Abel-summability of Fourier Series and its Derived Series

By Ng Tze Beng

In this article, we focus on the use of Poisson-Abel kernel to investigate the convergence of Fourier series. Abel-summability or A-summability is more general than (C,1) summability. We concentrate on A-summability and its application to Fourier series, Fourier integral and harmonic function. A-summability, like (C,1) summability, is also regular and so it too generalizes the usual notion of convergence in the sense of Cauchy. In section A, we give the definition of A-summability and its regularity property. In section B we define the Poisson kernel, the conjugate kernel, the Poisson integral and conjugate Poisson integral arising from a Lebesgue integrable function, show that they are harmonic in the open unit disk and list some useful properties of the Poisson kernel. In section C, we give the convergence theorems and some application to harmonic function, defined by the Poisson integral, state a special form of the maximum and minimum principle for harmonic function of the Poisson integral type and show that we can recover the continuous harmonic function defined on a closed unit disk from the values on the unit circle. In section D, we show that the derived series of a Fourier series is A-summable to its symmetric derivative if it exists. Through the use of the symmetric derivative of a primitive of Lebesgue integrable periodic function *f*, we show that its Fourier series is Lebesgue summable at θ to *c* if`, and only if, *f* satisfies condition ℓ_c at θ.

Section A A-summability

Definition 1. Let (a_n), n =0, 1, 2, 3, …. , be a sequence. Let $s_n = \sum_{k=0}^{n} a_k$ be the (*n*+1)-th partial sum of the series $\sum_{k=0}^{\infty} a_k$. The series $\sum_{k=0}^{\infty} a_k$ is convergent if the sequence (s_n) converges. Let $T_n = \sum_{k=0}^{n} s_k$ and σ_n be the arithmetic mean

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k .$$

The series $\sum_{k=0}^{\infty} a_k$ is said to be (C,1) *summable* to a (finite) value *s* if the sequence (σ_n) converges to *s*.

Consider the power series $\sum_{k=0}^{\infty} a_k x^k$. Suppose the power series converges for $0 \le x < 1$ or $|x| < 1$. If the limit $\lim_{x\to 1^-}\sum_{k=0}^{\infty} a_k x^k = s$, then we say the series $\sum_{k=0}^{\infty} a_k$ is *Abel summable* or *A-summable* to *s*.

We shall establish the regularity of A-summability as a consequence of the regularity of (C,1) summability.

Theorem 2. (C,1) summability is regular. If the series $\sum_{k=0}^{\infty} a_k$ converges to *s*, then it is (C,1) summable to *s*, i.e., the arithmetic mean $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k$ converges to *s*.

Proof. Note that $\sum_{k=0}^{\infty} a_k$ converges to *s* if and only if $(a_0 - s) + a_1 + \cdots$ converges to 0. $\sum_{k=0}^{\infty} a_k$ is (C,1) summable to *s* if and only if σ_n converges to *s* if, and only if, $\frac{1}{n+1}\sum_{k=0}^{n} s_k - s$ converges to 0 if, and only if, $\frac{1}{n+1}\sum_{k=0}^{n}(s_k - s)$ converges to 0 if, and only if, $(a_0 - s) + a_1 + \cdots$ is (C,1) summable to 0. Therefore, we may assume without loss of generality that $\sum_{k=0}^{\infty} a_k$ converges to 0.

Thus, $s_n \to 0$. So given any ε>0, there exists integer *N* such that

$$n \geq N \Rightarrow |s_n| < \varepsilon \Rightarrow -\varepsilon < s_n < \varepsilon . \qquad \text{--------------------} \quad (1)$$

Hence, for *n* > *N*, by using (1),

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k = \frac{1}{n+1}\sum_{k=0}^{N} s_k + \frac{1}{n+1}\sum_{k=N+1}^{n} s_k > \frac{1}{n+1}\sum_{k=0}^{N} s_k - \frac{1}{n+1}(n-N)\varepsilon .$$

It follows that

$$\liminf_{n\to\infty} \sigma_n \geq \liminf_{n\to\infty}\left(\frac{1}{n+1}\sum_{k=0}^{N} s_k - \frac{1}{n+1}(n-N)\varepsilon\right) = -\varepsilon .$$

Since ε is arbitrary, $\liminf_{n\to\infty} \sigma_n \geq 0$.

Also using (1) we get for *n* > *N*,

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k = \frac{1}{n+1}\sum_{k=0}^{N} s_k + \frac{1}{n+1}\sum_{k=N+1}^{n} s_k < \frac{1}{n+1}\sum_{k=0}^{N} s_k + \frac{1}{n+1}(n-N)\varepsilon .$$

Therefore,

$$\limsup_{n\to\infty}\sigma_n \le \limsup_{n\to\infty}\left(\frac{1}{n+1}\sum_{k=0}^{N}s_k + \frac{1}{n+1}(n-N)\varepsilon\right) = \varepsilon \ .$$

Thus, $\limsup_{n\to\infty}\sigma_n \le 0$ and we have $0 \le \liminf_{n\to\infty}\sigma_n \le \limsup_{n\to\infty}\sigma_n \le 0$. It follows that $\lim_{n\to\infty}\sigma_n = 0$. This proves Theorem 2.

Theorem 3. If the series $\sum_{k=0}^{\infty} a_k$ is (C,1) summable to *s*, then it is A-summable to *s*.

Proof. Note that $\sum_{k=0}^{\infty} a_k$ is A-summable to *s* if, and only if, $(a_0 - s) + a_1 + \cdots$ is A summable to 0. Thus, we may, as in the proof of Theorem 2, assume that $\sum_{k=0}^{\infty} a_k$ is (C,1) summable to 0. This means that $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k = \frac{1}{n+1}T_n \to 0$. It follows that given any $\varepsilon > 0$, there exists integer $N > 0$ such that

$$n \ge N \Rightarrow \left|\frac{T_n}{n+1}\right| < \varepsilon \Rightarrow |T_n| < (n+1)\varepsilon \ . \qquad \text{-------------------- (2)}$$

Therefore, using (2), we get

$$n > N \Rightarrow |s_n| = |T_n - T_{n-1}| \le |T_n| + |T_{n-1}| < 2(n+1)\varepsilon \,. \qquad \text{--------------- (3)}$$

It follows that

$$n > N+1 \Rightarrow |a_n| = |s_n - s_{n-1}| \le |s_n| + |s_{n-1}| < 4(n+1)\varepsilon \,. \qquad \text{---------------- (4)}$$

Since $\sum_{n=0}^{\infty}(n+1)x^n$ converges absolutely for $|x| < 1$, it follows by Comparison Test and (2), (3), and (4) that $\sum_{n=0}^{\infty} T_n x^n$, $\sum_{n=0}^{\infty} s_n x^n$ and $\sum_{n=0}^{\infty} a_n x^n$ all converges absolutely for $|x| < 1$.

We want to show that $\lim_{x\to 1^-}\sum_{k=0}^{\infty} a_k x^k = 0$. As a first step we shall rewrite the $(n+1)$-th partial sum $\sum_{k=0}^{n} a_k x^k$.

$$\sum_{k=0}^{n} a_k x^k = s_0 + \sum_{k=1}^{n}(s_k - s_{k-1})x^k = \sum_{k=1}^{n} s_k x^k - x\sum_{k=0}^{n-1} s_k x^k + s_0 = \sum_{k=0}^{n} s_k x^k - x\sum_{k=0}^{n-1} s_k x^k$$

$$= (1-x)\sum_{k=0}^{n} s_k x^k + s_n x^{n+1}$$

$$= (1-x)\left((1-x)\sum_{k=0}^{n} T_k x^k + T_n x^{n+1}\right) + s_n x^{n+1},$$

by applying similar derivation for $\sum_{k=0}^{n} s_k x^k$,

$$= (1-x)^2\sum_{k=0}^{n} T_k x^k + (1-x)T_n x^{n+1} + s_n x^{n+1}. \qquad \text{------------} \quad (5)$$

Now for $|x| < 1$, $(n+1)|x|^{n+1} \to 0$. Using this fact together with (2) and (3), we can easily deduce that $T_n x^{n+1} \to 0$ and $s_n x^{n+1} \to 0$ for $|x| < 1$.

Hence, using (5), we have that for $|x| < 1$,

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = (1-x)^2 \sum_{k=0}^{\infty} T_k x^k. \qquad \text{---------------} \quad (6)$$

Therefore, for $|x| < 1$,

$$|f(x)| = (1-x)^2\left|\sum_{k=0}^{\infty} T_k x^k\right| = (1-x)^2\left|\sum_{k=0}^{N} T_k x^k + \sum_{k=N+1}^{\infty} T_k x^k\right|$$

$$\le (1-x)^2\left|\sum_{k=0}^{N} T_k x^k\right| + (1-x)^2\left|\sum_{k=N+1}^{\infty} T_k x^k\right|$$

$$\le (1-x)^2\left|\sum_{k=0}^{N} T_k x^k\right| + \varepsilon(1-x)^2\sum_{k=N+1}^{\infty}(k+1)|x|^k, \text{ by using inequality (2).}$$

Hence for $0 \le x < 1$,

$$\left| f(x) \right| \le (1-x)^2 \left| \sum_{k=0}^{N} T_k x^k \right| + \varepsilon (1-x)^2 \sum_{k=N+1}^{\infty} (k+1) x^k$$

$$\le (1-x)^2 \left| \sum_{k=0}^{N} T_k x^k \right| + \varepsilon (1-x)^2 \frac{1}{(1-x)^2} = (1-x)^2 \left| \sum_{k=0}^{N} T_k x^k \right| + \varepsilon .$$

Thus, $\limsup_{x \to 1^-} \left| f(x) \right| \le \limsup_{x \to 1^-} \left((1-x)^2 \left| \sum_{k=0}^{N} T_k x^k \right| + \varepsilon \right) = \varepsilon$.

Since ε is arbitrary, $\limsup_{x \to 1^-} \left| f(x) \right| = 0$. Therefore, $\lim_{x \to 1^-} \left| f(x) \right| = 0$ and so $\lim_{x \to 1^-} f(x) = 0$. This means $\sum_{k=0}^{\infty} a_k$ is A-summable to 0. This completes the proof.

Theorem 2 and Theorem 3 give the following

Corollary 4. A-summability is regular.

Section B

Abel's Method, Poisson Kernel, Conjugate Poisson Kernel and Harmonic Function

To bring in circular functions, i.e., sine and cosine functions, we consider the complex power series

$$\frac{1}{2} + \sum_{k=1}^{\infty} r^k e^{ikx} = \frac{1}{2} + \sum_{k=1}^{\infty} \left(re^{ix} \right)^k . \qquad \text{------------------} \quad (7)$$

For $0 \le r < 1$ or $|r| < 1$, this power series converges absolutely for all x and

$$\frac{1}{2} + \sum_{k=1}^{\infty} r^k e^{ikx} = \frac{1}{2} + re^{ix} \frac{1}{1-re^{ix}} = \frac{1+re^{ix}}{2(1-re^{ix})}$$

$$= \frac{1-r^2+2i\, r\sin(x)}{2(1+r^2-2\, r\cos(x))} . \qquad \text{--------------------} \ (8)$$

Taking the real part of the power series we have:

$$\frac{1}{2}+\sum_{k=1}^{\infty} r^k \cos(kx) = \frac{1-r^2}{2(1+r^2-2\,r\cos(x))}\,. \qquad \text{----------} \ (9)$$

And taking the imaginary part we obtain:

$$\sum_{k=1}^{\infty} r^k \sin(kx) = \frac{r\sin(x)}{1+r^2-2\,r\cos(x)}\ . \qquad \text{---------------} \ (10)$$

Let $$P(r,x)=\frac{1-r^2}{2(1+r^2-2\,r\cos(x))} \qquad \text{----------------} \ (11)$$

and $$Q(r,x)=\frac{r\sin(x)}{1+r^2-2\,r\cos(x)}\ . \qquad \text{-----------------} \ (12)$$

$P(r,x)$ is called the Poisson kernel and $Q(r,x)$ the conjugate Poisson kernel.

Suppose $A_n(\theta)=a_n\cos(n\theta)+b_n(n\theta)\to 0$ or $(A_n(\theta))$ is bounded.

This is always satisfied if (a_n , b_n) are Fourier coefficients of a Lebesgue integrable periodic function f of period 2π and

$$\frac{1}{2}a_0+\sum_{k=1}^{\infty} A_k(\theta)$$

is its Fourier series.

Then

$$u(r,\theta)=\frac{1}{2}a_0+\sum_{k=1}^{\infty} r^k A_k(\theta) \qquad \text{-----------------------} \ (13)$$

converges (absolutely) for $|\,r| < 1$.

