Question 1

The function f is defined by

$$f(x)=\begin{cases} \frac{2}{3}x^3+\frac{1}{3}, & x>1 \\ x^2\sin(\frac{\pi}{2x}), & -1\le x\le 1 \text{ and } x\neq 0 \\ x^3-1, & x<-1 \\ 0, & x=0 \end{cases}.$$

(a) For $x < -1$, $f(x) = x^3 - 1 < -2$.

Also, for $x < -1$, $x^3 - 1 < -2 \Leftrightarrow x < -1$.

Thus f maps $(-\infty, -1)$ onto $(-\infty, -2)$.

(Because for any $y < -2$, we can take $x = \sqrt[3]{y+1}$ so that $f(x) = y$)

Also, for $-1 \le x \le 1$, $-1 \le f(x) \le 1$.

This is seen as follows.

For $-1 \le x \le 1$ and $x \ne 0$, $|f(x)| = \left| x^2 \sin(\frac{\pi}{2x}) \right| \le x^2 \le 1$.

Also we know $f(0) = 0$.

Thus $-1 \le f(x) \le 1$. Therefore, $f(1) = 1$ is the absolute maximum of f on $[-1, 1]$

and $f(-1) = -1$ is the absolute minimum of f on $[-1, 1]$.

Assuming that f is continuous on $[-1, 1]$
(as we shall show in part (d) below),
by the Intermediate Value Theorem)
f maps the interval $[-1,1]$ onto $[-1,1]$.

Finally for $x > 1$, $f(x) = \frac{2}{3}x^3 + \frac{1}{3} > 1$.
And for any $y > 1$, we can take
$x = \sqrt[3]{\frac{3y-1}{2}} > 1$ so that $f(x) = y$.
Hence f maps $(1, \infty)$ onto $(1, \infty)$.

Hence the range of f is
$(-\infty, -2) \cup [-1, 1] \cup (1, \infty) = (-\infty, -2) \cup [-1, \infty)$

(b) By part (a) Range(f) =
$(-\infty, -2) \cup [-1, \infty) \neq \mathbf{R}$ = codomain(f),
therefore f is not surjective.

(c) (i) By part (a)
1 is in the image of [-1, 1] under f .

Thus, to find the preimage we need to solve the equation

$$x^2 \sin(\frac{\pi}{2x}) = 1 \text{ for } x \text{ in } [-1, 1]-\{0\}.$$

For $x \neq 0$ and $-1 < x < 1$, $\left|x^2 \sin(\frac{\pi}{2x})\right| \leq x^2 < 1$.

Since we know $f(1) = 1$, and $f(-1) < 0$, $x = 1$.

(ii) From part (a)

- 2 is not in the range of f.

Thus, the solution of $f(x) = -2$ does not exist.

Therefore, there is no value of x such that $f(x) = -2$

(d) When $x < -1$, $f(x) = x^3 - 1$, f is continuous on $(-\infty, -1)$.

When $-1 < x < 1$ and $x \neq 0$, $f(x) = x^2 \sin(\frac{\pi}{2x})$.

Since $x^2 \sin(\frac{\pi}{2x})$ is continuous on $(-1, 0)$ and on $(0, 1)$, f is continuous on the union of these two intervals.

When $x > 1$, $f(x)$ is a polynomial function and so it is continuous for $x > 1$.

Thus it remains to check if f is continuous at $x = -1, 0$ or 1.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 \sin(\frac{\pi}{2x}) = 1^2 \sin(\frac{\pi}{2}) = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{2}{3}x^3 + \frac{1}{3} = \frac{2}{3} + \frac{1}{3} = 1 = f(1)$$

Therefore, $\lim_{x\to 1} f(x) = f(1)$ and so f is continuous at $x = 1$.

Now $\lim_{x\to (-1)^-} f(x) = \lim_{x\to (-1)^-} x^3 - 1 = -2$

$$\lim_{x\to (-1)^+} f(x) = \lim_{x\to (-1)^+} x^2 \sin(\frac{\pi}{2x}) = 1^2 \sin(-\frac{\pi}{2}) = -1$$

Thus the left and the right limits of f at x = -2 are not the same and so f is not continuous at x = -1.

Now $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin(\frac{\pi}{2x}) = 0$
by the Squeeze Theorem.

