\eta'}^\delta \frac{\sin(\lambda t)}{t}dt\right| \le 2DN_g(\delta). \text{ -------- (37).}$$

Consequently, using (36) and (37), we have

$$\left|\int_\tau^\delta h(t)\frac{\sin(\lambda t)}{t}dt\right| = \left|\int_\tau^\delta g(t)\frac{\sin(\lambda t)}{t}dt\right| = \left|\int_\tau^\delta \left(P_g(t)-N_g(t)\right)\frac{\sin(\lambda t)}{t}dt\right|$$

$$\le \left|\int_\tau^\delta P_g(t)\frac{\sin(\lambda t)}{t}dt\right| + \left|\int_\tau^\delta N_g(t)\frac{\sin(\lambda t)}{t}dt\right|$$

$$\le 2D\left(P_g(\delta)+N_g(\delta)\right) = KV_g[0,\delta] = KV_h(0,\delta),$$

where $K = 2D$.

This completes the proof of Theorem 15.

Suppose f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$ and therefore, integrable over any finite interval. Suppose

$$T(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}A_n(x)$$

is its Fourier series and $t_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}A_k(kx)$ is the sum of its (n+1) terms.

Let c be any real number, then from (23) we have

$$t_n(x) - c = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x+u) - c\right)D_n(u)du$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(f(x+u) - c\right)D_n(u)du + \frac{1}{\pi}\int_{-\pi}^{0}\left(f(x+u) - c\right)D_n(u)du$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(f(x+u) - c\right)D_n(u)du - \frac{1}{\pi}\int_{\pi}^{0}\left(f(x-s) - c\right)D_n(-s)ds$$, by a change of variable,

$$= \frac{1}{\pi}\int_0^{\pi}\left(f(x+u) - c\right)D_n(u)du + \frac{1}{\pi}\int_0^{\pi}\left(f(x-u) - c\right)D_n(-u)du$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(f(x+u) - c\right)D_n(u)du + \frac{1}{\pi}\int_0^{\pi}\left(f(x-u) - c\right)D_n(u)du,$$

since $D_n(u)$ is an even function,

$$= \frac{1}{\pi}\int_0^{\pi}\left(f(x+u) + f(x-u) - 2c\right)D_n(u)du$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(f(x+u) + f(x-u) - 2c\right)\frac{\sin\left((n+\frac{1}{2})u\right)}{2\sin(\frac{1}{2}u)}du. \quad \text{------------------------- (38).}$$

For a fixed c and x, let $g_c(t) = \frac{1}{2}\left(f(x+t) + f(x-t) - 2c\right)$

Then from (38) we obtain

$$t_n(x) - c = \frac{1}{\pi}\int_0^{\pi}g_c(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du. \quad \text{---------------------- (39)}$$

Hence a necessary and sufficient condition for the Fourier series at x to converge to c is

$$\frac{1}{\pi}\int_0^{\pi} g_c(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0 \text{ or, equivalently, } \int_0^{\pi} g_c(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0.$$

It follows that

$$t_n(x) \to c \Leftrightarrow \int_0^{\pi} g_c(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0 \text{ . --------- (40)}$$

Now, for $0 < \delta < \pi$,

$$\int_0^{\pi} g_c(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du = \int_0^{\delta} g_c(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du + \int_{\delta}^{\pi} g_c(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \text{ . ------- (41)}$$

For a fixed δ with $0 < \delta < \pi$,

$$\int_{\delta}^{\pi} g_c(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du = \int_{\delta}^{\pi} g_c(u)\cot(\tfrac{1}{2}u)\sin(nu)du + \int_{\delta}^{\pi} g_c(u)\cos(nu)du \text{ .-------- (42)}$$

By the Second Mean Value Theorem (Theorem 11), for some $\alpha(x)$ with $\delta < \alpha(x) < \pi$,

$$\int_{\delta}^{\pi} g_c(u)\cot(\tfrac{1}{2}u)\sin(nu)du = \cot(\tfrac{1}{2}\delta)\int_{\delta}^{\alpha(x)} g_c(u)\sin(nu)du \text{ . ----------------- (43)}$$

Now by the Riemann Lebesgue Theorem (Theorem 13),

$$\int_{\delta}^{\alpha(x)} g_c(u)\cos(nu)du \to 0 \text{ and } \int_{\delta}^{\pi} g_c(u)\sin(nu)du \to 0$$

boundedly and uniformly in n and x.

It follows then by using (42) and (43) that for any arbitrary δ with $0 < \delta < \pi$,

$\int_{\delta}^{\pi} g_c(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0$ as n tends to infinity.

Thus, we have proved the following theorem.

Theorem 16. A necessary and sufficient condition for the Fourier series $T(x)$ of the function f to converge at x to the value c is that there exists a fixed δ such that $0 < \delta < \pi$ and

$$\int_0^\delta g_c(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0\,.$$

Theorem 16 is about convergence at a point.

Now we investigate condition for bounded convergence and uniform convergence.

Let *E* be a subset of $[-\pi, \pi]$ and suppose *c* is a function defined on *E*. The Fourier series at each point *x* of *E* converges to *c*(*x*) if, and only if, there exists δ with $0 < \delta < \pi$, $\int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0$ pointwise on *E*.

Suppose now *c*(*x*) is bounded on *E* by *M*, i.e., $|c(x)| < M$ for all *x* in *E*. For all *x* in *E*,

$$t_n(x) \to c(x) \Leftrightarrow \int_0^\pi g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0. \quad \text{------------------ (44)}$$

Following (41) we obtain

$$\int_0^\pi g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du = \int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du + \int_\delta^\pi g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du\,.$$

-------------- (45)

We shall show next that the second integral on the right of (45) for a fixed $\delta > 0$ tends to 0 uniformly in *x* for *x* in *E*.

From (42) we get

$$\int_\delta^\pi g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du = \int_\delta^\pi g_{c(x)}(u)\cot(\tfrac{1}{2}u)\sin(nu)du + \int_\delta^\pi g_{c(x)}(u)\cos(nu)du\,. \quad \text{---- (46)}$$

The second integral in the right-hand side of (46),

$$\int_\delta^\pi g_{c(x)}(u)\cos(nu)du = \frac{1}{2}\int_\delta^\pi \left(f(x+u)+f(x-u)-2c(x)\right)\cos(nu)du$$

$$= \frac{1}{2}\int_\delta^\pi \left(f(x+u)+f(x-u)\right)\cos(nu)du - c(x)\int_\delta^\pi \cos(nu)du$$

$$= \frac{1}{2}\int_{\delta}^{\pi}\left(f(x+u)+f(x-u)\right)\cos(nu)du + \frac{c(x)}{n}\sin(n\delta).$$

Therefore,

$$\left|\int_{\delta}^{\pi} g_{c(x)}(u)\cos(nu)du\right| \le \frac{1}{2}\left|\int_{\delta}^{\pi}\left(f(x+u)+f(x-u)\right)\cos(nu)du\right| + \frac{1}{n}|c(x)|$$

$$\le \frac{1}{2}\left|\int_{\delta}^{\pi}\left(f(x+u)+f(x-u)\right)\cos(nu)du\right| + \frac{1}{n}M. \quad \text{---------------- (47)}$$

Moreover, $\left|\int_{\delta}^{\pi} g_{c(x)}(u)\cos(nu)du\right| \le \frac{1}{2}\int_{\delta}^{\pi}\left(|f(x+u)|+|f(x-u)|\right)du + \frac{1}{n}M \le \int_{0}^{\pi}|f(u)|du + M$.

Now by Riemann Lebesgue Theorem (Theorem 13),

$\left|\int_{\delta}^{\pi}\left(f(x+u)+f(x-u)\right)\cos(nu)du\right| \to 0$ as n tends to ∞ uniformly and boundedly in δ and x and so for a fixed $0 < \delta < \pi$,

$$\left|\int_{\delta}^{\pi}\left(f(x+u)+f(x-u)\right)\cos(nu)du\right| \to 0 \text{ uniformly and boundedly on } E.$$

-------------- (48)

It follows from (47) that for a fixed $0 < \delta < \pi$,

$$\int_{\delta}^{\pi} g_{c(x)}(u)\cos(nu)du \to 0 \text{ uniformly and boundedly on } E. \quad \text{---------- (49)}$$

For the first integral in the right-hand side of (46) following (43), we have

$$\int_{\delta}^{\pi} g_{c(x)}(u)\cot(\tfrac{1}{2}u)\sin(nu)du = \cot(\tfrac{1}{2}\delta)\int_{\delta}^{\alpha(x)} g_{c(x)}(u)\sin(nu)du, \quad \text{---------- (50)}$$

where $\delta < \alpha(x) < \pi$.

We can prove as above by invoking the Riemann Lebesgue Theorem (Theorem 11) that for a fixed $0 < \delta < \pi$, since $c(x)$ is bounded on E,

$$\int_{\delta}^{\alpha(x)} g_{c(x)}(u)\sin(nu)du \to 0 \quad \text{uniformly and boundedly on } E.$$

Thus, from (50) we see that

$$\int_{\delta}^{\pi} g_{c(x)}(u)\cot(\tfrac{1}{2}u)\sin(nu)du \to 0 \text{ uniformly and boundedly on } E.$$

Therefore, it follows from (46) that for a fixed $0 < \delta < \pi$,

$$\int_\delta^\pi g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0 \text{ uniformly and boundedly on } E. \quad \text{-------} \ (51)$$

It then follows from (45) and (51) that if $c(x)$ is bounded on E the Fourier series converges boundedly on E if, and only if, $\int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0$ boundedly on E and it converges uniformly on E if $\int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0$ uniformly on E. We have thus proved the following theorem.

Theorem 17. Let E be a subset of $[-\pi, \pi]$. Let c: $E \to \mathbb{R}$ be a finite function.

