The Boundedness Theorem, Extreme Value Theorem and Intermediate Value Theorem
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These are very important theorems. **They all have very close connection with the completeness property of the real numbers. They can be shown to be equivalent to the completeness property of the real numbers.**

Theorem 1 (Boundedness Theorem). If $f: [a, b] \to \mathbf{R}$ is continuous, then f is bounded.

Proof. If $a = b$, we have nothing to prove. We assume $a < b$, that is, $[a, b]$ is a non-trivial interval. Let $K = \{ x \in [a, b]$: the restriction of f to $[a, x]$, $f|_{[a, x]}:[a, x] \to \mathbf{R}$ is bounded $\}$. Then obviously K is bounded and K is non-empty for it obviously contains a. Let C = supremum of K, sup K. C exists by the completeness property of $\mathbf{R}$ because K is bounded above by b. By definition of C, $C \leq b$ because b is an upper bound for K. We claim that $C > a$. Why is this so? Since f is continuous at $x = a$, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any x in $[a, b]$ with $a \leq x < a + \delta$, we get $f(a) - \varepsilon < f(x) < f(a) + \varepsilon$. Now choose any $\delta' > 0$ with $\delta' < \min(\delta, b - a)$ then for any x with $a \leq x \leq a + \delta'$, we get then $f(a) - \varepsilon < f(x) < f(a) + \varepsilon$. That is to say $f|_{[a, a+\delta']}:[a, a + \delta'] \to \mathbf{R}$ is bounded. Hence $a + \delta' \in K$ and so $a < a + \delta' \leq \sup K = C$. This proves the claim.

We next claim that $C = b$. Suppose on the contrary that $C < b$, then $a < C < b$. Then since f is continuous at C, following the same argument as above we can show that there exists a $\delta > 0$ such that f is bounded on the interval $[C - \delta, C + \delta] \subseteq [a, b]$, where $C + \delta < b$. That means there exists a $M > 0$ such that for all x in $[C - \delta, C + \delta]$, $|f(x)| < M$. Now since $C - \delta < C$, *and* sup $K = C$, there exists a x in K such that $C - \delta < x \leq C$. That means f is bounded on $[a, x]$, say by N, and so f is bounded on $[a, C + \delta] = [a, x] \cup [C - \delta, C + \delta]$ by max (M, N). I.e., for all x in $[a, C + \delta]$, $|f(x)| <$ max (M, N). Therefore, $C + \delta \in K$. Hence $C + \delta \leq C$. contradicting $\delta > 0$. Hence $C = b$. (This means that for any x such that $a \leq x < b$, f is bounded on $[a, x]$. Why? The reason is as follows. There exists a y in K such that $a \leq x < y \leq b$ and f is bounded on $[a, y]$ and so f is bounded on $[a, x]$.)
Now we observe that by the continuity of f at b, for a fixed choice of an $\varepsilon > 0$, we have that there exists a $\delta > 0$ such that for any x in $[a, b]$ with $b - \delta < x \leq b$, we get $f(b) - \varepsilon < f(x) < f(b) + \varepsilon$. Hence f is bounded on $[x, b]$ by max $(|f(b) - \varepsilon|, |f(b) + \varepsilon|)$ for some x in K with $b - \delta < x \leq b$ since $b = C = \sup K$. Therefore, f is bounded on $[a, x]$ say by M_1. Therefore, f is bounded on $[a, b] = [a, x] \cup [x, b]$ by $max((|f(b) - \varepsilon|, |f(b) + \varepsilon|, M_1)$. This completes the proof.

Theorem 2 (Extreme Value Theorem). A continuous function on a closed and bounded interval attains its supremum and infimum. That is, if $f: [a, b] \to \mathbf{R}$ is continuous, then there exist c and d in $[a, b]$ such that $f(c)$ = supremum $f([a, b])$ and $f(d)$ = infimum $f([a, b])$.

Proof. Again if $a = b$, we have nothing to prove. Assume $a < b$. Consider the range $f([a, b])$. Let M = supremum $f([a, b])$. Suppose M does not lie in the range, i.e., there does not exist c in $[a, b]$ such that $f(c) = M$. Then $f(x) < M$ for all x in $[a, b]$. That means $M - f(x) > 0$. Since f is continuous on $[a, b]$, the function defined by $h(x) = M - f(x)$ for x in $[a, b]$ is also continuous and $h(x) > 0$ for all x in $[a, b]$. Therefore, since $h(x)$ is non-zero, $g(x) = 1/h(x)$ is also continuous on $[a, b]$. Hence by Theorem 1, g is bounded and so is bounded