Observe that, for $k\geq 1$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(k(t-\theta))dt=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\big(\cos(kt)\cos(k\theta)+\sin(kt)\sin(k\theta)\big)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\cos(kt)\right)dt\cos(k\theta) + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\sin(kt)\right)dt\sin(k\theta)$$

$$= a_k\cos(k\theta) + b_k\sin(k\theta) = A_k(\theta).$$

Therefore, for $0 \le r < 1$,

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} r^k \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(k(t-\theta))dt$$

$$= \frac{1}{2}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{k=1}^{\infty} r^k \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(k(t-\theta))dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{2} + \sum_{k=1}^{\infty} r^k\cos(k(t-\theta)\right\} f(t)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{2(1+r^2-2r\cos(t-\theta))} f(t)dt, \quad \text{------------------------} \quad (14)$$

by using (9),

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} P(r,t-\theta) f(t)dt. \quad \text{------------------------------} \quad (15)$$

The integral in (14) is called the Poisson integral of f and $P(r, t\text{-}\theta)$ is its associated Poisson kernel.

The series $\frac{1}{2}a_0 + \sum_{k=1}^{\infty} A_k(\theta)$ is A-summable to s if $\lim_{r\to 1^-} u(r,\theta) = s$ or, equivalently, if $\lim_{r\to 1^-}\frac{1}{\pi}\int_{-\pi}^{\pi} P(r,t-\theta) f(t)dt = s$.

Thus we can use the Poisson kernel to investigate A-summability.

We observe the following important properties of the Poisson kernel.

Note that for $|r| < 1$,

$$P(r,x) = \frac{1-r^2}{2(1+r^2-2r\cos(x))} = \frac{1-r^2}{2\left\{(1-r\cos(x))^2 + r^2(1-\cos^2(x))\right\}} \ge 0 \quad \text{----} \quad (16)$$

We can integrate $P(r, x)$, by integrating the power series expansion (9) of $P(r, x)$ term by term by invoking the Riemann Dominated Convergence Theorem since the power series is absolutely convergent and uniformly bounded by $\frac{1}{2}+|r|\frac{1}{1-|r|}$, to give, for $0 \le r < 1$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} P(r,x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}dx = 1. \qquad \text{-----------------------} \quad (17)$$

Consider the special cosine series:

(C) $$\frac{1}{2}+\cos(\theta)+\cos(2\theta)+\cdots$$

(C) is A-summable to 0 for θ not a multiple of 2π, since

$$\frac{1}{2}+\sum_{k=1}^{\infty} r^k \cos(k\theta) = \frac{1-r^2}{2(1+r^2-2\,r\cos(\theta))}$$

and $$\lim_{r\to 1^-}\frac{1-r^2}{2(1+r^2-2\,r\cos(\theta))} = \frac{0}{2(2-2\cos(\theta))} = 0 .$$

On the other hand, the (C,1) mean of (C) is given by the Fejer kernel,

$K_n(\theta) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(\theta) = \frac{2}{n+1}\left\{\frac{\sin(\frac{1}{2}(n+1)x)}{2\sin(\frac{1}{2}\theta)}\right\}^2$, which tends to 0 for θ not a multiple of 2π. So (C) is both (C,1) and A-summable to 0 for θ not a multiple of 2π. We can of course deduce the A-summability of (C) indirectly by invoking Theorem 3.

Now we turn to the special sine series

(S) $$\sin(\theta)+\sin(2\theta)+\sin(3\theta)+\cdots$$

For $|r| < 1$, from (10) we have $\sum_{k=1}^{\infty} r^k \sin(k\theta) = \frac{r\sin(\theta)}{1+r^2-2\,r\cos(\theta)}$ and for θ not a multiple of 2π, $\lim_{r\to 1^-}\frac{r\sin(\theta)}{1+r^2-2\,r\cos(\theta)} = \frac{\sin(\theta)}{2-2\cos(\theta)} = \frac{1}{2}\cot\left(\frac{\theta}{2}\right)$. This means

(S) is A-summable to $\frac{1}{2}\cot\left(\frac{\theta}{2}\right)$ for θ not a multiple of 2π. Of course, we can also deduce this by invoking Theorem 3 and that (S) is (C,1) summable to $\frac{1}{2}\cot\left(\frac{\theta}{2}\right)$ for θ not a multiple of 2π. The conjugate Fejer kernel is

$$K_n(\theta) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(\theta) = \frac{1}{n+1}\sum_{k=1}^{n} \frac{\cos(\frac{1}{2}\theta) - \cos(\frac{1}{2}(k+1)\theta)}{2\sin(\frac{1}{2}\theta)}$$

$$= \frac{1}{n+1}\left\{\frac{n}{2}\cot(\tfrac{1}{2}\theta) - \sum_{k=1}^{n}\frac{\cos(\frac{1}{2}(k+1)\theta)}{2\sin(\frac{1}{2}\theta)}\right\}$$

$$= \frac{1}{n+1}\left\{\frac{n}{2}\cot(\tfrac{1}{2}\theta) - \frac{1}{2}\cot(\tfrac{1}{2}\theta)\sum_{k=1}^{n}\cos(k\theta) + \frac{1}{2}\sum_{k=1}^{n}\sin(k\theta)\right\}$$

$$= \frac{1}{n+1}\left\{\frac{n}{2}\cot(\tfrac{1}{2}\theta) - \frac{1}{2}\cot(\tfrac{1}{2}\theta)\left(D_n(\theta) - \frac{1}{2}\right) + \frac{1}{2}D_n(\theta)\right\}$$

$$= \frac{1}{n+1}\left\{\frac{n}{2}\cot(\tfrac{1}{2}\theta) + \frac{1}{4}\cot(\tfrac{1}{2}\theta) - \frac{1}{2}\cot(\tfrac{1}{2}\theta)\frac{\sin((n+\frac{1}{2})\theta)}{2\sin(\frac{1}{2}\theta)} + \frac{1}{2}\frac{\cos(\frac{1}{2}\theta) - \cos((n+\frac{1}{2})\theta)}{2\sin(\frac{1}{2}\theta)}\right\}$$

$$= \frac{1}{n+1}\left\{\frac{n+1}{2}\cot(\tfrac{1}{2}\theta) - \frac{\cos(\frac{1}{2}\theta)\sin((n+\frac{1}{2})\theta) + \sin(\frac{1}{2}\theta)\cos((n+\frac{1}{2})\theta)}{4\sin^2(\frac{1}{2}\theta)}\right\}$$

$$= \frac{1}{2}\cot(\tfrac{1}{2}\theta) - \frac{1}{n+1}\left\{\frac{\sin((n+1)\theta)}{4\sin^2(\frac{1}{2}\theta)}\right\}.$$

Thus, $\lim_{n\to\infty} K_n(\theta) = \lim_{n\to\infty}\left\{\frac{1}{2}\cot(\tfrac{1}{2}\theta) - \frac{1}{n+1}\left\{\frac{\sin((n+1)\theta)}{4\sin^2(\frac{1}{2}\theta)}\right\}\right\} = \frac{1}{2}\cot(\tfrac{1}{2}\theta)$. This shows that (S) is (C,1) summable to $\frac{1}{2}\cot\left(\frac{\theta}{2}\right)$ for θ not a multiple of 2π.

Observe that (C) diverges for all θ and (S) converges only when it is a trivial series when θ is a multiple of π.

If $\frac{1}{2}a_0 + \sum_{k=1}^{\infty} A_k(\theta)$ is a trigonometric series, then its *conjugate* series is defined to be

$$\sum_{k=1}^{\infty} B_k(\theta)\ , \qquad \text{---------------------------} \quad (18)$$

where $B_k(\theta) = b_k \cos(k\theta) - a_k \sin(k\theta)$.

If $B_n(\theta) \to 0$ or when ($B_n(\theta)$) is bounded, then the series

$$v(r,\theta) = \sum_{k=1}^{\infty} r^k B_k(\theta) \quad \text{---------------------------------------} \quad (19)$$

converges absolutely for | r | < 1.

This is the case when (19) is the conjugate series of the Fourier series of a periodic Lebesgue integrable function f.

Assume now that (19) is such a series. Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(k(t-\theta))dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\big(\sin(kt)\cos(k\theta) - \cos(kt)\sin(k\theta)\big)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt\cos(k\theta) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\big(\cos(kt)\big)dt\sin(k\theta)$$

$$= b_k \cos(k\theta) - b_k \sin(k\theta) = B_k(\theta).$$

Therefore, for $0 \leq r < 1$,

$$v(r,\theta) = \sum_{k=1}^{\infty} r^k \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(k(t-\theta))dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \left\{ \sum_{k=1}^{\infty} r^k \sin(k(t-\theta)) \right\} f(t)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{r\sin(t-\theta)}{1+r^2-2r\cos(t-\theta)} f(t)dt, \quad \text{------------------------------} \quad (20)$$

by using (10),

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} Q(r,t-\theta) f(t)dt. \quad \text{----------------------------------} \quad (21)$$

The integral in (20) is called the conjugate Poisson integral of f and Q(r, t-θ) is its associated conjugate Poisson kernel.

For the conjugate Poisson kernel $Q(r, x)$ we have that

$$\int_{-\pi}^{\pi} Q(r,x)dx = 0. \quad \text{--------------------------} \quad (22)$$

We can use complex variable technique to evaluate this integral.

Let $f(z) = \dfrac{z}{(1-rz)(1-rz^{-1})}\dfrac{1}{iz} = \dfrac{z}{i(1-rz)(z-r)}$. Let $\gamma(x) = e^{ix}$, $-\pi \le x \le \pi$ parametrize the unit circle |z| = 1. Then

$$\int_{-\pi}^{\pi} \frac{r\sin(x)}{1+r^2-2\,r\cos(x)}dx = \operatorname{Im}\int_{\gamma} f(z)dz\,.$$

For $0 < r < 1$, f (z) has only one simple pole r in the unit disk |z| < 1. The residue of f at r is given by

$$res(f,r) = \lim_{z\to r}(z-r)\frac{z}{i(1-rz)(z-r)} = \lim_{z\to r}\frac{z}{i(1-rz)} = \frac{r}{i(1-r^2)}\ .$$

Therefore,

$$\int_{\gamma} f(z)dz = 2\pi i\frac{r}{i(1-r^2)} = \frac{2\pi r}{1-r^2},$$

and so $$\int_{-\pi}^{\pi} \frac{r\sin(x)}{1+r^2-2\,r\cos(x)}dx = 0$$

and $$\int_{-\pi}^{\pi} \frac{r\cos(x)}{1+r^2-2\,r\cos(x)}dx = \frac{2\pi r}{1-r^2}. \quad \text{------------------} \quad (22)^*$$

Consequently, $\operatorname{Im}\int_{\gamma} f(z)dz = 0$.

We may of course use the Dominated Riemann Convergence Theorem to integrate $Q(r, x)$ term by term to give 0.

Theorem 5. $u(r,\theta)$ and $v(r,\theta)$ are harmonic function and

$u(r,\theta) - iv(r,\theta)$ is analytic in the unit disk |z| < 1.

Proof. We shall show that both $u(r,\theta)$ and $v(r,\theta)$ satisfy the Laplace equation.

In polar co-ordinate the Laplace equation is

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0$$

Or $$\left(r\frac{\partial}{\partial r}\right)^2\phi+\frac{\partial^2\phi}{\partial\theta^2}=0. \quad \text{---------------------------} \quad (23)$$

We begin by writing $u(r,\theta)$ in a different form.

Define $a_{-n}=a_n$ and $b_{-n}=-b_n$ for $n>0$ and $b_0=0$. Let $c_n=\frac{1}{2}(a_n-ib_n)$.
Then $a_n=c_n+c_{-n}$ and $b_n=i(c_n-c_{-n})$. We now rewrite $u(r,\theta)$.

$$u(r,\theta)=\frac{1}{2}a_0+\sum_{k=1}^{\infty}r^kA_k(\theta)=c_0+\sum_{k=-\infty}^{\infty}c_kr^{|k|}e^{ki\theta}$$

$$=c_0+\sum_{k=1}^{\infty}c_kr^ke^{ki\theta}+\sum_{k=1}^{\infty}c_{-k}r^ke^{-ki\theta} \quad \text{----------------------------} \quad (24)$$

We assume as before that a_n and b_n are bounded and so $u(r,\theta)$ is absolutely convergent for $0\le r<1$.