Since $f(0) = 0$, f is continuous at $x = 0$.
Hence f is continuous at x for all $x \neq -1$.

(e) f is differentiable at $x = 1$. This is seen as follows.

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 \sin(\frac{\pi}{2x}) - 1}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{2x \sin(\frac{\pi}{2x}) - \frac{\pi}{2} \cos(\frac{\pi}{2x})}{1}$$ by L' Hôpital's Rule

$= 2$

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{\frac{2}{3}x^3+\frac{1}{3}-1}{x-1}$$

$$=\frac{2}{3}\lim_{x\to 1^+}\frac{x^3-1}{x-1}=2.$$

Thus f is differentiable at $x = 1$ and $f'(1) = 2$.

(f) Note that f is an odd function on the interval [-1, 1], since $f(-x) = x^2\sin(-\pi/(2x)) = -x^2\sin(\pi/(2x)) = -f(x)$.

$$\int_{-1}^{0} f(x)dx = -\int_{1}^{0} f(-t)dt \text{ where } t = -x$$

$$=\int_{1}^{0} f(t)dt = -\int_{0}^{1} f(t)dt$$

Therefore,

$$\int_{-1}^{1} f(x)dx = \int_{0}^{1} f(x)dx + \int_{-1}^{0} f(x)dx = 0.$$

Question 2

(a) $$\lim_{x\to\infty}\frac{61x^7+2x^3+1}{907x^7+7x^3+5x^2+7}$$

$$=\lim_{x\to\infty}\frac{61+\frac{2}{x^4}+\frac{1}{x^7}}{907+\frac{7}{x^4}+\frac{5}{x^5}+\frac{7}{x^7}}=\frac{61+0+0}{907+0+0+0}=\frac{61}{907}.$$

(b) $$\lim_{x\to 0}\frac{\sqrt{7x^2+121}-11}{14x^2}$$

$$=\lim_{x\to 0}\frac{7x^2}{14x^2(\sqrt{7x^2+121}+11)}$$

$$=\lim_{x\to 0}\frac{1}{2(\sqrt{7x^2+121}+11)}=\frac{1}{44}$$

(c) $\lim_{x\to\infty} \frac{x^5}{e^{x^2}} = \lim_{x\to\infty} \frac{5x^4}{2xe^{x^2}}$

$$= \frac{5}{2} \lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \frac{5}{2} \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}}$$

$$= \frac{15}{4} \lim_{x\to\infty} \frac{x}{e^{x^2}} = \frac{15}{4} \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0$$

by repeated use of L' Hôpital's rule and $\lim_{x\to\infty} 2xe^{x^2} = \infty$ so that the limit of the reciprocal function $\lim_{x\to\infty} \frac{1}{2xe^{x^2}}$ is 0.

(d) $$\lim_{x \to 0} \frac{\sin(\tan(x))}{\tan(\sin(x))}$$

$$= \lim_{x \to 0} \frac{\cos(\tan(x))\ \sec^2(x)}{\sec^2(\sin(x))\ \cos(x)} \text{ by L' Hôpital's rule}$$

$$= \frac{\cos(\tan(0))\ \sec^2(0)}{\sec^2(\sin(0))\ \cos(0)} = 1$$

(e) Let $y = (e^{(x^3)} + 3x^2)^{(1/x^2)}$.

Then $\ln(y) = \dfrac{1}{x^2}\ln(e^{(x^3)} + 3x^2)$.

Now $\lim\limits_{x\to 0}\ln(y) = \lim\limits_{x\to 0}\dfrac{\ln(e^{(x^3)} + 3x^2)}{x^2}$

$= \lim\limits_{x\to 0}\dfrac{3x^2 e^{(x^3)} + 6x}{2x(e^{(x^3)} + 3x^2)}$

$= \dfrac{3}{2}\,\dfrac{0+2}{1+0} = 3$

Therefore, $\lim\limits_{x\to 0} y = \lim\limits_{x\to 0} e^{\ln(y)} = e^{\lim\limits_{x\to 0}\ln(y)} = e^3$

Question 3

(a) $$\int \frac{dx}{(x^2+2)(x^2+3)} = \int (\frac{1}{x^2+2} - \frac{1}{x^2+3})dx$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}(\frac{x}{\sqrt{2}}) - \frac{1}{\sqrt{3}} \tan^{-1}(\frac{x}{\sqrt{3}}) + C$$