above. Hence the sup of its range exists by the completeness property of **R**. Let K = supremum $g([a, b])$. Because $h(x) > 0$ for all x in $[a, b]$, $g(x) > 0$ for all x in $[a, b]$. Thus $K > 0$ and so $0 < g(x) \le K$ for all x in $[a, b]$. Hence $h(x) = 1/g(x) \ge 1/K$ for all x in $[a, b]$. *That means* $M - f(x) \ge 1/K$ *and hence* $f(x) \le M - 1/K$. for all x in $[a, b]$. Therefore, $M - 1/K \ge M$ *since* M = supremum $f([a, b])$ and so $1/K \le 0$ contradicting $K > 0$.
We conclude that there exists c in $[a, b]$ such that $f(c) = M$.

Similarly if there does not exists d in $[a, b]$ such that $f(d) = N$ = infimum $f([a, b])$, we can arrive at a contradiction. We use $h(x) = f(x) - N > 0$. *Then* $g(x) = 1/h(x) > 0$ for all x in $[a, b]$ and g is continuous on $[a, b]$. Let L = supremum $g([a, b]) > 0$. L exists by Theorem 1. Thus, $0 < g(x) \le L$ for all x in $[a, b]$. Therefore, $h(x) = 1/g(x) \ge 1/L$ for all x in $[a, b]$. *That means* $f(x) - N \ge 1/L$ *and hence* $f(x) \ge N + 1/L$ for all x in $[a, b]$. Therefore, $N + 1/L \le N$ since N = infimum $f([a, b])$, contradicting $1/L > 0$. Hence, there exists a d in $[a, b]$ such that $f(d)$ = infimum $f([a, b])$. (See Figure 1 below for illustration.)

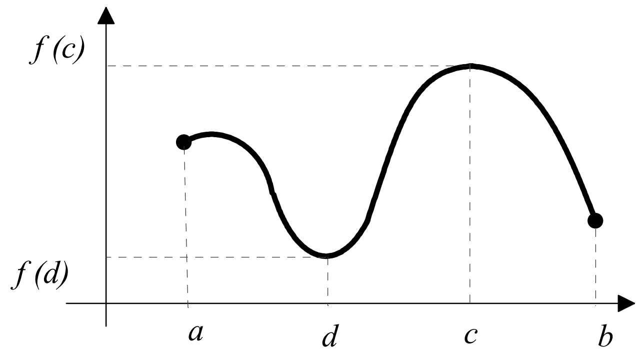

Fig. 1

Theorem 3 Intermediate Value Theorem. Suppose $f: [a, b] \to \mathbf{R}$ is continuous. If γ is an intermediate value between $f(a)$ and $f(b)$, i.e. either $f(a) \le \gamma \le f(b)$ or $f(b) \le \gamma \le f(a)$, then there exists c in $[a, b]$ such that $f(c) = \gamma$.

Proof. Again if $a = b$, we have nothing to prove. Assume $a < b$. Without loss of generality we may assume that $f(a) < f(b)$. If $\gamma = f(a)$ or $f(b)$, we have nothing to prove. Now take any γ such that $f(a) < \gamma < f(b)$. Then define $g : [a, b] \to \mathbf{R}$ by $g(x) = f(x) - \gamma$ for x in $[a, b]$. Then g is a continuous function, $g(a) < 0$ and $g(b) > 0$. We are going to find a point κ in $[a, b]$ such that $g(\kappa) = 0$. We do this by using the completeness property of the real numbers **R**. Let $\mathtt{F} = \{ x \in [a, b]: g(x) < 0\}$. Then $\mathtt{F} \ne \varnothing$ since $a \in \mathtt{F}$ because $g(a) < 0$. Obviously $\mathtt{F}$ is bounded above by b. Hence by the completeness property of **R**, supremum of $\mathtt{F}$ exists. Let $\kappa = \sup \mathtt{F}$. Since g is continuous at a and $g(a) < 0$, there exists $\delta > 0$ such that for all x with $a \le x < a + \delta < b$, $g(x) < 0$. (Take $\varepsilon = -g(a)/2$. By continuity of g at a, there exists $\delta_1 > 0$ such that for all x in $[a, b]$ with $a \le x < a + \delta_1$, $|g(x) - g(a)| < -g(a)/2$ or $3g(a)/2 < g(x) < g(a)/2 < 0$. Take $\delta = \min(\delta_1 , (b - a)/2)$.) This means $\kappa \ge a + \delta' > a$ for any δ' with $0 < \delta' < \delta$. Therefore, $\kappa > a$. Thus $a < \kappa \le b$. Now by the continuity of g at b and the fact that $g(b) > 0$, there exists $\delta_2 > 0$ such that for all x with $a < b - \delta_2 < x \le b$, $g(x) > 0$. That means for any k with $b - \delta_2 < k < b$, $k \notin \mathtt{F}$ and consequently any k with $b - \delta_2 < k < b$ is an upper bound for $\mathtt{F}$. Thus $\kappa = \sup \mathtt{F} \le b - \delta_2 < b$. *Hence* $a < \kappa < b$.
We now claim that $g(\kappa) = 0$. That is $f(\kappa) = \gamma$.

