Derivative and Derived Functions
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Comment on the use of the following result and related form.

It is often the case when a function is given as a piecewise defined function and the function in each piece of the domain has known derivative, students have a tendency to find the derivative by taking limits of the known derivative of each piece in some appropriate manner. This note is to show that some time this does give the right derivative at the point, where the function on the left and right of it are given by distinct differentiable functions.

Proposition 1. If f is a function defined by $f(x)=\begin{cases} g(x), x\le a \\ h(x), x>a \end{cases}$, where g and h are differentiable functions with $g(a) = h(a)$. Then f is differentiable at $x = a$ if and only if $g'(a) = h'(a)$.

Proof. Suppose f is differentiable at $x = a$. Then

$$\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}.$$

Hence, because

$$\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a^-}\frac{g(x)-g(a)}{x-a}=g'(a),$$ since g is differentiable at $x = a$ and

$$\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a^+}\frac{h(x)-h(a)}{x-a}=h'(a),$$ since h is differentiable at $x = a$,

$g'(a) = h'(a)$.

Conversely, suppose $g'(a) = h'(a)$.

Then because

$$\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a^-}\frac{g(x)-g(a)}{x-a}=g'(a) \text{ and } \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a^+}\frac{h(x)-h(a)}{x-a}=h'(a)$$

$$\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}.$$ Therefore, f is differentiable at $x = a$.

Note that we only require that g be left differentiable at $x = a$ and h be right differentiable at $x = a$. *We can replace* $g'(a) = h'(a)$ by the left derivativeof g at a = right derivative of h at a.

The result is used knowingly or unknowingly in the following form as follows:

(1) Since g and h are differentiable,

if $g'(a)\neq h'(a)$, then f is not differentiable at $x = a$.

To use this, you will have to state clearly that your function f is defined as in Proposition 1. That is, $f(x)=\begin{cases} g(x), x\le a \\ h(x), x>a \end{cases}$, where g and h are differentiable functions with $g(a) = h(a)$.

(i) For instance if $f(x)=\begin{cases} x^3-2x+1, x<0 \\ 1, x=0, \\ 1-12x+6x^2, x>0 \end{cases}$, since the derivative of x^3-2x+1 at $x = 0$ is -2 and the derivative of $1-12x+6x^2$ at $x = 0$ is -12 and are thus not the same, f is not differentiable at $x = 0$.

(ii) If $f(x)=\begin{cases} x^3-6x+1, x>0 \\ 1, x=0, \\ \sqrt[3]{x+1}\ , x<0 \end{cases}$, since the derivative of x^3-6x+1 at $x=0$ is -6 and the derivative of $\sqrt[3]{x+1}$ at $x=0$ is 1/3 and are thus not the same, f is not differentiable at $x=0$.

(iii) If $f(x)=\begin{cases} \frac{1}{\sqrt{1-x}}, x<0 \\ 1, x=0, \\ 1-12x+6x^2-x^3, x>0 \end{cases}$, since the derivative of $\frac{1}{\sqrt{1-x}}$ at $x=0$ is 1/2 and the derivative of $1-12x+6x^2-x^3$ at $x=0$ is -12 and are thus not the same, f is not differentiable at $x=0$.

Comment: I would advise using this result cautiously since you need to know very clearly when you can use it. Most students use this unaware of the condition in the proposition and may get confused. **Use the difference quotient as in the sample solutions for test 2. Although it is a little longer but you are actually working directly with the definition of the derivative and limit.**

Proposition 2. Suppose f is a function defined by $f(x)=\begin{cases} g(x), x<a \\ c, x=a \\ h(x), x>a \end{cases}$,where $\lim\limits_{x\to a^-} g(x)=\lim\limits_{x\to a^+} h(x)=c$, g is differentiable on $(-\infty, a)$ and h is differentiable on (a,∞).

(1) If $\lim\limits_{x\to a^-} g'(x)$ and $\lim\limits_{x\to a^+} h'(x)$ exist and $\lim\limits_{x\to a^-} g'(x)=\lim\limits_{x\to a^+} h'(x)=l$, then f is differentiable at $x=a$ and $f'(a)=l$.

(2) If $\lim\limits_{x\to a^-} g'(x)$ and $\lim\limits_{x\to a^+} h'(x)$ exist and $\lim\limits_{x\to a^-} g'(x)\neq\lim\limits_{x\to a^+} h'(x)$, then f is NOT differentiable at $x=a$.

Proof. $\lim\limits_{x\to a^-} g'(x)=L$ means that given any $\varepsilon>0$, there exist $\delta>0$ such that

$$a-\delta<x<a \text{ implies that } |g'(x)-L|<\varepsilon. \text{ ----- (1)}$$

Let $G(x)=g(x)$ for $x<a$ and $G(a)=c$. Then G is continuous on $(-\infty, a]$ and differentiable on $(-\infty, a)$ and $G'(x)=g'(x)$ for $x<a$. Let $H(x)=h(x)$ for $x>a$ and $H(a)=c$. Then H is continuous on $[a,\infty)$ and differentiable on (a,∞) and $H'(x)=h'(x)$ for $x>a$. Therefore $f(x):=\begin{cases} G(x), x\leq a \\ H(x), x>a \end{cases}$. Now for any x such that $a-\delta<x<a$, by the Mean Value Theorem, there exists y such that $x<y<a$ with $\dfrac{G(x)-G(a)}{x-a}=G'(y)=g'(y)$.

Therefore, $\left|\dfrac{G(x)-G(a)}{x-a}-L\right|=|g'(y)-L|<\varepsilon$ by (1). This means the left derivative of G at $x=a$ is L, i.e.,

$$\lim_{x\to a^-}\frac{G(x)-G(a)}{x-a}=L.$$

Similarly, $\lim\limits_{x\to a^+} h'(x)=l$ means that given any $\varepsilon>0$, there exist $\delta>0$ such that

$$a<x<a+\delta \text{ implies that } |h'(x)-l|<\varepsilon. \text{ ----- (2)}$$

Now for any x such that $a<x<a+\delta$, by the Mean Value Theorem, there exists y such that

$a < y < x$ with

$$\frac{H(x) - H(a)}{x - a} = H'(y) = h'(y).$$

Therefore, $\left|\frac{H(x) - H(a)}{x - a} - l\right| = |h'(y) - l| < \varepsilon$ by (2). This means the right derivative of H at $x = a$ is l, i.e.,

$$\lim_{x \to a^+} \frac{H(x) - H(a)}{x - a} = l.$$

(1) If $L = l$, by Proposition 1, f is differentiable at $x = a$ and $f'(a) = l$.
(2) If $L \neq l$, by Proposition 1, f is not differentiable at $x = a$.

Comment: Proposition 2 Part (2) tends to be used incorrectly. The condition " $\lim_{x \to a^-} g'(x)$ and $\lim_{x \to a^+} h'(x)$ exist " must be checked first and foremost.

For instance the function $f(x) := \begin{cases} x^2 \sin(\frac{1}{x}), x > 0 \\ x^3, x \leq 0 \end{cases}$ is differentiable at $x = 0$, although $\lim_{x \to 0^-} g'(x)$ exists where $g(x) = x^3$ and the limit $\lim_{x \to 0^+} h'(x)$ does not exist where $h(x) = x^2 \sin(\frac{1}{x})$, for $x > 0$.

Advise using Proposition 2 cautiously too since you are not directly working with the definition of the derivative.