(i) A necessary and sufficient condition for the Fourier series $T(x)$ of the function f to converge pointwise to $c(x)$ on E is that there exists a fixed δ such that $0 < \delta < \pi$ and

$$\int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0 \text{ pointwise on } E;$$

(ii) Suppose c is bounded on E. A necessary and sufficient condition for the Fourier series $T(x)$ of the function f to converge boundedly to $c(x)$ on E is that there exists a fixed δ such that $0 < \delta < \pi$ and

$$\int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0 \text{ boundedly on } E;$$

(iii) Suppose c is bounded on E. A necessary and sufficient condition for the Fourier series $T(x)$ of the function f to converge uniformly to $c(x)$ on E is that there exists a fixed δ such that $0 < \delta < \pi$ and

$$\int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du \to 0 \text{ uniformly on } E.$$

Our next goal will be to obtain simpler integrand for the integral in the last theorem. We shall do this in stages.

We would like to replace the integrand $g_{c(x)}(u)\dfrac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}$ by $g_{c(x)}(u)\dfrac{\sin\left((n+\frac{1}{2})u\right)}{u}$. Notice that the convergence behaviour of $\int_0^\delta g_{c(x)}(u)\dfrac{\sin\left((n+\frac{1}{2})u\right)}{\sin(\frac{1}{2}u)}du$ is the same as that of $\int_0^\delta g_{c(x)}(u)\dfrac{\sin\left((n+\frac{1}{2})u\right)}{2\sin(\frac{1}{2}u)}du=\int_0^\delta g_{c(x)}(u)D_n(u)du$.

Now,

$$\int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{2\sin(\frac{1}{2}u)}du-\int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{u}du$$

$$=\int_0^\delta g_{c(x)}(u)\left(\frac{1}{2\sin(\frac{1}{2}u)}-\frac{1}{u}\right)\sin\left((n+\tfrac{1}{2})u\right)du\,. \quad \text{------------------------ (52)}$$

Let h: $(0, \pi] \to \mathbf{R}$ be defined by $h(u)=\dfrac{1}{2\sin(\frac{1}{2}u)}-\dfrac{1}{u}$. Then for $0 < u < \pi$,

$$h'(u)=\frac{1}{u^2}-\frac{\cos(\frac{u}{2})}{4\sin^2(\frac{u}{2})}=\frac{4\sin^2(\frac{u}{2})-u^2\cos(\frac{u}{2})}{\sin^2(\frac{u}{2})u^2}$$

$$=\frac{16\tan^2(\frac{u}{4})-u^2+u^2\tan^4(\frac{u}{4})}{4(1+\tan^2(\frac{u}{4}))^2\sin^2(\frac{u}{2})u^2}>0, \quad \text{------------------ (53)}$$

by expressing sin(u/2) and cos(u/2) in terms of tan(u/4) and the fact that for $0 < u < \pi$, $2\tan(\frac{u}{2})>u$ so that $16\tan(\frac{u}{4})^2>u^2$. Note that

$$\lim_{u\to 0^+}h(u)=\lim_{u\to 0^+}\frac{u-2\sin(\frac{u}{2})}{2u\sin(\frac{u}{2})}=0 \text{ by L' Hôpital's Rule.}$$

Hence, h is strictly increasing on $[0, \pi]$, if we defined $h(0)$ to be equal to 0. By the Second Mean Value Theorem (Theorem11) (taking $A\le\lim_{u\to 0^+}h(u)=\lim_{u\to 0^+}\dfrac{u-2\sin(\frac{u}{2})}{2u\sin(\frac{u}{2})}=0$ and $B=1>h(\pi)=\dfrac{1}{2}-\dfrac{1}{\pi}$),

$$\int_0^\delta g_{c(x)}(u)\left(\frac{1}{2\sin(\frac{1}{2}u)}-\frac{1}{u}\right)\sin\left((n+\tfrac{1}{2})u\right)du=\int_{\beta(x)}^\delta g_{c(x)}(u)\sin\left((n+\tfrac{1}{2})u\right)du \quad \text{---- (54)}$$

for some $0 < \beta(x) < \delta$.

Now, $\int_{\beta(x)}^{\delta} g_{c(x)}(u)\sin\left((n+\frac{1}{2})u\right)du$

$$= \int_{\beta(x)}^{\delta} g_{c(x)}(u)\sin(nu)\cos\left(\tfrac{1}{2}u\right)du + \int_{\beta(x)}^{\delta} g_{c(x)}(u)\cos(nu)\sin\left(\tfrac{1}{2}u\right)du . \text{ ------- (55)}$$

By the Second Mean Value Theorem (Theorem11) (taking $A = 1 \geq \cos\left(\frac{1}{2}\beta(x)\right) \geq \cos(\delta) \geq 0 = B$),

$$\int_{\beta(x)}^{\delta} g_{c(x)}(u)\sin(nu)\cos\left(\tfrac{1}{2}u\right)du = \int_{\beta(x)}^{\beta''(x)} g_{c(x)}(u)\sin(nu)\,du \text{ ---------------- (56)}$$

and (taking $A = 0 \leq \sin\left(\frac{1}{2}\beta(x)\right) \leq \sin(\delta) \leq 1 = B$)

$$\int_{\beta(x)}^{\delta} g_{c(x)}(u)\cos(nu)\sin\left(\tfrac{1}{2}u\right)du = \int_{\beta'(x)}^{\delta} g_{c(x)}(u)\cos(nu)\,du \text{ , ---------------- (57))}$$

for some $\beta(x) < \beta'(x), \beta''(x) < \delta$.

Therefore, by the Riemann Lebesgue Theorem, for a fixed $\delta > 0$, it follows from (54), (55), (56) and (57) that

$$\int_0^{\delta} g_{c(x)}(u)\left(\frac{1}{2\sin(\frac{1}{2}u)} - \frac{1}{u}\right)\sin\left((n+\tfrac{1}{2})u\right)du \to 0 \text{ pointwise on } E$$

and if $c(x)$ is bounded on E,

$$\int_0^{\delta} g_{c(x)}(u)\left(\frac{1}{2\sin(\frac{1}{2}u)} - \frac{1}{u}\right)\sin\left((n+\tfrac{1}{2})u\right)du \to 0 \text{ uniformly and boundedly on } E.$$

It follows from (52) that convergence of the Fourier series $T(x)$ on E is equivalent to the convergence of the integral $\int_0^{\delta} g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{u}du$ to 0.

Hence, we have proved the next equivalent condition for convergence.

Theorem 18. Let E be a subset of $[-\pi, \pi]$. Let c: $E \to \mathbf{R}$ be a finite function.

(i) A necessary and sufficient condition for the Fourier series $T(x)$ of the function f to converge pointwise to $c(x)$ on E is that there exists a fixed δ such that $0 < \delta < \pi$ and

$$\int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{u}du \to 0 \text{ pointwise on } E;$$

(ii) Suppose c is bounded on E. A necessary and sufficient condition for the Fourier series $T(x)$ of the function f to converge boundedly to $c(x)$ on E is that there exists a fixed δ such that $0 < \delta < \pi$ and

$$\int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{u}du \to 0 \text{ boundedly on } E;$$

(iii) Suppose c is bounded on E. A necessary and sufficient condition for the Fourier series $T(x)$ of the function f to converge uniformly to $c(x)$ on E is that there exists a fixed δ such that $0 < \delta < \pi$ and

$$\int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{u}du \to 0 \text{ uniformly on } E.$$

Now consider a simpler integrand for the convergence problem, namely $g_{c(x)}(u)\frac{\sin(nu)}{u}$. We examine the difference in the same manner as above.

$$\int_0^\delta g_{c(x)}(u)\frac{\sin\left((n+\frac{1}{2})u\right)}{u}du - \int_0^\delta g_{c(x)}(u)\frac{\sin(nu)}{u}du$$

$$= \int_0^\delta g_{c(x)}(u)\left(\frac{\sin\left((n+\frac{1}{2})u\right)}{u} - \frac{\sin(nu)}{u}\right)du$$

$$= \int_0^\delta g_{c(x)}(u)\left(2\frac{\cos((n+\frac{1}{4})u)\sin(\frac{1}{4}u)}{u}\right)du = 2\int_0^\delta g_{c(x)}(u)\left(\frac{\sin(\frac{1}{4}u)}{u}\right)\cos((n+\frac{1}{4})u)du\,.$$

---------------- (58)

Now for $0 < u < \pi$, $\frac{d}{du}\frac{\sin(\frac{1}{4}u)}{u} = \frac{\cos(\frac{u}{4})\left(\frac{u}{4} - \tan\frac{u}{4}\right)}{u^2} < 0$ since $\tan(\frac{u}{4}) > \frac{u}{4}$ for $0 < u < \pi$.

Thus, $\frac{\sin(\frac{1}{4}u)}{u}$ is non-negative and decreasing on $(0, \pi]$. Note that $\lim_{u\to 0^+}\frac{\sin(\frac{1}{4}u)}{u} = \frac{1}{4}$. Therefore, by the Generalized Second Mean Value Theorem (Theorem11), (taking $A = ½$ and $B = 0$), there exists $0 < \delta'(x) < \delta$ such that

$$\int_0^{\delta} g_{c(x)}(u)\left(2\frac{\cos((n+\frac{1}{4})u)\sin(\frac{1}{4}u)}{u}\right)du = \int_0^{\delta'(x)} g_{c(x)}(u)\cos((n+\tfrac{1}{4})u)du \text{ . ------------- (59)}$$

As before as for the case of (55), by the Riemann Lebesgue Theorem, $\int_0^{\delta'(x)} g_{c(x)}(u)\cos((n+\frac{1}{4})u)du$ converges to 0 pointwise and if $c(x)$ is bounded on *E,* it converges to 0, boundedly and uniformly on *E*. It follows then from (58) and (59) that the same holds true for $\int_0^{\delta} g_{c(x)}(u)\frac{\sin((n+\frac{1}{2})u)}{u}du - \int_0^{\delta} g_{c(x)}(u)\frac{\sin(nu)}{u}du$.

