A Formula of Euler and Appreciating Calculus

A presentation brought to you by

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Introduction

- Euler produced a staggering 900 treatises, books and papers.
- His collected works are still being published under the title Opera Omnia.
- His work has touched on almost every field of mathematics including some which were his creation.

Young Euler

- Born in Basel, Switzerland in 1707 AD.
- Finished college at the age of 15.
- Studied with Johann Bernoulli.
- 4 years later won a prize from the Parisian Academy of Sciences for the optimum placement of masts upon a sailing ship.

Depth

- Most of Euler's works are still as fresh as when he first created them
  - a testimony to the eternal nature of mathematics as absolute truth.
- Some of Euler's works are deep. Among them is Euler number for a surface or indeed for a combinatorial manifold. The invariance of Euler number was proved only when homology theory was invented.

Appreciation

- We shall describe a famous formula of Euler with a view to appreciate the beauty and nature of mathematics
  - in this case

Calculus.

The Formula is

$$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} + \dots = \frac{\pi^2}{6}$$

- This series had baffled Leibniz and the Bernoulli brothers. Euler gave this summation in 1734 AD.
- This series is still regarded too difficult to be included in most calculus text books.

The Connection

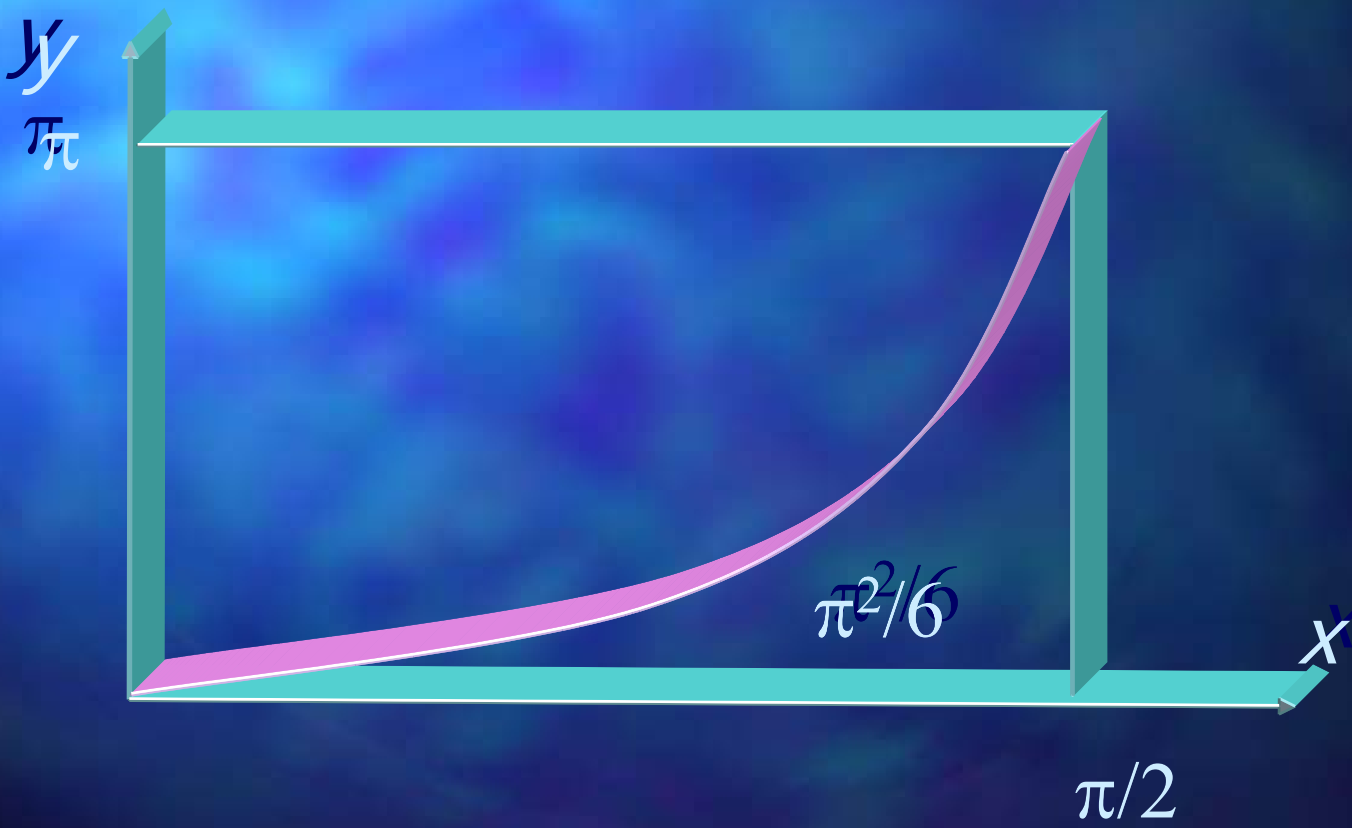
$$\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}+\ldots=\frac{4}{\pi}\int_0^{\frac{\pi}{2}} x^2 dx$$