(b) $$\int \sin^{-1}(4x)dx = x\sin^{-1}(4x) - \int 4x\frac{1}{\sqrt{1-16x^2}}dx$$

by integration by parts

$$= x\sin^{-1}(4x) + \tfrac{1}{8}\int \frac{-32x}{\sqrt{1-16x^2}}dx$$

$$= x\sin^{-1}(4x) + \tfrac{1}{4}\sqrt{1-16x^2} + C$$

by change of variable or substitution using e.g. $u = \sqrt{(1-16x^2)}$

$$\text{(c)} \quad \int e^{2x}\sin(5x)dx = \tfrac{1}{2}e^{2x}\sin(5x) - \tfrac{1}{2}\int e^{2x}5\cos(5x)dx$$

$$= \tfrac{1}{2}e^{2x}\sin(5x) - \tfrac{5}{2}[\tfrac{1}{2}e^{2x}\cos(5x) + \tfrac{1}{2}\int e^{2x}5\sin(5x)dx]$$

$$= \tfrac{1}{2}e^{2x}\sin(5x) - \tfrac{5}{4}e^{2x}\cos(5x) - \tfrac{25}{4}\int e^{2x}\sin(5x)dx]$$

by integration by parts.

Therefore,

$$\tfrac{29}{4}\int e^{2x}\sin(5x)dx = \tfrac{1}{2}e^{2x}\sin(5x) - \tfrac{5}{4}e^{2x}\cos(5x) + C.$$

Thus

$$\int e^{2x}\sin(5x)dx = \tfrac{2}{29}e^{2x}\sin(5x) - \tfrac{5}{29}e^{2x}\cos(5x) + C'.$$

Therefore,

$$\int_0^{\frac{\pi}{5}} e^{2x}\sin(5x)dx = \frac{1}{29}[2e^{2x}\sin(5x) - 5e^{2x}\cos(5x)]_0^{\frac{\pi}{5}}$$

$$= \frac{1}{29}\left[5e^0\cos(0) - 5e^{\frac{2\pi}{5}}\cos(\pi)\right]$$

$$= \frac{5}{29}(1 + e^{\frac{2\pi}{5}}).$$

(d) $\int \frac{x+3}{x^2+2x+4}dx = \int\left(\frac{1}{2}\frac{2x+2}{x^2+2x+4} + \frac{2}{x^2+2x+4}\right)dx$

$= \frac{1}{2}\ln(x^2+2x+4) + 2\int \frac{1}{(x+1)^2+3}dx$

$= \frac{1}{2}\ln(x^2+2x+4) + \frac{2}{\sqrt{3}}\tan^{-1}(\frac{x+1}{\sqrt{3}}) + C$

Therefore,

$\int_0^2 \frac{x+3}{x^2+2x+4}dx = \left[\frac{1}{2}\ln(x^2+2x+4) + \frac{2}{\sqrt{3}}\tan^{-1}(\frac{x+1}{\sqrt{3}})\right]_0^2$

$= \frac{1}{2}((\ln(12) - \ln(4)) + \frac{2}{\sqrt{3}}(\tan^{-1}(\sqrt{3}) - \tan^{-1}(\frac{1}{\sqrt{3}}))$

$= \frac{1}{2}\ln(3) + \frac{2}{\sqrt{3}}(\frac{\pi}{3} - \frac{\pi}{6}) = \frac{1}{2}\ln(3) + \frac{\pi}{3\sqrt{3}}$

Question 4

(a) Let $f(x) = 3x - 2$. First note that
$|f(x) - 1| = |3x - 2 - 1| = 3|x-1|$.

Therefore,
given any $\varepsilon > 0$, take $\delta = \varepsilon/3$

Thus,

$$0 < |x - 1| < \delta \Rightarrow |f(x) - 1| = 3|x-1| < 3\delta = \varepsilon.$$

Therefore, by the definition of limit,

$$\lim_{x\to 1} f(x) = 1.$$

(b) First note that

f is differentiable at *x* for *x* in (-∞, -π) or (π, ∞) since on these intervals the function is the same as sine(*x*) and sine (*x*) is differentiable on these intervals.

Now for *x* such that -π < *x* < π, *f* (*x*) is given by a polynomial and any polynomial is differentiable on the interval (-π , π).

Then a necessary condition for *f* to be differentiable at π is that *f* be continuous at π.