Observe that,

$$\left(r\frac{\partial}{\partial r}\right)u(r,\theta)=\sum_{k=1}^{\infty}c_kk\,r^ke^{ki\theta}+\sum_{k=1}^{\infty}c_{-k}k\,r^ke^{-ki\theta} ,$$

$$\left(r\frac{\partial}{\partial r}\right)^2u(r,\theta)=\sum_{k=1}^{\infty}c_kk^2\,r^ke^{ki\theta}+\sum_{k=1}^{\infty}c_{-k}k^2\,r^ke^{-ki\theta} ,$$

$$\frac{\partial u}{\partial\theta}(r,\theta)=\sum_{k=1}^{\infty}c_kik\,r^ke^{ki\theta}-\sum_{k=1}^{\infty}c_{-k}ik\,r^ke^{-ki\theta} \text{ and}$$

$$\frac{\partial^2u}{\partial\theta^2}(r,\theta)=-\sum_{k=1}^{\infty}c_k\,k^2\,r^ke^{ki\theta}-\sum_{k=1}^{\infty}c_{-k}\,k^2\,r^ke^{-ki\theta}.$$

It follows that $\left(r\dfrac{\partial}{\partial r}\right)^2 u+\dfrac{\partial^2 u}{\partial\theta^2}=0$. Thus $u(r,\theta)$ is harmonic in the open unit disk.

Let $z=re^{i\theta}$. Then from (24),

$$u(r,\theta)=c_0+\sum_{k=1}^{\infty}c_k z^k+\sum_{k=1}^{\infty}c_{-k}\overline{z}^{k}=c_0+\sum_{k=1}^{\infty}c_k z^k+\sum_{k=1}^{\infty}\overline{c}_k\overline{z}^k \quad \text{----------} (25)$$

since $\overline{c}_k=c_{-k}$.

Now let $F(z)=c_0+2\sum_{k=1}^{\infty}c_k z^k$. Plainly, F(z) is analytic in the open unit disk.

Observe that

$$u(r,\theta)=\frac{1}{2}(F(z)+\overline{F(z)})=\operatorname{Re}F(z). \quad \text{----------------------} \ (26)$$

Hence $u(r,\theta)$ is harmonic in the open unit disk.

Note that the imaginary part of F(z) is

$$V(z)=-\frac{i}{2}(F(z)-\overline{F(z)}) \quad \text{-----------------------} (27)$$

Observe that

$$v(r,\theta)=i\left(\sum_{k=1}^{\infty}c_k z^k-\sum_{k=1}^{\infty}c_{-k}\overline{z}^{k}\right)=i\left(\sum_{k=1}^{\infty}c_k z^k-\sum_{k=1}^{\infty}\overline{c}_k\overline{z}^k\right)=i\,\frac{F(z)-\overline{F(z)}}{2}.$$

Hence, $v(r,\theta)=-V(z)$. Since V(z), being the imaginary part of an analytic function in the open unit disk, is harmonic in the open unit disk, $v(r,\theta)$ is harmonic in the open unit disk. Note that V(z) is the harmonic conjugate of $u(r,\theta)$ in the sense of harmonic function. But $v(r,\theta)=-V(z)$ is conjugate function of $u(r,\theta)$ in the Fourier series sense.

We list some properties of these harmonic functions.

(1) $F(z)=u(r,\theta)-i\,v(r,\theta)$,

(2) $F(z)e^{-im\theta} = (u(r,\theta) - iv(r,\theta))e^{-im\theta} = c_0 e^{-im\theta} + 2\sum_{k=1}^{\infty} c_k r^k e^{i(k-m)\theta}$,

(3) $\frac{1}{2\pi}\int_{-\pi}^{\pi}(u(r,\theta) - iv(r,\theta))e^{-im\theta}d\theta = \begin{cases} 2c_m r^m & \text{if } m > 0 \\ c_0 & \text{if } m = 0 \end{cases}$,

(4) $\frac{1}{2\pi}\int_{-\pi}^{\pi}(u(r,\theta) - iv(r,\theta))e^{im\theta}d\theta = 0$ for $m > 0$.

Similarly,

(5) $\frac{1}{2\pi}\int_{-\pi}^{\pi}(u(r,\theta) + iv(r,\theta))e^{-im\theta}d\theta = 0$ for $m > 0$,

(6) $\frac{1}{\pi}\int_{-\pi}^{\pi}u(r,\theta)e^{-im\theta}d\theta = 2c_m r^m$ for $m > 0$,

(7) $\frac{-i}{\pi}\int_{-\pi}^{\pi}v(r,\theta)e^{-im\theta}d\theta = 2c_m r^m$ for $m > 0$.

Properties of Poisson Kernel

(1) $P(r,\theta) = \frac{1-r^2}{2(1+r^2-2r\cos(\theta))}$ is even in θ.

(2) For $0 < r < 1$, maximum of $P(r, \theta)$ occurs at $\theta = 0$ and is equal to $\frac{1+r}{2(1-r)}$.

(3) For $0 < r < 1$, $P(r, \theta)$ is monotone decreasing in $0 \le \theta \le \pi$ with a maximum at $\theta = 0$ and a minimum of $\frac{1-r}{2(1+r)}$ at $\theta = \pi$. Thus, $P(r, \theta) > 0$ for $0 \le r < 1$.

(4) For $0 \le r < 1$, $\frac{1}{\pi}\int_{-\pi}^{\pi}P(r,x)dx = 1$.

Let $g_r(\theta) = \frac{1-r^2}{2(1+r^2-2r\cos(\theta))}$. Then $\frac{\partial}{\partial\theta}g_r(\theta) = \frac{-r(1-r^2)\sin(\theta)}{(1+r^2-2r\cos(\theta))^2}$.

Therefore, $\frac{\partial}{\partial\theta} g_r(\theta) < 0$ for $0 < r < 1$ and $0 < \theta < \pi$. Hence, $g_r(\theta)$ is decreasing in $[0, \pi]$ and increasing in $[-\pi, 0]$. Maximum value of $g_r(\theta)$ occurs at $\theta = 0$ and is equal to $\frac{1+r}{2(1-r)}$ and minimum value of $\frac{1-r}{2(1+r)}$ occurring at $\theta = \pi$.

Section C

Convergence Theorems

Before we state the theorem on the A-summability of Fourier series, we prepare some preliminary definition.

We shall assume f is a Lebesgue integrable periodic function of period 2π. Let

$$g_c(t) = \frac{1}{2}\big(f(\theta+t) + f(\theta-t) - 2c \big).$$

The function f is said to satisfy condition ℓ_c if $\lim_{t\to 0}\frac{1}{t}\int_0^t g_c(x)dx = 0$. The function f is said to satisfy the stronger condition L_c if $\lim_{t\to 0}\frac{1}{t}\int_0^t |g_c(x)|dx = 0$. Plainly, L_c implies ℓ_c.

Our main theorem is:

Theorem 6. Suppose f is a Lebesgue integrable periodic function of period 2π. The Fourier series of $f(t)$, for $t = \theta$, is A-summable to $f(\theta)$ at a point of continuity and to $\frac{1}{2}\big(f(\theta+) + f(\theta-) \big)$ at a point of jump discontinuity and is uniformly A-summable in any closed interval of continuity. It is A-summable to c at any point θ at which f satisfies ℓ_c and to $f(\theta)$ almost everywhere. Moreover, $u(r,\theta) \to f(\theta)$ in the L^1 norm.

It is useful to use a closed form of the Poisson integral,

$$\int \frac{1-r^2}{2(1+r^2-2\,r\cos(\theta))}\,d\theta,$$

to investigate behaviour of the harmonic function $u(r,\ \theta)$ near the boundary of the unit disk.

We shall use trigonometric substitution to do this. Let $u=\tan\left(\frac{\theta}{2}\right)$. Then

$$\frac{du}{d\theta}=\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)=\frac{1}{2}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)=\frac{1}{2}(1+u^2)$$

and

$$\int \frac{1-r^2}{2(1+r^2-2\,r\cos(\theta))}\,d\theta=\int \frac{1-r^2}{2(1+r^2-2\,r\,\frac{1-u^2}{1+u^2})}\,\frac{2}{1+u^2}\,du$$

$$=\int \frac{1-r^2}{(1+r^2)(1+u^2)-2\,r(1-u^2)}\,du=\int \frac{1-r^2}{(1+r)^2u^2+(1-r)^2}\,du$$

$$=\frac{1-r^2}{(1+r)^2}\int \frac{1}{u^2+(\frac{1-r}{1+r})^2}\,du=\frac{1-r^2}{(1+r)^2}\,\frac{1+r}{1-r}\tan^{-1}\left(u\frac{1+r}{1-r}\right)+C$$

$$=\tan^{-1}\left(u\frac{1+r}{1-r}\right)+C=\tan^{-1}\left(\tan(\tfrac{\theta}{2})\frac{1+r}{1-r}\right)+C\,.$$

Thus, for any $h>0$,

$$\int_0^h P(r,\theta)d\theta=\int_0^h \frac{1-r^2}{2(1+r^2-2\,r\cos(\theta))}\,d\theta=\tan^{-1}\left(\tan(\tfrac{h}{2})\frac{1+r}{1-r}\right). \quad \text{----------}\ (28)$$

If $h_r\to 0$ and $\frac{h_r}{1-r}\to a$ as $r\to 1^-$, then

$$\tan(\tfrac{h_r}{2})\frac{1+r}{1-r}=\frac{1}{2}\,\frac{\sin(\frac{h_r}{2})}{(\frac{h_r}{2})}\,\frac{h_r}{1-r}\,\frac{1+r}{\cos(\frac{h_r}{2})}\to a \quad \text{as} \quad r\to 1^- \,.$$

Hence, we have from (28), the next theorem.

Theorem 7. If $h_r \to 0$ and $\dfrac{h_r}{1-r} \to a$ as $r \to 1^-$, then

$$\int_0^{h_r} P(r,\theta)d\theta = \int_0^{h_r} \frac{1-r^2}{2(1+r^2-2\,r\cos(\theta))}d\theta \to \tan^{-1}(a) \text{ as } r \to 1^- .$$

We now examine the Poisson integral of *f* to write it in a more useful form.

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} P(r,t-\theta)\,f(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{2(1+r^2-2\,r\cos(t-\theta))}\,f(t)dt .$$

$$= \frac{1}{\pi}\int_{-\pi-\theta}^{\pi-\theta} \frac{1-r^2}{2(1+r^2-2\,r\cos(u))}\,f(u+\theta)du \text{, by a change of variable } u = t-\theta,$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{2(1+r^2-2\,r\cos(u))}\,f(u+\theta)du \text{, by periodicity of } f,$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \frac{1-r^2}{2(1+r^2-2\,r\cos(u))}\,f(u+\theta)du + \frac{1}{\pi}\int_{-\pi}^{0} \frac{1-r^2}{2(1+r^2-2\,r\cos(u))}\,f(u+\theta)du$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \frac{1-r^2}{2(1+r^2-2\,r\cos(u))}\,f(u+\theta)du + \frac{1}{\pi}\int_{0}^{\pi} \frac{1-r^2}{2(1+r^2-2\,r\cos(-s))}\,f(-s+\theta)ds$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \frac{1-r^2}{2(1+r^2-2\,r\cos(u))}\,f(u+\theta)du + \frac{1}{\pi}\int_{0}^{\pi} \frac{1-r^2}{2(1+r^2-2\,r\cos(u)}\,f(-u+\theta)du$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \frac{1-r^2}{2(1+r^2-2\,r\cos(u))}\left\{f(u+\theta)+f(\theta-u)\right\}du$$

$$= \frac{1}{\pi}\int_{0}^{\pi} P(r,u)\left\{f(u+\theta)+f(\theta-u)\right\}du . \qquad \text{-------------------------} \quad (29)$$

Then

$$u(r,\theta)-c_\theta = \frac{1}{\pi}\int_{0}^{\pi} P(r,u)\left\{f(u+\theta)+f(\theta-u)\right\}du - \frac{1}{\pi}\int_{-\pi}^{\pi} c_\theta P(r,u)du ,$$

$$\text{since } \frac{1}{\pi}\int_{-\pi}^{\pi} P(r,u)du = 1,$$

$$=\frac{1}{\pi}\int_0^{\pi} P(r,u)\{f(u+\theta)+f(\theta-u)-2c_\theta\}du$$, since $P(r, u)$ is even in u,

$$=\frac{2}{\pi}\int_0^{\pi} P(r,u)\left\{\frac{f(u+\theta)+f(\theta-u)}{2}-c_\theta\right\}du. \qquad \text{---------------------------} \quad (30)$$

Let $\phi(\theta,u)=\dfrac{f(\theta+u)+f(\theta-u)}{2}$ and $g_{c_\theta}(u)=\phi(\theta,u)-c_\theta$. Then

$$u(r,\theta)-c_\theta=\frac{2}{\pi}\int_0^{\pi} P(r,u)g_{c_\theta}(u)du. \qquad \text{--------------------------} \ (31)$$

Thus,

$$\lim_{r\to 1^-} u(r,\theta)=c_\theta \Leftrightarrow \lim_{r\to 1^-}\frac{2}{\pi}\int_0^{\pi} P(r,u)g_{c_\theta}(u)du=0 \ . \quad \text{-----------------} \ (32)$$

We shall now investigate the behaviour of the Poisson integral,

$$\frac{2}{\pi}\int_0^{\pi} P(r,u)g_{c_\theta}(u)du.$$

Take a fixed $\delta > 0$ such that $0 < \delta < \pi$ and consider splitting the Poisson integral

$$u(r,\theta)-c_\theta=\frac{2}{\pi}\int_0^{\delta} P(r,u)g_{c_\theta}(u)du+\frac{2}{\pi}\int_\delta^{\pi} P(r,u)g_{c_\theta}(u)du. \ \text{----------} \quad (33)$$

Lemma 8. $\int_\delta^{\pi} P(r,u)g_{c_\theta}(u)du \to 0$ as $r\to 1^-$. If c_θ is bounded in the interval (a, b), then the convergence is uniform in θ in (a, b) as $r\to 1^-$.