As a consequence, we have the next convergence theorem.

Theorem 19. Let E be a subset of $[-\pi, \pi]$. Let $c: E \to R$ is a finite function.

(i) A necessary and sufficient condition for the Fourier series $T(x)$ of the function f to converge pointwise to $c(x)$ on E is that there exists a fixed δ such that $0 < \delta < \pi$ and

$$\int_0^{\delta} g_{c(x)}(u)\frac{\sin(nu)}{u}du \to 0 \text{ pointwise on } E;$$

(ii) Suppose c is bounded on E. A necessary and sufficient condition for the Fourier series $T(x)$ of the function f to converge boundedly to $c(x)$ on E is that there exists a fixed δ such that $0 < \delta < \pi$ and

$$\int_0^{\delta} g_{c(x)}(u)\frac{\sin(nu)}{u}du \to 0 \text{ boundedly on } E;$$

(iii) Suppose c is bounded on E. A necessary and sufficient condition for the Fourier series $T(x)$ of the function f to converge uniformly to $c(x)$ on E is that there exists a fixed δ such that $0 < \delta < \pi$ and

$$\int_0^{\delta} g_{c(x)}(u)\frac{\sin(nu)}{u}du \to 0 \text{ uniformly on } E.$$

A summary of the above theorems is as follows.

Theorem 20. Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Let $t_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} A_k(kx)$ be the $(n+1)$ partial sum of its Fourier series $T(x)$. Then the behaviour of $t_n(x)$ for large n depends on the behaviour of $f(u)$ for values of u in $(x-\delta, x+\delta)$.

Proof. Taking $c = 0$, by Theorem 19, the behaviour of $t_n(x)$ for large n depends on the behaviour of $g(u) = \frac{1}{2}\left(f(x+u) + f(x-u)\right)$ which obviously involves values of f in $(x-\delta, x+\delta)$.

Theorem 21. Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Suppose at x, $\lim_{t \to x^+} f(t) = f(x+)$ and $\lim_{t \to x^-} f(t) = f(x-)$. Let $c = \frac{1}{2}\left(f(x+) + f(x-)\right)$. Then a necessary and sufficient condition for $t_n(x)$ to converge to c is that there exists a δ with $0 < \delta < \pi$, $\int_0^\delta g_c(u) \frac{\sin(nu)}{u} du \to 0$, where $g_c(u) = \frac{1}{2}\left(f(x+u) + f(x-u) - 2c\right)$.

Proof. Immediate from Theorem 19.

We can formulate a version of Theorem 21 in terms of the behaviour of $g_0(u)$ below.

Theorem 22. Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Let x be in $(-\pi, \pi)$. Suppose the function $\phi(u) = \frac{1}{2}\left(f(x+u) + f(x-u)\right)$ is of bounded variation in $(0, \eta)$. Let $\lim_{u \to 0^+} \phi(u) = \phi(0+)$. Then $t_n(x) \to \phi(0+)$.

Proof. Suppose ϕ is of bounded variation in the interval $(0, \eta)$. Then $\lim_{t \to 0^+} \phi(t) = \phi(0+)$ exists. By Theorem 19, $t_n(x) \to \phi(0+)$ if and only if $\int_0^\delta \left(\phi(u) - \phi(0+)\right) \frac{\sin(nu)}{u} du \to 0$.

$$\int_0^{\delta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du=\int_0^{\eta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du+\int_{\eta}^{\delta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du .$$

And so,

$$\left|\int_0^{\delta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du\right|$$

$$\leq\left|\int_0^{\eta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du\right|+\left|\int_{\eta}^{\delta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du\right| . \text{ -------- } (60)$$

By Theorem 15,

$$\left|\int_0^{\eta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du\right|\leq KV_{\phi}(0,\eta) , \text{ ----------------- } (61)$$

where $V_{\phi}(0,\eta)$ is the total variation of ϕ on the open interval (0, δ). Note that the total variation $V_{\phi}(0,\delta)$ is equal to the total variation of $h(u)=\begin{cases}\phi(u)-\phi(0+), & u>0\\ 0, & u=0\end{cases}$, which is continuous at 0 and so $\lim_{u\to 0^+}V_{\phi}(0,u)=\lim_{u\to 0^+}V_h[0,u]=0$. Hence, given any $\varepsilon > 0$, there exists $\tau > 0$ such that $0<\eta<\tau \Rightarrow V_{\phi}(0,\eta)<\frac{\varepsilon}{2K}$. Thus, for all $0 < \eta < \tau$ and $0 < \eta < \delta$, we have then from (60) and (61), that for any integer $n > 0$,

$$\left|\int_0^{\delta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du\right|\leq\frac{\varepsilon}{2}+\left|\int_{\eta}^{\delta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du\right| . \text{ -------- } (62)$$

But by the Riemann Lebesgue Theorem, $\left|\int_{\eta}^{\delta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du\right|\to 0$ as n tends to ∞, and so there exists N such that $n \geq N$ implies that

$$\left|\int_{\eta}^{\delta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du\right|<\frac{\varepsilon}{2} . \text{ ----------------- } (63)$$

It follows from (62) and (63), that for $n \geq N$,

$$\left|\int_0^{\delta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon .$$

This means $\int_0^{\delta}(\phi(u)-\phi(0+))\frac{\sin(nu)}{u}du \to 0$ and so $t_n(x)\to\phi(0+)$.

Convergence Theorems

We now expand our domain of convergence,

Theorem 23. Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Suppose f is of bounded variation in $(a, b) \subseteq [-\pi, \pi]$. Then its Fourier series $T(x)$ converges boundedly to $\frac{1}{2}\left(f(x+)+f(x-)\right)$ in any closed interval $[a',b']$ in (a, b). If in addition, f is also continuous on (a, b), then the series $T(x)$ converges uniformly on $[a',b']$.

Proof. Since $[a',b']\subseteq(a,b)$, there exists a $\delta > 0$ so that for any x in $[a',b']$, $[x-\delta, x+\delta]\subseteq(a,b)$. Since f is of bounded variation in (a, b), the left and right limits, $f(x-)$ and $f(x+)$, exist at each point x in (a, b).

Let $c(x)=\frac{1}{2}\left(f(x+)+f(x-)\right)$ for x in (a, b). Then $c(x)$ is bounded on (a, b) since f is bounded on (a, b).

Let $\phi(u)=\frac{1}{2}\left(f(x+u)+f(x-u)\right)$. For x in (a, b),

$$\phi(x+)=\lim_{u\to 0^+}\phi(u)=\frac{1}{2}\left(f(x+)+f(x-)=c(x)\right).$$

Then by Theorem 22, $T(x)$ converges pointwise to $c(x)$ for any x in (a, b).

We next show that the convergence is boundedly on $[a',b']$.

For each x in (a, b), define

$$f_x^+(u)=f(x+u)-f(x+) \text{ and } f_x^-(u)=f(x-u)-f(x-). \text{ ----- (64)}$$

Then $f_x^+(u)=f(x+u)-f(x+)$ and $f_x^-(u)=f(x-u)-f(x-)$ are of bounded variation in $[x-\delta, x+\delta]\subseteq(a,b)$.

By Theorem 19, $T(x)$ converges to $c(x)$ boundedly on $[a',b']$ if, and only if, $\int_0^\delta g_{c(x)}(u)\frac{\sin(nu)}{u}du \to 0$ boundedly on $[a',b']$, where

$$g_{c(x)}(u)=\frac{1}{2}\left(f(x+u)+f(x-u)-2c(x)\right)=\frac{1}{2}\left(f_x^+(u)+f_x^-(u)\right) \text{ ----- (65)}$$

for $0 < u < \delta$ is of bounded variation on [0, δ]. In particular, the total variation of $g_{c(x)}(u)$ in (0, δ), $V_{g_{c(x)}}(0,\delta)$ satisfies

$$V_{g_{c(x)}}(0,\delta)\le\frac{1}{2}\left(V_{f_x^+}(0,\delta)+V_{f_x^-}(0,\delta)\right)\le\frac{1}{2}\left(V_f(a,b)+V_f(a,b)\right)=V_f(a,b), \text{ --- (66)}$$

where $V_{f_x^+}(0,\delta)$ and $V_{f_x^-}(0,\delta)$ are respectively the total variations of $f_x^+(u)$ and $f_x^-(u)$ in (0, δ) and $V_f(a,b)$ is the total variations of f in (a, b).

We already knew that $\int_0^\delta g_{c(x)}(u)\frac{\sin(nu)}{u}du \to 0$ pointwise on $[a',b']$. By Theorem 15, for all integer $n > 0$,

$$\int_0^\delta g_{c(x)}(u)\frac{\sin(nu)}{u}du \le V_{g_{c(x)}}(0,\delta)K \le V_f(a,b)K. \qquad \text{--------------- (67)}$$

This shows that $\int_0^\delta g_{c(x)}(u)\frac{\sin(nu)}{u}du$ is uniformly bounded and so the convergence is boundedly.

Suppose now that f is also continuous in (a, b). Then $c(x) = f(x)$ for all x in (a, b) and $\int_0^\delta g_{c(x)}(u)\frac{\sin(nu)}{u}du \to 0$ pointwise on $[a',b']$. Let a'' be such that $a < a'' < a'$. Since f is continuous and of bounded variation on (a, b), the function $T_f(x)=V_f[a'',x]$ for x in (a, b) is continuous on (a, b) and so is continuous on $[a',b']$. Therefore, $T_f(x)$ is uniformly continuous on $[a',b']$.