That is, $\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^+} f(x) = f(\pi).$

Now $\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} ax^3 + bx = a\pi^3 + b\pi = f(\pi)$

and $\lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^+} \sin(x) = \sin(\pi) = 0.$

And so our first condition is

$a\pi^2 + b = 0$ ------------------- (1)

Since we know the derivative of $\sin(x)$ is $\cos(x)$, that is

$$\lim_{y\to x} \frac{\sin(y) - \sin(x)}{y - x} = \cos(x),$$

$$\lim_{x\to\pi^+} \frac{f(x) - f(\pi)}{x - \pi} = \lim_{x\to\pi^+} \frac{\sin(x) - 0}{x - \pi}$$

$$= \lim_{x\to\pi} \frac{\sin(x) - \sin(\pi)}{x - \pi} = \cos(\pi) = -1.$$

Similarly,

$$\lim_{x\to\pi^-} \frac{f(x)-f(\pi)}{x-\pi} = \lim_{x\to\pi^-} \frac{ax^3+bx-(a\pi^3+b\pi)}{x-\pi}$$

$$= 3a\pi^2 + b.$$

Therefore, in addition to equation (1), for differentiability at π, we must have

$$3a\pi^2 + b = -1 \quad \text{----------------} \quad (2)$$

Solving (1) and (2) gives $b=\frac{1}{2}$ and $a=-\frac{1}{2\pi^2}$.

For differentiability at - π,
we get the same equations (1) and (2) above.
Thus the same values for *a* and *b* above will guarantee differentiability at - π too.

(c) Let $f(x) = 2x^3 + 3x + 1 - 3\sin(x)\cos(x)$

$$= 2x^3 + 3x + 1 - \tfrac{3}{2}\sin(2x).$$

Then $f'(x) = 6x^2 + 3 - 3\cos(2x)$

$$= 6x^2 + 3(1 - \cos^2(x) + \sin^2(x))$$

$$= 6(x^2 + \sin^2(x)).$$

Therefore, $f'(x) > 0$ for $x \neq 0$.

Since f is continuous on **R**, f is continuous at $x = 0$. Thus f is increasing on (-∞, 0] and on [0, ∞) and so it is increasing on **R**. Therefore, f is injective.

- Now $f(0) = 1 > 0$ and $f(-\pi) = -2\pi^2 - 3\pi + 1 < 0$.
- Therefore, by the *Intermediate Value Theorem*, there exists a point c in **R** such that $f(c) = 0$.
- That is, f has a root in **R**.
- Since f is injective, it has exactly one real root.

Question 5

Observe that

$$f(x)=\begin{cases}\dfrac{2x|x|}{1+x^2}, & x<1\\ \dfrac{1}{x}, & x\ge 1\end{cases} = \begin{cases}-\dfrac{2x^2}{1+x^2}, & x<0\\ \dfrac{2x^2}{1+x^2}, & 0\le x<1\\ \dfrac{1}{x}, & x\ge 1\end{cases}.$$

We note that

1. f is continuous on $(1, \infty)$ because f is a rational function on $(1, \infty)$.
2. f is continuous on $(-\infty, 1)$ because f is a product of a rational function and $|x|$ and $|x|$ is a continuous function .

Now $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x} = 1 = f(1)$ and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{2x|x|}{1+x^2} = 1.$$

Therefore, $\lim_{x\to 1} f(x) = f(1)$ and so f is continuous at $x = 1$.

Thus f is continuous on **R**.

Then

$$f'(x) = \begin{cases} -\dfrac{4x}{(1+x^2)^2}, & x < 0 \\ \dfrac{4x}{(1+x^2)^2}, & 0 < x < 1 \\ -\dfrac{1}{x^2}, & x > 1 \end{cases} \qquad \text{---------} \quad (1)$$

$$f''(x) = \begin{cases} 4\dfrac{(3x^2-1)}{(1+x^2)^3}, & x<0 \\ -4\dfrac{(3x^2-1)}{(1+x^2)^3}, & 0<x<1 \\ \dfrac{2}{x^3}, & x>1 \end{cases}$$

$$= \begin{cases} 12\dfrac{(x^2-\frac{1}{3})}{(1+x^2)^3}, & x<0 \\ -12\dfrac{(x^2-\frac{1}{3})}{(1+x^2)^3}, & 0<x<1 \\ \dfrac{2}{x^3}, & x>1 \end{cases} \quad \text{-----------} \; (2)$$

(a) For $x < 0$, $-4x > 0$ and so from (1)

$$f'(x) = \frac{-4x}{(1+x^2)^2} > 0 \quad \text{for } x \text{ in } (-\infty, 0)$$

since $(1 + x^2) > 0$.