Proof.

Since $P(r, \theta)$ is non-negative and monotone decreasing in $0 \le \theta \le \pi$,

$$\left|\int_\delta^{\pi} P(r,u)g_{c_\theta}(u)du\right| \le \int_\delta^{\pi} P(r,u)\left|g_{c_\theta}(u)\right|du \le P(r,\delta)\int_\delta^{\pi}\left|g_{c_\theta}(u)\right|du. \ \text{-------} \ (34)$$

Now, $\displaystyle\int_\delta^{\pi}\left|g_{c_\theta}(u)\right|du=\int_\delta^{\pi}\left|\frac{f(u+\theta)+f(\theta-u)}{2}-c_\theta\right|du$

$$\leq \int_{-\pi}^{\pi} |f(u)| du + \pi |c_\theta| . \qquad \text{-----------------------} \quad (35)$$

Since $P(r,\delta) = \dfrac{1-r^2}{2(1+r^2-2r\cos(\delta))} \to 0$ as $r \to 1^-$, it follows from (34) and (35) that $\left| \int_\delta^\pi P(r,u) g_{c_\theta}(u) du \right| \to 0$ as $r \to 1^-$. Consequently

$$\int_\delta^\pi P(r,u) g_{c_\theta}(u) du \to 0 \text{ as } r \to 1^- .$$

If c_θ is bounded in the interval (a, b), then from (35), $\int_\delta^\pi |g_{c_\theta}(u)| du$ is uniformly bounded in θ in (a, b). Hence $\int_\delta^\pi P(r,u) g_{c_\theta}(u) du \to 0$ as $r \to 1^-$ uniformly in θ in (a, b).

It follows from Lemma 8 the following A-summable criterion.

Theorem 9. Suppose the function *f* is a Lebesgue integrable periodic function of period 2π. Then $u(r,\theta) \to c_\theta$ as $r \to 1^-$ if, and only if, there exists $0 < \delta < \pi$ such that $\dfrac{2}{\pi}\int_0^\delta P(r,u) g_{c_\theta}(u) du \to 0$ as $r \to 1^-$. If c_θ is bounded in the set E, then $u(r,\theta) \to c_\theta$ uniformly in E as $r \to 1^-$ if, and only if,

$$\frac{2}{\pi}\int_0^\delta P(r,u) g_{c_\theta}(u) du \to 0 \text{ uniformly in } E \text{ as } r \to 1^- .$$

The Fourier series of $f(t)$, for $t = \theta$, is A-summable to c_θ if there exists $0 < \delta < \pi$ such that

$$\frac{2}{\pi}\int_0^\delta P(r,u) g_{c_\theta}(u) du \to 0 \text{ as } r \to 1^- .$$

If c_θ is bounded in the set E, then the Fourier series of $f(t)$, for $t = \theta$, is uniformly A-summable to c_θ in E if there exists $0 < \delta < \pi$ such that

$$\frac{2}{\pi}\int_0^\delta P(r,u) g_{c_\theta}(u) du \to 0 \text{ uniformly in } E \text{ as } r \to 1^- .$$

We now look at situation when A-summability is possible.

Theorem 10. Suppose the function *f* is a Lebesgue integrable periodic function of period 2π.

(1) If $\lim_{u\to 0^+}\phi(\theta,u)=\lim_{u\to 0^+}\frac{f(\theta+u)+f(\theta-u)}{2}$ exists and equals $\phi(\theta+)$, then $u(r,\theta)\to\phi(\theta+)$ as $r\to 1^-$. That is, the Fourier series of *f* is A-summable to $\phi(\theta+)$.

Consequently:

(2) The Fourier series of $f(t)$ at $t=\theta$ is A-summable to $f(\theta)$ at a point of continuity and to $\frac{f(\theta+)+f(\theta-)}{2}$ at a point of jump discontinuity. The Fourier series of *f* is uniformly A-summable to *f* in any closed interval of continuity.

(3) If *f* is bounded in $(a, b)\subseteq[-\pi, \pi]$, then $u(r,\theta)$ is bounded in any closed interval in (a, b). Thus, if $u(r,\theta)$ converges in a subset *E* in a closed interval in (a, b) as $r\to 1^-$, it converges boundedly in *E*. That is to say, if the Fourier series is A- summable in *E*, then it is boundedly A-summable in *E*.

Proof.

Parts (1) and (2):

Choose a $\delta>0$ such that $\delta<\pi$. Since

$$\lim_{u\to 0^+}\phi(\theta,u)=\lim_{u\to 0^+}\frac{f(\theta+u)+f(\theta-u)}{2}=\phi(\theta+),$$

there exists positive η such that $\eta<\pi$ and

$$\left|g_{c_\theta}(u)\right|=\left|\phi(\theta,u)-c_\theta\right|<\varepsilon \text{ for } |u|<\eta, \qquad \text{----------------} \quad (36)$$

where $c_\theta=\phi(\theta+)$.

Then

$$\left|\frac{2}{\pi}\int_0^{\delta} P(r,u)g_{c_\theta}(u)du\right| \le \frac{2}{\pi}\int_0^{\eta} P(r,u)\left|g_{c_\theta}(u)\right|du + \frac{2}{\pi}\left|\int_{\eta}^{\delta} P(r,u)g_{c_\theta}(u)du\right|$$

$$\le \frac{2}{\pi}\varepsilon\int_{-\pi}^{\pi} P(r,u)du + \frac{2}{\pi}\left|\int_{\eta}^{\delta} P(r,u)g_{c_\theta}(u)du\right| < \varepsilon + \frac{2}{\pi}\left|\int_{\eta}^{\delta} P(r,u)g_{c_\theta}(u)du\right|. \quad \text{-- (37)}$$

As in the proof of Lemma 8, $\int_{\eta}^{\delta} P(r,u)g_{c_\theta}(u)du \to 0$ as $r \to 1^-$. It follows then from (37),

$$\limsup_{r\to 1^-}\left|\frac{2}{\pi}\int_0^{\delta} P(r,u)g_{c_\theta}(u)du\right| \le \varepsilon .$$

Since ε is arbitrary, $\lim_{r\to 1^-}\frac{2}{\pi}\int_0^{\delta} P(r,u)g_{c_\theta}(u)du = 0$ and so $u(r,\theta) \to \phi(\theta+)$ as $r \to 1^-$.

At a point of continuity $c_\theta = \phi(\theta+) = f(\theta)$ and at a point of jump discontinuity,

$\phi(\theta+) = \dfrac{f(\theta+) + f(\theta-)}{2}$. Therefore, the Fourier series of $f(t)$ at $t = \theta$ is A-summable to $f(\theta)$ at a point of continuity and to $\dfrac{f(\theta+) + f(\theta-)}{2}$ at a point of jump discontinuity. If f is continuous in $[a, b]$, then f is uniformly continuous in $[a, b]$ and $c_\theta = \phi(\theta+) = f(\theta)$ is bounded in $[a, b]$. We may thus choose the same η for all θ in $[a, b]$ so that

$$\frac{2}{\pi}\int_{\eta}^{\delta} P(r,u)g_{c_\theta}(u)du \to 0 \text{ uniformly in } \theta \text{ in } [a,b] \text{ as } r \to 1^- .$$

It follows then from (37), that

$$u(r,\theta) \to f(\theta) \text{ uniformly in } \theta \text{ in } [a,\ b] \text{ as } r \to 1^- .$$

Thus the Fourier series of f is uniformly A-summable to f in any closed interval of continuity.

Part (3):

We now take a different perspective. Suppose f is bounded in (a, b) or $[a, b]$.

Assume that $|f(x)| \le M$ for all x in (a, b).

Let $[a_1, b_1] \subseteq (a, b)$. Let $d =$ ½ min { distance of a to $[a_1, b_1]$, distance of b to $[a_1, b_1]$}. Let $0 < \eta < \min(\delta, d)$. Thus, for any θ in $[a_1, b_1]$,

$$[\theta - d, \theta + d] \subseteq (a,b) \ .$$

In particular, for any θ in $[a_1, b_1]$,

$$|u| < \eta \Rightarrow \theta + u, \theta - u \in [\theta - \eta, \theta + \eta] \subseteq (a,b) .$$

Thus for all θ in $[a_1, b_1]$ and $|u| < \eta$,

$$\left|\phi(\theta,u)\right| = \left|\frac{f(\theta+u)+f(\theta-u)}{2}\right| \le \left|\frac{f(\theta+u)}{2}\right| + \left|\frac{f(\theta-u)}{2}\right| \le M \ . \text{ ----- } (38)$$

Hence, for all θ in $[a_1, b_1]$,

$$\left|u(r,\theta)\right| = \frac{2}{\pi}\left|\int_0^\pi P(r,u)\phi(u,\theta)du\right| \le \frac{2}{\pi}\left|\int_0^\eta P(r,u)\phi(u,\theta)du\right| + \frac{2}{\pi}\left|\int_\eta^\pi P(r,u)\phi(u,\theta)du\right|$$

$$\le \frac{2}{\pi}\int_0^\eta P(r,u)\left|\phi(u,\theta)\right|du + \frac{2}{\pi}\int_\eta^\pi P(r,u)\left|\phi(u,\theta)\right|du$$

$$\le M\,\frac{2}{\pi}\int_0^\eta P(r,u)du + \frac{2}{\pi}P(r,\eta)\int_\eta^\pi \left|\phi(u,\theta)\right|du$$

$$\le M\,\frac{2}{\pi}\int_0^\pi P(r,u)du + \frac{2}{\pi}P(r,\eta)\int_{-\pi}^\pi \left|f(u)\right|du$$

$$\le M + \frac{2}{\pi}P(r,\eta)\int_{-\pi}^\pi \left|f(u)\right|du . \text{ ----------------------------- } (39)$$

But $P(r,\eta) = \dfrac{1-r^2}{2(1+r^2-2r\cos(\eta))} = \dfrac{1-r^2}{2((r-\cos(\eta))^2+\sin^2(\eta))} \le \dfrac{1}{2\sin^2(\eta)}$.

Hence, for all θ in $[a_1, b_1]$,

$$\left|u(r,\theta)\right| \le M + \frac{1}{\pi\sin^2(\eta)}\int_{-\pi}^\pi \left|f(u)\right|du .$$

This means $u(r, \theta)$ is uniformly bounded in r and in θ in $[a_1, b_1]$. Therefore, if the limit of $u(r, \theta)$ as $r \to 1^-$ exists on a subset E contained in any closed interval in (a, b), then the convergence is boundedly. The corresponding statement about bounded A-summability in E follows.

Next we have that $u(r, \theta)$ converges to $f(\theta)$ in the L^1 norm. Note that in general $u(r, \theta)$ need not converge (pointwise) to $f(\theta)$.

Theorem 11. Suppose the function f is a Lebesgue integrable periodic function of period 2π. Then $u(r,\theta) \to f(\theta)$ in the L^1 norm.

The next result gives some insight into the behaviour of $u(r, \theta)$ at the boundary of the unit disk.

Theorem 12. If $h_r \to 0$ as $r \to 1^-$, then $u(r,\theta + h_r) \to f(\theta)$ at a point of continuity θ, and at a point of jump discontinuity,

$$u(r,\theta + h_r) - \frac{1}{\pi}\left(f(\theta+) - f(\theta-)\right)\int_0^{h_r} P(r,u)du \to \frac{1}{2}\left(f(\theta+) + f(\theta-)\right).$$

Proof.