Since $g_{c(x)}(u)=\frac{1}{2}\left(f(x+u)+f(x-u)-2f(x)\right)$, the total variation of $g_{c(x)}(u)$ on (0, δ) is less than or equal to half the total variation of f on (x, $x+\delta$) + the total variation of f on ($x-\delta$, x), i.e.,

$$V_{g_{c(x)}}(0,\delta)\le\frac{1}{2}\left(V_f(x,x+\delta)+V_f(x-\delta,x)\right)$$

$$\le \frac{1}{2}\left(T_f(x+\delta)-T_f(x)+T_f(x)-T_f(x-\delta)\right) \text{ ----------------- (68)}$$

Since $T_f(x)$ is uniformly continuous on $[a',b']$, given $\varepsilon > 0$, there exists $\alpha > 0$ such that for all x in $[a',b']$, $|t| < \alpha$ implies $\left|T_f(x+t)-T_f(x)\right| < \frac{\varepsilon}{2K}$. Therefore, for all x in $[a',b']$, for any $0 < \delta' < \alpha$,

$$V_{g_{c(x)}}(0,\delta') \le \frac{1}{2}\left(\frac{\varepsilon}{2K}+\frac{\varepsilon}{2K}\right) = \frac{\varepsilon}{2K}. \qquad \text{----------------- (69)}$$

Choose a τ such that $0 < \tau < \alpha$ and $0 < \tau < \delta$. Then

$$\int_0^\delta g_{c(x)}(u)\frac{\sin(nu)}{u}du = \int_0^\tau g_{c(x)}(u)\frac{\sin(nu)}{u}du + \int_\tau^\delta g_{c(x)}(u)\frac{\sin(nu)}{u}du. \text{ -------- (70)}$$

By Theorem 15, for all integer $n > 0$ and for all x in $[a',b']$,

$$\left|\int_0^\tau g_{c(x)}(u)\frac{\sin(nu)}{u}du\right| \le V_{g_{c(x)}}(0,\tau)K \le \frac{\varepsilon}{2K}K = \frac{\varepsilon}{2}, \text{ ---------------- (71)}$$

by using (67).

Now, we claim that $\int_\tau^\delta g_{c(x)}(u)\frac{\sin(nu)}{u}du$ converges to 0 uniformly and boundedly on $[a',b']$. By the Mean Value Theorem (Theorem 11), we get

$$\int_\tau^\delta g_{c(x)}(u)\frac{\sin(nu)}{u}du = \frac{1}{\tau}\int_\tau^{\gamma(x)} g_{c(x)}(u)\sin(nu)du \qquad \text{-------------------- (72)}$$

for some $\tau < \gamma(x) < \delta$.

Since $c(x)$ is bounded on E, by the Riemann Lebesgue Theorem, $\int_\tau^{\gamma(x)} g_{c(x)}(u)\sin(nu)du$ tends to 0 boundedly and uniformly on $[a',b']$.

Hence from (72),

$$\frac{1}{\tau}\int_\tau^{\gamma(x)} g_{c(x)}(u)\sin(nu)du \to 0. \text{ -------------------- (73)}$$

Therefore, by (72) and (73),

$$\int_{\tau}^{\delta} g_{c(x)}(u)\frac{\sin(nu)}{u}du \to 0 \text{ ---------------------------- (74)}$$

boundedly and uniformly on $[a',b']$.

Hence, given $\varepsilon > 0$, there exists integer N such that for all x in $[a',b']$,

$$n \geq N \Rightarrow \left|\int_{\tau}^{\delta} g_{c(x)}(u)\frac{\sin(nu)}{u}du\right| < \frac{\varepsilon}{2}. \text{ ------------------- (75)}$$

It then follows from (68), (69) and (75) that for all x in $[a',b']$,

$$n \geq N \Rightarrow \left|\int_{0}^{\delta} g_{c(x)}(u)\frac{\sin(nu)}{u}du\right|$$

$$\leq \left|\int_{0}^{\tau} g_{c(x)}(u)\frac{\sin(nu)}{u}du\right| + \left|\int_{\tau}^{\delta} g_{c(x)}(u)\frac{\sin(nu)}{u}du\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

This means $\int_{0}^{\delta} g_{c(x)}(u)\frac{\sin(nu)}{u}du \to 0$ uniformly on $[a',b']$.

Therefore, by Theorem 19, the Fourier series $T(x)$ converges uniformly to $f(x)$ on $[a',b']$.

This completes the proof of Theorem 23.

Remark. The proof above also shows that if $c(x)$ is bounded, then

$\int_{\tau}^{\delta} g_{c(x)}(u)\frac{\sin(nu)}{u}du \to 0$ uniformly and boundedly.

If we take the interval (a, b) in Theorem 23 to be $[-\pi, \pi]$, then we have immediately the following:

Theorem 24. Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Suppose f is of bounded variation in $[-\pi, \pi]$. Then its Fourier series $T(x)$ converges boundedly to $\frac{1}{2}\left(f(x+)+f(x-)\right)$ in $[-\pi, \pi]$. If, in addition, f is also continuous on $[-\pi, \pi]$, then the series $T(x)$ converges uniformly to $f(x)$ on $[-\pi, \pi]$.

Proof. Note that it is not essential that $(a, b) \subseteq [-\pi, \pi]$ in Theorem 23. One can replace the interval in Theorem 23 simply by a bounded interval (a, b) in **R**. Therefore, if f is of bounded variation on $[-\pi, \pi]$, then f is of bounded variation on $[-2\pi, 2\pi]$, and we can take (a, b) to be any open interval containing $[-2\pi, 2\pi]$. Theorem 24 is then a corollary of Theorem 23.

We next describe a test of Dini.

Theorem 25 (Dini Test). Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Let c be a real number and $x \in [-\pi, \pi]$. Let $g_c(u) = \frac{1}{2}\left(f(x+u) + f(x-u) - 2c\right)$. If $\frac{g_c(u)}{u}$ is Lebesgue integrable, then the Fourier series of f at x, $T(x)$, converges to c.

If f has a jump discontinuity at x and $\frac{g_c(u)}{u}$ is Lebesgue integrable, then c is necessarily equal to $\frac{1}{2}\left((f(x+) + f(x-)\right)$.

Proof. By Theorem 19, $T(x)$ converges to c if, and only if, for some $0 < \delta < \pi$,

$$\int_0^\delta g_c(u)\frac{\sin(nu)}{u}du \to 0.$$

But by the Riemann Lebesgue Theorem, since $\frac{g_c(u)}{u}$ is Lebesgue integrable, $\int_0^\delta g_c(u)\frac{\sin(nu)}{u}du \to 0$ and so the theorem follows.

If f has a jump discontinuity at x, then by Féjer's Theorem (Theorem 12 in *Fourier Cosine and Sine Series or* Theorem 34 this note), the $(C,1)$ mean of the Fourier series at x converges to $\frac{1}{2}\left((f(x+) + f(x-)\right)$. Since the Fourier series converges at x to c, c must be equal to $\frac{1}{2}\left((f(x+) + f(x-)\right)$ by the regularity of $(C,1)$ convergence.

Sectionally Continuous Function

A function f : $[a, b] \to \mathbf{R}$ is said to be *sectionally continuous* if there is a *finite* partition, $a = x_0 < x_1 < \cdots < x_n = b$ such that f is bounded and continuous on each open interval (x_{i-1}, x_i), $i = 1,2, \ldots, n$.

Therefore, a sectionally continuous function on $[a, b]$ can have at most a finite number of discontinuity. Note that they need not be jump discontinuity.

Lemma 26. Suppose at x_0 f is differentiable in an open neighbourhood containing x_0, except possibly at x_0 , i.e., there exists a $\delta > 0$ such that f is differentiable in $(x_0 - \delta, x_0 + \delta) - \{ x_0 \}$. Suppose $\lim_{x \to x_0^+} f'(x) = f'(x_0+)$ and $\lim_{x \to x_0^-} f'(x) = f'(x_0-)$ both exist and are finite. Then f is necessarily continuous at x_0 or has both left and right limits at x_0 .

Proof. Take any sequence (a_n) in $(x_0, x_0 + \delta)$ such that $a_n > x_0$ and $a_n \to x_0$ . . Since $\lim_{x \to x_0^+} f'(x) = f'(x_0+)$, there exists $\delta_1 > 0$ such that

$$x_0 < x < x_0 + \delta_1 \Rightarrow \left| f'(x) - f'(x_0+) \right| < 1. \quad \text{------------------} \quad (76)$$

There exists an integer N such that

$$n \geq N \Rightarrow | a_n - x_0 | < \frac{1}{2} \min \left\{ \delta_1, \frac{\varepsilon}{1 + \left| f'(x_0+) \right|} \right\}. \quad \text{----------------} \quad (77)$$

For all n, $m \geq N$, if $a_n \neq a_m$, by the Mean Value Theorem,

$$\frac{f(a_n) - f(a_m)}{a_n - a_m} = f'(\eta) \quad ,$$

for some η strictly between a_n and a_m.. This means $| \eta - x_0 | < \delta_1$.

Hence

$$\left| f(a_n) - f(a_m) \right| = \left| f'(\eta) \right| \left| a_n - a_m \right| \leq \left(1 + \left| f'(x_0+) \right| \right) \left| a_n - a_m \right|,$$

because $|\eta - x_0| < \delta_1$ so that by (76), $\left|f'(\eta)\right| < \left(1+\left|f'(x_0+)\right|\right)$,

$$< \left(1+\left|f'(x_0+)\right|\right)\frac{\varepsilon}{1+\left|f'(x_0+)\right|} = \varepsilon \,. \quad \text{-----------------} \quad (78)$$

If $a_n = a_m$, (78) holds true trivially.

Therefore, $(f(a_n))$ is a Cauchy sequence and so is convergent. This means $\lim_{x \to x_0^+} f(x) = f(x_0+)$ exists. Similarly, we can show that $\lim_{x \to x_0^-} f(x) = f(x_0-)$ exists. Note that even if both the left and right limits $f'(x_0+)$ and $f'(x_0-)$ at x_0 are the same, f need not be continuous at x_0.