Thus f is increasing on the interval $(-\infty, 0]$ since f is continuous at $x = 0$.

Now for x in $(0, 1)$ $f'(x) = \dfrac{4x}{(1+x^2)^2} > 0$.

Therefore, f is increasing on $[0, 1]$ since f is continuous at $x = 0$ and at $x = 1$.

Thus f is increasing on the interval $(-\infty, 1]$.

For $x > 1$, $f'(x) = -\frac{1}{x^2} < 0$.

Thus f is decreasing on the interval $[1, \infty)$ since f is continuous at $x = 1$.

(b) Now $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x} = 0$

and so the line $y = 0$ is a horizontal asymptote of the graph of f.

Next we check the following limit.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -\frac{2x^2}{1+x^2} = \lim_{x\to-\infty} -\frac{2}{1+\frac{1}{x^2}} = -2$$

Therefore, the line $y = -2$ is another horizontal asymptote of the graph of f.

(c) When $x < -\frac{1}{\sqrt{3}}$, from (2),

$$f''(x) = 12\frac{(x^2 - \frac{1}{3})}{(1+x^2)^3} > 0 \text{ since } (x^2 - \tfrac{1}{3}) > 0.$$

Hence the graph of f is concave upward on the interval $(-\infty, -\frac{1}{\sqrt{3}})$.

Also from (2), when $-\frac{1}{\sqrt{3}} < x < 0$,

$$f''(x) = 12\frac{(x^2 - \frac{1}{3})}{(1+x^2)^3} < 0 .$$

Therefore, the graph of f is concave downward on the interval $(-\frac{1}{\sqrt{3}}, 0)$.

Again from (2), for $0 < x < \frac{1}{\sqrt{3}} (< 1)$,

$$f''(x) = -12\frac{(x^2 - \frac{1}{3})}{(1+x^2)^3} > 0 \text{ since } (x^2 - \tfrac{1}{3}) < 0.$$

Therefore, the graph of f is concave upward on $(0, \frac{1}{\sqrt{3}})$.

For $\frac{1}{\sqrt{3}} < x < 1$, $f''(x) = -12\frac{(x^2 - \frac{1}{3})}{(1+x^2)^3} < 0$

and therefore the graph of f is concave downward on $(\frac{1}{\sqrt{3}}, 1)$.

Finally for $x > 1$, $f''(x) = \frac{2}{x^3} > 0$ and so the graph of f is concave upward on $(1, \infty)$.

(d) Since from part (a) f is increasing on $(-\infty, 1]$ and decreasing on $[1, \infty)$, f has a relative maximum value at $x = 1$.

- Indeed the relative maximum value is $f(1) = 1$.
  Since f is increasing on $(-\infty, 1]$, f has no relative minimum in $(-\infty, 1]$.
- Likewise since f is decreasing on $[1, \infty)$, f has no relative minimum value in $[1, \infty)$, .

Therefore f has no relative minimum value.

(e)

From part (c), there are changes of concavity before and after the following points in the graph:

$$(-\frac{1}{\sqrt{3}}, f(-\frac{1}{\sqrt{3}})) = (-\frac{1}{\sqrt{3}}, -\frac{1}{2}),$$