Suppose $g(\kappa) < 0$. Then by the continuity of g at κ, there exists $\delta_3 > 0$ such that for any x with $a < \kappa - \delta_3 \le x \le \kappa + \delta_3 < b$, we have $g(x) < 0$. This means $\kappa + \delta_3 \in F$. Thus $\kappa + \delta_3 \le \sup F = \kappa$, and $\delta_3 \le 0$ contradicting $\delta_3 > 0$. Hence $g(\kappa) \ge 0$. Similarly if $g(\kappa) > 0$, then by the continuity of g at κ, there exists $\delta_4 > 0$ such that for any x with $a < \kappa - \delta_4 \le x \le \kappa + \delta_4 < b$, we have $g(x) > 0$. Then any x in $[a, b]$ and $g(x) < 0$ would imply that $x < \kappa - \delta_4$. Thus $\kappa - \delta_4$ is an upper bound for F and hence $\kappa \le \kappa - \delta_4$ giving $\delta_4 < 0$ contradicting $\delta_4 > 0$. Hence $g(\kappa) = 0$. We now take $c = \kappa$ and $f(c) = \gamma$.

If $f(a) > f(b)$, then multiply by -1, we get $-f(a) > -f(b)$. Replace f above by $-f$, γ by $-\gamma$ and the proof proceeds in exactly the same manner as above to obtain a c in $[a, b]$ such that $-f(c) = -\gamma$ and that is the same as $f(c) = \gamma$. This completes the proof.

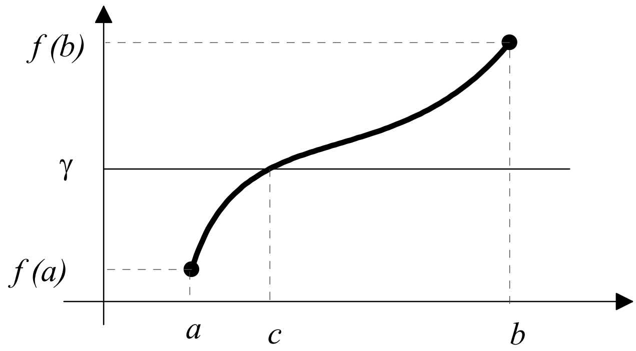

Fig. 2

Theorem 4. (Compactness Theorem). If $f: [a, b] \to \mathbf{R}$ is a continuous function defined on a closed and bounded interval $[a, b]$, then its range is also a closed and bounded interval.

Proof. By the Extreme Value Theorem, Theorem 2 above, the range of f, $f([a, b]) \subseteq [\inf f([a, b]), \sup f([a, b])] = [f(c), f(d)]$ for some c and d in $[a, b]$. But by the Intermediate Value Theorem, Theorem 3 above, $[f(c), f(d)] \subseteq f([a, b])$. Hence $f([a, b]) = [f(c), f(d)]$ and $[f(c), f(d)]$ is a closed and bounded interval. That means the range $f([a, b])$ is a closed and bounded interval. (See figure 3 below.)

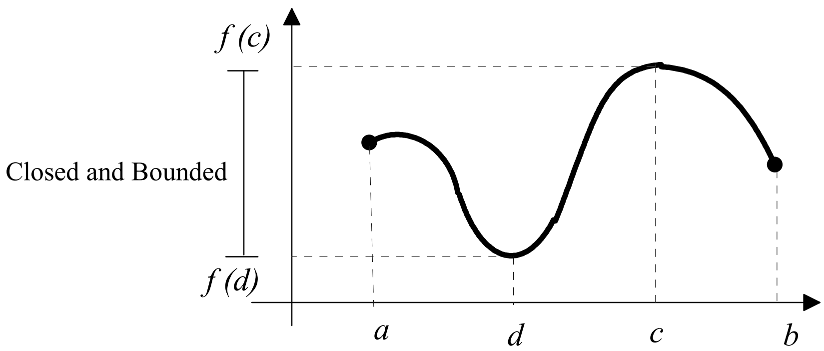

Fig. 3