Now we examine the local behaviour of f at such an x_0. By the hypothesis of Lemma 26, f is continuous in $(x_0, x_0 + \delta)$. Since $\lim_{x \to x_0^+} f(x)$ exists, by taking a smaller value of δ, we may assume that f is bounded in $(x_0, x_0 + \delta)$. Therefore, $g(h) = \dfrac{f(x_0+h) - f(x_0+)}{h}$ is defined and continuous on $(0, \delta)$.

Moreover, by the Mean Value Theorem, for $0 < h < \delta$,

$$\frac{f(x_0+h) - f(x_0+)}{h} = f'(x_0 + \ell) \text{ for some } 0 < \ell < h$$

and so

$$\left|\frac{f(x_0+h) - f(x_0+)}{h} - f'(x_0+)\right| = \left|f'(x_0+\ell) - f'(x_0+)\right| \,. \quad \text{---------} \quad (79)$$

Given $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$x_0 < x < x_0 + \delta_2 \Rightarrow \left|f'(x) - f'(x_0+)\right| < \varepsilon \,.$$

Thus, from (79) we have

$$0 < h < \min\{\delta_2, \delta\} \Rightarrow \left|\frac{f(x_0+h) - f(x_0+)}{h} - f'(x_0+)\right| < \varepsilon \,. \quad \text{--------} \quad (80)$$

That is to say, $\lim_{h\to 0^+}\dfrac{f(x_0+h)-f(x_0+)}{h}=f'(x_0+)$.

This means $g(h)$ is Lebesgue integrable on $(0, \delta)$.

Similarly, we can show that $\lim_{h\to 0^+}\dfrac{f(x_0-h)-f(x_0-)}{-h}=f'(x_0-)$ and $k(h)=\dfrac{f(x_0-h)-f(x_0-)}{h}$ is Lebesgue integrable on $(0, \delta)$.

It follows that

$$\frac{g_c(h)}{h}=\frac{1}{2}\big(g(h)+k(h)\big)=\frac{f(x_0+h)+f(x_0-h)-f(x_0+)-f(x_0-)}{2h}$$

with $c=\dfrac{f(x_0+)+f(x_0-)}{2}$ is Lebesgue integrable on $(0, \delta)$. Therefore, by Theorem 25, the Fourier series of f at x_0 converges to $\dfrac{f(x_0+)+f(x_0-)}{2}$.

We have thus proved the following

Theorem 27. Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Let x_0 be in $(-\pi, \pi)$. Suppose $f'(x_0+)$ and $f'(x_0-)$ exist, i.e., there exists a $\delta > 0$ such that f is differentiable in $(x_0-\delta, x_0+\delta)-\{x_0\}$ and $\lim_{x\to x_0^+} f'(x)=f'(x_0+)$ and $\lim_{x\to x_0^-} f'(x)=f'(x_0-)$ both exist and are finite. Then f necessarily has a jump discontinuity at x_0 or is continuous at x_0 and the Fourier series at x_0 converges to $\dfrac{f(x_0+)+f(x_0-)}{2}$. If f is differentiable at x_0, then the Fourier series at x_0 converges to $f(x_0)$.

Proof. We only need to prove the last statement. If f is differentiable at x_0, then $\lim_{h\to 0}\dfrac{f(x_0+h)+f(x_0-h)-2f(x_0)}{2h}=0$ and so the function $\dfrac{f(x_0+h)+f(x_0-h)-2f(x_0)}{2h}$ is Lebesgue integrable. It then follows by Theorem 25 that the Fourier series at x_0 converges to $f(x_0)$.

Corollary 28. Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Suppose f is sectionally continuous on $[-\pi, \pi]$. Let x_0 be in $(-\pi, \pi)$. Suppose the left and right limits of f at x_0 exist. If $\lim_{h\to 0^+}\frac{f(x_0+h)-f(x_0+)}{h}$ and $\lim_{h\to 0^+}\frac{f(x_0-h)-f(x_0-)}{-h}$ both exist, then the Fourier series at x_0 converges to $\frac{f(x_0+)+f(x_0-)}{2}$.

Proof. The Lebesgue integrability of $g_c(h)=\frac{f(x_0+h)+f(x_0+h)-f(x_0+)-f(x_0-)}{2h}$ is a consequence of sectional continuity and the existence of $\lim_{h\to 0^+}\frac{f(x_0+h)-f(x_0+)}{h}$ and $\lim_{h\to 0^+}\frac{f(x_0-h)-f(x_0-)}{-h}$ as in Theorem 27. The result then follows from Theorem 25.

The next result is about absolute convergence of Fourier series.

Theorem 29. Suppose the function f is periodic of period 2π and is continuous on $[-\pi, \pi]$. Suppose f' is sectionally continuous on $[-\pi, \pi]$ with jump discontinuity. That is to say, there is a *finite* partition, $-\pi = x_0 < x_1 < \cdots < x_n = \pi$ such that f is differentiable on each open interval (x_{i-1}, x_i) so that f' is continuous and bounded on (x_{i-1}, x_i), $i = 1, 2, \ldots, n$, and at each point x of discontinuity of f', both the left and right limits of f' exist. Then the Fourier series of f converges uniformly and absolutely to f on $[-\pi, \pi]$.

Proof. Since f' is continuous and bounded on (x_{i-1}, x_i) and f is continuous on $[x_{i-1}, x_i]$, by the Mean Value Theorem For Derivative, for any x, y in $[x_{i-1}, x_i]$, $|f(x)-f(y)| \le C|x-y|$, where C is a bound for f' in (x_{i-1}, x_i). It follows that f is Lipschitz on $[x_{i-1}, x_i]$ and so f is continuous and of bounded variation on $[x_{i-1}, x_i]$. Since there is only a finite number of such interval in the partition, f is continuous and of bounded variation on $[-\pi, \pi]$. Then by Theorem 24, the Fourier series of f converges uniformly to f on $[-\pi, \pi]$. More is true, since f is Lipschitz on each $[x_{i-1}, x_i]$ and there is only a finite number of these intervals in the partition, f is Lipschitz on $[-\pi, \pi]$ and so f is absolutely continuous on $[-\pi, \pi]$.

We shall show next that the Fourier series of f converges absolutely and uniformly to f on $[-\pi, \pi]$.

Here we digress to bring in *Parseval inequality* for square integrable function.

Suppose g is square integrable on $[-\pi, \pi]$, that is, g^2 is Lebesgue integrable. Since $[-\pi, \pi]$ is of finite measure, g is also Lebesgue integrable. Suppose $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$ is the Fourier series of g. Now for each integer $m \geq 1$,

$$\int_{-\pi}^{\pi}\left|g(x) - \sum_{n=1}^{m}\left(a_n\cos(nx) + b_n\sin(nx)\right)\right|^2 dx \geq 0. \quad \text{------------------------ (81)}$$

On the other hand,

$$\int_{-\pi}^{\pi}\left|g(x) - \sum_{n=1}^{m}\left(a_n\cos(nx) + b_n\sin(nx)\right)\right|^2 dx = \int_{-\pi}^{\pi}\left(g(x)\right)^2 dx + \int_{-\pi}^{\pi}\left(\sum_{n=1}^{m}\left(a_n\cos(nx) + b_n\sin(nx)\right)\right)^2 dx$$

$$-2\sum_{n=1}^{m}\int_{-\pi}^{\pi}a_n g(x)\cos(nx)dx - 2\sum_{n=1}^{m}\int_{-\pi}^{\pi}b_n g(x)\sin(nx)dx$$

$$= \int_{-\pi}^{\pi}\left(g(x)\right)^2 dx + \pi\sum_{n=1}^{m}\left(a_n^2 + b_n^2\right) - 2\sum_{n=1}^{m}a_n\pi a_n - 2\sum_{n=1}^{m}b_n\pi b_n$$

$$= \int_{-\pi}^{\pi}\left(g(x)\right)^2 dx - \pi\sum_{n=1}^{m}\left(a_n^2 + b_n^2\right).$$ Therefore, $\pi\sum_{n=1}^{m}\left(a_n^2 + b_n^2\right) \leq \int_{-\pi}^{\pi}\left(g(x)\right)^2 dx < \infty$.

It follows that $\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) < \infty$.

This means that if g is square integrable, then its Fourier coefficients satisfy

$$\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) < \infty. \quad \text{------------------------- (82)}$$

Now we return to our function f. Note that f is absolutely continuous on $[-\pi, \pi]$. Thus, we may write f as an integral, i.e.,

$$f(x)=\int_0^x f'(t)dt+f(0).$$

Then

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx=\frac{1}{\pi}\int_0^{2\pi}\left(\int_0^x f'(t)dt+f(0)\right)\cos(nx)dx$$

$$=\frac{1}{\pi}\left[\left(\int_0^x f'(t)dt+f(0)\right)\frac{\sin(nx)}{n}\right]_0^{2\pi}-\frac{1}{\pi}\int_0^{2\pi}\frac{f'(x)\sin(nx)}{n}dx$$, by integration by parts,

$$=-\frac{1}{n}B_n,$$ ------------------------------------- (83)

where $\frac{1}{2}A_0+\sum_{n=1}^{\infty}\left(A_n\cos(nx)+B_n\sin(nx)\right)$ is the Fourier series of f'.