If f is continuous at θ, then given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|u| < \delta \Rightarrow | f(\theta + u) - f(\theta)| < \varepsilon. \text{ ---------------------- (40)}$$

If $h_r \to 0$ as $r \to 1^-$, then for this value of δ, there exists $\delta_1 > 0$ such that

$$1 - \delta_1 < r < 1 \Rightarrow | h_r | < \frac{\delta}{2} . \text{ ---------------------------- (41)}$$

Therefore, for $1 - \delta_1 < r < 1$,

$$|u| < \frac{\delta}{2} \Rightarrow |\phi(\theta + h_r, u) - f(\theta)| = \frac{1}{2}\left| f(\theta + h_r + u) + f(\theta + h_r - u) - 2f(\theta)\right|$$

$$\leq \frac{1}{2}\left(\left| f(\theta + h_r + u) - f(\theta)\right| + \left| f(\theta + h_r - u) - f(\theta)\right|\right) < \frac{1}{2}\left(2\varepsilon\right) = \varepsilon . \text{ ----- (42)}$$

Therefore, for $1-\delta_1 < r < 1$,

$$|u(r,\theta+h_r)-f(\theta)| = \frac{2}{\pi}\left|\int_0^{\pi} P(r,u)\{\phi(\theta+h_r,u)-f(\theta)\}du\right|$$

$$\le \frac{2}{\pi}\left|\int_{\delta/2}^{\pi} P(r,u)\{\phi(\theta+h_r,u)-f(\theta)\}du\right| + \frac{2}{\pi}\left|\int_0^{\delta/2} P(r,u)\{\phi(\theta+h_r,u)-f(\theta)\}du\right|$$

$$\le \frac{2}{\pi}\int_{\delta/2}^{\pi} P(r,u)|\phi(\theta+h_r,u)-f(\theta)|du + \frac{2}{\pi}\int_0^{\delta/2} P(r,u)|\phi(\theta+h_r,u)-f(\theta)|du$$

$$\le \frac{2}{\pi}P(r,\frac{\delta}{2})\int_{\delta/2}^{\pi}|\phi(\theta+h_r,u)-f(\theta)|du + \frac{2}{\pi}\int_0^{\delta/2} P(r,u)\varepsilon du$$

$$\le \frac{2}{\pi}P(r,\frac{\delta}{2})\left\{\int_{-\pi}^{\pi}|f(u)|du + \pi|f(\theta)|\right\} + \varepsilon . \quad \text{------------------------------------} \quad (43)$$

Since $P(r,\delta/2)\to 0$ as $r\to 1^-$, it follows from (43) that

$$\limsup_{r\to 1^-}|u(r,\theta+h_r)-f(\theta)| \le \varepsilon .$$

Consequently, since ε is arbitrary, $\lim\limits_{r\to 1^-}\big(u(r,\theta+h_r)-f(\theta)\big)=0$. This means

$u(r,\theta+h_r)\to f(\theta)$ as $r\to 1^-$.

Suppose now *f* has a jump discontinuity at θ. By redefining *f* at θ, we may assume that $f(\theta)=\frac{1}{2}\big(f(\theta+)+f(\theta-)\big)$.

Let $d = f(\theta+)-f(\theta-)$. Let $\ell(t-\theta)=\sum\limits_{n=1}^{\infty}\frac{\sin(n(t-\theta))}{n}$.

Let $g(t)=f(t)-\frac{d}{\pi}\ell(t-\theta)=f(t)-k(t)$, $k(t)=\frac{d}{\pi}\ell(t-\theta)$.

Observe that $\ell(t-\theta)=\frac{1}{2}(\pi-(t-\theta))$ for $0<t-\theta<2\pi$. Note that $\sum\limits_{n=1}^{\infty}\frac{\sin(n\,t)}{n}$ converges boundedly to the function $\ell(t)=\frac{1}{2}(\pi-t)$ for $0 < t < 2\pi$ and $\ell(0)=0$. It converges uniformly in any closed interval free from multiples of 2π. (See

Integrating a Fourier series in *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric Series, page 34*.)

(See also Theorem 9 and Theorem 14 in *Fourier Cosine and Sine Series, page 45*.)

$\ell(t)$ is odd and has a jump of π at 0.

Note that $g(\theta) = f(\theta) = \frac{1}{2}\left(f(\theta+) + f(\theta-)\right)$. Observe that

$$\lim_{t\to\theta^+} g(t) = f(\theta+) - \frac{d}{\pi}\lim_{t\to\theta^+}\ell(t-\theta) = f(\theta+) - \frac{d}{\pi}\frac{\pi}{2}$$

$$= f(\theta+) - \frac{1}{2}\left(f(\theta+) - f(\theta-)\right) = \frac{1}{2}\left(f(\theta+) + f(\theta-)\right) \text{ and}$$

$$\lim_{t\to\theta^-} g(t) = f(\theta-) - \frac{d}{\pi}\lim_{t\to\theta^-}\ell(t-\theta) = f(\theta-) + \frac{d}{\pi}\frac{\pi}{2}$$

$$= f(\theta-) + \frac{1}{2}\left(f(\theta+) - f(\theta-)\right) = \frac{1}{2}\left(f(\theta+) + f(\theta-)\right).$$

Hence, $\lim_{t\to\theta} g(t) = \frac{1}{2}\left(f(\theta+) + f(\theta-)\right) = g(\theta)$ and g is continuous at θ.

Plainly, *g* is periodic and Lebesgue integrable and so has a Fourier series. Therefore, by what we have just proved for a point of continuity,

$$u(r,\theta+h_r,g) \to g(\theta) = \frac{1}{2}\left(f(\theta+) + f(\theta-)\right) \text{ as } r \to 1^- . \text{ ---------------- (44)}$$

Here, we add an additional third parameter to the notation $u(r,\theta,g)$ to include a reference to the function *g* whose Fourier series is used.

Now we consider the function *k*, which is of course periodic and Lebesgue integrable.

$$u(r,\theta+h_r,k) = \frac{2}{\pi}\int_0^\pi P(r,t)\left\{\frac{k(\theta+h_r+t)+k(\theta+h_r-t)}{2}\right\}dt$$

$$= \frac{2}{\pi}\frac{d}{\pi}\int_0^\pi P(r,t)\left\{\frac{\ell(h_r+t)+\ell(h_r-t)}{2}\right\}dt \quad . \text{ ----------------- (45)}$$

Now,

$$\ell(h_r+t)+\ell(h_r-t)=\begin{cases}\frac{1}{2}(\pi-h_r-t)+\frac{1}{2}(\pi-h_r+t),\ 0<t<h_r\\ \frac{1}{2}(\pi-h_r-t)-\frac{1}{2}(\pi-(t-h_r)),\ h_r<t<\pi\end{cases}$$

$$=\begin{cases}\pi-h_r,\ 0<t<h_r\\ -h_r,\ h_r<t<\pi\end{cases}\quad . \quad \text{-------------------------} \quad (46)$$

Therefore, in view of (45) and (46), we have,

$$u(r,\theta+h_r,k)$$
$$=\frac{d}{\pi^2}\int_0^{h_r}P(r,t)\left\{\ell(h_r+t)+\ell(h_r-t)\right\}dt+\frac{d}{\pi^2}\int_{h_r}^{\pi}P(r,t)\left\{\ell(h_r+t)+\ell(h_r-t)\right\}dt$$

$$=\frac{d}{\pi^2}\int_0^{h_r}P(r,t)\left(\pi-h_r\right)dt-\frac{d}{\pi^2}\int_{h_r}^{\pi}P(r,t)h_r dt$$

$$=\frac{d}{\pi}\int_0^{h_r}P(r,t)dt-\frac{d}{\pi^2}h_r\int_0^{\pi}P(r,t)dt=\frac{d}{\pi}\int_0^{h_r}P(r,t)dt-\frac{d}{2\pi}h_r. \quad \text{-----} \quad (47)$$

Now, $u(r,\theta+h_r,f)=u(r,\theta+h_r,g)+u(r,\theta+h_r,k)$

$$=u(r,\theta+h_r,g)+\frac{d}{\pi}\int_0^{h_r}P(r,t)dt-\frac{d}{2\pi}h_r.$$

Therefore,

$$u(r,\theta+h_r,f)-\frac{d}{\pi}\int_0^{h_r}P(r,t)dt=u(r,\theta+h_r,g)-\frac{d}{2\pi}h_r\to g(\theta)\text{ as }\ r\to 1^-.$$

This completes the proof.

The following gives the bounds for the harmonic function $u(r,\ \theta)$ a kind of "maximum and minimum principle" for such a harmonic function.

Lemma 13. Suppose the function f is Lebesgue integrable and of period 2π.

(1) If max $\{\, | f(\theta) | : \theta \in [-\pi, \pi]\}$ exists, then

$$|u(r,\theta)| \le \max\{|f(\theta)| : \theta \in [-\pi,\pi]\} \quad .$$

(2) $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} |u(r,\theta)|\, d\theta \le \dfrac{1}{\pi}\int_{-\pi}^{\pi} |f(\theta)|\, d\theta$.

Proof.

(1) If max $\{\, | f(\theta) | : \theta \in [-\pi, \pi]\}$ exists and equals K, then

$$|u(r,\theta)| = \frac{1}{\pi}\left|\int_{-\pi}^{\pi} P(r,u)\, f(u+\theta)du\right| \le \frac{1}{\pi} K \int_{-\pi}^{\pi} P(r,u)du = K ,$$

since $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} P(r,u)du = 1$.

(2) $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} |u(r,\theta)|\, d\theta \le \dfrac{1}{\pi^2}\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi} P(r,u)\, f(u+\theta)du\right| d\theta$

$$\le \frac{1}{\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} P(r,u)|f(u+\theta)|du\right)d\theta$$

$$= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} P(r,u)|f(u+\theta)|d\theta\right)du ,$$

by Fubini Theorem for non-negative functions,

$$= \frac{1}{\pi^2}\int_{-\pi}^{\pi} P(r,u)\left(\int_{-\pi}^{\pi} |f(u+\theta)|d\theta\right)du$$

$$= \frac{1}{\pi^2}\int_{-\pi}^{\pi} P(r,u)du \int_{-\pi}^{\pi} |f(u+\theta)|d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |f(\theta)|d\theta ,$$

since f is periodic of period 2π and $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} P(r,u)du = 1$.

Proof of Theorem 11.

The proof is similar to the proof of that the sequence of (C,1) means of the Fourier series of f converges to f in the L^1 norm as in Theorem 12* in *Fourier Cosine and Sine Series*. Indeed the proof can be carried out in exactly the same manner as in the proof of Theorem 12* in *Fourier Cosine and Sine Series*, replacing the Fejér kernel by the Poisson kernel and taking limits as r tends to 1 on the left. The following proof differs in detail by not using explicitly the continuity of the function $\eta(u)=\int_{-\pi}^{\pi}|f(x+u)-f(x)|dx$ and that the Fourier series of η at u = 0 is A-summable to 0. (Theorem 10 part 2). The proof below may be applied to summation method with kernel having properties similar to Fejér kernel. We may of course prove Theorem 12* using the method below.

By Theorem 13 of *Fourier Cosine and Sine Series*, we may take a continuous approximation f * of f such that

$$\int_{-\pi}^{\pi}|f^*(t)-f(t)|dt<\frac{\varepsilon}{3} \quad . \text{ ---------------------- (48)}$$

We can extend f * to a periodic function on $\mathbb{R}$.

Then $$\int_{-\pi}^{\pi}|f(\theta+u)-f^*(\theta+u)|du=\int_{-\pi}^{\pi}|f(u)-f^*(u)|du<\frac{\varepsilon}{3}. \text{ -------- (49)}$$

$$|u(r,\theta,f)-u(r,\theta,f^*)|=\frac{1}{\pi}\left|\int_{-\pi}^{\pi}P(r,u)f(u+\theta)du-\int_{-\pi}^{\pi}P(r,u)f^*(u+\theta)du\right|$$

$$=\frac{1}{\pi}\left|\int_{-\pi}^{\pi}P(r,u)\big(f(u+\theta)-f^*(\theta+u)\big)du\right|$$

$$\leq\frac{1}{\pi}\int_{-\pi}^{\pi}P(r,u)|f(u+\theta)-f^*(\theta+u)|du. \qquad \text{---------------- (50)}$$

Therefore,

$$\int_{-\pi}^{\pi}|u(r,\theta,f)-u(r,\theta,f^*)|d\theta\leq\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}P(r,u)|f(u+\theta)-f^*(\theta+u)|du\right)d\theta$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}P(r,u)|f(u+\theta)-f^*(\theta+u)|d\theta\right)du,$$

by Fubini Theorem,

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} P(r,u)du \int_{-\pi}^{\pi} \left| f(u+\theta) - f^*(\theta+u) \right| d\theta$$

$$= \int_{-\pi}^{\pi} \left| f(u+\theta) - f^*(\theta+u) \right| d\theta < \frac{\varepsilon}{3}, \quad \text{------- (51)}$$

by (49).

$$\int_{-\pi}^{\pi} \left| u(r,\theta,f) - f(\theta) \right| d\theta \le \int_{-\pi}^{\pi} \left| u(r,\theta,f) - u(r,\theta,f^*) \right| d\theta$$

$$+ \int_{-\pi}^{\pi} \left| u(r,\theta,f^*) - f^*(\theta) \right| d\theta + \int_{-\pi}^{\pi} \left| f(\theta) - f^*(\theta) \right| d\theta$$

$$\le \int_{-\pi}^{\pi} \left| u(r,\theta,f^*) - f^*(\theta) \right| d\theta + \frac{2}{3}\varepsilon, \text{ by (49) and (51).}$$

------------------ (52).