This reminds us of

The Fundamental Theorem of Calculus.

CLUE?

- The graph of $y = 4x^2/\pi$



Vanishing Clue?

- The only clue or red herring is

$$\frac{4}{\pi}\int_0^{\frac{\pi}{2}} x^2\,dx = \frac{4}{\pi}\left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{6}$$

Inspiration?

- This will come from

$$I_N = \int_0^{\frac{\pi}{2}} \cos^{2N}(x)\,dx$$

- To evaluate this we shall need integration by parts:

$$\int_a^b F(x)G'(x)\,dx = \big[F(x)G(x)\big]_a^b - \int_a^b F'(x)G(x)\,dx$$

The Initial Computation

$$I_N = \int_0^{\frac{\pi}{2}} \cos^{2N-1}(x)\cos(x)\,dx$$

$$= \frac{2N-1}{2N}\int_0^{\frac{\pi}{2}} \cos^{2N-2}(x)\,dx$$

Done by integration by parts .

I_N

Repeated use of I_N yields

$$I_N = \frac{(2N-1)(2N-3)\cdots 1}{2N(2n-2)\cdots 2} I_0$$

$$= \frac{(2N-1)(2N-3)\cdots 1}{2N(2n-2)\cdots 2}\frac{\pi}{2}$$

$$= \frac{(2N-1)!!}{(2N)!!}\frac{\pi}{2}$$

The Other Way to I_N

$$I_N = \int_0^{\frac{\pi}{2}} 1 \bullet \cos^{2N}(x)\,dx$$

$$= N(2N-1)\int_0^{\frac{\pi}{2}} x^2 \cos^{2N-2}(x)\,dx$$

$$-2N^2 \int_0^{\frac{\pi}{2}} x^2 \cos^{2N}(x)\,dx$$

Then comes the realisation.....

$$N(2N-1)\int_0^{\frac{\pi}{2}} x^2 \cos^{2N-2}(x)\,dx$$

$$-2N^2\int_0^{\frac{\pi}{2}} x^2 \cos^{2N}(x)\,dx$$

$$\equiv \frac{(2N-1)!!}{(2N)!!}\frac{\pi}{2}$$

- Shoo! $1/N^2$ is somewhere here.

The Difference!

$$\frac{4}{\pi}\frac{(2N-2)!!}{(2N-3)!!}\int_0^{\frac{\pi}{2}} x^2\cos^{2N-2}(x)\,dx$$

$$-\frac{4}{\pi}\frac{(2N)!!}{(2N-1)!!}\int_0^{\frac{\pi}{2}} x^2\cos^{2N}(x)\,dx$$

$$=\frac{1}{N^2}$$

- This is true for $N > 1$.

Ah! We define

$$J_N = \frac{4}{\pi}\frac{(2N)!!}{(2N-1)!!}\int_0^{\frac{\pi}{2}} x^2 \cos^{2N}(x)\,dx$$

for $N \geq 1$ and

$$J_0 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x^2\,dx = \frac{\pi^2}{6}$$

- Then things look simpler!

The connection in place ....

For $N \geq 1$,

$$J_{N-1} - J_N = \frac{1}{N^2} \quad \text{-----} \quad (A)$$

- Ah! Ahhh! $J_0 = ?$

You guess it!

$$J_0 = (J_0 - J_1) + (J_1 - J_2) + \cdots + (J_{N-1} - J_N)$$
$$+ J_N$$
$$= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2} + J_N$$

---------(B)

- This we see is derived from (A).