Similarly,

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx=\frac{1}{\pi}\int_0^{2\pi}\left(\int_0^x f'(t)dt+f(0)\right)\sin(nx)dx$$

$$=\frac{1}{\pi}\left[-\left(\int_0^x f'(t)dt+f(0)\right)\frac{\cos(nx)}{n}\right]_0^{2\pi}+\frac{1}{\pi}\int_0^{2\pi}\frac{f'(x)\cos(nx)}{n}dx$$, by integration by parts,

$$=\frac{1}{\pi}\left(\frac{f(0)}{n}-\frac{f(2\pi)}{n}\right)+\frac{1}{n}A_n=\frac{1}{n}A_n.$$ -------------------------------- (84)

By the Cauchy Schwartz's inequality,

$$\sum_{k=1}^{n}\left|\frac{B_k}{k}\right|=\sum_{k=1}^{n}\sqrt{\frac{B_k^2}{k^2}}\le\left\{\left(\sum_{k=1}^{n}B_k^2\right)\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)\right\}^{1/2}.$$ -------------------- (85)

Note that since f' is sectionally continuous, it is bounded almost everywhere and so $\left(f'\right)^2$ is integrable and so its coefficients satisfy the Parseval inequality (82). It follows that $\sum_{k=1}^{\infty}B_k^2<\infty$. Since $\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty$, it follows from (85) that $\sum_{k=1}^{\infty}\left|\frac{B_k}{k}\right|<\infty$. Then by (83) $\sum_{k=1}^{\infty}\left|a_k\right|<\infty$. Similarly, we can deduce that $\sum_{k=1}^{\infty}\left|b_k\right|=\sum_{k=1}^{\infty}\left|\frac{A_k}{k}\right|<\infty$. It then follows by Weierstrass M-Test that the Fourier series of *f* converges uniformly and absolutely.

Differentiation of Fourier Series

Theorem 30. Suppose the function f is periodic of period 2π and continuous on $[-\pi, \pi]$. Suppose f' is sectionally continuous on $[-\pi, \pi]$ with jump discontinuity. Let x_0 be in $[-\pi, \pi]$. Suppose $f''(x_0)$ exists. Then the Fourier series of f may be differentiated term by term to give $f'(x_0)$ and the resultant series is the Fourier series of f' at x_0 .

Proof. Since f' is sectionally continuous on $[-\pi, \pi]$, f is absolutely continuous on $[-\pi, \pi]$. Hence, both f and f' are Lebesgue integrable on $[-\pi, \pi]$. Suppose the Fourier series of f' is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos(nx) + B_n \sin(nx)\right). \quad \text{--------------} \quad (86)$$

And the Fourier series of f is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right). \quad \text{---------------} \quad (87)$$

Note that $A_0 = \int_{-\pi}^{\pi} f'(t)dt = f(\pi) - f(-\pi) = 0$ and the coefficients of the two series are related by (83) and (84), i.e.,

$$B_n = -na_n \text{ and } A_n = nb_n. \quad \text{-----------------------} \ (88)$$

By Theorem 27, the Fourier series of f' at x_0 converges to $f'(x_0)$.

By Theorem 29, $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right) \to f(x)$. In particular,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx_0) + b_n \sin(nx_0)\right) \to f(x_0).$$

If we differentiate (87) term by term we get (86) on account of (88).

This completes the proof.

Theorem 31. Suppose the function f is periodic of period 2π and is absolutely continuous on $[-\pi, \pi]$. Then f and f' are Lebesgue integrable on $[-\pi, \pi]$. If $f \sim (a_n, b_n)$, then $f' \sim (nb_n, -na_n)$.

Proof. Immediate from the proof of Theorem 29.

(C,1) Summability of Fourier Series.

Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Suppose $t_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} A_k(x)$ is sum of the $(n+1)$ terms of the Fourier series of f. Then the $(C,1)$ mean of the Fourier series is

$$\sigma_{n+1}(x) = \frac{1}{n+1}\left(t_0(x) + t_1(x) + \cdots + t_n(x)\right).$$

Recall from (10) that

$$\sigma_{n+1}(x) = \frac{1}{n+1}\left(t_0(x) + t_1(x) + \cdots + t_n(x)\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\frac{1}{n+1}\sum_{k=0}^{n} D_k(u)du$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)K_n(u)du,$$

where $K_n(u) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(u) = \frac{2}{n+1}\left\{\frac{\sin(\frac{1}{2}(n+1)x)}{2\sin(\frac{1}{2}x)}\right\}^2$. (See (15).)

Let c be a real number. Then recall from (24)

$$\sigma_{n+1}(x) - c = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x+u) - c\right)K_n(u)du$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left(f(x+u) - c\right)K_n(u)du + \frac{1}{\pi}\int_{-\pi}^{0}\left(f(x+u) - c\right)K_n(u)du$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left(f(x+u) - c\right)K_n(u)du - \frac{1}{\pi}\int_{\pi}^{0}\left(f(x-s) - c\right)K_n(-s)ds$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left(f(x+u) - c\right)K_n(u)du + \frac{1}{\pi}\int_{0}^{\pi}\left(f(x-u) - c\right)K_n(u)du,$$ since K_n is even,

$$= \frac{1}{\pi} \int_0^{\pi} \left(f(x+u) + f(x-u) - 2c \right) K_n(u) du \qquad \text{---------------------- (89)}$$

$$= \frac{1}{2(n+1)\pi} \int_0^{\pi} \left(f(x+u) + f(x-u) - 2c \right) \frac{\sin^2(\frac{1}{2}(n+1)u)}{\sin^2(\frac{1}{2}u)} du \,. \text{ --------------- (90)}$$

With (90) we get immediately:

Theorem 32. Suppose the function *f* is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Then the $(C,1)$ mean of the Fourier series at *x* converges to *c* if, and only if,

$$\frac{1}{n} \int_0^{\pi} g_c(u) \frac{\sin^2(\frac{1}{2}nu)}{\sin^2(\frac{1}{2}u)} du \to 0 \,,$$

where $g_c(u) = \frac{1}{2}\left(f(x+u) + f(x-u) - 2c \right)$.

Take a fixed $0 < \delta < \pi$.

$$\frac{1}{n} \int_{\delta}^{\pi} g_c(u) \frac{\sin^2(\frac{1}{2}nu)}{\sin^2(\frac{1}{2}u)} du = \frac{1}{n} \frac{1}{\sin^2(\frac{1}{2}\delta)} \int_{\delta}^{\alpha(x)} g_c(u) \sin^2(\frac{1}{2}nu) du \,,$$

for some $\delta < \alpha(x) < \pi$, --------- (91)

by the Second Mean Value Theorem.

Therefore, $\left| \frac{1}{n} \int_{\delta}^{\pi} g_c(u) \frac{\sin^2(\frac{1}{2}nu)}{\sin^2(\frac{1}{2}u)} du \right| \le \frac{1}{n} \frac{1}{\sin^2(\frac{1}{2}\delta)} \int_{\delta}^{\alpha(x)} |g_c(u)| du$. -------------- (92)

Now $\int_{\delta}^{\alpha(x)} |g_c(u)| du \le \int_0^{\pi} | f(u) | du + \pi |c|$.

It then follows from (92) that $\frac{1}{n} \int_{\delta}^{\pi} g_c(u) \frac{\sin^2(\frac{1}{2}nu)}{\sin^2(\frac{1}{2}u)} du$ tends to 0 uniformly and boundedly in *x*. Hence, we have proved the following:

Theorem 32. Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Then the $(C,1)$ mean of the Fourier series at x converges to c if, and only if, for some $0 < \delta < \pi$, $\frac{1}{n}\int_0^{\delta} g_c(u)\frac{\sin^2(\frac{1}{2}nu)}{\sin^2(\frac{1}{2}u)}du \to 0$.

Suppose E is a subset of $[-\pi, \pi]$ and $c:E \to \mathbb{R}$ is a finite function. Suppose c is bounded. Then the $(C,1)$ mean of the Fourier series at x converges uniformly to $c(x)$ on E if, and only if, for some $0 < \delta < \pi$, $\frac{1}{n}\int_0^{\delta} g_c(u)\frac{\sin^2(\frac{1}{2}nu)}{\sin^2(\frac{1}{2}u)}du \to 0$ uniformly on E.

Assume c: $E \to \mathbb{R}$ is bounded. Consider the difference,

$$\int_0^{\delta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{\sin^2(\frac{1}{2}u)}du - \int_0^{\delta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{(\frac{1}{2}u)^2}du$$

$$= \int_0^{\delta} g_{c(x)}(u)\sin^2(\tfrac{1}{2}nu)\left(\frac{1}{\sin^2(\frac{1}{2}u)} - \frac{1}{\left(\frac{1}{2}u\right)^2}\right)du .$$

Now $h(u) = \frac{1}{\sin^2(u)} - \frac{1}{u^2}$ is strictly increasing on $(0, \frac{\pi}{2}]$. Observe that for u in $(0, \frac{\pi}{2})$, $h'(u) = -\frac{\cos(u)}{\sin^3(u)} + \frac{1}{u^3} > 0 \Leftrightarrow \frac{\sin(u)}{(\cos(u))^{1/3}} - u > 0$. Note that $k(u) = \frac{\sin(u)}{(\cos(u))^{1/3}} - u$ is increasing on $[0, \frac{\pi}{2})$ by observing that $k'(0) = 0$ and $k''(u) > 0$ for u in $(0, \frac{\pi}{2})$.