By Theorem 10 part (2), since f^* is continuous,

$$u(r,\theta,f^*) \to f^*(\theta) \text{ uniformly in } \theta \text{ as } r \to 1^-.$$

This means there exists $\delta_2 > 0$ such that for all θ,

$$0 < r - 1 < \delta_2 \quad \Rightarrow \left| u(r,\theta,f^*) - f^*(\theta) \right| < \frac{\varepsilon}{6\pi} . \quad \text{--------------- (53)}$$

It follows then from (52) and (53), that

$$\int_{-\pi}^{\pi} \left| u(r,\theta,f) - f(\theta) \right| d\theta < 2\pi \frac{\varepsilon}{6\pi} + \frac{2}{3}\varepsilon = \varepsilon$$

for $1 - \delta_2 < r < 1$. That is to say, $u(r,\theta) \to f(\theta)$ in the L^1 norm.

Theorem 14. Suppose the function f is a Lebesgue integrable periodic function of period 2π.

(1) Suppose at a point θ, f satisfies ℓ_c. That is, $\lim_{t\to 0}\frac{1}{t}\int_0^t g_c(x)dx = 0$, where

$g_c(t) = \frac{1}{2}\big(f(\theta+t) + f(\theta-t) - 2c\big)$. Then

$$u(r,\theta) \to c \text{ as } r \to 1^- .$$

This means the Fourier series of f is A-summable at a point θ where f satisfies ℓ_c.

(2) Since f is Lebesgue integrable, almost every point is a Lebesgue point, i.e., at almost every point θ, f satisfies $\ell_{f(\theta)}$. The Fourier series of f is A-summable to $f(\theta)$ almost everywhere.

Proof.

(1) Observe that the partial derivative of the Poisson kernel,

$$P'(r,t) = \frac{\partial P}{\partial t}(r,t) \le 0$$

for $0 \le r < 1$ and $0 \le t \le \pi$. (See properties of $P(r,t)$.)

By integration by parts,

$$\int_0^\pi tP'(r,t)dt = \big[tP(r,t)\big]_0^\pi - \int_0^\pi P(r,t)dt$$

$$= \pi P(r,\pi) - \frac{\pi}{2} = \frac{\pi}{2}\frac{1-r}{1+r} - \frac{\pi}{2} = -\pi\frac{r}{1+r} \text{ , ------------ } (54)$$

$$\text{since } \frac{2}{\pi}\int_0^\pi P(r,t)dt = 1.$$

Therefore, for $0 \le r < 1$,

$$\int_0^\pi t\left|P'(r,t)\right|dt = -\int_0^\pi tP'(r,t)dt = \pi\frac{r}{1+r} < \frac{\pi}{2}. \text{ ---------------- } (55)$$

Since $\lim_{t\to 0}\frac{1}{t}\int_0^t g_c(x)dx = 0$, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < t \le \delta \Rightarrow \left| \frac{1}{t} \int_0^t g_c(u)du \right| < \varepsilon \Rightarrow \left| \int_0^t g_c(u)du \right| < t\varepsilon . \qquad \text{-------------} \ (56)$$

For this value of δ,

$$\int_0^\delta P(r,t)g_c(t)dt = \left[\Phi(t)P(r,t)\right]_0^\delta - \int_0^\delta \Phi(t)P'(r,t)dt ,$$

where $\Phi(t) = \int_0^t g_c(u)du$,

$$= \Phi(\delta)P(r,\delta) - \int_0^\delta \Phi(t)P'(r,t)dt . \qquad \text{--------------------} \ (57)$$

Therefore,

$$\left| \int_0^\delta P(r,t)g_c(t)dt \right| \le \left| \Phi(\delta)P(r,\delta) \right| + \left| \int_0^\delta \Phi(t)P'(r,t)dt \right|$$

$$\le P(r,\delta) \left| \int_0^\delta g_c(u)du \right| + \int_0^\delta \left|\Phi(t)\right| \left|P'(r,t)\right| dt$$

$$\le P(r,\delta)\varepsilon\delta + \int_0^\delta \varepsilon t \left|P'(r,t)\right| dt\varepsilon , \ \text{by (56)}$$

$$< \varepsilon\delta \frac{1-r^2}{2(1+r^2-2\,r\cos(\delta))} + \varepsilon \frac{\pi}{2} \ \text{ by (55).}$$

Therefore,

$$\limsup_{r \to 1^-} \left| \int_0^\delta P(r,t)g_c(t)dt \right| \le \frac{\pi}{2}\varepsilon .$$

Since ε is arbitrary, $\lim_{r \to 1^-} \int_0^\delta P(r,t)g_c(t)dt = 0$.

Hence, by Theorem 8, the Fourier series is A-summable to *c*.

(2). Suppose f is Lebesgue integrable. Then by Lemma 10 of *Convergence of Fourier Series* (*Page 9*), almost every point is a Lebesgue point of f.

That is to say, for almost every θ in $[-\pi, \pi]$,

$$\lim_{h\to 0}\frac{1}{h}\int_0^h \left| f(\theta+x)-f(\theta)\right| dx = 0. \qquad \text{------------} \quad (58)$$

Plainly, this implies $\lim_{h\to 0}\frac{1}{h}\int_0^h \left| f(\theta-x)-f(\theta)\right| dx = 0$ for almost every θ in $[-\pi, \pi]$.

Now, for $h \neq 0$,

$$\left|\frac{1}{h}\int_0^h g_{f(\theta)}(x)dx\right| = \left|\frac{1}{h}\int_0^h \frac{1}{2}\left(f(\theta+x)+f(\theta-x)-2f(\theta)\right)dx\right|$$

$$\leq \frac{1}{2}\left|\frac{1}{h}\int_0^h \left(f(\theta+x)-f(\theta)\right)dx\right| + \frac{1}{2}\left|\frac{1}{h}\int_0^h \left(f(\theta-x)-f(\theta)\right)dx\right|$$

$$\leq \frac{1}{2}\left|\frac{1}{h}\int_0^h \left| f(\theta+x)-f(\theta)\right| dx\right| + \frac{1}{2}\left|\frac{1}{h}\int_0^h \left| f(\theta-x)-f(\theta)\right| dx\right|. \quad \text{------------} \ (59)$$

Therefore, by the Comparison Test and using (58) and (59), we deduce that, for almost all θ, $\lim_{h\to 0}\left|\frac{1}{h}\int_0^h g_{f(\theta)}(x)dx\right| = 0$ and so $\lim_{h\to 0}\frac{1}{h}\int_0^h g_{f(\theta)}(x)dx = 0$.

Hence, f satisfies condition $\ell_{f(\theta)}$ at θ for almost all θ.

Therefore, by part (1), for almost all θ, $u(r,\theta) \to f(\theta)$ as $r \to 1^-$. That is, for almost all θ, the Fourier series of $f(\theta)$ is A-summable to $f(\theta)$.

This completes the proof.

Proof of Theorem 6.

Theorem 6 is now a consequence of Theorem 10, Theorem 11 and Theorem 14.

Application to Harmonic Functions

Theorem 15 . Uniqueness Theorem

Suppose H is a function continuous in the closed unit disk and harmonic in the open unit disk. Let $f(\theta) = H(1, \theta)$, where H is expressed in polar coordinate.

Then the function $u(r,\theta) \to f(\theta)$ as $r \to 1^-$. $u(r, \theta)$ is harmonic in the open unit disk and $u(r,\theta) = H(r,\theta)$ identically in the closed unit disk. Moreover, the Fourier series of $f(\theta)$ is A –summable to $f(\theta)$.

Proof.

We know by Theorem 5 that $u(r, \theta)$ is harmonic in the open unit disk. Moreover, since f is continuous, by Theorem 12, we can extend $u(r, \theta)$ to the boundary of the unit disk to obtain a continuous function on the closed unit disk. By Theorem 10 part (2), $u(r,\theta) \to f(\theta) = H(1,\theta)$ as $r \to 1^-$ for all θ. Since now the extended function agrees with H on the boundary of the closed unit disk, $u(r,\theta) = H(r,\theta)$ identically in the closed unit disk. Of course the Fourier series of $f(\theta)$ is A –summable to $f(\theta)$.

So long as f is Lebesgue integrable, $u(r,\theta)$ is harmonic in the open unit disk.

If f is continuous at two different points with two different values, then $u(r,\theta)$ tends to these two different values as $r \to 1^-$. Consequently, $u(r,\theta)$ is non-constant. Hence, by the maximum-minimum principle of harmonic function, $u(r,\theta)$ cannot attain its maximum nor minimum in the open unit disk.

Thus, we have the following special case of harmonic function derived from the Poisson integral.

Theorem 16. Suppose the function f is a Lebesgue integrable periodic function of period 2π. Suppose f is continuous at two distinct points in $[-\pi, \pi]$ with two distinct values. Then $u(r,\theta)$ is harmonic in the open unit disk and does not have an absolute maximum nor minimum in the open unit disk.

Suppose f is a Lebesgue integrable periodic function of period 2π. Suppose f is not equal to a constant function almost everywhere. Suppose f attains its maximum M and minimum m at θ_1 and θ_2 with $m < M$. Since almost every point in $[-\pi, \pi]$ is a Lebesgue point and f is non-constant almost everywhere, there must exist two points ζ_1 and ζ_2 in $[-\pi, \pi]$ such that $f(\zeta_1) \neq f(\zeta_2)$ with

$u(r,\zeta_1) \to f(\zeta_1)$ and $u(r,\zeta_2) \to f(\zeta_2)$ as $r \to 1^-$. Consequently $u(r,\theta)$ is non-constant in the unit open disk. Moreover,

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} P(r,u) f(u+\theta)du \le \frac{1}{\pi} M \int_{-\pi}^{\pi} P(r,u)du = M = f(\theta_1)$$

and $$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} P(r,u) f(u+\theta)du \ge \frac{1}{\pi} m \int_{-\pi}^{\pi} P(r,u)du = m = f(\theta_2).$$

Hence, we have:

Theorem 16. Suppose f is a Lebesgue integrable periodic function of period 2π and f is not equal to a constant function almost everywhere. Then

(1) $u(r,\theta)$ is harmonic in the open unit disk and does not have an absolute maximum nor minimum in the open unit disk.

(2) If f attains its maximum M and minimum m at θ_1 and θ_2 with $m < M$, then $m = f(\theta_2) < u(r,\theta) < f(\theta_1) = M$ for all $0 \le r < 1$ and $-\pi \le \theta \le \pi$.

If f satisfies $\ell_{f(\theta_1)}$ at θ_1, then the supremum of $u(r,\theta)$ over the open unit disk is $f(\theta_1)$. If f satisfies $\ell_{f(\theta_2)}$ at θ_2, then the infimum of $u(r,\theta)$ over the open unit disk is $f(\theta_2)$. Consequently, if f is continuous at θ_1, then the supremum of $u(r,\theta)$ over the open unit disk is $f(\theta_1)$ and if f is continuous at θ_2, then the infimum of $u(r,\theta)$ over the open unit disk is $f(\theta_2)$

Proof.

We only need to prove the last few statements in part (2).

If f satisfies $\ell_{f(\theta_1)}$ at θ_1, then $u(r,\theta_1) \to f(\theta_1)$ as $r \to 1^-$. Therefore, it follows from the fact that $u(r,\theta) < f(\theta_1)$, the supremum of $u(r,\theta)$ over the open unit disk is $f(\theta_1)$. Similarly, if f satisfies $\ell_{f(\theta_2)}$ at θ_2, then $u(r,\theta_2) \to f(\theta_2)$ as $r \to 1^-$ and the infimum of $u(r,\theta)$ over the open unit disk is $f(\theta_2)$. If f is continuous at θ, then f satisfies $L_{f(\theta)}$ and a fortiori, $\ell_{f(\theta)}$ at θ. Therefore, the last statement in part (2) follows from the previous statement.

In general, we may not be able to extend $u(r, \theta)$ to the boundary of the unit disk to define a meaningful value at $(1, \theta)$. Suppose f has a jump discontinuity of $d = f(\theta+) - f(\theta-)$ at θ. Then by Theorem 12, if $h_r \to 0$ as $r \to 1^-$,

$$u(r,\theta+h_r) - \frac{1}{\pi}\big(f(\theta+) - f(\theta-)\big)\int_0^{h_r} P(r,u)du \to \frac{1}{2}\big(f(\theta+) + f(\theta-)\big).$$

And if we take any value *a and* choose a path $(r, h_r + \theta)$ making an angle of $\tan^{-1}(a)$ with the radial vector connecting the origin to $(1, \theta)$ *in such a way that* $h_r \to 0$ and $\dfrac{h_r}{1-r} \to a$ as $r \to 1^-$, connecting say $(1/2, h_{1/2} + \theta)$ to $(1, \theta)$, then by Theorem 7,

$$\int_0^{h_r} P(r,\theta)d\theta = \int_0^{h_r} \frac{1-r^2}{2(1+r^2-2\,r\cos(\theta))}d\theta \to \tan^{-1}(a) \text{ as } r \to 1^- .$$

It then follows that

$$u(r,\theta+h_r) \to \frac{1}{\pi}\big(f(\theta+) - f(\theta-)\big)\tan^{-1}(a) + \frac{1}{2}\big(f(\theta+) + f(\theta-)\big),$$

i.e.,

$$u(r,\theta+h_r) \to \frac{d}{\pi}\tan^{-1}(a) + \frac{1}{2}\big(f(\theta+) + f(\theta-)\big).$$

as $r \to 1^-$.