Precision

- Equation (B) is not much good if J_N does not get closer and closer to 0 as N gets larger and larger.
- So we need to scrutinise J_N more closely.
- We want to show that

$$J_N \to 0 \text{ as } N \to \infty.$$

Getting acquainted with J_N

For $N \geq 1$,

$$0 \leq J_N = \frac{4}{\pi} \frac{(2N)!!}{(2N-1)!!} \int_0^{\frac{\pi}{2}} x^2 \cos^{2N}(x)\,dx$$

$$\leq \frac{8}{\pi} \int_0^{\frac{\pi}{2}} N x^2 \cos^{2N}(x)\,dx = \frac{8}{\pi} K_N \quad \text{----------- (C)}$$

- Like all inquisitive children, we should try to understand more about K_N.

Ah.... the limit we go to ....

- We hope to show that

$$K_N = \int_0^{\frac{\pi}{2}} N x^2 \cos^{2N}(x)\,dx \to 0$$

as $N \to \infty$.

- That is,

$$\lim_{N\to\infty} K_N = 0.$$

Getting to know K_N……..

- We shall look at the integrand of K_N. It is, for each integer $N > 0$,

$$g_N(x) = Nx^2 \cos^{2N}(x) \text{ and}$$

$$K_N = \int_0^{\frac{\pi}{2}} g_N(x)dx$$

Good old Fermat.........

- g_N is a function with domain the closed interval $[0, \pi/2]$.
- We should try to know a few things about g_N as a function.
  - g_N is continuous on $[0, \pi/2]$.
  - g_N is differentiable on $(0, \pi/2)$.
  - g_N is non-negative on $[0, \pi/2]$.
- What is its absolute maximum value?

Going to the peak ....

- The derivative of g_N for $N > 0$, on $(0, \pi/2)$,

$$g'_N : (0, \tfrac{\pi}{2}) \to \mathbf{R}$$

is given by

$$g'_N(x) = 2N^2 x \cos^{2N}(x)\left(\tfrac{1}{N} - x\tan(x)\right)$$

Locating the critical point

- There is only one critical point of g_N in $(0, \pi/2)$ given by

$$t_N \text{ in the interval } (0, \frac{\pi}{2})$$

defined by the equation

$$t_N \tan(t_N) = \frac{1}{N} \text{-----(D)}$$

The peak and a new function

- Since g_N is non-negative on $[0, \pi/2]$ and $g_N(0) = g_N(\pi/2)=0$,

  $g_N(t_N)$ is the absolute maximum value on the closed interval $[0, \pi/2]$.
- To track the absolute maximum, we shall look at the function involved in (D). We shall call it f .

“The hills are not insurmountable.”

For $N \geq 1$,

$$g_N(t_N) = N t_N^{\,2} \cos^{2N}(t_N)$$

$$= \frac{t_N}{\tan(t_n)} \cos^{2N}(t_N) \quad \text{by (D)}$$

$$\leq \cos^{2N}(t_N) \leq 1 \text{---(E)}$$

- The last inequality is deduced by using the mean value theorem.

Going back to (E)

- Since $t_N > 0$, and all conditions for the Mean Value Theorem are met, by the theorem, we have a point c in $(0, t_N)$ such that

$$\frac{\tan(t_N) - \tan(0)}{t_N} = \sec^2(c) \geq 1.$$

- This is what is required to proved (E).

Taking stock!

- Recall the function appearing in (D) is

$$f : [0, \tfrac{\pi}{2}) \to \mathbf{R}$$

given by $f(x) = x\tan(x)$.

- Then its derivative

$$f'(x) = \tan(x) + x\sec^2(x) > 0$$

$$\text{for } 0 < x < \tfrac{\pi}{2}.$$

Another look at t_N ....

- Thus f is increasing on $[0, \pi/2)$.
- Since

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} x \tan(x) = \infty,$$

a moment's reflection with the help of the Intermediate Value Theorem, will convince us, that f is a bijection onto the interval $[0, \infty)$.

Beginning to see the hill........

- For $0 < x < t_N$, we have

$$f(x) = x\tan(x) < f(t_N) = 1/N$$

since f is increasing.

- For x in $(0, \pi/2)$,

$$g'_N(x) > 0 \text{ if and only if } 1/N - f(x) > 0.$$

- Thus g_N is increasing on $[0, t_N]$ and
- g_N is decreasing on $[t_N, \pi/2]$

The inverse of f

- The inverse of f

$$f^{-1}:[0,\infty)\to[0,\tfrac{\pi}{2})$$

is a continuous, increasing function.