Thus, $\frac{1}{\sin^2(\frac{1}{2}u)} - \frac{1}{\left(\frac{1}{2}u\right)^2}$ is strictly increasing on $(0, \pi]$ and so by the Second Mean Value Theorem, since $\lim_{u\to 0^+} \frac{1}{\sin^2(\frac{1}{2}u)} - \frac{1}{\left(\frac{1}{2}u\right)^2} = \frac{1}{3} > 0$, there exists, $0 < \alpha(x) < \delta$ such that

$$\int_0^{\delta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{\sin^2(\frac{1}{2}u)}du - \int_0^{\delta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{(\frac{1}{2}u)^2}du$$

$$= \left(\frac{1}{\sin^2(\frac{1}{2}\delta)} - \frac{1}{\left(\frac{1}{2}\delta\right)^2}\right)\int_{\alpha(x)}^{\delta} g_{c(x)}(u)\sin^2(\tfrac{1}{2}nu)du .$$

Now,

$$\left|\int_{\alpha(x)}^{\delta} g_{c(x)}(u)\sin^2(\tfrac{1}{2}nu)du\right| \le \int_{\alpha(x)}^{\delta}\left|g_{c(x)}(u)\right|du \le \int_0^{\pi}\left|g_{c(x)}(u)\right|du \le \int_0^{\pi}\left|f(u)\right|du + \left|c(x)\right| . \text{ ----- (92)}$$

Since $c(x)$ is bounded on E, it follows that

$$\frac{1}{n}\int_0^{\delta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{\sin^2(\frac{1}{2}u)}du - \frac{1}{n}\int_0^{\delta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{(\frac{1}{2}u)^2}du \to 0$$

uniformly and boundedly on E. We have then by Theorem 32 the following:

Theorem 33. Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Then the $(C,1)$ mean of the Fourier series at x converges to c if, and only if, for some $0 < \delta < \pi$, $\frac{1}{n}\int_0^{\delta} g_c(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du \to 0$.

Suppose E is a subset of $[-\pi, \pi]$ and c: $E \to \mathbb{R}$ is a finite function. Suppose c is bounded. Then the $(C,1)$ mean of the Fourier series at x converges uniformly to $c(x)$ on E if and only if for some $0 < \delta < \pi$, $\frac{1}{n}\int_0^{\delta} g_c(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du \to 0$ uniformly on E.

Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Suppose at each point x of a subset E of $[-\pi, \pi]$, the left and right limit of f exists. Suppose further that $c(x) = \frac{1}{2}\left(f(x+) + f(x-)\right)$ is bounded on E. For instance, $c(x)$ is bounded if E is a finite set.

Take a fixed $0 < \delta < \pi$. Let $0 < \eta < \delta$. Then

$$\frac{1}{n}\int_0^{\delta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du = \frac{1}{n}\int_0^{\eta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du + \frac{1}{n}\int_{\eta}^{\delta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du . \text{ --- (93)}$$

For each x in E, $\lim_{u\to 0} g_{c(x)}(u) = \lim_{u\to 0}\frac{1}{2}\left(f(x+u) + f(x-u) - f(x+) - f(x-)\right) = 0$.

Therefore, given $\varepsilon > 0$, there exists $\eta(x)$ depending on x, with $0 < \eta(x) < \delta < \pi$ such that

$$|u| \le \eta(x) \Rightarrow |g_{c(x)}(u)| < \frac{2\varepsilon}{\pi}. \quad \text{---------------------------} \quad (94)$$

Therefore,

$$\frac{1}{n}\left|\int_0^{\eta(x)} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du\right| \le \frac{1}{n}\int_0^{\eta(x)}|g_{c(x)}(u)|\frac{\sin^2(\frac{1}{2}nu)}{u^2}du$$

$$\le \frac{2\varepsilon}{n\pi}\int_0^{\eta(x)}\frac{\sin^2(\frac{1}{2}nu)}{u^2}du = \frac{2\varepsilon}{n\pi}\int_0^{\frac{1}{2}n\eta(x)}\frac{\sin^2(v)}{\left(\frac{2v}{n}\right)^2}\frac{2}{n}dv = \frac{\varepsilon}{\pi}\int_0^{\frac{1}{2}n\eta(x)}\frac{\sin^2(v)}{v^2}dv$$

$$< \frac{\varepsilon}{\pi}\int_0^{\infty}\frac{\sin^2(v)}{v^2}dv = \frac{\varepsilon}{\pi}\frac{\pi}{2} = \frac{\varepsilon}{2}, \quad \text{-------------------------} \quad (95)$$

since $\int_0^{\infty}\frac{\sin^2(v)}{v^2}dv = \frac{\pi}{2}$.

By the Second Mean Value Theorem, there exists, $\eta(x) < \beta(x) < \delta$, such that

$$\int_{\eta(x)}^{\delta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du = \frac{1}{(\eta(x))^2}\int_{\eta(x)}^{\beta(x)} g_{c(x)}(u)\sin^2(\tfrac{1}{2}nu)du . \quad \text{----------------} \quad (96)$$

Now,

$$\left|\int_{\eta(x)}^{\beta(x)} g_{c(x)}(u)\sin^2(\tfrac{1}{2}nu)du\right| \le \int_{\eta(x)}^{\beta(x)}|g_{c(x)}(u)|du \le \int_0^{\pi}|g_{c(x)}(u)|du \le \int_0^{\pi}|f(u)|du + \pi|c(x)|.$$

It follows that if $c(x)$ is bounded on E and that $\eta(x)$ is bounded below on E by a positive constant, then

$\int_{\eta(x)}^{\delta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du = \frac{1}{(\eta(x))^2}\int_{\eta(x)}^{\beta(x)} g_{c(x)}(u)\sin^2(\tfrac{1}{2}nu)du$ is uniformly bounded.

Therefore, $\frac{1}{n}\int_{\eta(x)}^{\delta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du$ tends to 0 pointwise and if $\eta(x)$ is bounded below on E by a positive constant $\frac{1}{n}\int_{\eta(x)}^{\delta} g_{c(x)}(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du \to 0$ uniformly and boundedly on E.

Hence there exists an integer N such that

$$n \ge N \Rightarrow \left| \frac{1}{n} \int_{\eta(x)}^{\delta} g_{c(x)}(u) \frac{\sin^2(\frac{1}{2}nu)}{u^2} du \right| < \frac{\varepsilon}{2} \, . \quad \text{-------------------} \quad (97)$$

If $\eta(x)$ is bounded below on E by a positive constant, then there exists an integer N so that (97) holds for all x in E.

It follows then from (93), (95) and (97),

$$n \ge N \Rightarrow \left| \frac{1}{n} \int_{0}^{\delta} g_{c(x)}(u) \frac{\sin^2(\frac{1}{2}nu)}{u^2} du \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \, .$$

This means $\frac{1}{n} \int_{0}^{\delta} g_{c(x)}(u) \frac{\sin^2(\frac{1}{2}nu)}{u^2} du \to 0$ pointwise on E .

If f is uniformly continuous and bounded on E, then the left and right limits at x in E are the same and equal to $f(x)$. Thus $c(x) = f(x)$ on E. We may choose the same $\eta(x)$ for all x in E by uniform continuity so that $\eta(x)$ is bounded below on E and that (94) and (97) holds for all x in E. Hence (95) and (97) implies that $\frac{1}{n} \int_{0}^{\delta} g_{c(x)}(u) \frac{\sin^2(\frac{1}{2}nu)}{u^2} du \to 0$ uniformly and boundedly on E.

We have thus proved the following theorem.

Theorem 34. Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Suppose at each point x of a subset E of $[-\pi, \pi]$, the left and right limits of f exist.

(i) At each point x of E, the $(C,1)$ mean of the Fourier series at x converges to $c(x) = \frac{1}{2}\left(f(x+) + f(x-) \right)$.

(ii) Suppose f is uniformly continuous and bounded on E. Then the $(C,1)$ mean of the Fourier series converges uniformly and boundedly to $f(x)$ on E.

(ii) If f is continuous on $[-\pi, \pi]$, then the $(C,1)$ mean of the Fourier series converges uniformly and boundedly to $f(x)$ on $[-\pi, \pi]$.

Our next result is a more general sufficient condition for summability. This is the Féjer Lebesgue theorem.

Theorem 35 (Féjer Lebesgue). Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Then the Fourier series of f is $(C, 1)$ summable at every point x for which $\lim_{t\to 0}\frac{1}{t}\int_0^t |f(x+u)-f(x)|du = 0$, to $f(x)$.

More generally, if f satisfies condition L_c at θ, i.e., if $\lim_{t\to 0}\frac{1}{t}\int_0^t |g_c(u)|du = 0$, where $g_c(u) = \frac{f(\theta+u)+f(\theta-u)}{2} - c$, then the Fourier series of $f(x)$ is $(C, 1)$ summable to c at $x = \theta$.

Before we prove Theorem 35, we state a useful consequence.

Theorem 36. Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Then the Fourier series of f is $(C, 1)$ summable almost everywhere to $f(x)$.

Proof. By Lemma 10, almost every point of $[-\pi, \pi]$ is a Lebesgue point of f.

That is to say, for all x in $[-\pi, \pi]$ except for a null set,

$$\lim_{t\to 0}\frac{1}{t}\int_0^t |f(x+u)-f(x)|du = 0.$$

Therefore, by Theorem 35, the Fourier series of f at x is $(C, 1)$ is summable to $f(x)$ for all x outside a set of measure zero.

Proof of Theorem 35.

Fix a δ with $0 < \delta < \pi$. Let $g_x(u) = \frac{1}{2}\left(f(x+u)+f(x-u)-2f(x)\right)$.

Then for $t > 0$,

$$\frac{1}{t}\int_0^t |g_x(u)|du = \frac{1}{t}\int_0^t \left|\frac{1}{2}\left(f(x+u)+f(x-u)-2f(x)\right)\right|du$$

$$\le \frac{1}{2}\left(\frac{1}{t}\int_0^t |f(x+u)-f(x)|du + \frac{1}{t}\int_0^t |f(x-u)-f(x)|du\right)$$

$$\le \frac{1}{2}\left(\frac{1}{t}\int_0^t |f(x+u)-f(x)|du + \frac{1}{-t}\int_0^{-t} |f(x+u)-f(x)|du\right). \quad \text{--------- (98)}$$

Since $\lim_{t\to 0}\frac{1}{t}\int_0^t |f(x+u)-f(x)|du = 0$, it follows that $\lim_{t\to 0^+}\frac{1}{t}\int_0^t |g_x(u)|du = 0$.