Thus, for different values of a, $u(r,\theta+h_r)$ tends to different values.

In particular when the path is tangential to the unit circle at $(1, \theta)$, when $a = \infty$, $u(r,\theta+h_r) \to \dfrac{d}{\pi}\dfrac{\pi}{2} + \dfrac{1}{2}\big(f(\theta+) + f(\theta-)\big) = f(\theta+)$ as $r \to 1^-$. Along the radial line, $u(r,\theta+h_r) \to \dfrac{1}{2}\big(f(\theta+) + f(\theta-)\big)$ as $r \to 1^-$. Therefore, we cannot extend the function $u(r, \theta)$ to the boundary of the unit disk at $(1, \theta)$ meaningfully.

Section D

Derived Series of Fourier Series

Definition 17. Suppose g is a real valued-function defined in an open interval I. Let θ be in I. Suppose the limit

$$\lim_{h\to 0}\frac{g(\theta+h)-g(\theta-h)}{2h}$$

exists. Then we call this the *generalised symmetric derivative* of g at θ and denote it by $Dg(\theta)$.

Suppose f is a periodic Lebesgue integrable function of period 2π. Let $k(t)=\int_0^t f(u)du$. Then k is absolutely continuous and differentiable almost everywhere.

Suppose c is a real number and let $g_c(t)=\frac{1}{2}\big(f(\theta+t)+f(\theta-t)-2c\big)$. Recall that f satisfies condition ℓ_c at θ, if $\lim_{t\to 0}\frac{1}{t}\int_0^t g_c(x)dx=0$. The following is a sufficient condition for f to satisfy condition ℓ_c at θ.

Lemma 18. Let $k(t)=\int_0^t f(u)du$. Then the *generalised symmetric derivative* of f at θ, $Dk(\theta)$ exists and is equal to c if, and only if, f satisfies condition ℓ_c at θ. Consequently if $Dk(\theta)=c$, the Fourier series of f at θ is A-summable to c.

Proof.

$$\frac{1}{t}\int_0^t g_c(u)du=\frac{1}{t}\int_0^t\frac{1}{2}\big(f(\theta+u)+f(\theta-u)-2c\big)du$$

$$=\frac{1}{t}\int_0^t \frac{1}{2}\left(f(\theta+u)+f(\theta-u)\right)du-c$$

$$=\frac{1}{2t}\int_0^t f(\theta+u)du+\frac{1}{2t}\int_0^t f(\theta-u)du-c$$

$$=\frac{1}{2t}\int_\theta^{\theta+t} f(s)ds-\frac{1}{2t}\int_\theta^{\theta-t} f(s)ds-c$$

$$=\frac{1}{2t}\int_0^{\theta+t} f(s)ds-\frac{1}{2t}\int_0^{\theta-t} f(s)ds-c$$

$$=\frac{1}{2t}\int_0^{\theta+t} f(s)ds-\frac{1}{2t}\int_0^{\theta-t} f(s)ds-c$$

$$=\frac{k(\theta+t)-k(\theta-t)}{2t}-c$$

Thus, $\lim\limits_{t\to 0}\frac{1}{t}\int_0^t g_c(x)dx=0 \Leftrightarrow \lim\limits_{t\to 0}\frac{k(\theta+t)-k(\theta-t)}{2t}=c$.

Hence, f satisfies condition ℓ_c at θ if, and only if, the symmetric derivative of $k(t)$ exists and is equal to c at θ. Therefore, by Theorem 6, the Fourier series of f at θ is A-summable to c.

Now we look at the situation with the derived series of the Fourier series of a Lebesgue integrable function. We can deduce a similar conclusion, if f has a generalized symmetric derivative at θ, then the derived series of the Fourier series at θ is A summable to $Df(\theta)$.

Suppose

$$\frac{1}{2}a_0+\sum_{k=1}^{\infty}A_k(\theta)=\frac{1}{2}a_0+\sum_{k=1}^{\infty}\left(a_k\cos(k\theta)+b_k\sin(k\theta)\right). \text{ --------- } (60)$$

Then its derived series is given by a formal differentiation term by term of (60),

$$\sum_{k=1}^{\infty}k\left(b_k\cos(k\theta)-a_k\sin(k\theta)\right)=\sum_{k=1}^{\infty}kB_k(\theta)\ , \text{ ------------------ } (61).$$

where $B_k(\theta)=b_k\cos(k\theta)-a_k\sin(k\theta)$.

We shall show that, if the generalized symmetric derivative at θ, $Df(\theta)$, exists, then $\sum_{k=1}^{\infty}k\left(b_k\cos(k\theta)-a_k\sin(k\theta)\right)r^k\to Df(\theta)$ as $r\to 1^-$.

We begin with

$$u(r,\theta)=\frac{1}{2}a_0+\sum_{k=1}^{\infty}A_k(\theta)r^k=\frac{1}{2}a_0+\sum_{k=1}^{\infty}\left(a_k\cos(k\theta)+b_k\sin(k\theta)\right)r^k$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}P(r,t-\theta)f(t)dt=\frac{1}{\pi}\int_{-\pi}^{\pi}P(r,t)f(\theta+t)dt\,.$$

$$\frac{\partial}{\partial\theta}u(r,\theta)=\frac{\partial}{\partial\theta}\frac{1}{\pi}\int_{-\pi}^{\pi}P(r,t-\theta)f(t)dt$$

$$=-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial t}P(r,t-\theta)f(t)dt=-\frac{1}{\pi}\int_{-\pi-\theta}^{\pi-\theta}\frac{\partial}{\partial t}P(r,s)f(s+\theta)ds$$

$$=-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial t}P(r,s)f(s+\theta)ds\text{ ,by periodicity,}$$

$$=-\frac{1}{\pi}\int_{0}^{\pi}\frac{\partial}{\partial t}P(r,s)f(s+\theta)ds-\frac{1}{\pi}\int_{-\pi}^{0}\frac{\partial}{\partial t}P(r,s)f(s+\theta)ds$$

$$=-\frac{1}{\pi}\int_{0}^{\pi}\frac{\partial}{\partial t}P(r,s)f(s+\theta)ds+\frac{1}{\pi}\int_{\pi}^{0}\frac{\partial}{\partial t}P(r,-s)f(-s+\theta)ds\,. \quad \text{------- (62)}$$

But $\frac{\partial}{\partial t}P(r,t)=\frac{-r(1-r^2)\sin(t)}{(1+r^2-2r\cos(t))^2}$ so that $\frac{\partial}{\partial t}P(r,-s)=-\frac{\partial}{\partial t}P(r,s)$ and so it follows from (62) that

$$\frac{\partial}{\partial\theta}u(r,\theta)=-\frac{1}{\pi}\int_{0}^{\pi}\frac{\partial}{\partial t}P(r,s)f(s+\theta)ds+\frac{1}{\pi}\int_{0}^{\pi}\frac{\partial}{\partial t}P(r,s)f(-s+\theta)ds$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)\right)ds$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\frac{r(1-r^2)\sin(s)}{(1+r^2-2r\cos(s))^2}\left(f(\theta+s)-f(\theta-s)\right)ds\,. \quad \text{------- (63)}$$

Note that $-\dfrac{\partial}{\partial t}P(r,s)\sin(s)=\dfrac{r(1-r^2)\sin^2(s)}{(1+r^2-2\,r\cos(s))^2}\geq 0$.

Now, for $0\leq r<1$,

$$\frac{1}{\pi}\int_0^{\pi}2\sin(s)\frac{\partial}{\partial t}P(r,s)ds=\left[2\sin(s)P(r,s)\right]_{s=0}^{s=\pi}-\frac{1}{\pi}\int_0^{\pi}2\cos(s)P(r,s)ds\,,$$

by integration by parts,

$$=-\frac{1}{\pi}\int_0^{\pi}2\cos(s)P(r,s)ds$$

$$=-\frac{1}{\pi}\int_0^{\pi}\frac{\cos(s)(1-r^2)}{1+r^2-2r\cos(s)}ds$$

$$=-\frac{1-r^2}{\pi}\int_0^{\pi}\frac{\cos(s)}{1+r^2-2r\cos(s)}ds=-\frac{1-r^2}{\pi}\frac{\pi r}{1-r^2}=-r\,,\text{ by (22)*.}$$

Therefore,

$$\frac{\partial}{\partial\theta}u(r,\theta)-Df(\theta)==\frac{1}{\pi}\int_0^{\pi}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)\right)ds-Df(\theta)$$

$$=\frac{1}{\pi}\int_0^{\pi}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)\right)ds+\frac{1}{\pi}\int_0^{\pi}2\sin(s)\frac{\partial}{\partial t}P(r,s)\frac{Df(\theta)}{r}ds$$

$$=\frac{1}{\pi}\int_0^{\pi}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds\,.\quad\text{----------- (64)}$$

Hence, $\lim\limits_{r\to 1^-}\dfrac{\partial}{\partial\theta}u(r,\theta)-Df(\theta)=0$

$$\Leftrightarrow\lim_{r\to 1^-}\frac{1}{\pi}\int_0^{\pi}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds=0\,.\quad\text{-- (65)}$$

Since $\displaystyle\lim_{s\to 0}\frac{f(\theta+s)-f(\theta-s)}{2\sin(s)}=\lim_{s\to 0}\frac{f(\theta+s)-f(\theta-s)}{2s}\lim_{s\to 0}\frac{s}{\sin(s)}$

$$=\lim_{s\to 0}\frac{f(\theta+s)-f(\theta-s)}{2s}\cdot 1=Df(\theta)\,,$$

given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0<|s|<\delta \Rightarrow \left|\frac{f(\theta+s)-f(\theta-s)}{2\sin(s)}-Df(\theta)\right|<\varepsilon . \quad \text{-------------} (66)$$

In view of (66), we write

$$\frac{1}{\pi}\int_0^{\pi}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds$$

$$=\frac{1}{\pi}\int_0^{\delta}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds$$

$$+\frac{1}{\pi}\int_{\delta}^{\pi}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds . \text{---} (67)$$

We may choose δ in (66) to satisfy $0 < \delta < \pi/2$.

Note that for $\delta \le s \le \pi$ *and* $0 \le r < 1$,

$$\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)$$

$$=\frac{r(1-r^2)\sin(s)}{(1+r^2-2r\cos(s))^2}\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right).$$

For $\delta \le s \le \pi-\delta$ and $0 \le r < 1$,

$$\left|\frac{r(1-r^2)\sin(s)}{(1+r^2-2r\cos(s))^2}\right|=\left|\frac{r(1-r^2)\sin(s)}{((r-\cos(s))^2+\sin^2(s))^2}\right|\le\frac{r(1-r^2)}{\sin^4(\delta)} .$$

For $\pi-\delta \le s \le \pi$ and $0 \le r < 1$,

$$\left|\frac{r(1-r^2)\sin(s)}{(1+r^2-2r\cos(s))^2}\right|\le\left|r(1-r^2)\right|\le\frac{r(1-r^2)}{\sin^4(\delta)} .$$

Hence, for $\delta \le s \le \pi$ *and* $0 \le r < 1$,

$$\left|\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)\right|$$

$$\le \frac{r(1-r^2)}{\sin^4(\delta)}\left(\left|f(\theta+s)\right|+\left|f(\theta-s)\right|+2\frac{\left|Df(\theta)\right|}{r}\right).$$

And so, for $0 < r < 1$,

$$\left|\frac{1}{\pi}\int_\delta^\pi\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds\right|$$

$$\le \frac{1}{\pi}\int_\delta^\pi\left|\left(-\frac{\partial}{\partial t}P(r,s)\right)\right|\left|\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)\right|ds$$

$$\le \frac{r(1-r^2)}{\sin^4(\delta)}\frac{1}{\pi}\int_\delta^\pi\left(\left|f(\theta+s)\right|+\left|f(\theta-s)\right|+2\frac{\left|Df(\theta)\right|}{r}\right)ds$$

$$\le \frac{1}{\pi}\frac{r(1-r^2)}{\sin^4(\delta)}\left(2\int_{-\pi}^{\pi}\left|f(s)\right|ds+\frac{2\pi}{r}\left|Df(\theta)\right|\right). \text{ ------------------ (68)}$$