- From (D)

$$t_N = f^{-1}\left(\tfrac{1}{N}\right) > 0.$$

Diminishing t_N ....

- For positive integers N and M, if $N > M$, then $1/N < 1/M$ and since f^{-1} is increasing,

$$t_N = f^{-1}\left(\frac{1}{N}\right) < f^{-1}\left(\frac{1}{M}\right) = t_M .$$

- Thus, $\{ t_N : N$ is a natural number$\}$ is a decreasing sequence.

In praise of continuity.........

$$\lim_{N\to\infty} t_N = \lim_{N\to\infty} f^{-1}\left(\tfrac{1}{N}\right).$$

$$= f^{-1}\left(\lim_{N\to\infty}\tfrac{1}{N}\right) = f^{-1}(0) = 0$$

$$- - - - \; (F)$$

because f^{-1} is continuous at $x = 0$.

- Hey! The foot of the peak marches towards 0.

A little goes a long way........

Given any $\varepsilon > 0$,

there exists an integer N_0 such that

if $N \geq N_0$, then $0 < t_N < \frac{\varepsilon}{2}$

------------(G)

- Shoo, we are going to make use of t_N .

Returning to K_N ....

$$\text{For } N \ge N_0, K_N = \int_0^{\frac{\pi}{2}} g_N(x)dx$$

$$= \int_0^{t_{N_0}} g_N(x)dx + \int_{t_{N_0}}^{\frac{\pi}{2}} g_N(x)dx$$

$$\le t_{N_0} + \int_{t_{N_0}}^{\frac{\pi}{2}} g_N(x)dx$$

because of (E)

$$\le \frac{\varepsilon}{2} + \int_{t_{N_0}}^{\frac{\pi}{2}} g_N(x)dx \text{ - - - - (H)}$$

because of (G)

Still on K_N ……………

For $N \geq N_0, t_N \leq t_{N_0}$, and so

g_N is decreasing on $[t_{N_0}, \frac{\pi}{2}]$.

- Thus from (H) we get, for $N \geq N_0$,

$$K_N < \frac{\varepsilon}{2} + \int_{t_{N_0}}^{\frac{\pi}{2}} g_N(t_{N_0})dx$$

$$= \frac{\varepsilon}{2} + g_N(t_{N_0})(\frac{\pi}{2} - t_{N_0}) \text{ --- (I)}$$

Vanish we will, eventually……

- Take a number *a* > 1, then

$$\lim_{x\to\infty}\frac{x}{a^x} = \lim_{x\to\infty}\frac{1}{\ln(a)a^x}$$

by L' Hôpital's Rule

$$= 0 \text{ since } \lim_{x\to\infty} a^x = \infty.$$

- - - - - - - (J)

Still vanishing............

Now $0 < t_{N_0} < \frac{\pi}{2}$, and so $\cos^2(t_{N_0}) < 1.$

$$\lim_{N\to\infty} g_N(t_{N_0}) = \lim_{N\to\infty} t_{N_0}^2 N \cos^{2N}(t_{N_0})$$

$$= t_{N_0}^2 \lim_{N\to\infty} \frac{N}{(\sec^2(t_{N_0}))^N}$$

$$= t_{N_0}^2 \cdot 0 = 0 \text{-----(K)}$$

by (J).

Being precise is close to the truth

- (K) says for any $\varepsilon > 0$,

  there exists an integer M_0 such that

  if $N \geq M_0$, then

$$g_N(t_{N_0}) < \frac{\varepsilon}{2} \cdot \frac{1}{\left(\frac{\pi}{2} - t_{N_0}\right)}$$

-------------- (L)

Returning to K_N ....

- Thus from (I) and (L), for any $\varepsilon > 0$, if $N \geq \max(N_0, M_0)$, then

$$0 < K_N < \frac{\varepsilon}{2} + g_N(t_{N_0})\left(\frac{\pi}{2} - t_{N_0}\right)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \cdot \frac{1}{(\frac{\pi}{2} - t_{N_0})}\left(\frac{\pi}{2} - t_{N_0}\right) = \varepsilon.$$

Thus $\lim_{N \to \infty} K_N = 0.$

At long last...........