We can deduce similarly that $\lim_{t\to 0^-}\frac{1}{t}\int_0^t |g_x(u)|du = 0$ Therefore,

$\lim_{t\to 0}\frac{1}{t}\int_0^t |g_x(u)|du = 0$. This means *f* satisfies condition $L_{f(x)}$ at *x*. Thus, we may proceed the proof with the more general case taking *c* in place of *f* (*x*) and write θ in place of *x*. We assume that $\lim_{t\to 0}\frac{1}{t}\int_0^t |g_c(u)|du = 0$, where

$g_c(u) = \dfrac{f(\theta+u)+f(\theta-u)}{2} - c$.

Let $\Psi(t) = \int_0^t |g_c(u)|du$ for $t > 0$. Then given $\varepsilon > 0$, there exists η such that for $0 < \eta < \delta$ and for $0 < t \le \eta$,

$$\Psi(t) < t\frac{\varepsilon}{4}. \quad \text{-------------------------------- (99)}$$

Choose $n > \frac{1}{\eta}$, i.e., $\frac{1}{n} < \eta$.

$$\int_0^\delta g_c(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du = \int_0^{\frac{1}{n}} g_c(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du + \int_{\frac{1}{n}}^{\eta} g_c(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du + \int_\eta^\delta g_c(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}du$$

$$= I_1 + I_2 + I_3 . \quad \text{----------------------------- (100)}$$

Now

$$|I_1| \le \int_0^{\frac{1}{n}} \left| g_c(u)\frac{\sin^2(\frac{1}{2}nu)}{u^2}\right| du \le \int_0^{\frac{1}{n}} |g_c(u)|\frac{n^2}{4}du \le \frac{n^2}{4}\varepsilon\frac{1}{4n} = \frac{n\varepsilon}{16}, \quad \text{--------- (101)}$$

since $\frac{1}{n} < \eta$.

$$|I_2| \le \int_{\frac{1}{n}}^{\eta} \left| g_c(u) \frac{\sin^2(\frac{1}{2}nu)}{u^2} \right| du \le \int_{\frac{1}{n}}^{\eta} \frac{|g_c(u)|}{u^2} du = \left[\frac{\Psi(u)}{u^2} \right]_{\frac{1}{n}}^{\eta} + 2\int_{\frac{1}{n}}^{\eta} \frac{\Psi(u)}{u^3} du ,$$

by integration by parts,

$$\le \frac{\Psi(\eta)}{\eta^2} - n^2\Psi(\tfrac{1}{n}) + 2\int_{\frac{1}{n}}^{\eta} \frac{\varepsilon u}{4u^3} du < \frac{\varepsilon}{4\eta} + \frac{\varepsilon}{2}\left(n - \frac{1}{\eta} \right) < \varepsilon \left(\frac{1}{4\eta} + \frac{n}{2} \right). \quad \text{--------------- (102)}$$

$$|I_3| \le \int_{\eta}^{\delta} \left| g_c(u) \frac{\sin^2(\frac{1}{2}nu)}{u^2} \right| du \le \int_{\eta}^{\delta} |g_c(u)| \frac{1}{u^2} du \le \frac{1}{\eta^2} \int_{\eta}^{\delta} |g_c(u)| du \le \frac{1}{\eta^2} \int_{0}^{\delta} |g_c(u)| du . \quad \text{---- (103)}$$

Therefore, given $\varepsilon > 0$, for a fixed η as above, by (100)-(103),

$$\frac{1}{n} \left| \int_0^{\delta} g_c(u) \frac{\sin^2(\frac{1}{2}nu)}{u^2} du \right| < \frac{1}{n} \left(\frac{n\varepsilon}{16} + n\frac{\varepsilon}{2} + \frac{\varepsilon}{4\eta} + \frac{1}{\eta^2} \int_0^{\delta} |g_c(u)| du \right). \quad \text{----------------- (104)}$$

Thus, given $\varepsilon > 0$, fix an η, with $0 < \eta < \delta$ such that for $0 < t \le \eta$, (99) holds.

Then take n sufficiently large so that

$$\frac{\varepsilon}{16} + \frac{\varepsilon}{2} + \frac{1}{n} \left(\frac{\varepsilon}{4\eta} + \frac{1}{\eta^2} \int_0^{\delta} |g_c(u)| du \right) < \varepsilon .$$

This means $\frac{1}{n} \int_0^{\delta} g_c(u) \frac{\sin^2(\frac{1}{2}nu)}{u^2} du \to 0$.

Therefore, by Theorem 33, the Fourier series of f at $x = \theta$, is (C, 1) summable to c. This proves the more general case and hence the special case when $\lim_{t \to 0} \frac{1}{t} \int_0^t |f(x+u) - f(x)| du = 0$.

Theorem 37. Suppose the function f is periodic of period 2π and is Lebesgue integrable on $[-\pi, \pi]$. Suppose the function $\phi(u) = \frac{1}{2}\big(f(x+u) + f(x-u) \big)$ has a limit at $u = 0$, i.e., $\lim_{u \to 0^+} \phi(u) = \phi(0+)$ exists, then the Fourier series of f at x is (C, 1) summable to $\phi(0+)$. In particular, if f is continuous at x, then the Fourier series of f at x is (C, 1) summable to $f(x)$.

Proof. Note that if $\lim_{u \to 0^+} \phi(u) = \phi(0+)$, then *f* satisfies condition L_c at *x* with $c = \phi(0+)$. Theorem 37 then follows from Theorem 35. Observe that if *f* is continuous at *x*, then $\lim_{u \to 0^+} \phi(u) = \phi(0+) = f(x)$.

Convergence Almost Everywhere

We now discuss some results concerning convergence almost everywhere. A most useful result is the important Riesz-Fischer Theorem.

Theorem 38. If $\sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right) < \infty$, then the trigonometric series

$$\sum_{n=1}^{\infty} A_n(x) = \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

is the Fourier series of a square integrable function *F*.

The following summability result of Hardy is very useful in passing from summability to normal convergence.

Theorem 39. Notation as in Theorem 38. If $A_n(x) = O(\frac{1}{n})$, then $\sum_{n=1}^{\infty} A_n(x)$ is (C,1) summable to *s* if, and only, if $\sum_{n=1}^{\infty} A_n(x)$ converges to *s*.

The next two results are about convergence almost everywhere.

Theorem 40. If $\sum_{n=1}^{\infty} A_n(x) = \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$ is the Fourier series of a square integrable function *F,* then the series

$$\sum_{n=2}^{\infty} \left(\frac{a_n \cos(nx) + b_n \sin(nx)}{\left(\ln(n) \right)^{\frac{1}{2}}} \right)$$

converges almost everywhere on $\mathbb{R}$.

Theorem 41. Suppose a_n, b_n are Fourier coefficients of an integrable function f, i.e., $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$ is a Fourier series, then both series

$$\sum_{n=2}^{\infty}\left(\frac{a_n \cos(nx) + b_n \sin(nx)}{\ln(n)}\right) \text{ and } \sum_{n=2}^{\infty}\left(\frac{a_n \sin(nx) - b_n \cos(nx)}{\ln(n)}\right)$$

converge almost everywhere.

The proofs of theorems 38 to 41 can be found in Zygmund's monumental work *Trigonometric Series*.

We now state some consequences of these theorems.

Theorem 42. If $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$ is a trigonometric series with $a_n \to 0$ and $b_n \to 0$, then the series $\frac{1}{2}a_0 x - \sum_{n=1}^{\infty}\left(\frac{b_n \cos(nx) - a_n \sin(nx)}{n}\right)$ converges almost everywhere to some function.

Proof. $a_n \to 0$ and $b_n \to 0$ implies that both sequences (a_n) and (b_n) are bounded and as a consequence $\sum_{n=1}^{\infty}\left(\left(\frac{a_n}{n}\right)^2 + \left(\frac{b_n}{n}\right)^2\right) < \infty$. Therefore, by Theorem 38, $\sum_{n=1}^{\infty}\left(\frac{b_n \cos(nx) - a_n \sin(nx)}{n}\right)$ is the Fourier series of a square integrable function F. Then F is of course integrable. By Theorem 36, this series is $(C,1)$ summable almost everywhere to $F(x)$. As $a_n \to 0$ and $b_n \to 0$, $\frac{b_n \cos(nx) - a_n \sin(nx)}{n}$ is $o\left(\frac{1}{n}\right)$. Therefore, by Hardy's Theorem, Theorem 39, this series converges almost everywhere to $F(x)$. Consequently, the series $\frac{1}{2}a_0 x - \sum_{n=1}^{\infty}\left(\frac{b_n \cos(nx) - a_n \sin(nx)}{n}\right)$ obtained by a formal integration of the trigonometric series term by term, converges almost everywhere.

Theorem 43. The trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$ converges almost everywhere if $\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\left(\ln(n)\right)^2 < \infty$.

Proof. By Theorem 38, $\sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)\ln(n)$ is the Fourier series of some square integrable function *F*. It follows then by Theorem 41 that

$$\sum_{n=2}^{\infty}\frac{\left(a_n \cos(nx) + b_n \sin(nx)\right)\ln(n)}{\ln(n)} = \sum_{n=2}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$$

converges almost everywhere to some function. Consequently, the series converges almost everywhere.

Theorem 44. The trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$ converges almost everywhere if $\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\ln(n) < \infty$.

Proof. By Theorem 38, $\sum_{n=2}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)\sqrt{\ln(n)}$ is the Fourier series of some square integrable function *F*. It follows then by Theorem 40 that

$$\sum_{n=2}^{\infty}\frac{\left(a_n \cos(nx) + b_n \sin(nx)\right)\sqrt{\ln(n)}}{\sqrt{\ln(n)}} = \sum_{n=2}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$$

converges almost everywhere to some function. Hence the trigonometric series converges almost everywhere.