It follows then from (68) that

$$\lim_{r\to1^-}\left|\frac{1}{\pi}\int_\delta^\pi\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds\right|=0. \text{ ---- (69)}$$

Now, in view of (66),

$$\frac{1}{\pi}\int_0^\delta\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds$$

$$=\frac{1}{\pi}\int_0^\delta\left(-2\sin(s)\frac{\partial}{\partial t}P(r,s)\right)\left(\frac{f(\theta+s)-f(\theta-s)}{2\sin(s)}-\frac{Df(\theta)}{r}\right)ds$$

$$=\frac{1}{\pi}\int_0^\delta\left(-2\sin(s)\frac{\partial}{\partial t}P(r,s)\right)\left(\frac{f(\theta+s)-f(\theta-s)}{2\sin(s)}-Df(\theta)+\left(1-\frac{1}{r}\right)Df(\theta)\right)ds.$$

$$=\frac{1}{\pi}\int_0^\delta\left(-2\sin(s)\frac{\partial}{\partial t}P(r,s)\right)\left(\frac{f(\theta+s)-f(\theta-s)}{2\sin(s)}-Df(\theta)\right)ds$$

$$+\frac{1}{\pi}\int_0^\delta\left(-2\sin(s)\frac{\partial}{\partial t}P(r,s)\right)\left(1-\frac{1}{r}\right)Df(\theta)ds.$$

Therefore,

$$\left|\frac{1}{\pi}\int_0^{\delta}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds\right|$$

$$\leq\frac{1}{\pi}\int_0^{\delta}\left|\left(-2\sin(s)\frac{\partial}{\partial t}P(r,s)\right)\right|\left|\left(\frac{f(\theta+s)-f(\theta-s)}{2\sin(s)}-Df(\theta)\right)\right|ds$$

$$+\frac{1}{\pi}\int_0^{\delta}\left(-2\sin(s)\frac{\partial}{\partial t}P(r,s)\right)\left(1-\frac{1}{r}\right)ds\left|Df(\theta)\right|$$

$$\leq\frac{1}{\pi}\varepsilon\int_0^{\delta}\left(-2\sin(s)\frac{\partial}{\partial t}P(r,s)\right)ds+\frac{1}{\pi}\int_0^{\delta}\left(-2\sin(s)\frac{\partial}{\partial t}P(r,s)\right)ds\left(1-\frac{1}{r}\right)\left|Df(\theta)\right|$$

$$\leq\frac{1}{\pi}\varepsilon\int_0^{\pi}\left(-2\sin(s)\frac{\partial}{\partial t}P(r,s)\right)ds+\frac{1}{\pi}\int_0^{\pi}\left(-2\sin(s)\frac{\partial}{\partial t}P(r,s)\right)ds\left(1-\frac{1}{r}\right)\left|Df(\theta)\right|$$

$$\leq\varepsilon r+\left(r-1\right)\left|Df(\theta)\right|. \qquad \text{----------------------} \quad (70)$$

Therefore, it follows from (70) that

$$\limsup_{r\to1^-}\left|\frac{1}{\pi}\int_0^{\delta}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds\right|\leq\varepsilon\,.$$

----- (71)

Therefore, from (67),

$$\limsup_{r\to1^-}\left|\frac{1}{\pi}\int_0^{\pi}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds\right|$$

$$\leq\limsup_{r\to1^-}\left|\frac{1}{\pi}\int_{\delta}^{\pi}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds\right|$$

$$+\limsup_{r\to1^-}\left|\frac{1}{\pi}\int_0^{\delta}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds\right|$$

$\leq 0+\varepsilon=\varepsilon$, by (69) and (71).

Since ε is arbitrary,

$$\limsup_{r\to 1^-}\left|\frac{1}{\pi}\int_0^{\pi}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds\right|=0$$

and so $\lim_{r\to 1^-}\frac{1}{\pi}\int_0^{\pi}\left(-\frac{\partial}{\partial t}P(r,s)\right)\left(f(\theta+s)-f(\theta-s)-2\sin(s)\frac{Df(\theta)}{r}\right)ds=0$.

It follows from (65) that $\lim_{r\to 1^-}\frac{\partial}{\partial\theta}u(r,\theta)=Df(\theta)$.

That is, $\frac{\partial}{\partial\theta}u(r,\theta)=\sum_{k=1}^{\infty}k\left(b_k\cos(k\theta)-a_k\sin(k\theta)\right)r^k\to Df(\theta)$ as $r\to 1^-$.

This means the derived series of the Fourier series of f at θ is A-summable to $Df(\theta)$.

We have thus proved the following theorem.

Theorem 19. Suppose f is a periodic Lebesgue integrable function of period 2π. If f has a generalized symmetric derivative $Df(\theta)$ at θ, then the derived series of the Fourier series of f at θ,

$$\sum_{k=1}^{\infty}k\left(b_k\cos(k\theta)-a_k\sin(k\theta)\right)=\sum_{k=1}^{\infty}kB_k(\theta)$$

is A-summable to $Df(\theta)$.

Remark. Note that the derived series of the Fourier series of f at θ may not be a Fourier series.

Observe that if f is a periodic Lebesgue integrable function of period 2π, then by Theorem 30 of *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and trigonometric series*, the series

$$\frac{1}{2}a_0x+\sum_{n=1}^{\infty}\frac{b_n}{n}-\sum_{n=1}^{\infty}\frac{b_n\cos(nx)-a_n\sin(nx)}{n}=\frac{1}{2}a_0x+\sum_{n=1}^{\infty}\frac{b_n}{n}-\sum_{n=1}^{\infty}\frac{B_n(x)}{n},$$

where (a_n,b_n) are Fourier coefficients of *f*, converges uniformly to $\int_0^x f(t)dt$.

Thus, $\sum_{n=1}^{\infty}\frac{b_n}{n}-\sum_{n=1}^{\infty}\frac{b_n\cos(nx)-a_n\sin(nx)}{n}=\sum_{n=1}^{\infty}\frac{b_n}{n}-\sum_{n=1}^{\infty}\frac{B_n(x)}{n}$ is the Fourier series of

$\int_0^x f(t)dt-\frac{1}{2}a_0x$. Then

$$\frac{\int_0^{x+h} f(t)dt-\int_0^{x-h} f(t)dt}{2h}=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)\frac{\sin(nh)}{nh} \quad .\text{---} \quad (72)$$

So, the symmetric derivative $Dk(x)$ of $k(x)=\int_0^x f(t)dt$ exists if, and only if,

$$\lim_{h\to 0}\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)\frac{\sin(nh)}{nh} \quad \text{exists.}$$

Thus, by Theorem 19, if $Dk(x)$ exists, then

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)r^n \to Dk(x) \quad \text{as } r\to 1^- .$$

If $\lim_{h\to 0}\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)\frac{\sin(nh)}{nh}=c$, then we say the series

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right) \quad \text{--------------------- } (*)$$

is *Lebesgue summable* or *L-summable* to c . We have shown that if the Fourier series is *L*-summable to *c* then it is A-summable to *c*. The converse is generally not true. Zygmund gave a sufficient condition for the convergence of a trigonometric series to imply Lebesgue summability. There are other sufficient conditions for the converse. For instance, if a_n and b_n are of $O(1/n)$, then the series is convergent if, and only if, it is Lebesgue summable. If $\sum_{k=1}^{n}k\sqrt{a_k^2+b_k^2}=O(n)$, then A-summable to *c* implies Lebesgue summable to *c* (Vindas Theorem 3 in *On the relation between Lebesgue summability and some other summation methods*, J Math Anal Appl vol 411 2014 page 75-82.)

It is worth noting that for Fourier series, Lebesgue summable to *c* implies by Lemma 18 and Theorem 32 of *Ideas of Lebesgue and Perron Integration in Uniqueness of Fourier and Trigonometric series*, that the Fourier series is Riemann summable to the same value *c*. Thus we can also conclude by Theorem 20 below that if f is a Lebesgue integrable periodic function, then its Fourier series is Riemann summable almost everywhere to f.

If $\lim_{u\to 0^+}\phi(\theta,u)=\lim_{u\to 0^+}\dfrac{f(\theta+u)+f(\theta-u)}{2}$ exists and is equal to *c*, then *f* satisfies condition ℓ_c at θ. We have the following Lebesgue summability Theorem.

Theorem 20. Suppose *f* is a periodic Lebesgue integrable function of period 2π.

(1) If $\lim_{t\to 0^+}\phi(\theta,t)=\lim_{t\to 0^+}\dfrac{f(\theta+t)+f(\theta-t)}{2}$ exists and equals $\phi(\theta+)$, then the Fourier series of *f* is Lebesgue-summable to $\phi(\theta+)$ **.**

Consequently:

(2) The Fourier series of $f(t)$ at $t=\theta$ is Lebesgue-summable to $f(\theta)$ at a point of continuity and to $\dfrac{f(\theta+)+f(\theta-)}{2}$ at a point of jump discontinuity.

(3) The Fourier series of f is uniformly Lebesgue summable to $f(\theta)$ in any closed interval of continuity.

(4) The Fourier series of f is Lebesgue summable almost everywhere to $f(\theta)$.

Proof.

Note that we have shown in the proof of Lemma 18 that

$$\frac{\int_0^{\theta+h} f(t)dt-\int_0^{\theta-h} f(t)dt}{2h}=\frac{1}{h}\int_0^h \frac{1}{2}\big(f(\theta+t)+f(\theta-t)\big)dt$$

and so

$$\frac{\int_0^{\theta+h} f(t)dt-\int_0^{\theta-h} f(t)dt}{2h}-c=\frac{1}{h}\int_0^h \frac{1}{2}\big(f(\theta+t)+f(\theta-t)-2c\big)dt=\frac{1}{h}\int_0^h g_c(t)dt\,.$$

It follows from (72) that

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}\big(a_n\cos(n\theta)+b_n\sin(n\theta)\big)\frac{\sin(nh)}{nh}-c=\frac{1}{h}\int_0^h g_c(t)dt\,. \quad \text{------------ (73)}$$

Therefore, the Fourier series of f is Lebesgue summable at any point θ, where

$\lim_{h\to 0}\frac{1}{h}\int_0^h g_c(t)dt = \lim_{h\to 0}\frac{1}{h}\int_0^h \frac{1}{2}\left(f(\theta+t)+f(\theta-t)-2c\right)dt = 0$, that is to say, f satisfies condition ℓ_c .

Since $\lim_{t\to 0^+}\phi(\theta,t) = \lim_{t\to 0^+}\frac{f(\theta+t)+f(\theta-t)}{2} = \phi(\theta+)$, given $\varepsilon > 0$

there exists $\delta > 0$ such that

$$\left|g_{c_\theta}(t)\right| = \left|\phi(\theta,t)-c_\theta\right| < \varepsilon \text{ for } 0 < |t| < \delta, \qquad \text{-------------------- (74)}$$

where $c_\theta = \phi(\theta+)$.

Thus for $0 < h < \delta$,

$$\left|\frac{1}{h}\int_0^h \frac{1}{2}\left(f(\theta+t)+f(\theta-t)-2c_\theta\right)dt\right| = \left|\frac{1}{h}\int_0^h \left(\phi(\theta,t)-c_\theta\right)dt\right|$$

$$\le \frac{1}{h}\int_0^h \left|\phi(\theta,t)-c_\theta\right|dt \le \frac{1}{h}\int_0^h \varepsilon dt = \varepsilon .$$

Since ε is arbitrary, this shows that

$$\lim_{h\to 0}\frac{1}{h}\int_0^h g_c(t)dt = \lim_{h\to 0}\frac{1}{h}\int_0^h \frac{1}{2}\left(f(\theta+t)+f(\theta-t)-2c\right)dt = 0 .$$

Hence, the Fourier series of f is Lebesgue summable at θ.

This proves part (1).

Part (2) follows from part (1) since at a point of continuity θ, $\phi(\theta+) = f(\theta)$ and at a point of jump discontinuity, $\phi(\theta+) = \frac{1}{2}\left(f(\theta+)+f(\theta-)\right)$.

If f is continuous in $[a, b]$, then f is uniformly continuous in $[a, b]$ and so we can choose the same $\delta > 0$ in (74) for all θ in $[a, b]$. This implies that

$\frac{1}{h}\int_0^h g_{f(\theta)}(t)dt \to 0$ uniformly in θ in $[a, b]$. Consequently, the Fourier series of f is uniformly Lebesgue summable to $f(\theta)$ in $[a, b]$. This proves part (3).

Since f is Lebesgue integrable, almost every point is a Lebesgue point of f. Hence, as we have shown in the proof of Theorem 14, f satisfies condition $\ell_{f(\theta)}$ at θ for almost all θ. Therefore, the Fourier series of f is Lebesgue summable almost everywhere to $f(\theta)$.