- Recall inequality (C):

$$\text{For } N \geq 1, 0 \leq J_N \leq \frac{8}{\pi} K_N.$$

Therefore, by the Squeeze Theorem,

$$\lim_{N\to\infty} J_N = 0.$$

The End

Fundamental Theorem of Calculus

$$\int_a^b f'(x)\,dx = \left[f(x)\right]_a^b$$

A Theorem of Beauty and Greatness

The Mean Value Theorem

If $h:[a,b]\to\mathbf{R}$ is

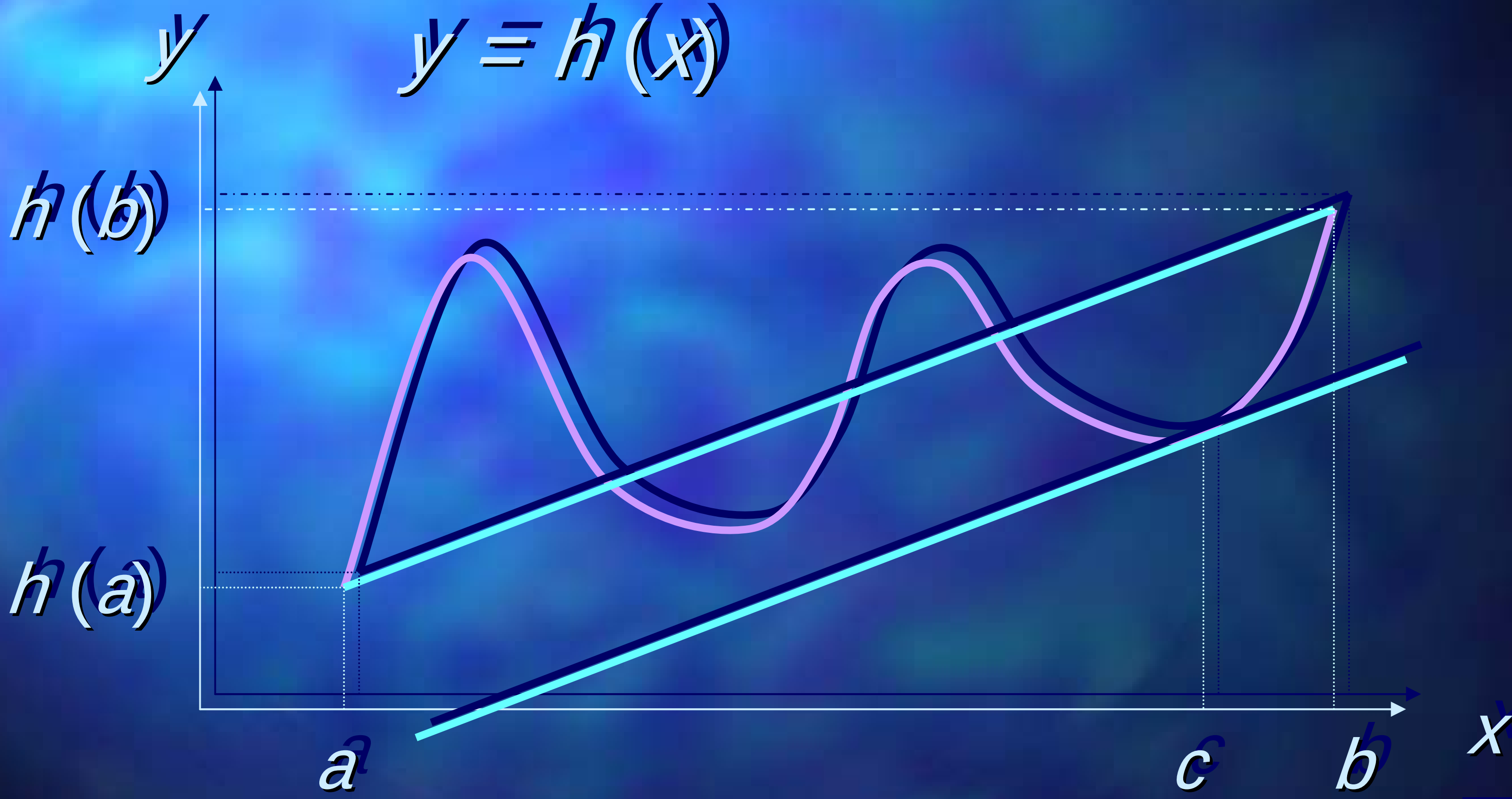
(1) continuous on $[a,b]$, and