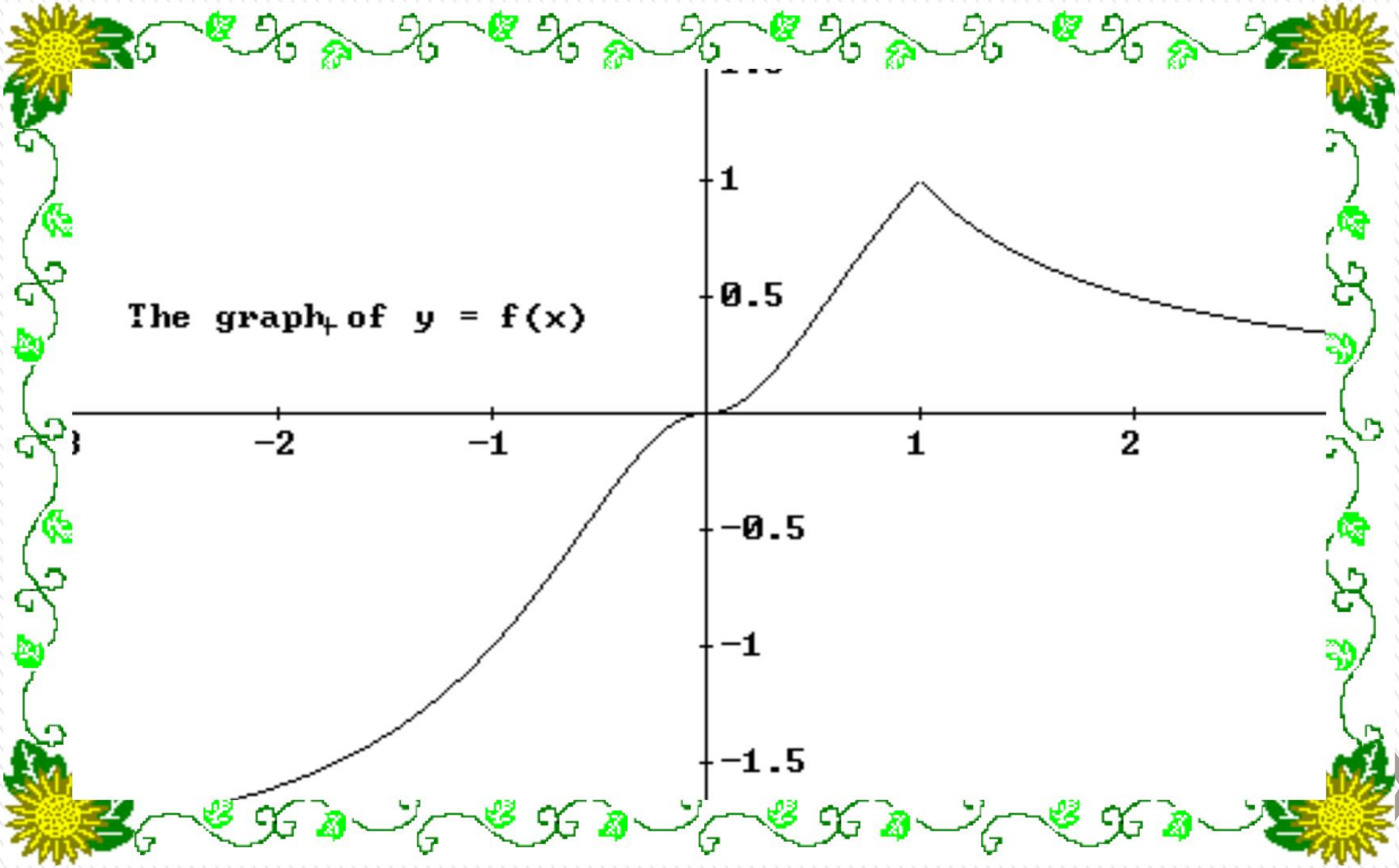
$(0, f(0)) = (0, 0),$

$$(\frac{1}{\sqrt{3}}, f(\frac{1}{\sqrt{3}})) = (\frac{1}{\sqrt{3}}, \frac{1}{2}))$$

and $(1, f(1)) = (1, 1)$.

Therefore, these are the points of inflection.

(f) *The graph of* f (not drawn to scale)



Question 6

(a) $g(x) = \int_{-x}^{x^3} \frac{1}{1+\sin^2(2t)+t^2}\,dt$

$$= \int_0^{x^3} \frac{1}{1+\sin^2(2t)+t^2}\,dt + \int_{-x}^{0} \frac{1}{1+\sin^2(2t)+t^2}\,dt$$

$$= \int_0^{x^3} \frac{1}{1+\sin^2(2t)+t^2}\,dt - \int_0^{-x} \frac{1}{1+\sin^2(2t)+t^2}\,dt$$

$$= F(x^3) - F(-x)$$

where $F(x) = \int_0^{x} \frac{1}{1+\sin^2(2t)+t^2}\,dt$

Therefore,

$$g'(x) = F'(x^3) \cdot 3x^2 - F'(-x) \cdot (-1)$$

by the *Chain Rule*

$$= \frac{3x^2}{1 + \sin^2(2x^3) + x^6} + \frac{1}{1 + \sin^2(-2x) + x^2}$$

by the FTC

(b) (i)

Since $k(x)=\int_1^x \frac{1}{\sqrt{1+t^4}}dt$, by the FTC,

$$k'(x)=\frac{1}{\sqrt{1+x^4}}>0 \text{ since } 1+x^4>0.$$

Therefore, k is increasing on the whole of $\mathbf{R}$.
Thus k is injective.