(2) differentiable on (a,b), then

there is a point c in (a,b) such that

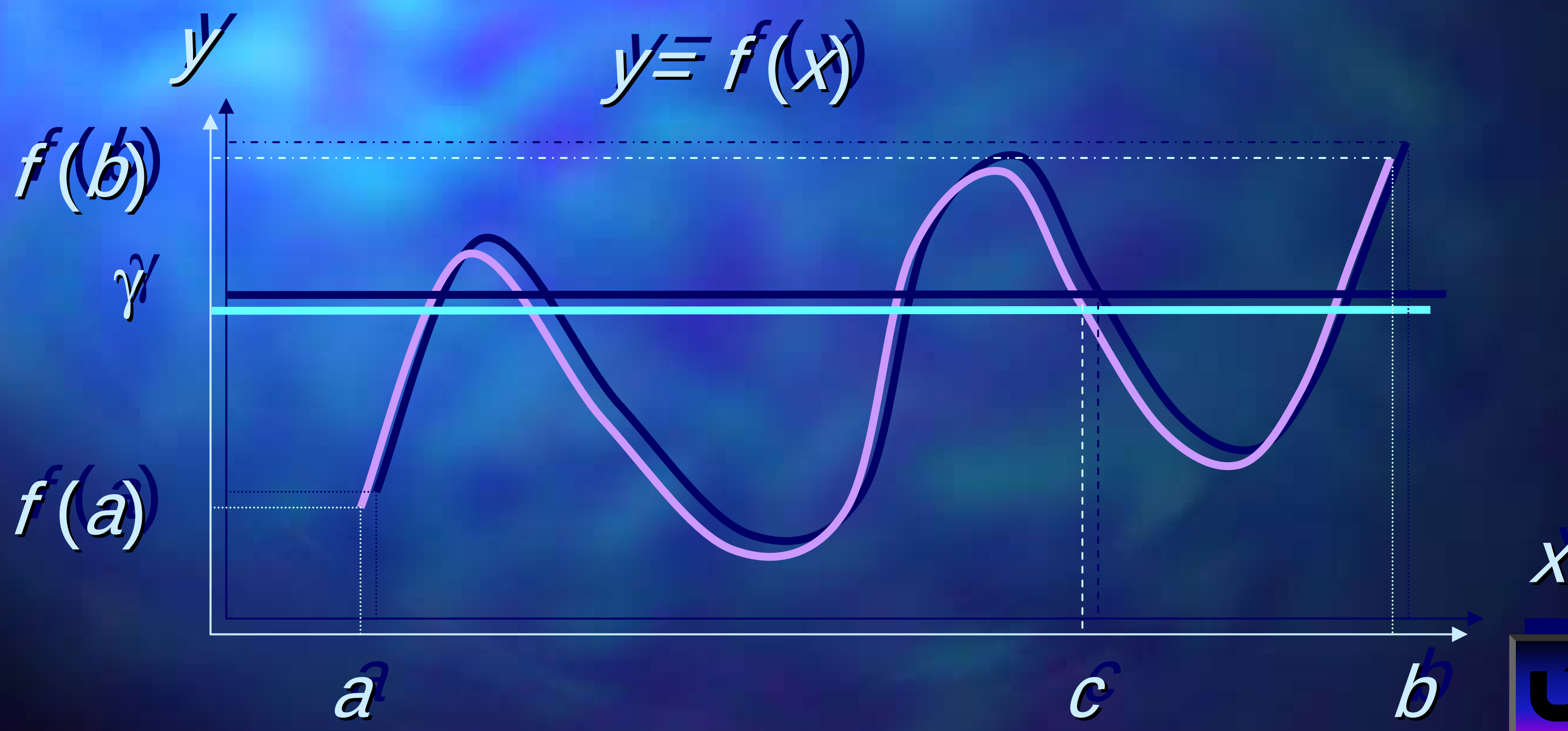
$$h'(c)=\frac{h(b)-h(a)}{b-a}.$$

$$h'(c) = \frac{h(b) - h(a)}{b - a}$$



Intermediate Value Theorem

If $f:[a,b] \to \mathbf{R}$ is continuous on the close interval $[a,b]$, then for any value γ between $f(a)$ and $f(b)$, there is a point c in $[a,b]$ such that $f(c)=\gamma$



Continuity of *f* at the point *a*