(ii) $(k^{-1})'(0) = \dfrac{1}{k'(k^{-1}(0))}$.

- Need to know the value of $k^{-1}(0)$.

$$k^{-1}(0) = x \Leftrightarrow k(x) = 0 \Leftrightarrow \int_1^x \frac{1}{\sqrt{1+t^4}}dt = 0$$

Since $k(1) = \int_1^1 \frac{1}{\sqrt{1+t^4}}dt = 0$ and

k is injective, $x = 1$.

Therefore,

$$(k^{-1})'(0) = \frac{1}{k'(k^{-1}(0))} = \frac{1}{k'(1)} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

(c) (i)

Let $f(x) = \int_a^x h(t)dt + \int_{h(a)}^{h(x)} h^{-1}(s)ds - xh(x)$

- We want to show that this is a constant function.
- *At this point, it is reasonable to make some assumption that allows us to proceed to show that this is true under this assumption. We then assume that h is differentiable. This is to make sure that the function f is differentiable.*
- Notice that f is continuous on $[a, b]$.

With this assumption, by the Fundamental Theorem of Calculus, f is indeed differentiable and

$$f'(x) = h(x) + h^{-1}(h(x))h'(x) - (h(x) + xh'(x))$$

$$= h(x) + x\,h'(x) - h(x) - x\,h'(x) = 0$$

Therefore,

$$f(x) = C \text{ for some constant } C.$$

Thus $C = f(a) = -a\,h(a)$.

Hence $\int_a^x h(t)dt = xh(x) - ah(a) - \int_{h(a)}^{h(x)} h^{-1}(s)ds$

In particular

$$\int_a^b h(t)dt = bh(b) - ah(a) - \int_{h(a)}^{h(b)} h^{-1}(s)ds$$

The solution to this part without assuming the differentiability of *h* is given at the end of the page

http://www.math.nus.edu.sg/~matngtb/Calculus/test_paper/99ex1.pdf

(ii)

Let $h(x) = \sqrt{1 + (x-1)^{\frac{1}{3}}}$ for x in [0,1].

$h(x) = y$ if and only if

$1 + (x-1)^{\frac{1}{3}} = y^2 \Leftrightarrow (x-1)^{\frac{1}{3}} = y^2 - 1$ so that

$x = (y^2-1)^3 + 1 = y^6 - 3y^4 + 3y^2$

Therefore $h^{-1}(y) = y^6 - 3y^4 + 3y^2$

Now $h(0) = 0$ and $h(1) = 1$.

- Before we use part (i), note that in part (i) we only require that h be differentiable on (a, b).

Hence by part (i),

$$\int_0^1 \sqrt{1+(x-1)^{\frac{1}{3}}}\,dx = h(1) - \int_0^1 (y^6 - 3y^4 + 3y^2)\,dy$$

$$= 1 - [\tfrac{1}{7}y^7 - \tfrac{3}{5}y^5 + y^3]_0^1 = 1 - (1 + \tfrac{1}{7} - \tfrac{3}{5}) = \tfrac{16}{35}.$$

Or use substitution $u = 1 + (x-1)^{\frac{1}{3}}.$

Then $x = u^3 - 3u^2 + 3u.$

$$\int_0^1 \sqrt{1 + (x-1)^{\frac{1}{3}}}\,dx = \int_0^1 (3u^{\frac{5}{2}} - 6u^{\frac{3}{2}} + 3u^{\frac{1}{2}})du$$

$$= 3\left[\tfrac{2}{7}u^{\frac{7}{2}} - \tfrac{4}{5}u^{\frac{5}{2}} + \tfrac{2}{3}u^{\frac{3}{2}}\right]_0^1$$

$$= \frac{6}{7} - \frac{12}{5} + 2 = \frac{16}{35}$$